

Determination of functions by metric slopes

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Abstract. We show that in a metric space, any continuous function with compact sublevel sets and finite metric slope is uniquely determined by the slope and its critical values.

Key words Metric slope, critical value, determination of a function.

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1 Introduction

Let (M, d) be a metric space. The *metric slope* (also known as strong slope) of a function $f : M \rightarrow \mathbb{R}$ at a point $x \in M$ is given by

$$|\nabla f|(x) = \begin{cases} 0, & \text{if } x \text{ is an isolated point,} \\ \limsup_{y \rightarrow x} \frac{(f(x) - f(y))^+}{d(x, y)}, & \text{otherwise,} \end{cases} \quad (1)$$

where $(f(x) - f(y))^+ = \max\{f(x) - f(y), 0\}$ is the standard asymmetric norm in \mathbb{R} . This notion was first introduced by De Giorgi, Marino and Tosques in [4] to extend steepest descent curves to metric spaces. Since then, the metric slope has been popularized to study descent curves in abstract settings: see the monograph [1] and references therein, or the more recent works [5, 11]. The metric slope has also been widely used in nonsmooth analysis in relation with concepts like error bounds ([2] *e.g.*) or metric regularity and subdifferential calculus (see *e.g.* monograph [9] and references therein).

One of the main properties that makes the metric slope a suitable generalization from vector spaces to metric spaces is that it coincides with the norm of steepest descent vectors in both, the convex and the smooth settings. Specifically,

- Whenever X is a normed space and $f : X \rightarrow \mathbb{R}$ is a differentiable function at $x \in X$, one has that $|\nabla f|(x) = \|\nabla f(x)\|$ (see, *e.g.* [9, Chapter 3.1.2]).
- Whenever X is a Banach space and $f : X \rightarrow \mathbb{R}$ is a continuous convex function, one has that

$$|\nabla f|(x) = \|\partial f(x)^\circ\|, \quad \forall x \in X,$$

where $\partial f(x)^\circ$ is the element of minimal norm of the Moreau-Rockafellar subdifferential of f at x (see, *e.g.*, [6, Theorem 5]).

Recently, an ostensibly unexpected result was discovered: In Banach spaces, the metric slope contains sufficient information to determine lower semicontinuous convex functions that are bounded from below. This determination theorem was first established for convex functions of class \mathcal{C}^2 in [3, Theorem 3.8], then for convex functions of class \mathcal{C}^1 for which (global) minima

exist¹ and it has eventually been extended to Hilbert spaces in [12, Corollary 3.2] and to Banach spaces in [13, Theorem 5.1]. The main result can be summarized as follows:

Theorem 1.1 (slope determination – convex case). *Let X be a Banach space and $f, g : X \rightarrow \mathbb{R}$ be two convex continuous and bounded from below functions such that*

$$|\nabla f|(x) = |\nabla g|(x), \quad \forall x \in X.$$

Then there exists $c \in \mathbb{R}$ such that $f = g + c$.

The main strategy to obtain the above determination result when X is a Hilbert space was to control the difference of the values of the convex functions f, g at an arbitrary point x by the limiting value $\inf f - \inf g$ by means of a suitable steepest descent curve issued from x . The argument uses the fact that the only possible critical value of a convex function is its infimum together with the existence of subgradient descent curves issued from any point of the space.

The extension of Theorem 1.1 to Banach spaces has instead been obtained through an abstract determination result formulated in metric spaces and based on the notion of *global slope* (see, e.g., [1, Definition 1.2.4]). Concretely, it has been shown in [13, Corollary 4.1] that two continuous, bounded from below functions are equal up to a constant, provided they have the same (finite) global slope at every point. The notion of global slope is difficult to handle (it is based on global information, instead of local), but for convex functions in Banach spaces it does coincide with the (local notion of) metric slope and as a consequence Theorem 1.1 follows.

Focusing on the metric slope, our goal in this work is to provide minimal sufficient conditions, beyond the convex case, under which for two functions $f, g : M \rightarrow \mathbb{R}$ over a metric space M the following implication holds:

$$|\nabla f|(x) = |\nabla g|(x), \quad \forall x \in M \implies f = g \text{ up to an additive constant ?} \quad (2)$$

Although the aforementioned strategies fail in the nonconvex case even for bounded, (globally) Lipschitz real-analytic functions defined in \mathbb{R} (see Example 3.1), in this work we show that neither convexity nor the lineal structure of the ambient space are necessary conditions for establishing such a determination result of local nature. The main idea is to replace the steepest descent curves, used in [12], by discrete descent paths, constructed via transfinite induction over the ordinals, that lead any point of the space towards a critical point for the slope. These latter points serve as points of comparison for the values of the functions f and g . To this end, we consider functions for which the construction of such discrete paths is possible and the existence of critical points is ensured: these requirements are fulfilled by the class of continuous functions with finite slope and compact sublevel sets. Quite notably, the topology τ on the space M under which the function is continuous and the sublevel sets are compact does not necessarily have to be its metric topology: it can be stronger, weaker or even not comparable.

Let us now fix our setting and terminology. From now on, we denote by M a metric space endowed with a distance function d . For a function $f : M \rightarrow \mathbb{R}$ over M , we denote by $[f \leq \alpha]$ its sublevel set at the value $\alpha \in \mathbb{R}$, that is,

$$[f \leq \alpha] = \{x \in M : f(x) \leq \alpha\}.$$

¹J.-B. Baillon, Personal communication in 2018

We recall by (1) the definition of the metric slope $|\nabla f|$ and we denote by $\text{Crit}(f)$ the set of (metric) critical points of f with respect to the slope, that is,

$$\text{Crit}(f) = \{x \in M : |\nabla f|(x) = 0\}.$$

Let further τ be any topology on M (which may or may not be its metric topology). For instance, M can be a Banach space and τ its weak-topology.

Definition 1.2 (τ -coercive function). A function $f : M \rightarrow \mathbb{R}$ is called τ -coercive if the sublevel sets $[f \leq \alpha]$ are τ -compact for all $\alpha < \sup_{x \in M} f(x)$.

Notice that τ -coercivity ensures the existence of global minimizers, and therefore of metric critical points. If the whole space M is itself τ -compact, then every τ -lower semicontinuous function is τ -coercive.

2 Main results

In this section we show that in any metric space (M, d) , any τ -continuous τ -coercive function (for some topology τ on M) can be determined by its slope (provided it is everywhere finite) and its critical values.

Lemma 2.1 (key lemma). *Let $f, g : M \rightarrow \mathbb{R}$ be two real-valued functions such that*

$$|\nabla f|(x) > |\nabla g|(x), \text{ for every } x \in M \setminus \text{Crit}(f).$$

Then, for every $x \in M \setminus \text{Crit}(f)$, there exists $z \in M$ such that

$$(f - g)(x) > (f - g)(z) \quad \text{and} \quad f(x) > f(z).$$

Proof. Let $x \in M \setminus \text{Crit}(f)$ and pick $\varepsilon > 0$ such that $|\nabla f|(x) > |\nabla g|(x) + 2\varepsilon$. Then $|\nabla f|(x) > 0$ (therefore, x is not an isolated point in M) and $|\nabla g|(x) < +\infty$. By (1) there exists $\delta > 0$ such that

$$\sup_{y \in B(x, \delta)} \frac{(g(x) - g(y))^+}{d(x, y)} < |\nabla g|(x) + \varepsilon. \quad (3)$$

If $|\nabla f|(x) < +\infty$ there exists $z \in B(x, \delta)$ such that

$$\frac{(f(x) - f(z))^+}{d(x, z)} > |\nabla f|(x) - \varepsilon.$$

Since $|\nabla f|(x) - \varepsilon > |\nabla g|(x) + \varepsilon > 0$, it follows that

$$f(x) - f(z) = (f(x) - f(z))^+ > d(x, z) (|\nabla g|(x) + \varepsilon).$$

The above inequality holds true also if $|\nabla f|(x) = +\infty$ and yields $f(x) - f(z) > 0$. Combining with (3) we obtain

$$f(x) - f(z) > d(x, z) (|\nabla g|(x) + \varepsilon) > (g(x) - g(z))^+ \geq g(x) - g(z)$$

and the statement of the lemma follows. \square

Proposition 2.2 (strict comparison). *Let $f, g : M \rightarrow \mathbb{R}$ be τ -continuous, τ -coercive functions (for some topology τ in M) and*

$$|\nabla f|(x) > |\nabla g|(x), \text{ for every } x \in M \setminus \text{Crit}(f).$$

Then, $\text{Crit}(f) \neq \emptyset$ and for every $x \in M \setminus \text{Crit}(f)$

$$f(x) - g(x) > m(x) := \inf_{z \in [f \leq f(x)] \cap \text{Crit}(f)} (f - g)(z). \quad (4)$$

Proof. Apply Lemma 2.1 to obtain $x_0 \in M$ such that $(f - g)(x) > (f - g)(x_0)$ and $f(x) > f(x_0)$. If $x_0 \in \text{Crit}(f)$, then $(f - g)(x) > (f - g)(x_0) \geq m(x)$ and (4) holds. If $x_0 \in M \setminus \text{Crit}(f)$, then we can apply again Lemma 2.1 and obtain $x_1 \in M$ such that $(f - g)(x_0) > (f - g)(x_1)$ and $f(x_0) > f(x_1)$. As long as we do not meet a critical point of f we build a generalized sequence $\{x_\alpha\}_\alpha \subset [f \leq f(x_0)]$ over the ordinals as follows: for every ordinal λ for which $\{x_\alpha\}_{\alpha < \lambda} \subset M \setminus \text{Crit}(f)$ is defined:

- (i). If $\lambda = \beta + 1$ is a successor ordinal and $x_\beta \in M \setminus \text{Crit}(f)$ has been defined, then we apply Lemma 2.1 for $x = x_\beta$ and set $x_{\beta+1} := z$. Then

$$(f - g)(x_\beta) > (f - g)(x_{\beta+1}) \quad \text{and} \quad f(x_\beta) > f(x_{\beta+1}). \quad (5)$$

- (ii). Assume now that λ is a limit ordinal and $\{x_\beta\}_{\beta < \lambda} \subset [f \leq f(x_0)]$ has been defined such that (5) holds true. Since $[f \leq f(x_0)]$ is τ -compact, the set

$$A := \bigcap_{\beta < \lambda} \overline{\{x_\alpha : \beta \leq \alpha < \lambda\}}^\tau$$

is nonempty. Pick any $x_\lambda \in A$. Then, for every $\beta < \lambda$, by τ -continuity of the function $f - g$ we have

$$\begin{aligned} (f - g)(x_\beta) &= \sup \left\{ (f - g)(x) : x \in \overline{\{x_\alpha : \beta \leq \alpha < \lambda\}}^\tau \right\} \\ &\geq (f - g)(x_\lambda). \end{aligned}$$

The above construction will necessarily end up to a critical point of f , that is, we eventually obtain $x_\lambda \in \text{Crit}(f)$ for some ordinal λ . Indeed, if this does not happen, then reaching ordinals of arbitrary cardinality (in particular, bigger than the cardinality $|M|$ of the set M) we deduce the existence of two ordinals $\beta < \alpha$ for which $x_\beta = x_\alpha$. Then our construction yields

$$(f - g)(x_\beta) > (f - g)(x_{\beta+1}) \geq (f - g)(x_\alpha),$$

which is obviously a contradiction. Therefore, $\text{Crit}(f) \neq \emptyset$. Moreover, $\{x_\alpha\}_\alpha \subset [f < f(x_0)]$ and for every x_α we have

$$(f - g)(x) > (f - g)(x_\alpha) \geq \inf_{z \in [f \leq f(x_\alpha)] \cap \text{Crit}(f)} (f - g)(z).$$

The proof is complete. □

Proposition 2.3 (Comparison Principle). *Let $f, g : M \rightarrow \mathbb{R}$ be τ -continuous, τ -coercive functions. If*

- (i) $|\nabla g|(x) \leq |\nabla f|(x) < +\infty$ for all $x \in M$, and
- (ii) *There exists $c \in \mathbb{R}$ such that $g(x) - f(x) \leq c$ for all $x \in \text{Crit}(f)$,*

then $g \leq f + c$.

Proof. The conclusion is trivial if $\text{Crit}(f) = M$. If $\text{Crit}(f) \neq M$, set $f_\varepsilon = (1+\varepsilon)f$ and notice that f_ε is τ -continuous and τ -coercive. Then, for every $x \in M$ we know that $|\nabla f_\varepsilon|(x) = (1+\varepsilon)|\nabla f|(x)$ (see, e.g. [9, Proposition 3.3]). Thus, we have that $\text{Crit}(f_\varepsilon) = \text{Crit}(f)$ and $|\nabla g|(x) = |\nabla f|(x) < |\nabla f_\varepsilon|(x)$ for every $x \in M \setminus \text{Crit}(f_\varepsilon)$. By Proposition 2.2, we have

$$\begin{aligned} g(x) &< f_\varepsilon(x) - \inf_{z \in [f_\varepsilon \leq f_\varepsilon(x)] \cap \text{Crit}(f)} (f_\varepsilon - g)(z) \\ &\leq f_\varepsilon(x) + c - \inf_{z \in [f_\varepsilon \leq f_\varepsilon(x)] \cap \text{Crit}(f)} \varepsilon f(z) \\ &= f(x) + \varepsilon \left[\underbrace{f(x) - \inf_{z \in [f_\varepsilon \leq f_\varepsilon(x)] \cap \text{Crit}(f)} f(z)}_{\text{uniformly bounded for all } \varepsilon} \right] + c. \end{aligned}$$

By taking $\varepsilon \searrow 0$, we conclude that $g(x) \leq f(x) + c$. The proof is complete. \square

By applying twice Proposition 2.3, we can deduce the main result of this work.

Theorem 2.4 (Determination Theorem). *Let M be a metric space and τ be a topology over M . Let $f, g : M \rightarrow \mathbb{R}$ be two τ -continuous and τ -coercive functions such that*

- (i) $|\nabla f|(x) = |\nabla g|(x) < +\infty$ for every $x \in M$.
- (ii) *There exists $c \in \mathbb{R}$ such that $g(x) - f(x) = c$ for all $x \in \text{Crit}(f)$.*

Then, $g = f + c$.

A direct albeit important corollary is that, whenever M is a compact metric space, the determination theorem applies for continuous functions with finite slope, by taking τ to be the topology defined by the metric d .

Corollary 2.5. *Let M be a compact metric space and let $f, g : M \rightarrow \mathbb{R}$ be two continuous functions with finite slope such that*

- (i) $|\nabla f|(x) = |\nabla g|(x)$ for all $x \in M$.
- (ii) *There exists $c \in \mathbb{R}$ such that $g(x) - f(x) = c$ for all $x \in \text{Crit}(f)$.*

Then, $g = f + c$.

The distinction between the metric d for which the slope is being computed, and the topology τ over which continuity and coercivity are considered, yields the following corollary over (possibly infinite-dimensional) reflexive Banach spaces.

Corollary 2.6. *Let X be a reflexive Banach space and let $f, g : X \rightarrow \mathbb{R}$ be two weak-continuous and weak-coercive functions such that*

- (i) $|\nabla f|(x) < +\infty$ for every $x \in M$.
- (ii) $|\nabla f|(x) = |\nabla g|(x)$ for all $x \in M$.
- (iii) There exists $c \in \mathbb{R}$ such that $g(x) - f(x) = c$ for all $x \in \text{Crit}(f)$.

Then, $g = f + c$.

3 Pertinence of the assumptions and illustrative examples

The slope, by its own, does not provide enough information to verify (2) for arbitrary continuous functions, unless additional assumptions are imposed. In this section we illustrate the relevance of our assumptions by means of counterexamples.

Let us first observe that even in the framework of Theorem 1.1 (convex case), the assumption that the convex functions are bounded from below is pertinent: indeed, without this assumption, one could obtain an effortless counterexample by considering two different linear functionals (elements of the dual space) of the same norm. The simplest concrete example is to consider the 1-dimensional case $X = \mathbb{R}$ and the (linear) functions $f(t) = t$ and $g(t) = -t$ for all $t \in \mathbb{R}$.

Concurrently, the following example shows that mere boundedness from below without convexity is not enough, even for real-analytic, bounded functions. This example illustrates, in case of absence of convexity, the need to assume existence of global minima (or at least, of critical points). We recall that in the framework of Theorem 2.4, this is ensured by the coercivity assumption.

Example 3.1. Let us consider the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t) = \arctan(t) \quad \text{and} \quad g(t) = -\arctan(t)$$

Since both functions are continuously differentiable, we have that

$$|\nabla f|(t) = |f'(t)| = \frac{1}{t^2 + 1} = |g'(t)| = |\nabla g|(t).$$

Nevertheless, the desired implication (2) does not hold (see Figure 1). □

The following example shows that for (nonconvex) smooth coercive functions in \mathbb{R} , implication (2) might fail if we lack information on the behavior of the functions on the set of critical values.

Example 3.2. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} (x + \pi/2)^2 - 1, & \text{if } x < -\pi/2 \\ \sin(x), & \text{if } x \in [-\pi/2, \pi/2] \\ (x - \pi/2)^2 + 1, & \text{if } x > \pi/2 \end{cases} \quad (6)$$

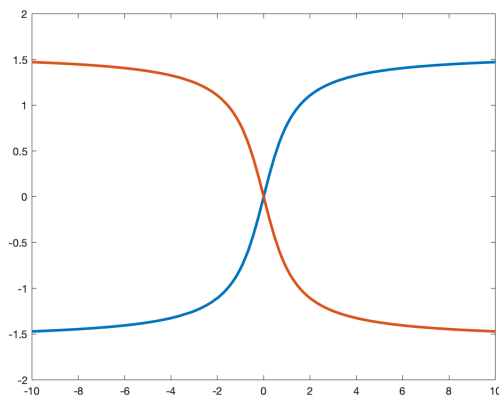


Figure 1: The functions $f(t) = \arctan(t)$ (in blue) and $g(t) = -\arctan(t)$ (in red).

It is not hard to see that f is of class \mathcal{C}^1 and coercive, with

$$f'(x) = \begin{cases} 2(x + \frac{\pi}{2}), & \text{if } x < -\frac{\pi}{2} \\ \cos(x), & \text{if } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 2(x - \frac{\pi}{2}), & \text{if } x > \frac{\pi}{2} \end{cases} \quad \text{and} \quad |\nabla f|(x) = \begin{cases} \cos(x), & \text{if } x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 2(|x| - \frac{\pi}{2}), & \text{if } x \notin [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

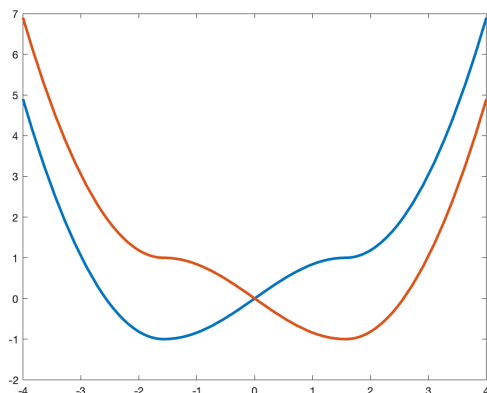


Figure 2: The functions $f(x)$ given by (6) (in blue) and $g(x) = f(-x)$ (in red).

By considering $g : \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = f(-x)$, we get that $|\nabla f|(x) = |\nabla g|(x)$ for every $x \in \mathbb{R}$, but (2) fails, as illustrates Figure 2. \square

The last example illustrates the problem of allowing the slope to take the value $+\infty$ quite often. In this example, the functions are continuous, have compact sublevel sets and the same critical values, yet (2) fails.

Example 3.3. Let us consider the classical Cantor Staircase function $\mathfrak{c} : [0, 1] \rightarrow [0, 1]$, which has directional derivatives equal to either 0 or $+\infty$.

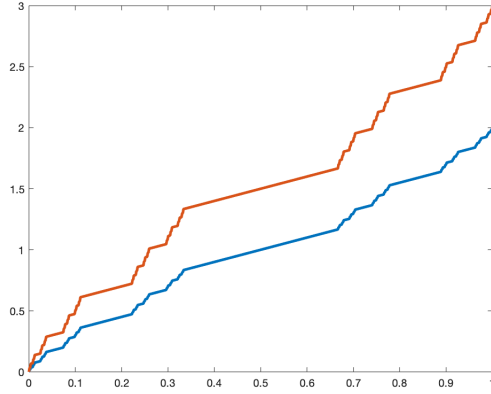


Figure 3: The functions $f(t) = \mathbf{c}(t) + t$ (in blue) and $g(t) = 2\mathbf{c}(t) + t$ (in red).

Define $f, g : [0, 1] \rightarrow \mathbb{R}$ by

$$f(t) = \mathbf{c}(t) + t \quad \text{and} \quad g(t) = 2\mathbf{c}(t) + t.$$

Both functions are continuous and coercive and it is not hard to see that

$$|\nabla f|(t) = |\nabla g|(t) \in \{1, +\infty\}, \quad \text{for all } t \in (0, 1]$$

and

$$|\nabla f|(0) = |\nabla g|(0) = 0.$$

Therefore, $\text{Crit}(f) = \text{Crit}(g) = \{0\}$ and $f(0) = g(0) = 0$. Nevertheless, the determination theorem (2) fails to hold, precisely because the slope is not finite everywhere. \square

Final conclusion. In this work, we established a determination result for functions in a general framework, based on three complementary hypotheses: Finite slope (at each point), continuity and coercivity for a suitable topology and knowledge of the critical values. Our main result shows that in this case the metric slope contains sufficient first-order information to determine the function. The first hypothesis seems to be necessary, since if the slope can take infinite values, one cannot control the variation of the functions over those points, as Example 3.3 illustrates. The third hypothesis acts as a boundary condition: since the slope does not provide information over the set of critical points, in order to verify (2), it is necessary to impose that the functions are equal (up to an additive constant) over this set, which is nonempty thanks to the second assumption. Nonwithstanding, it seems there is room for improvement (weakening) of this second assumption. A perspective of this work is to provide an alternative condition over spaces where compactness is not present, like general Banach spaces.

Last but not least, as pointed out by the referee, Theorem 2.4 and Corollary 2.5 can be reformulated as uniqueness results for the metric eikonal equation $|\nabla u| = \psi$ under a Dirichlet boundary condition (data on the critical set S). This relates to recent works on eikonal equation and Hamilton–Jacobi equations on metric spaces, see [7, 8, 10] and can possibly inspire new connections and ideas. In particular, the comparison principle and uniqueness result in the aforementioned works can be compared with the main result herein.

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