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Hölder Regularity in Bang-Bang Type Affine Optimal Control Problems*

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Abstract

This paper revisits the issue of Hölder Strong Metric sub-Regularity (HSMs-R) of the optimality system associated with ODE optimal control problems that are affine with respect to the control. The main contributions are as follows. First, the metric in the control space, introduced in this paper, differs from the ones used so far in the literature in that it allows to take into consideration the bang-bang structure of the optimal control functions. This is especially important in the analysis of Model Predictive Control algorithms. Second, the obtained sufficient conditions for HSMs-R extend the known ones in a way which makes them applicable to some problems which are non-linear in the state variable and the Hölder exponent is smaller than one (that is, the regularity is not Lipschitz).

Keywords :optimal control, affine problems, Hölder metric sub-regularity
AMS Clasification: 49J40, 49J53

1 Introduction

Consider the following affine optimal control problem

$$\min_{u\in\mathcal{U}}\left\{l(x(T))+\int_0^T\Big[w\big(t,x(t)\big)+\langle s\big(t,x(t)\big),u(t)\rangle\Big]\,dt\right\}, \tag{1}$$

subject to

$$\dot{x}(t)=a\big(t,x(t)\big)+B\big(t,x(t)\big)u(t),\quad x(0)=x_0. \tag{2}$$

Here the state vector $x(t)$ belongs to $\mathbb{R}^n$ and the control function belongs to the set $\mathcal{U}$ of all Lebesgue measurable functions $u:[0,T]\to U$, where $U\subset\mathbb{R}^m$. Correspondingly, $l:\mathbb{R}^n\to\mathbb{R}$ and $w:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are real-valued functions, $s:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ and $a:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ are

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vector-valued functions, and B is an $(n \times m)$- matrix-valued function taking values in $\mathbb{R} \times \mathbb{R}^n$. The initial state $x_0 \in \mathbb{R}^n$ and the final time $T > 0$ are fixed.

We make the following basic assumption.

Assumption 1 *The set U is a convex compact polyhedron. The functions $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ given by*

$$f(t,x,u) := a(t,x) + B(t,x)u, \quad g(t,x,u) := w(t,x) + \langle s(t,x), u \rangle,$$

and $l : \mathbb{R}^n \to \mathbb{R}$ are measurable and bounded in t, locally uniformly in (x,u), and differentiable in x. Moreover, these functions and their first derivatives in x are Lipschitz continuous in x, uniformly in $(t,u) \in [0,T] \times U$.

With the usual definition of the Hamiltonian

$$H(t,x,u,p) := g(t,x,u) + \langle p, f(t,x,u) \rangle,$$

the local form of the Pontryagin principle for problem (1)-(2) can be represented by the following *optimality system* for x, u and an absolutely continuous function $p : [0,T] \to \mathbb{R}^n$: for almost every $t \in [0,T]$

$$\begin{aligned}
0 &= \dot{x}(t) - f(t, x(t), u(t)), && (3)\\
0 &= x(0) - x_0, && (4)\\
0 &= \dot{p}(t) + \nabla_x H(t, x(t), p(t), u(t)), && (5)\\
0 &= p(T) - \nabla l\big(x(T)\big), && (6)\\
0 &\in \nabla_u H(t, x(t), p(t), u(t)) + N_U(u(t)), && (7)
\end{aligned}$$

where $N_U(u)$ is the usual normal cone to the convex set U at $u \in \mathbb{R}^m$. The optimality system can be recast as a generalized equation

$$0 \in \varPhi(x,p,u), \tag{8}$$

where the *optimality map* Φ is defined as

$$\Phi(x,p,u) := \begin{pmatrix} -\dot{x} + f(\cdot,x,u) \\ x(0) - x_0 \\ \dot{p} + \nabla_x H(\cdot,x,p,u) \\ p(T) - \nabla l(x(T)) \\ \nabla_u H(\cdot,x,p) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{9}$$

We remind the general definition of the property of Hölder Strong Metric sub-Regularity (HSMs-R) of a map, introduced under this name in [6] and appearing earlier in [4] (see the recent paper [2] for a comprehensive analysis of this property).

Definition 1 *Let* $(\mathcal{Y}, d_{\mathcal{Y}})$ *and* $(\mathcal{Z}, d_{\mathcal{Z}})$ *be metric spaces. A set-valued map* $\Phi : \mathcal{Y} \to \mathcal{Z}$ *is strongly Hölder sub-regular at* $\hat{y} \in \mathcal{Y}$ *for* $\hat{z} \in Z$ *with exponent* $\theta > 0$ *if* $\hat{z} \in \Phi(\hat{y})$ *and there exist positive numbers* a, b *and* κ *such that if* $y \in \mathcal{Y}$ *and* $z \in \mathcal{Z}$ *satisfy*

$$i) \quad z \in \Phi(y) \qquad ii) \quad d_{\mathcal{Y}}(y, \hat{y}) \leq a, \qquad iii) \quad d_{\mathcal{Z}}(z, \hat{z}) \leq b,$$

then

$$d_{\mathcal{Y}}(y, \hat{y}) \leq \kappa d_{\mathcal{Z}}(z, \hat{z})^{\theta}.$$

We call a, b *and* κ *parameters of strong Hölder sub-regularity. If* $\theta = 1$*, then the property is called SMs-R.*

In this paper, we reconsider this property for the optimality map Φ, with an appropriate definition of the metric space where $y = (x, p, u)$ takes values and of the image space. It is well known that the HSMs-R property of the optimality map plays an important role in the analysis of stability of the solutions and of approximation methods in optimization, in general. We refer to [2] for general references, and to [10], where more bibliography on the utilization of the HSMs-R property in the error analysis of optimal control problems is provided. We mention that a sufficient condition for SMs-R follows from the fundamental paper [5], but it does not apply to affine problems.

The paper contains two main contributions.

(i) Usually in the investigations of regularity of the optimality map for affine problems (see [9] and the bibliography therein) the metric in the control space is related to the L^1-norm, which does not give information about the structure of the control function even if the optimal control is of bang-bang type, as assumed later in this paper. The metric in $\mathcal{U}$ that we define in the present paper captures some structural similarities of the controls, thus the regularity property in this metric is closer to (but weaker then) the so called *structural stability*, investigated in e.g. [7, 8]. The SMs-R or HSMs-R properties of the optimality map Φ in this metric is especially important in the analysis of Model Predictive Control algorithms.

(ii) The obtained sufficient conditions for HSMs-R extend the known ones (e.g. [1, 9]) in a way which makes them applicable to some problems which are non-linear in the state variable and the Hölder exponent θ is smaller than one.

2 The main result

First of all we define the metric spaces $\mathcal{Y}$ and $\mathcal{Z}$ of definition and images of the set-valued map Φ in (8), (9). For that we introduce some notations.

Using geometric (rather than analytic) terminology, we denote by V the set of vertices of U, and by E the set of all unit vectors $e \in \mathbb{R}^m$ that are parallel to some edge of U. Let Z be a fixed non-empty subset of $[0, T]$. For $\varepsilon \geq 0$ and for $u_1, u_2 \in \mathcal{U}$ denote $\Sigma(\varepsilon) := [0, T] \setminus (Z + [-\varepsilon, \varepsilon])$, and for $u_1, u_2 \in \mathcal{U}$ define

$$d^*(u_1, u_2) := \inf \{\varepsilon > 0 \,:\, u_1(t) = u_2(t) \text{ for a.e. } t \in \Sigma(\varepsilon)\}.$$

For $Z = \emptyset$ we formally define $d^*(u_1, u_2) = 0$ if $u_1 = u_2$ a.e., and $d^*(u_1, u_2) = T$ else. It is easy to check that d^* is a shift-invariant metric in $\mathcal{U}$. For a shift-invariant metric d in any metric space we shorten the notation $d(y_1, y_2)$ as $d(y_1 - y_2)$. Then we define the spaces

$$\begin{aligned}\mathcal{Y} &:= W^{1,1}\big([0,T];\mathbb{R}^n\big) \times W^{1,1}\big([0,T];\mathbb{R}^n\big) \times \mathcal{U},\\ \mathcal{Z} &:= L^1\big([0,T];\mathbb{R}^n\big) \times \mathbb{R}^n \times L^1\big([0,T];\mathbb{R}^n\big) \times \mathbb{R}^n \times L^\infty\big([0,T];\mathbb{R}^n\big)\end{aligned}$$

with the metrics

$$d_{\mathcal{Y}}(x,p,u) := \|x\|_{1,1} + \|p\|_{1,1} + d^*(u),$$

$$d_{\mathcal{Z}}(\xi,\eta,\pi,\zeta,\rho) := \|\xi\|_1 + |\eta| + \|\pi\|_1 + |\zeta| + \|\rho\|_\infty.$$

The particular set Z in the definition of d^* will be defined in the next lines. The map Φ defined in (9) is now considered as a map acting on $\mathcal{Y}$ with images in $\mathcal{Z}$. The normal cone $N_{\mathcal{U}}(u)$ to the closed convex set $\mathcal{U} \subset L^1\big([0,T];\mathbb{R}^m\big)$ that appears in (9) is a subset of the dual space $L^\infty\big([0,T];\mathbb{R}^m\big)$, which can be equivalently defined as

$$N_{\mathcal{U}}(u) := \begin{cases} \emptyset & \text{if } u \notin \mathcal{U} \\ \{v \in L^\infty\big([0,T];\mathbb{R}^m\big) :\ v(t) \in N_U(u(t)) \text{ for a.e. } t \in [\tau, T]\} & \text{if } u \in \mathcal{U}. \end{cases}$$

By a standard argument, problem (1)–(2) has a solution, hence system (8) has a solution, as well. Let $\hat{y} = (\hat{x}, \hat{p}, \hat{u}) \in \mathcal{Y}$ be a reference solution of the optimality system (8). Denote by

$$\hat{\sigma} := \nabla_u H(\cdot, \hat{x}, \hat{p}) = B(\cdot, \hat{x})^\top \hat{p} + s(\cdot, \hat{x})$$

the so-called *switching function* corresponding to the triple $(\hat{x}, \hat{p}, \hat{u})$. We extend the definition of the switching function in the following way. For any $u \in \mathcal{U}$ define the function

$$[0,T] \ni t \mapsto \sigma[u](t) := B(t, x(t))^\top p(t) + s(t, x(t)),$$

where (x, p) solves the system (3)–(6) for the given u.

Assumption 2 *There exists numbers $\gamma_0 > 0$, $\alpha_0 > 0$ and $\nu \geq 1$ such that*

$$\int_0^T \langle \sigma[u](t), u(t) - \hat{u}(t) \rangle \, \mathrm{d}t \geq \gamma_0 \|u - \hat{u}\|_1^{\nu+1}$$

for all $u \in \mathcal{U}$ with $\|u - \hat{u}\|_1 \leq \alpha_0$.

This following assumption is standard in the literature on affine optimal control problems, see e.g. [3, 7, 9].

Assumption 3 *There exist numbers $\tau > 0$ and $\mu > 0$ such that if $s \in [0,T]$ is a zero of $\langle \hat{\sigma}, e \rangle$ for some $e \in E$, then*

$$|\langle \hat{\sigma}(t), e \rangle| \geq \mu |t - s|^\nu,$$

for all $t \in [s - \tau, s + \tau] \cap [0,T]$. Here ν is the number from Assumption 2.

Assumption 1 implies, in particular, that the set

$$\hat{Z} := \{s \in [0,T] \,:\, \langle \hat{\sigma}(s), e \rangle = 0 \;\text{ for some }\; e \in E\}$$

is finite. In what follows the set Z in the definition of the metric d^* will be fixed as $Z = \hat{Z}$.

The following result is well-known for a box-like set U; under the present assumptions it is proved in [9, Proposition 4.1].

Lemma 1 *Under Assumptions 1 and 3, $\hat{u}$ is (equivalent to) a piecewise constant function with values in the set V of vertices of U. Moreover, there exists a number $\gamma > 0$ such that*

$$\int_0^T \langle \hat{\sigma}(t), u(t) - \hat{u}(t) \rangle \, dt \geq \gamma \|u - \hat{u}\|_1^{\nu+1}$$

for all $u \in \mathcal{U}$.

As a consequence of the lemma, Assumption 2 is implied by Assumption 3, provided that there exists $\gamma_1 < \gamma$ such that

$$\int_0^T \langle \sigma[\hat{u} + v](t) - \sigma[\hat{u}](t), v(t) \rangle \, \mathrm{d}t \geq -\gamma_1 \|v\|_1^{\nu+1} \tag{10}$$

for all $v \in \mathcal{U} - \hat{u}$ with $\|v\|_1 \leq \alpha_0$. Notice that in the case of a linear-quadratic problem, condition (10) reduces to the one in [9, Corollary 3.1].

The main result in this paper follows.

Theorem 1 *Let Assumption 1–3 be fulfilled. There exist positive numbers a, b and κ such that if $y = (x, p, u) \in \mathcal{Y}$ and $z \in \mathcal{Z}$ satisfy*

$$i) \quad z \in \Phi(x, p, u) \qquad ii) \quad \|u - \hat{u}\|_1 \leq a, \qquad iii) \quad d_{\mathcal{Z}}(z, \hat{z}) \leq b,$$

then

$$d_{\mathcal{Y}}(y, \hat{y}) \leq \kappa d_{\mathcal{Z}}(z, \hat{z})^{-\nu^2}.$$

In particular, the optimality map $\Phi : \mathcal{Y} \to \mathcal{Z}$ is Hölder strongly metrically sub-regular at $\hat{y} = (\hat{x}, \hat{p}, \hat{u})$ for zero with exponent $1/\nu^2$.

3 Proof of the theorem

We begin with two lemmas.

Lemma 2 *There exists a positive number κ_0 such that for every $\varepsilon \in (0, T)$, $t \in \Sigma(\varepsilon)$, and $e \in E$ it holds that*

$$|\langle \hat{\sigma}(t), e \rangle| \geq \kappa_0 \varepsilon^{\nu}.$$

Proof. For brevity we use the notations

$$\hat{\sigma}_e(t) := \langle \hat{\sigma}(t), e \rangle, \qquad e \in E,$$

$$\delta := \inf\{|\hat{\sigma}_e(t)| \,:\, e \in E,\, t \in \Sigma(\tau)\} > 0.$$

Let $t \in \Sigma(\tau)$. Then

$$|\hat{\sigma}_e(t)| \geq \delta = \frac{\delta}{\varepsilon^\nu}\varepsilon^\nu \geq \frac{\delta}{T^\nu}\varepsilon^\nu.$$

Now let $t \in \Sigma(\varepsilon) \setminus \Sigma(\tau)$. This implies, in particular, that the set $Z = \hat{Z}$ is non-empty, since $\Sigma(\tau) = [0, T]$ if $\hat{Z} = \emptyset$. Then for some $s \in \hat{Z}$ and $e \in E$ with $\varepsilon \leq |t - s| \leq \tau$ it is fulfilled that

$$|\hat{\sigma}_e(t)| \geq \mu |t - s|^\nu \geq \mu \varepsilon^\nu.$$

This implies the claim of the lemma with $\kappa_0 = \min\{\mu, \delta/T^\nu\}$. □

Lemma 3 *There exist positive numbers κ_1 and ρ_1 such that for every functions $\sigma \in L^\infty$ with $\|\hat{\sigma} - \sigma\|_\infty \leq \rho_0$ and $u \in \mathcal{U}$ with $\sigma(t) + N_U(u(t)) \ni 0$ for a.e. $t \in [0, T]$ it holds that*

$$u(t) = \hat{u}(t) \;\; \textit{for a.e.} \;\; t \in \Sigma\big(\kappa_1 \|\sigma - \hat{\sigma}\|_\infty^{\frac{1}{\nu}}\big).$$

Proof. Consider the case $\|\sigma - \hat{\sigma}\|_\infty > 0$ and $\hat{Z} \neq 0$, the other cases are similar. For a vertex $v \in V$ we denote

$$E(v) = \left\{ \frac{v' - v}{|v' - v|} \,:\, v' \text{ is a neighboring vertex to } v \right\} \subset E.$$

From (7) applied to $(\hat{x}, \hat{p}, \hat{u})$, we obtain that $\langle \hat{\sigma}(t), v - \hat{u}(t) \rangle \geq 0$ for a.e. t and for every $v \in V$. From Lemma 1 we know that $\hat{u}(t) \in V$ for a.e. t. This implies that for a.e. t it holds that $\hat{\sigma}_e(t) \geq 0$ for all $e \in E(\hat{u}(t))$. Let us fix such a t which, moreover, belongs to $\Sigma\big(\kappa_1 \|\sigma - \hat{\sigma}\|_\infty^{\frac{1}{\nu}}\big)$; the number κ_1 will be defined in the next lines. Then according to Lemma 2 we have that

$$\hat{\sigma}_e(t) \geq \kappa_0 \Big(\kappa_1 \|\sigma - \hat{\sigma}\|_\infty^{\frac{1}{\nu}}\Big)^\nu = \kappa_0 (\kappa_1)^\nu \|\sigma - \hat{\sigma}\|_\infty.$$

Let us choose κ_1 and ρ_1 such that $\kappa_0 \kappa_1^\nu > 1$ and $\rho_1 < (T/\kappa_1)^\nu$. Then

$$\sigma_e(t) := \langle \sigma(t), e \rangle = \hat{\sigma}_e(t) + (\sigma_e(t) - \hat{\sigma}_e(t)) > \|\sigma - \hat{\sigma}\|_\infty - \|\sigma - \hat{\sigma}\|_\infty = 0.$$

Thus we obtain that

$$\langle \sigma(t), v - \hat{u}(t) \rangle > 0 \quad \text{for every } v \in V \setminus \{\hat{u}(t)\}.$$

This implies that $\hat{u}(t)$ is the unique solution of $\sigma(t) + N_U(u) \ni 0$, hence $u(t) = \hat{u}(t)$. □

Proof of Theorem 1. In the proof we use the constants involved in the assumptions and in the lemmas above. Let $z = (\xi, \eta, \pi, \upsilon, \rho) \in \mathcal{Z}$ and $y = (\tilde{x}, \tilde{\lambda}, \tilde{u}) \in \mathcal{Y}$ such that $z \in \Phi(y)$. Denote $\Delta := \rho + [\sigma_\tau[\tilde{u}] - \nabla_u H(\bar{p}, \tilde{x}, \tilde{\lambda})]$. Using the Grönwall's inequality, we can find constants c_1 and c_2 (indepent of y and z) such that $\|\Delta\|_\infty \leq c_1 \|z\|_{\mathcal{Z}}$ and $\|y - \hat{y}\|_{\mathcal{Y}} \leq c_2 \|\tilde{u} - \hat{u}\|_1$. Let $a := \alpha_0$, since $\Delta - \sigma[\tilde{u}] = \rho - \nabla_u H(\bar{p}, \tilde{x}, \tilde{\lambda}) \in N_{\mathcal{U}}(\tilde{u})$, we have

$$\int_0^T \langle \Delta - \sigma[\tilde{u}], \hat{u} - \tilde{u} \rangle \leq 0.$$

Now, by Assumption 2

$$\begin{aligned} 0 \geq \int_0^T \langle \Delta - \sigma[\tilde{u}], \hat{u} - \tilde{u} \rangle &= \int_0^T \langle \sigma[\tilde{u}], \tilde{u} - \hat{u} \rangle + \int_0^T \langle \Delta, \hat{u} - \tilde{u} \rangle \\ &\geq \gamma_0 \left(\int_0^T |\tilde{u} - \hat{u}| \right)^{\nu+1} - \|\Delta\|_\infty \int_0^T |\tilde{u} - \hat{u}|. \end{aligned}$$

Hence,

$$\|\tilde{u} - \hat{u}\|_1 \leq \frac{1}{\gamma_0^{\frac{1}{\nu}}} \|\Delta\|_\infty^{\frac{1}{\nu}} \leq \frac{c_1^{\frac{1}{\nu}}}{\gamma_0^{\frac{1}{\nu}}} \|z\|_{\mathcal{Z}_\tau}^{\frac{1}{\nu}}.$$

With $\kappa' := c_2 c_1^{\frac{1}{\nu}} \gamma_0^{-\frac{1}{\nu}}$ we obtain that

$$\|y - \hat{y}\|_{\mathcal{Y}} \leq \kappa' \|z\|_{\mathcal{Z}}^{\frac{1}{\nu}}.$$

There exists a constant $c_3 > 0$ such that $\|\sigma[\tilde{u}] - \sigma[\hat{u}]\|_\infty \leq c_3 \|\tilde{u} - \hat{u}\|_1$, hence

$$\|\sigma[\tilde{u}] - \rho - \sigma[\hat{u}]\|_\infty \leq c_3 \kappa' \|z\|_{\mathcal{Z}}^{\frac{1}{\nu}} + \|\rho\|_\infty \leq (c_3 \kappa' + 1) \|z\|_{\mathcal{Z}}^{\frac{1}{\nu}}.$$

Let b small enough so Lemma 3 holds with $\sigma = \sigma[\tilde{u}] - \rho$. We get $d^*(\tilde{u}, \hat{u}) \leq \left[(c_3 \kappa' + 1) \|z\|_{\mathcal{Z}}^{\frac{1}{\nu}} \right]^{\frac{1}{\nu}} = (c_3 \kappa' + 1)^{\frac{1}{\nu}} \|z\|_{\mathcal{Z}}^{\frac{1}{\nu^2}}$. Finally,

$$d_{\mathcal{Y}}(y, \hat{y}) \leq (c_3 \kappa' + 1)^{\frac{1}{\nu}} \|z\|_{\mathcal{Z}}^{\frac{1}{\nu^2}} + k' \|z\|_{\mathcal{Z}}^{\frac{1}{\nu}} \leq \kappa \|z\|_{\mathcal{Z}}^{\frac{1}{\nu^2}},$$

where $\kappa := (c_3 \kappa' + 1) + \kappa'$. □

4 An example

Let $\alpha : [0, +\infty) \to \mathbb{R}$ be a differentiable function with Lipschitz derivative such that α attains its minimum at zero and $\alpha'(x) \geq 0$ for all $x \in [0, +\infty)$, e.g., $\alpha(x) = x^2$ or $\alpha(x) = 1 - e^{-x^2}$. Moreover, let T be a positive number and let $\beta : [0, T] \to \mathbb{R}$ be ν-times differentiable ($\nu \geq 1$)

and satisfy $\beta(t) > 0$ for $t > 0$, $\beta(0) = \ldots = \beta^{(\nu-1)}(0) = 0$, $\beta^{(\nu)}(0) \neq 0$. Consider the following optimal control problem

$$\min_{u \in \mathcal{U}} \Big\{ \int_0^T \Big[\alpha\big(x(t)\big) + \beta(t) u(t) \Big] \,\mathrm{d}t \Big\}, \tag{11}$$

subject to

$$\dot{x}(t) = u(t), \quad x(0) = 0, \quad u(t) \in [0,1] \quad \text{a.e. in } [0,T]. \tag{12}$$

The optimality system of problem (11)-(12) is given by

$$0 = \dot{x}(t) - u(t), \quad x(0) = 0, \tag{13}$$
$$0 = \dot{p}(t) + \alpha'\big(x(t)\big), \quad p(T) = 0, \tag{14}$$
$$0 \in \beta(t) + p(t) + N_{[0,1]}(u(t)). \tag{15}$$

Hence the switching function corresponding to each control $u \in \mathcal{U}$ is given by $\sigma[u](t) := \beta(t) + p[u](t)$, where $p[u](t) = \int_t^T \alpha'\big(x[u](s)\big)\,\mathrm{d}s$ and $x[u](t) = \int_0^t u(s)\,\mathrm{d}s$, $t \in [0,T]$. It is clear that the unique minimizer of problem (11)-(12) is $(\hat{x}, \hat{p}, \hat{u}) = (0,0,0)$, and consequently its switching function is given by $\hat{\sigma}(t) = \beta(t)$. Since $\hat{\sigma}$ satisfies $\hat{\sigma}(0) = \ldots = \hat{\sigma}^{(\nu-1)}(0) = 0$ and $\hat{\sigma}^{(\nu)}(0) \neq 0$, we have that Assumption 3 is satisfied with the same number ν but not with $\nu - 1$. Now, observe that for $v \in \mathcal{U} - \hat{u}$ we have

$$\int_0^T \langle \sigma[\hat{u}+v](t) - \sigma[\hat{u}](t), v(t) \rangle \,\mathrm{d}t = \int_0^T \langle p[\hat{u}+v](t), v(t) \rangle \,\mathrm{d}t \geq 0.$$

Thus, in accordance with (10), Assumption 2 is satisfied.

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