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On the Matthew Effect on Individual Investments in Skills in Arts, Sports and Science

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Abstract

The paper describes the process of capital accumulation subject to the following characteristics: (i) convex returns to (human) capital; (ii) the need to self-finance the investment. This setup is applicable to explain some peculiarities in arts, sports and science, inter alia, the "Matthew effect" coined in Merton (1968) to explain why prominent researchers get disproportional credit for their work. The potential young artist's (sportsman's or scientist's) optimal strategies include quitting, or continuing and even expanding one's human capital in a profession. Both outcomes are separated by a threshold level in human capital. In addition, it can be optimal to stay in business although consumption falls and stays at the subsistence level forever (we call this outcome a "Sisyphus point"). This possibility is also interesting from a theoretical point of view, as the optimal control problem may turn "abnormal", i.e., the objective does not enter the Hamiltonian.

Keywords: Human capital accumulation, Abnormal control problem, Convex returns, Threshold, Matthew effect, Sisyphus point.

JEL Classification C61, E20, I24, I26, Z11.

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1 Introduction

This paper presents a variation of the Ramsey model of optimal investment with the following special features: convex returns to capital and convex opportunity costs for investment. In addition, we impose the constraint of no debt at any point in time which allows for complex and interesting dynamics including multiple equilibria separated by a threshold. One goal of the proposed model is to show the possibility of a threshold of the *Sisyphus* type, i.e., the optimality of an outcome at the boundary of zero consumption, which simultaneously separates an interior equilibrium from an attractor to the origin (quitting the business, art, sport, science etc.), which explains, e.g., why a very high fraction of researchers have none or very few publications with none or at best very few citations. Moreover, this outcome leads to the mathematically interesting and, in the context of intertemporal optimization in economics, (very) rare but necessary concern about the normality of an optimal control problem, compare the example of Halkin (1974) and El-Hodiri (2012).

However, this framework is not only of formal interest, but allows for a number of interesting interpretations and moreover complex dynamics and thresholds. Convex returns address the earnings of an exceptional talent or celebrity often coined the Matthew effect: "For to every one who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away", Matthew 25:29. This point was first addressed by the sociologist Robert K. Merton (1968), the "father of the economist", who pursued the question why eminent scientists get disproportionately more credit for their contributions, while relatively unknowns get disproportionately little. A famous example from economics is the familiar Solow-Swan model (Solow, 1956 and Swan, 1956).[1]

Following Merton (1968), many studies have investigated this so-called Matthew effect, both analytically and numerically, and it is empirically documented in many fields including less obvious ones like education (reading and math). The Matthew effect holds most clearly in the arts, in the past but also presently, in particular for painters, musicians and poets. Bask and Bask (2015) argue that the cumulative

[1]For which Solow got all (and still gets most of) the credit (including a Nobel prize) although Swan developed the model independently and published it around the same time (but in a less prestigious journal).

advantage is an intra-individual while the Matthew effect is an inter-individual phenomenon and that this difference in phenomena has consequences for the modeling of socio-economic processes and either ones are detected in data. Since it is very difficult to measure quality and thus precludes convincing empirical assessments of the magnitude of status effects, Azoulay, Stuart & Wang (2014) address this problem by examining the impact of a major status-conferring prize (becoming a Howard Hughes Medical Institute (HHMI) Investigator) that shifts actors' positions in a prestige ordering. They find only small and short-lived evidence of a post-appointment citation boost but prize winners are of (relatively) low status gain. The review of Matjaz (2014) shows that the Matthew effect, labelled as "the concept of preferential attachment", is ubiquitous across social and natural sciences and is related to the power law. It affects patterns of scientific collaboration, the growth of socio-technical and biological networks, the propagation of citations, scientific progress and impact, career longevity, the evolution of the most common words and phrases, education, as well as many other aspects of human culture. The recent prominence of the Matthew effect is largely due to the rise of network science and the concept of preferential attachment.

The paper starts with the model (section 2), which is then complemented by economic interpretations (section 3) before its analysis, theoretical (section 4) and numerical (section 5). The paper's contribution also extends to theory as it analyzes and applies how one can treat abnormality in optimal control problems. This part is, however and due to the mathematical complexity relegated to an Appendix.

2 The Model

Aside from the objective to model the consequences of the Matthew effect on individual decisions, the goal is to develop a tractable and economically meaningful model with a steady state, which is optimal but simultaneously separating the basins of attractions between high and low equilibria, and which is characterized by zero consumption. We call such a threshold one of the Sisyphus type because of continuing an activity yet obtaining nothing. The presence of such a point creates an attractor to the origin (i.e., quitting art, science, etc.), which explains why such a high fraction of researchers have just 0 or 1 publication and 0 or 1 citation, to use the example from the work of Merton (1968), or why many start a career in arts but end up in different professions (often as teachers, e.g., for music or a particular instrument).

For this purpose, we propose the following variation of the Ramsey model of maximizing intertemporal (using the constant discount rate $r > 0$) utility (u) from consumption (c), amended for a stock effect $v(k)$,[2]

$$\max \int_0^\infty e^{-rt}(u(c) + v(k))dt.$$

Consumption is, as in the Ramsey model, the difference between output $f(k)$ and investment (i),

$$c = f(k) - i.$$

The crucial deviation from the usual Ramsey setup is the assumed convexity of the production function, $f' > 0$ and $f'' \geq 0$, which will be economically justified below; Skiba (1978) and many follow-ups, e.g., see Brock & Dechert (1985) who consider convex-concave production functions. Capital accumulation is defined as usual but with the twist that investments are subject to diminishing returns, ($\alpha'' > 0$),

$$\dot{k} = i - \alpha(i) - \delta k, \ k(0) = k_0,$$

because too large investments are less effective in expanding the capital stock; $\delta > 0$ denotes the deprecation rate. Preceeding the interpretations of the model below and with reference to our own profession, purchasing software licenses (say Mathematica, MATLAB, SPSS) and books (e.g., for this paper, Barro and Sala-i-Martin, 1995, works of other economists as well as of sociologists including Merton) at once cannot all be put into effective use immediately. That is, the speed of learning is limited due to constraints, in particular of time, so that a piecemeal strategy will be more effective in turning investment into productive human capital.

In order to simplify as much possible and to allow for explicit, at least nu-

[2]As, e.g., in Hof & Wirl (2008) who show that stock spillovers are crucial for thresholds in concave set ups of the Ramsey model based on Barro & Sala-i-Martin (1995).

merical, calculations we assume linear and quadratic specifications leading to the following model:

$$\max_{i(t)\geq 0} \int_0^\infty e^{-rt}((mk^2 + bk - i) + hk)dt, \tag{1}$$

$$s.t. \qquad \dot{k} = i - ai^2 - \delta k,\ k(0) = k_0,\ k \geq 0, \tag{2}$$

$$c := mk^2 + bk - i \geq 0. \tag{3}$$

This model appears similar to Hartl & Kort (2004) but has the following crucial differences: (i) the adjustment costs associated with large investments appear in the state equation instead of in the objective, (ii) investment must be paid from current revenues (no debt), and (iii) consumption must be non-negative. Last but not least, (iv), the Hartl and Kort model does not allow for the kind of dynamics that we are interested in and that seem relevant for many fields.

As mentioned, a crucial point of the paper is the existence of Sisyphus points, denoted k_s, i.e., a level of (human) capital at which consumption turns zero and therefore all initial conditions to the left of it must end up in the origin, $k \to 0$. More precisely, departing from the constraint, $c \geq 0$, we can define the maximal level of feasible investment,

$$i \leq i^{\max} := mk^2 + bk. \tag{4}$$

Assuming in addition that capital does not decline and investing at the maximal level subject to $c \geq 0$, yields a fourth order polynomial for $\dot{k} = 0$ that can be reduced to one of the order 3,

$$\psi(k) := (b - d + mk - ak(b + mk)^2) = 0, \tag{5}$$

since $k = 0$ is one of the roots. Or arguing differently, we can define the minimal investment $i^{\min}$ that is necessary in order to avoid a decline in human capital. More precisely, $\dot{k} \geq 0$, iff

$$\frac{1 + \sqrt{1 - 4a\delta k}}{2a} \geq i \geq i^{\min} := \frac{1 - \sqrt{1 - 4a\delta k}}{2a}. \tag{6}$$

Definition The root $k_s > 0$ at which

$$i^{\max}(k_s) = i^{\min}(k_s) < \frac{1}{2a}$$

is called a *Sisyphus point*. Therefore, $\dot{k} < 0$ inevitably for $k \in (0, k_s)$ since $i^{\max}(k) < i^{\min}(k)$, see Fig. 1 (and thus also for $k \in (k^{\max}, \infty)$).

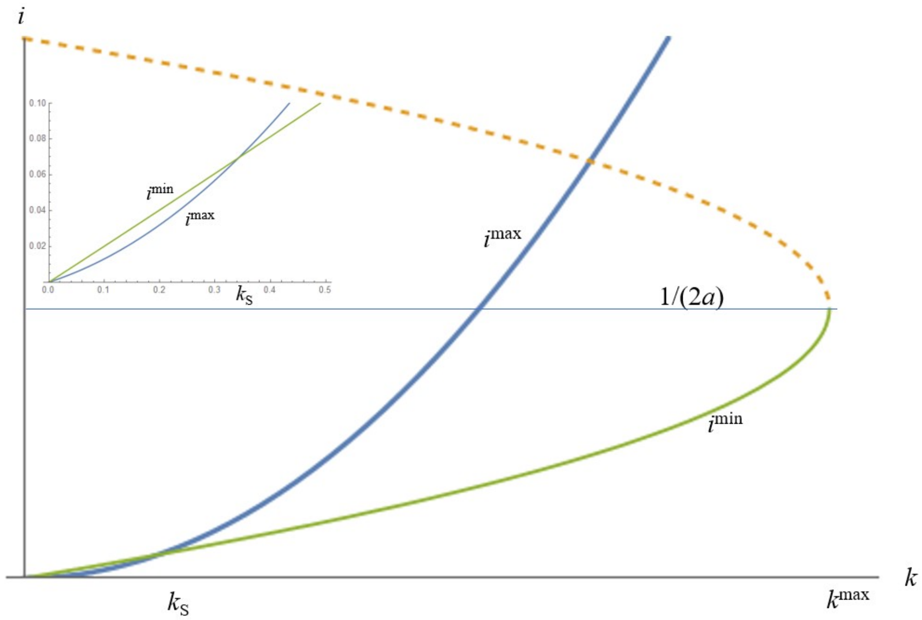


Figure 1: *Sketch of the curves $i^{\min}(k)$ and $i^{\max}(k)$. Only the area $k > k_s$ is feasible for interior solutions. The enlargement shows the neighborhood of the Sisyphus point for the reference parameters in (16) and $(m, \delta) = (0.3, 0.2)$.*

Fig. 1 plots the crucial terms, $i^{\min}$ and $i^{\max}$ with their intersection determining k_s (also magnified). The dashed line shows the larger root of the equation $\dot{k} = 0$ (the term on the left-hand side of (6)) with $i > 1/(2a)$. This part is irrelevant, because gross capital formation is declining for too large investments and thus dominated by investments,

$$i \leq \frac{1}{2a} = \arg\max_i i - ai^2. \tag{7}$$

Therefore, no solution with $i \geq 1/(2a)$ can be a candidate for maximizing (1) even if it satisfied the first-order optimality conditions. As a consequence,

$$k^{\max} = \frac{1}{4a\delta} \tag{8}$$

defines the upper bound of the capital stock (see Fig. 1).

The existence of $k_s > 0$ follows easily by considering numerical examples as well as from the limiting case of small a so that the quadratic term ai^2 can be neglected. A necessary (and sufficient) condition for $k_s > 0$ is that

$$\frac{di^{\max}}{dk} < \frac{di^{\min}}{dk} \text{ at } k = 0,$$

which implies

$$b < \delta. \tag{9}$$

Therefore, inequality (9) is assumed in the following. Interesting are the cases in which a steady state k_∞ exists such that

$$0 < k_s < k_\infty < k^{\max}.$$

Application of the implicit function theorem to

$$mk^2 + bk - \frac{1 - \sqrt{1 - 4a\delta k}}{2a} = 0,$$
$$mk^2 + bk - \frac{a}{2}\left(mk^2 + bk\right)^2 - \delta k = 0,$$

implies that a larger value of the parameter m leads to a decline of the Sisyphus point and to an increase in $k^{\max}$ and thus to an expansion of the area $k_s < k < k^{\max}$ in both directions. This leads to the conjecture that k_∞ increases w.r.t. m too but this requires further analysis.

By definition, the Sisyphus point k_s is located at the intersection of the two curves $i^{\max}(k)$ and $i^{\min}(k)$ and thus at the intersection of $\dot{k} \geq 0$ and $c \geq 0$. Any trajectory passing through k_s implies $\dot{i} \leq 0$ to its left while both $\dot{i} \geq 0$ and $\dot{i} < 0$ are possible to its right. Since $k = 0$ is always a feasible solution, some optimal trajectories can pass through the Sisyphus point on their way to $k = 0$. Another property of the Sisyphus point is that it can be optimal to stay there forever. This is the standard outcome for thresholds in concave optimization problems (compare Wirl & Feichtinger, 2005) but almost entirely ignored in dynamic optimization problems with convex-concave objectives (Hartl & Kort, 2004 draw attention to the possibility of a continuous policy function although the Hamiltonian is convex with respect to the state). Although nothing is consumed at the Sisyphus point ($c = 0$) since everything is invested ($i^{\min} = i^{\max}$, is required in order to avoid the decline, $k \to 0$), the payoff (i.e., the integrand in (1)) can be positive, if it includes a direct benefit from the state (hk). Therefore, if $h > 0$, then the Sisyphus point can be optimal. However, this is not necessary as the examples below show.

If an agent has no access to credit in order to expand his/her human capital starting at k_s or below, it will eventually converge to zero. That is, all paths, $0 < k(0) < k_s$, must end up in the origin. Contrary to usual thresholds, this attraction of the origin applies not only to optimal but to all feasible paths. This suggests an analogy to what is called in physics a 'black hole', because there is no way to avoid this limiting outcome ($k \to 0$) once the 'horizon' $k_s > 0$ is crossed to the left, given the constraints that the agent faces. On the other hand, trajectories that expand human capital can and do exist on the right-hand side in the neighborhood of the Sisyphus point (but need not be optimal).

3 Economic interpretations

Although the model is so far introduced only formally, it captures features that are crucial in different fields in which individual talent can lead to a very unequal distribution of incomes. Familiar examples are: sports, arts and also science according to Merton (1968). The evaluation of performance is most objective in sports but highly subjective in arts and thus also depends on luck, advertising and access (to media, markets, compare Yegorov et al. 2016). The situation in science is presumably in between thee other two.

A crucial observation in all these examples (explained in more detail below) is that the individual reward is linear in own human capital (k) but convex in prominence or fame, i.e., relative to the competitors in a particular field. We assume for simplicity that the reward per unit of human capital is affine,

$$\tilde{m}\frac{k}{K}+b,$$

in which b describes the individual productivity per unit of individual human capital and the first term accounts for the increasing returns due to prominence or fame. More precisely, the relative position of individual human capital (talent, ability, visibility, etc.) matters with respect to a reference point denoted by K. For example, K could describe the average over all other actors active in a field and is thus exogenously given at the individual level. Treating K as a constant,[3] we define a reward coefficient,

$$\rho\left(k\right):=mk+b,\ m:=\frac{\tilde{m}}{K}>0,\ \delta>b>0 \tag{10}$$

and obtain linear quadratic revenues (y) with respect to individual human capital,

$$y(k)=k\rho(k)=mk^2+bk,$$

as stipulated in (1). In traditional industries, $m=0$, yet $m>0$ in branches in which recognition, talent or prominence lead to excessive returns. The additional payoff term, hk, accounts for individual satisfaction from acquiring a certain status of human capital (whether in absolute or in relative terms does not matter given our assumption about K).

For a given population of sportsmen, artists, scientists or small businesses, with the same initial human capital $k(0)$ but different $\tilde{m}$ (or respectively, m), their personal Sisyphus points and long run attainments will differ. As a consequence, some of them will have to leave the market (those with low m and low k_0), while others with the same (or higher) m and $k_0>k_s$ will persist. Therefore, success is unevenly distributed leading to the Matthew effect.

The assumption of no debt at any point in time accounts for the uncertainties banks face about the skills of an applicant (e.g., a young painter asking for credit to travel to and learn from a famous master or at a foreign academy). Therefore,

[3]The extension for a competitive equilibrium of agents having different abilities and starting from different initial conditions is left for future research.

the applicant will not be offered credit, or if then only at prohibitively expensive terms. High bankruptcy rates characteristic for certain kinds of businesses provide another reason for credit restrictions.

3.1 Science

We start with science as our first example due to the original and stimulating work of Merton (1968) who coins and relates the Matthew effect to cumulative advantage: eminent scientists get disproportionately credit for their contributions to their field, while relatively unknown ones get disproportionately little. Stephan (1996) notes that compensation in science consists of two parts: one is paid regardless of an individual's success, the other (including prestige, journalistic citations, paid speaking invitations, and other such rewards) reflects the contribution to science.[4] Therefore, the recognition for scientific work is skewed in favor of established scientists and additional factors reinforce the process of cumulative advantage: differences in individual capabilities, inequality in access to resources, inequality of peer recognition, and inequality of scientific productivity.

In our notation (k), the scientific human capital of an individual researcher, is the only production factor. $\rho(k)$ denotes the scientists payoff from a particular piece of work (say a paper) accounting for the non-linear Matthew or recognition effect. The reward (y) can be used for consumption (c) and investment (i). The additional term (hk) accounts how a researcher values own achievements irrespective of their public evaluation (compare Benabou & Tirole, 2006 for the consequences of such intrinsic motives).

3.2 Sports

The Matthew effect is noticeable in many kinds of sports, because the winners take a disproportionately large share of the pie (but not all as competition is a *conditio sine qua non* for winning), monetarily but even more so in terms of fame. Indeed, everyone knows the name of the winner, say of the Tour de France, but only few know the ones ending in second place, except for the time when Poulidor

[4]And since scientists value both income and prestige, industrial scientists are willing to give up some income in exchange for greater ability to publish according to Stern (2004).

finished several times second. Sports provides also a good example linking the individual Matthew effect with the aggregate. Considering individual talents for different kinds of sports, e.g., in Austria, entering alpine skiing brings about fierce competition and thus a large K, while entering a related field like ski jumping will allow one to face a lower K; of course, payoffs are also larger in fields populated by many competitors (in the US football or baseball versus soccer). And sports is full of anecdotes, where people invest not only time but also their own money to make it to the top: in skiing (the Kostelic sisters were coached by their father), racing (Niki Lauda, three times World Champion of Formula 1 racing, spent his/her own money in order to be able to enter racing events), tennis (from Steffi Graf to the Williams sisters were also coached by their fathers). And those who did not make it quit (with countless but unknown examples).

The examples from sports are not limited to individuals but also include collectives. Recently, *The Economist* (2020) reports on the unequal situation between the Premier League and lower league professional football teams in England. For example, Bury FC, just north of Manchester, was kicked out of professional football after it failed to service its debt, while nearby Manchester City's Emirati owners generously paid the players' salaries, by far exceeding the club's revenues. Zoë Hitchen, a fan of Bury, said, "The system ... always lets people down at the bottom. It never lets down the people at the top."[5]

3.3 Art

Similar to sports, and maybe even larger, are the uneven returns in many disciplines of arts. For an example, in 2018 David Hockney earned above 90 million dollars for a single painting, while many painters earn just a few dollars for their work with a "value" in most cases for sure below $1/10^6$ of Hockney's painting (Callum, 2019). Yet, in May 2019 Jeff Koon's sculpture of a rabbit was bought for 91 million dollars (Kazakina, 2019). Moving back in history, only a handful of painters (e.g., Giotto, Dürer, da Vinci, Raffael, Picasso, etc.) out of thousands, if not millions, account for an overwhelming share of the total value of all paintings; similarly for composers, of which few remain known and played. A further

[5]However, UEFA expelled Manchester City from Europe's football contests for the following two years because of that.

characteristic of arts, again in particular for paintings, sculptures and related activities, is public contribution (directly or indirectly by publicly owned museums participating in auctions), but also in other fields, e.g., paying for the superstars among opera singers and conductors appearing in publicly financed opera houses and concert halls (the *status quo* for all big European houses).

Yegorov et al. (2016) present a model that addresses the different opportunities artists have when they try to get access to a market. This depends on cultural specifics (like language for writers, taste for the kind of music for musicians and composers, also for paintings) that can affect an artist's career choices. This explains, inter alia, the skewed distribution of authors, since writing in English offers immediate access to a much larger market than, say, writing in Albanian.[6]

3.4 Other examples

As already mentioned, the returns to small businesses and even start-ups can be highly skewed and this return to prominence seems to be increasing over time due to search engines like Google (the power law describes the distribution of visits to homepages in many fields). Another topical application is to self-employment in service sectors in the era of digitization (e.g., as an Amazon *Turk*). As industrial employment shrinks due to robotization, many people move to the service sector and may offer new and traditional types of services, like yoga, eastern healing, massage, even writing articles and theses. In such cases, skills can at best be observed imperfectly (evaluation by others, but which may mean little for one's specific task) and demand grows via word-of-mouth and nowadays via social networks. More talent or better advertising and/or initial luck can earn above the normal return to one's talent in a particular area. Then, the market return consists of two components, (i) proportional to skills, bk, and (ii) the gain due to prominence and marketing (access and ability to work in social networks). Let m measures those marketing skills compared to the average (because if all rivals advertise equally, they have costs, but the return is the same). Then, for a homogeneous distribution of skills (m) the returns will be disproportionately distributed where the term mk^2 captures the induced revenues.

[6]Nevertheless, Isaac Bashevis Singer received the Nobel prize for literature albeit writing in Yiddish. However, he lived in New York and was readily translated.

Drug dealing is another example that fits our crucial assumptions: it is self-financed (it is hard to get credit to finance one's career) and the returns are highly skewed: those at the top have luxury apartments and sports cars, while those selling the drugs in the street earn less than the minimum wage. That is, they stay at or close to what we call the Sisyphus point and this in spite of the implied risk in the hope to move up on the career ladder, see Levitt & Venkatesh (2000).[7]

4 Optimality conditions

We define the (current value) Hamiltonian of the optimal control problem (1) - (3),

$$H = \lambda_0(mk^2 + (b+h)k - i) + \lambda(i - ai^2 - \delta k) \tag{11}$$

and set, as is usual, $\lambda_0 = 1$. However, we have to return to this implicit assumption of normality when deriving the optimal paths, because the case $\lambda_0 = 0$ cannot be ruled out and the corresponding abnormal solutions are derived in the Appendix. Maximizing the Hamiltonian (H) with respect to the control and accounting for the constraints, $c = mk^2 + bk - i \geq 0$ and $i \geq 0$, yields

$$i^* = \left\{ \begin{array}{c} mk^2 + bk \\ \frac{\lambda-1}{2a\lambda} \\ 0 \end{array} \Leftrightarrow \begin{array}{c} \frac{\lambda-1}{2a\lambda} > mk^2 + bk, \\ 0 < \frac{\lambda-1}{2a\lambda} < mk^2 + bk, \\ \frac{\lambda-1}{2a\lambda} < 0. \end{array} \right\} \tag{12}$$

The other first-order condition determines the evolution of the costate (λ) according to (13), which together with the state equation after substituting the optimal control (14) yields the canonical equation system. This system is given below for the interior solution of (12):

$$\dot{\lambda} = \lambda(r+\delta) - 2mk - b - h, \tag{13}$$

$$\dot{k} = \frac{\lambda-1}{2a\lambda} - a\left(\frac{\lambda-1}{2a\lambda}\right)^2 - \delta k. \tag{14}$$

[7] Tragler et al. (2001) analyze the intertemporal trade-off between the social costs of illicit drug consumption and the expenditures for controlling the US-cocaine epidemics.

Equating the time derivatives to zero, we get the following algebraic system to determine the steady state(s) of the above dynamic system:

$$k = \frac{\lambda(r+\delta) - b - h}{2m},$$
$$k = \frac{\lambda^2 - 1}{4\delta a \lambda^2}.$$

The first equation defines a straight line in the (λ, k) plane with positive slope and a root at $\lambda = (b+h)/(r+\delta)$. The second function $k(\lambda)$ has a singularity at $\lambda = 0$ and two roots at $\lambda = \pm 1$. Equating the above two equations (and thus eliminating k) we get the following cubic equation in λ characterizing any steady state:

$$g(\lambda) := 4\delta a(r+\delta)\lambda^3 - \lambda^2\left((2m + 4\delta a(b+h)\right) + 2m. \tag{15}$$

Only the positive out of the three roots are of interest. Since all parameters are positive, $g \to -\infty$ for $\lambda \to -\infty$, $g \to +\infty$ for $\lambda \to +\infty$. Furthermore, $g(0) = 2m > 0$ is a local maximum and the other local extremum (a minimum) is at $\lambda > 0$. Therefore, $g(\lambda)$ must have one and only one negative root and the remaining two roots must be either positive or a pair of conjugate complex numbers.

5 Results

5.1 Bifurcation Diagrams

We fix the following parameters,

$$r = 0.03, a = 0.2, b = 0.1, h = 0.1, \tag{16}$$

i.e., the subjective discount rate is 3% per annum, the adjustment cost parameter limits investment to $i < 2.5$, and the linear earning term and the direct benefit parameter are both set to 0.1. Numerical means are necessary because it is impossible to determine by analytical means first the steady states and then which of the paths is optimal for a given initial condition. We derive the different cases varying the parameter which governs the Matthew effect (m) and the rate of depreciation (δ).

Fig. 2 depicts a typical phase portrait of the canonical equations but shows the control, investment i, instead of the costate (λ) assuming $m = 0.3$ and $\delta = 0.2$ in addition to the parameters in (16). The (unconstrained) system has a negative and stable steady state which is irrelevant due to $k \geq 0$ and is replaced by the corner solution $k \to 0$ as a possible long run outcome. The other two steady states are positive of which the lower one is a repelling focus (complex eigenvalues with positive real parts) and the third and largest steady state is a saddle point. Fig. 2 shows the isoclines, $\dot{i} = 0$ and $\dot{k} = 0$, the stable manifold for the unconstrained problem in bold (its extension to the left requires $c < 0$), the feasible set, $c \geq 0$, and how the impossibility of getting credit (i.e., of non-negative consumption) affects the policy heading towards the high steady state: in order to make up for the low but steeply increasing investment along the boundary ($c = 0$), investment is then flat in the interior along the saddle point path and close to the maximum due to the large Matthew effect. What this figure cannot tell us is which of the strategies, following the saddle point path to the high steady state or going to the origin (or to somewhere else) is optimal and for which initial conditions. All we can determine is that initial conditions $k_0 < k_s$ must end up at $k = 0$.

Fig. 3 shows how the positive steady states of (13) and (14) depend on the parameter m, measuring what we call the Matthew effect. Positive roots of (15) require at least $m > 0.02...$ so that an enterprise with only weakly convex returns but convex investment costs is doomed to fail. Of the two positive steady states the upper one is a saddle point with an asymptote at $k_\infty \to 6.25$ for $m \to \infty$. The lower one vanishes for large Matthew effects, i.e., $k_s \to 0$ for $m \to \infty$. The line between the two (positive) steady states (in blue) shows at which capital stocks the constraint $c \geq 0$ is binding, i.e., the Sisyphus point, k_s, depending on m. Therefore, the lower steady state is always located in the infeasible domain $c < 0$. The Sisyphus line intersects the upper steady state line in the area of the corner equilibria.[8] The gray curves refer to steady states of the canonical system that do not correspond to equilibria of the optimal solution (corresponding to the dominated part shown in Fig. 1 by the dashed curve). The colored curves refer to optimal long run outcomes. For $m < m_{bif}$ (= the bifurcation value)

[8]The term 'corner' refers to the active constraint $c = 0$.

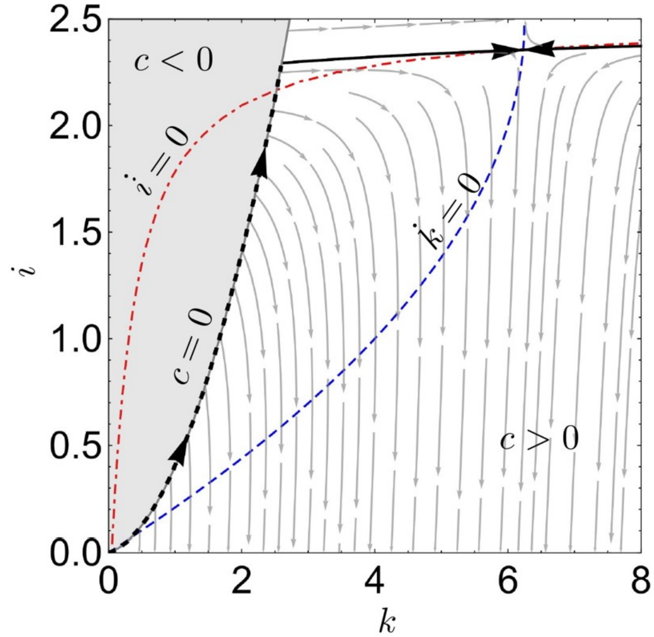


Figure 2: *The vector field, the isoclines and the feasible zone* $c > 0$ *for a rather low Sisyphus point and a large saddle point path, more precisely for (16) and* $(\delta, m) =$ *(0.2,0.3). The path from the Sisyphus point to the saddle point path (solid black) includes the corner solution (*$c = 0$*) unless* k_0 *is sufficiently large.*

only the origin is feasible and thus optimal as the only long run outcome. The Sisyphus point appears at $m = m_{bif}$ and the Sisyphus curve separates the interior outcomes ($c \geq 0$ is not binding and identified in Fig. 3 by the blue curve) from the corner solutions, $k \to 0$. A magnification shows the novel outcome of an interior equilibrium at which $c = 0$ is binding yet staying at the corresponding Sisyphus point is optimal.

Many, if not most, applied papers follow Skiba (1978) and others (e.g., Brock & Dechert, 1985; Dechert & Nishimura, 1983 and more recently applications to shallow lakes, Mäler 2000; Mäler et al. 2003), sketch phase diagrams and infer from there the existence of multiple long run outcomes separated by a threshold in the state space. This reasoning is insufficient, because one branch can be, and in many cases (also in our model) is, the globally optimal one. Therefore, instead of the local and geometric analysis, a global one is necessary; Wagener (2003) and later also supplemented by Kiseleva & Wagener (2010) are the first that check explicitly for the optimality of different paths in shallow lake models. Also see

Antoci et al. (2011) for a global analysis of a growth model with indeterminacy in the long run outcome. In fact, the determination of the optimal policy in our model faces additional complexities (elaborated in the Appendix) and thus requires advanced numerical techniques similar to the methods sketched in Grass et al. (2008) and in Grass (2012).

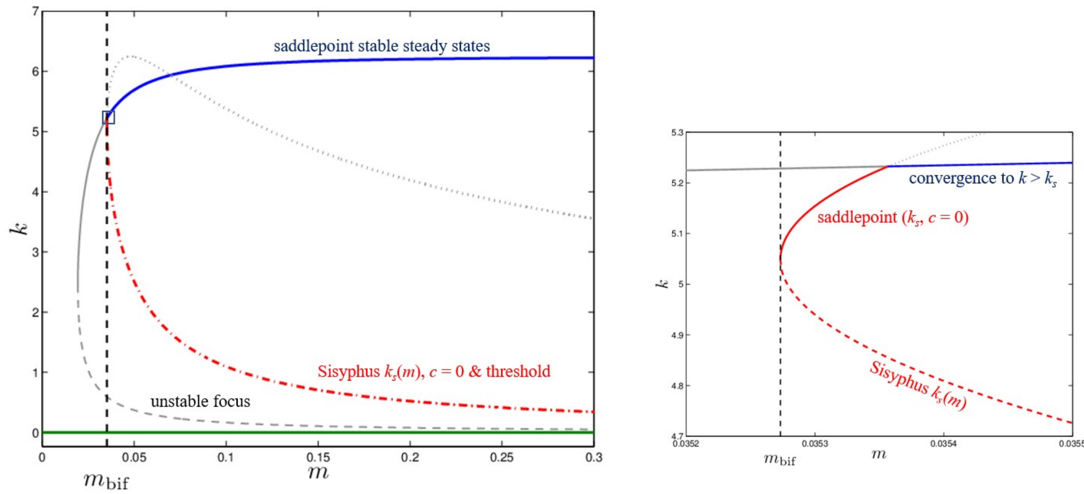


Figure 3: *The bifurcation diagram with respect to m for* (16) *and $\delta = 0.2$. The gray curves denote equilibria of the canonical system, which do not correspond to steady states of the optimal solution. Colored curves show equilibria that are attractors of the optimal solution, where blue denotes saddles in the interior of the control constraint, whereas for the red and green curves the control constraint is active. The green curve depicts the Sisyphus point and the red curve the equilibria with an active constraint but not satisfying $i^{\min} = i^{\max}$. The dashed part of the green curve (better visible in the left panel) corresponds to the normal case, see Prop. A2 and the solid part of the green curve corresponds to the abnormal case, see Prop. A3 in the Appendix.*

Fig. 4 shows the bifurcation diagram for δ and a pattern similar to the one in Fig. 3: a high and (saddle point) stable steady state and a low steady state, which is not only unstable but is located in the infeasible domain. Lowering δ below b (here 0.1) leads to an increase in the value of the upper saddle.[9] No positive roots exist for too large depreciation rates, $\delta > 0.6$, rendering again the origin as the only possible long run outcome. Fig. 4 includes also an identification of

[9]The set of feasible saddles is not bounded for $\delta \geq 0$. Contrary to the case of Fig. 3 this means that for a certain low depreciation the final outcome can be arbitrarily large.

the optimal policies conditional on the bifurcation parameter (δ) and the initial condition ($k = k_0$). For $\delta < 0.5063...$ and sufficiently large initial capital, $k_0 > 2.149...$, the saddle point path heading towards the high steady state is the optimal policy (indicated by the blue line). The enlargement shows that for slightly larger depreciation rates and lower capital stocks, first the boundary policy ($c = 0$ and $k = k_s$, red) and then heading towards the origin ($k \to 0$, green) is optimal.

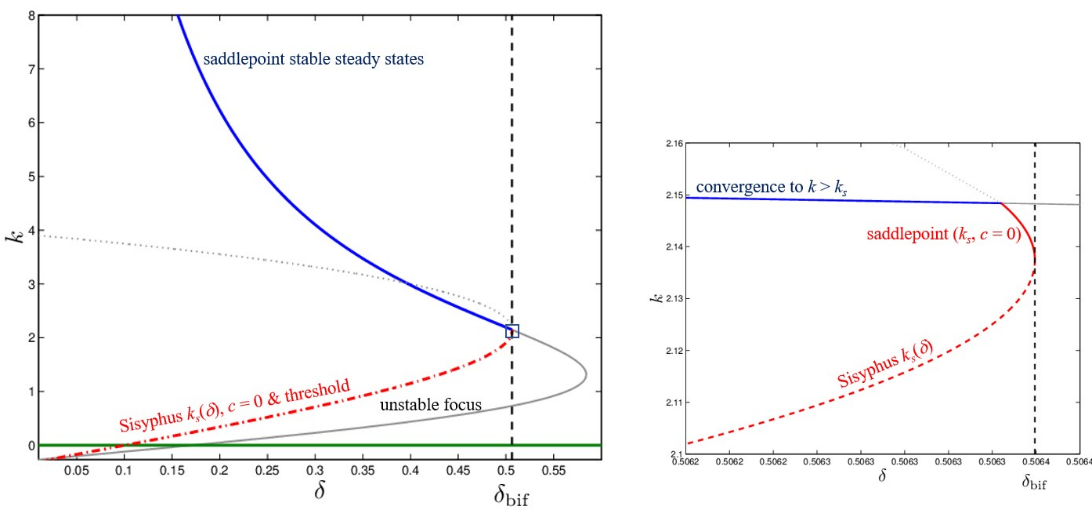


Figure 4: *The bifurcation diagram w.r.t. δ for* (16) *and $m = 0.3$. The colors indicate the optimal paths: convergence to the saddle point (blue), staying at the Sisyphus point (red, only visible in the enlargement on the right-hand side) and converging to the origin (green, for all points below $k_s(\delta)$, red dashed).*

5.2 The Optimal Strategies

Fig. 5a shows the optimal strategies in the (k, i) space for $m = 0.05$ (small convexity, but the generic case) and all other parameters as in (16) and Fig. 3. There are three possible long run outcomes depending on initial conditions. As conjectured, sufficiently large initial endowments with human capital lead to the high steady state as shown in the bifurcation diagram in Fig. 3. Low initial capital requires one to leave the profession. However, if placed at the Sisyphus point, then it is optimal to stay there forever! Two trajectories emerging from the Sisyphus point $k_s \approx 2.5$ move either to $k = 0$ or to the upper saddle point equilibrium (denoted $k_\infty \approx 5.7$). However, both strategies start and evolve over a wide range of human capital along the constraint $c = 0$ (shown in red) but end up very differently either at the saddle point path (at least close to the steady state k_∞ shown in blue) or at another border solution ($i = 0$ close to $k = 0$, in green). Fig. 5b shows the value function $V(k)$ for the same case highlighting the steep, actually infinite, slope $V' = \infty$ at k_s. This property is important from both an economic (see the discussion in the following subsection) and a mathematical point of view because of the violation of the assumption of normality, more precisely, the usually to 1 normalized coefficient λ_0 of the objective in the Hamiltonian (11) turns 0 and the costate λ diverges to infinity. The details, including the numerical treatment of the abnormal case, are given in the Appendix. The Sisyphus point k_s serves as a threshold that separates the two attractors, $k \to 0$ and $k \to k_\infty > 0$, and ensures continuity of the control (investment) instead of the jump typically linked to such non-concave dynamic optimization problems. Furthermore, even if starting to the right-hand side of the Sisyphus point and thus continuing with one's profession, it is optimal to do so with minimal consumption (i.e., at the boundary, $c = 0$) in order to invest the maximum possible subject to the impossibility of getting credit.

Fig. 6 shows the structural changes. At $m = 0.03$, which is below the case discussed above in Fig. 5, the optimal policy is to move to $k = 0$ (to give up) for all initial conditions. An increase of the Matthew parameter to $m = 0.03527$ leads to the emergence of a Sisyphus point along the optimal path, but it is then only passed on the way to the origin. Further increases of m render the high steady state feasible, which coincides with the Sisyphus point for $m = 0.03531$. That is, a positive steady state is optimal at least for initial capital exceeding the Sisyphus point, $k_0 > k_s$, but still at the boundary, i.e., $c = 0$. Further increases of m move the higher steady state into the interior allowing for $c > 0$, as shown in the example in Fig. 5. Even larger Matthew effects, such as $m = 0.3$ corresponding to the phase portrait in Fig. 2, render the high steady state attracting over a much wider range of initial conditions but may still require living at the boundary (see Fig. 2) unless endowed with large human capital.

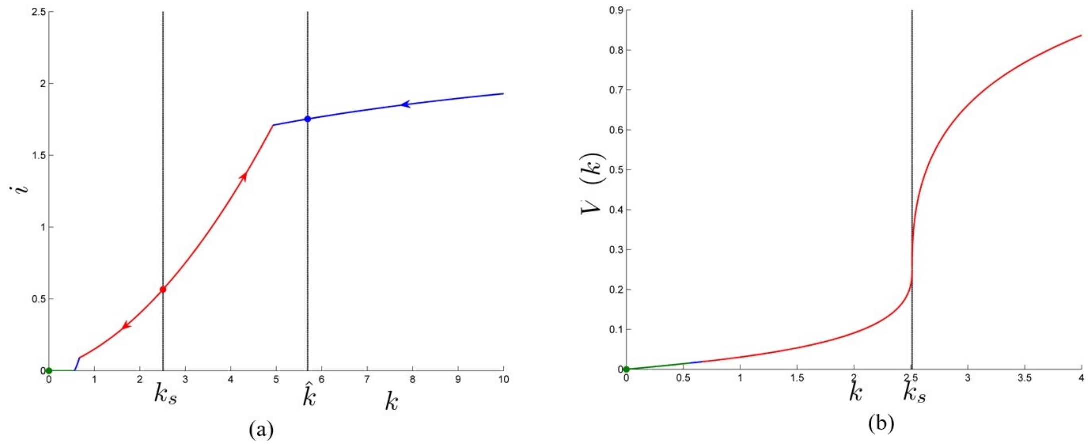


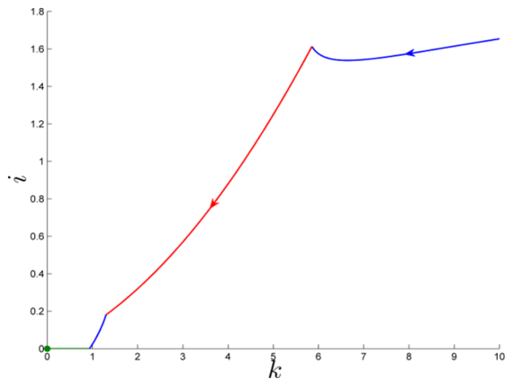
Figure 5: *The generic optimal policy for* (16) *and* $(m, \delta) = (0.05, 0.2)$*. The state space (panel a) is separated by a threshold at the Sisyphus point* k_s*: to the left, the optimal solution converges to zero and to the right it is optimal to move to an interior equilibrium* k_∞*. For* $k(0) = k_s$ *it is optimal to stay put. The circles denote the equilibria and the color code refers to different (in)active constraints: blue no constraint is active, red if* $c = 0$ *and green of* $i = 0$*. Panel (b) shows the value function* $V(k)$*.*

5.3 Economic Implications

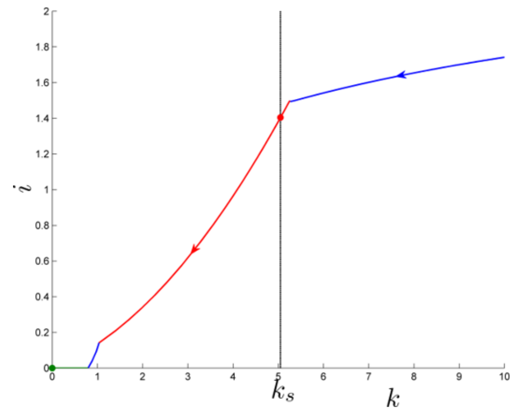
The model and our simulations describe the returns to talents accounting for individual accumulation of human capital and the possibility of (very) high returns or none at all. Therefore, we use the above results to derive a few economic implications for arts and sports, and to some extent also for science.

1. There always exists an area of initial conditions, $k_0 \in [0, k_s]$, for which it is impossible to grow. This explains why most do not enter particular fields of arts, sports and science and why many of those who enter give up.

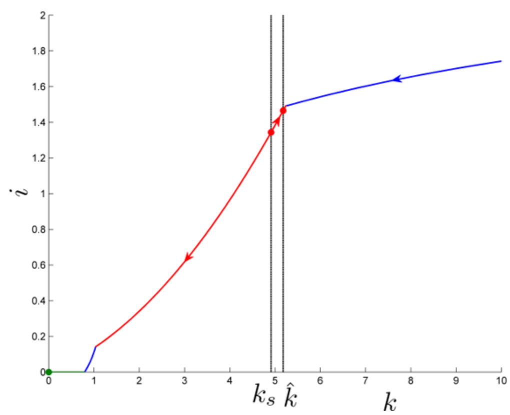
2. It also suggests the need for scholarships in order to foster young talents, artists, sportsmen and scientists, before they can make a living from the market's returns. If the duration of the scholarship is short relative to the applicant's maturity, he or she gives up (actually, has to) and the talent is lost (over time). Hence, the share of lost talents depends on competition in a sector and on the availability of scholarships (public or private). This explains the dominance of the former Socialist economies in particular when



(a) $m = 0.03$, $k \longrightarrow 0$ is the only solution



(b) $m = 0.03527$, the Sisyphus point exists and the only way is $k \longrightarrow 0$



(c) $m = 0.03531$, the Sisyphus point and an interior equilbrium appear

Figure 6: *Structural change in the* (k, i) *space for* (16) *and* $\delta = 0.2$ *and for* m *in the neighborhood of the lowest Sisyphus point (compare Fig.3). Circles refer to equilibria, the colors to the different (in)active constraints: blue means no constraint is active, red indicates that the constraint* $c = 0$ *is active and green* $i = 0$ *is active.*

the German Democratic Republic, GDR, beat the Federal Republic of Germany, FRG, in terms of medals at many Olympic games (with a population less than a third).

3. Even if a talent is able to surpass the Sisyphus point, the following period is hard, because Fig. 5 and also Fig. 2 (for a different scenario) suggest that all returns have to be invested ($c = 0$) at this stage for quite some time.

The reason is that the return to one's human capital is very large in this domain (actually infinite at the Sisyphus point). This ascent is followed by a period of slow growth (blue line) towards the steady state along which the agent can already enjoy the fruits of his/her work and talent. The story of Martin Eden (by Jack London) explains this phenomenon very well. Many of the impressionists did not become famous during their creative time – and of those many only posthumously. This explains also why in the past many artists and scientists came from wealthy and often noble families or depended on rich patrons or already famous men (e.g., Giotto on Cimabue) because of the need to self-finance education.[10]

4. The bifurcation diagram w.r.t. the depreciation δ in Fig. 4 implies: a very low depreciation allows for unbounded growth (as in the AK-model of Rebelo, 1991), while no other steady state than the boundary solution, $k \to 0$, exists for high depreciation rates. This can explain that only few become successful, contrary to the majority ending up with low productivity and no fame. Depreciation depends not only on personal abilities but also on trends (in science) and fashions (in arts, recall the fate of good naturalistic painters during the 20th century) that can depreciate one's particular kind of human capital. While in some fields (e.g., in some branches of mathematics) it is possible to use old knowledge, the need for particular skills changes very rapidly in others, for example, in informatics (people are forced to learn new versions of software every few years).[11] Therefore, working in a field characterized by a low erosion of the usage of particular techniques renders a comparative advantage. However, this effect may be countered by excessive entry of young talents who prefer to enter a stable rather than a volatile field, just think of the many students flocking to literature and political science. The parameter b captures just the opposite of δ, in fact, only $b - \delta$ matters.

5. The bifurcation diagram in Fig. 3 w.r.t. the parameter m, which accounts for disproportional returns to human capital (and thus, reflects the Matthew

[10] However, DeVereaux (2018) suggests that artists who earn a higher wage will work fewer hours.

[11] And that is why LaTeX is so successful.

effect), and the analysis of the corresponding optimal paths in Figs. 5 and 6 show: i) the location of the Sisyphus point shrinks as m grows; ii) the location of the upper saddle grows, but only slowly, and so does its domain of attraction. However, the first effect is stronger. Indeed, given the optimality of the high steady state, the existence of the Sisyphus point ensures that $k \to 0$ is optimal for sufficiently low endowments, at least for $k_0 < k_s$. Therefore, creators characterized by a large value of m are able to survive even if starting at relatively low initial human capital. A high value of m need not be due to human capital itself but could arise from exceptional ability to sell or market one's talent or to have access to networks and to large markets.

6. The parameter h accounts for non-monetary utilities. It accounts and explains why some artists really go for their topic even if that means fighting for survival due to the lack of sufficient proceeds from their work.

6 Conclusions

We have formulated an intertemporal optimization problem about the career decisions of an individual talent (in arts, sports or science) accounting for the difficulty (more precisely, impossibility in our model) to take credit and the convex returns to human capital that capture the Matthew effect observed in different areas. The proposed model leads to multiple steady states and thus to thresholds due to a non-concave maximization problem. Only sufficiently high initial values of human capital allow for convergence to the high (saddle point) equilibrium. Therefore, whether one should pursue or stop depends on initial conditions. The novel feature is the appearance of a Sisyphus point, i.e., a point at which consumption is at the subsistence level ($c = 0$), because all proceeds must be invested in order to avoid the decline of human capital. It turns out that this point can indeed be optimal and can determine the location of the threshold. Furthermore, the no-debt constraint affects the outcome substantially: first, it eliminates otherwise feasible interior solutions; second, even if it is optimal staying in business, it determines investment. This finding is in line with Caucutt & Lochner (2020) who show in an entirely different context that (life-cycle) borrowing constraints severely limit investments into human capital.

The model differs from previous socio-economic studies of thresholds by the emergence of new and special properties at a point that we call a Sisyphus point because it is associated with zero consumption. In particular, it can be optimal to stay at this point forever (but not necessarily), if the intrinsic benefit (the parameter h) is sufficiently large and can serve as a threshold between attractors to leaving (convergence to the origin of the state space) or to attaining a profitable outcome in one's profession. Therefore, Sisyphus points provide a sharp differentiation about the career prospects and how they depend on initial human capital, e.g., in science after receiving a Ph.D. If the path to $k = 0$ is the optimal (or the only viable) outcome for many, then it describes the Matthew effect and explains why some never publish a paper and quit science. This phenomenon is also observable in the arts (with teaching as an exit option for painters and musicians), in sports (e.g., offering tennis or skiing lessons after exiting) and for small businesses. The reason for this separation of outcomes is as follows. The reward grows always at k^2. The linear term dominates the necessary investments for capital expansion (i.e., $i > i^{min}$) at small levels of k while the quadratic term dominates for large values k (see Fig. 1).

An initial jump, $k_0 > k_s$, corresponds to the following situation, e.g., in research: a young researcher publishes a paper already prior to receiving the Ph.D., which helps in the post-graduate job market and thereby reinforces further growth (e.g., landing at a famous institution).[12] Those less lucky, can choose between staying at their Sisyphus point or quitting (human capital converges to zero). Indeed, many young scientists face this problem and even relatively established scientists can temporarily find themselves in a Sisyphus trap; such outcomes are even more frequent in the arts.

The model presented in this paper can be applied and extended into many directions. Our treatment of abnormality in an optimal control problem should be applicable to other problems lacking the property of normality. In terms of theory, a possible extension is for uncertainty (continuous as well as jumps) due to

[12]However, things need not be that straightforward, because a recent article in "The Economist (2019)" argues that "if at first you do not succeed, try, try, try again".

the presence of luck, in particular, in arts and also in science, but less so in sports; another one is to construct alternative formulations of both, the objective and the dynamic constraint. In terms of applications, the presented or related frameworks may provide insights into other fields and can lead to similar complex dynamic patterns including the possibility of Sisyphus points. In terms of economic policies, the existence of Sisyphus points can create socially unfavorable outcomes if too many talents cannot finance investment into their human capital due to credit constraints.

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A Appendix

Sisyphus point as an abnormal solution

We will explain the concepts of normality and abnormality in optimization problems. Therefore we start with a short excursion of a general finite dimensional constraint optimization problem, which in its simplest form writes as

$$\max_{x \in \mathbb{R}} f(x) \tag{17a}$$

$$g(x) \geq 0, \tag{17b}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions. To identify a maximizer of problem (17) a necessary optimality condition is the Kuhn-Tucker criterion (Kuhn & Tucker, 1950). Let x^* be a maximizer of problem (17) and let

$$L(x, \lambda) \coloneqq f(x) + \lambda^\top g(x) \tag{18a}$$

be the Lagrangian function, then there exists $\lambda \in \mathbb{R}^m$ satisfying,

$$L_x(x^*, \lambda) = 0, \tag{18b}$$

$$\lambda^\top g(x^*) = 0, \tag{18c}$$

$$\lambda \geq 0. \tag{18d}$$

But in general, without any further assumptions, so called constraint qualifications, the conditions (18c)-(18d) may fail. For the formulation of necessary optimality conditions without constraint qualifications conditions (18a)-(18d) hold only for the extended Lagrangian,

$$L(x, \lambda, \lambda_0) \coloneqq \lambda_0 f(x) + \lambda^\top g(x). \tag{19}$$

Due to the linearity of the Lagrangian in the Lagrange-multipliers (λ_0, λ) we can divide by λ_0 if $\lambda_0 \neq 0$. Thus, two cases can be distinguished, where either $\lambda_0 = 1$ also called the *normal* case or $\lambda_0 = 0$ the *abnormal* case.

The concepts of normality and abnormality of a problem also pass to the infinite dimensional case of optimal control problems. Halkin (1974) for example showed that for infinite time horizon problems with free end state the problem may not be normal.

Next we derive the necessary optimality conditions for the active constraint $mk^2 + bk - i \geq 0$, Eq. (3). Therefore we consider the Lagrangian,

$$L(k, i, \lambda, \nu, \lambda_0) = H(k, i, \lambda, \lambda_0) + \nu(mk^2 + bk - i), \tag{20}$$

where $H(k, i, \lambda, \lambda_0)$ is the Hamiltonian (11). Then an optimal solution $(k^*(\cdot), i^*(\cdot))$, with active constraint, has to satisfy,

$$i^*(t) = i^\circ(k^*(t))$$

with

$$i^\circ(k) = mk^2 + bk. \tag{21}$$

The condition $L_i(k, i, \lambda, \nu, \lambda_0) = 0$ yields the Lagrangian multiplier,

$$\nu(k, \lambda, \lambda_0) = -\lambda_0 + \lambda(1 - 2a(mk^2 + bk)). \tag{22}$$

The canonical system writes as

$$\begin{aligned}
\dot{k}(t) &= i^\circ(t) - ai^\circ(t)^2 - \delta k(t), && (23)\\
\dot{\lambda}(t) &= r\lambda - L_k(k(t), i^\circ(k(t)), \lambda(t), \nu(k(t), \lambda(t), \lambda_0), \lambda_0) \\
&= (r + \delta)\lambda(t) - \lambda_0(2k(t)m + b + h) - && (24)\\
&\qquad \nu(k(t), \lambda(t), \lambda_0)(2mk(t) + b),
\end{aligned}$$

together with the boundary conditions,

$$k(0) = k_0 \quad \text{and} \quad \lim_{t\to\infty} \mathrm{e}^{-rt}\lambda(t) = 0. \tag{25}$$

Let $\lambda(\cdot)$ satisfy the canonical system Eqs. (23)–(25) at the optimal solution $(k^*(\cdot), i^*(t))$. Then additionally the Lagrangian multiplier has to satisfy the complementary slackness condition,

$$\nu(k^*(t), \lambda(t), \lambda_0)(mk^*(t)^2 + bk^*(t) - i^*(t)) = 0$$

and the non-negativity condition,

$$\nu(k^*(t), \lambda(t), \lambda_0) \geq 0. \tag{26}$$

Subsequently, we consider the solution behavior in the vicinity of the Sisyphus point k_s, see Fig. 3a. Therefore we analyze the properties of the corresponding equilibrium (k_s, λ_s) in the state-costate space with

$$\lambda_s = \frac{h}{r + \delta - (b + 2k_s m)(1 - 2ak_s(b + mk_s))}, \tag{27}$$

the zero of Eq. (24).

Then we can distinguish two different cases. Firstly, if (k_s, λ_s) is an unstable node and secondly where it is a saddle. In the bifurcation diagram Fig. 3a the first case is depicted by a dashed (red) curve and the second case is depicted by a dashed-dotted (red) line.

The Lagrange multiplier (26) evaluated at the equilibrium (k_s, λ_s) is of particular importance. For simplicity we consider the case where (k_s, λ_s) is a *hyperbolic* equilibrium of the canonical system Eqs. (23)–(24). An equilibrium is *hyperbolic* if no (real part of the) eigenvalue of the Jacobian, evaluated at the equilibrium, is zero.

Proposition A1 *Let $(k_s, \lambda_s) \in \mathbb{R}^2$ be a hyperbolic equilibrium of the canonical system Eqs. (23)–(24) and $\lambda_0 = 1$. The equilibrium (k_s, λ_s) is a saddle iff the Lagrangian multiplier satisfies*

$$\nu(k_s, \lambda_s, \lambda_0) < 0. \tag{28}$$

The eigenvalues $\xi_{1,2}$ of the Jacobian at (k_s, λ_s) are

$$\xi_1 = (b + 2k_s m)(1 - 2ak_s(b + mk_s)) - \delta > 0 \tag{29}$$

and

$$\xi_2 = r + \delta - (b + 2k_s m)(1 - 2ak_s(b + mk_s)) < 0 \tag{30}$$

and the eigenvector related to ξ_2 *is*

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{31}$$

Proof The Jacobian J_s at (k_s, λ_s) of the canonical system Eqs. (23)–(24) is of the form

$$J_s = \begin{pmatrix} \Gamma(k_s) & 0 \\ \Omega(k_s, \lambda_s) & r - \Gamma(k_s) \end{pmatrix}$$

with

$$\Gamma(k) := (b + 2km)(1 - 2ak(b + mk)) - \delta.$$

The eigenvalues $\xi_{1,2}$ and corresponding eigenvectors $v_{1,2}$ of J_s are

$$\begin{aligned} &\xi_1 = \Gamma(k_s), \quad \xi_2 = r - \Gamma(k_s), \\ &v_1 = \begin{cases} \begin{pmatrix} \dfrac{r}{\Omega(k_s, \lambda_s)} \\ 1 \end{pmatrix} & \text{for} \quad \Omega(k_s, \lambda_s) \neq 0, \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for} \quad \Omega(k_s, \lambda_s) = 0, \end{cases} \\ &v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{32}$$

First we show that $\Gamma(k_s) > 0$. Assume to the contrary that $\Gamma(k_s) < 0$, then $r - \Gamma(k_s) > 0$ and (k_s, λ_s) is a saddle. In that case we find that due to Eq. (32) the stable path has a nonzero k component. Specifically, this means that there exist $k_0 < k_s$ and $\tilde{\lambda}$ such that the path $(k(\cdot), \lambda(\cdot))$ with $k(0) = k_0$ and $\lambda(0) = \tilde{\lambda}$ satisfying the canonical system Eqs. (23)–(24) converge to (k_s, λ_s). Since $k_0 < k_s$

this implies $\dot{k}\mid_{k_0} > 0$ contradicting the definition of k_s. Thus, we find $\Gamma(k_s) > 0$. Therefore, (k_s, λ_s) is a saddle iff $r - \Gamma(k_s) < 0$. Equation (6) implies that

$$i^{\max}(k_s) = mk_s^2 + bk_s < \frac{1}{2a}$$

and hence

$$1 - 2a(mk_s^2 + bk_s) > 0.$$

Using the expression of the equilibrium costate λ_s given by (27) and the expression for the Lagrangian multiplier (22) we find. The expression $r - \Gamma(k_s) < 0$ iff $\lambda_s < 0$ and hence

$$\nu(k_s, \lambda_s, 1) = -1 + \lambda_s(1 - 2a(mk_s^2 + bk_s)) < 0.$$

This finishes the proof. ■

Next, we define the optimized value function $V^*(\cdot)$. Let

$$V(k(\cdot), i(\cdot)) \coloneqq \int_0^\infty \mathrm{e}^{-rt}(mk(t)^2 + bk(t) - i(t) + hk(t))\,\mathrm{d}t,$$

then

$$\begin{aligned} &V^*(k_0) \coloneqq \max_{i(\cdot)} V(k(\cdot), i(\cdot)),\\ &0 \le i(t) \le mk(t)^2 + bk(t), \quad t \ge 0, \end{aligned}$$

where $k(\cdot)$ and $i(\cdot)$ satisfy the state dynamics (2), is called the *optimized value function* of problem (1)–(3).
We will also make use of the specific expression for the control value in the interior of the control region given by Eq. (12) and therefore define

$$i^{\mathrm{int}}(\lambda) \coloneqq \frac{\lambda - 1}{2a\lambda}. \tag{33}$$

Proposition A2 *Let $(k^*(\cdot), i^*(\cdot)) \equiv (k_s, i^\circ(k_s))$ be the optimal solution for problem (1)–(3) with $k(0) = k_s$ and let (k_s, λ_s) be a hyperbolic node for $\lambda_0 = 1$ and* *let*

$$i^{\text{int}}(\lambda_s) > i^{\max}(k_s). \tag{34}$$

Then the problem is normal, i.e. $\lambda_0 = 1$. *Let* $V^*(k)$ *be the optimized value function, then* $V^*(\cdot)$ *is continuously differentiable in* k_s. *Specifically it satisfies*

$$\lim_{k \to k_s} \frac{\partial}{\partial k} V^*(k) = \lambda_s. \tag{35}$$

Remark Property (34) states that the interior control value, given by the term (34), is not admissible at the Sisyphus point k_s.

Proof We make use of the following property for an optimal solution of problem (1)–(3). Let

$$(k^*(k_0, \cdot), i^*(k_0, \cdot)),$$

denote the optimal solution of problem (1)–(3) for $k(0) = k_0$. Then the following property holds,

$$\lim_{k_0 \to k_s} (k^*(k_0, \cdot), i^*(k_0, \cdot)) = (k^*(k_s, \cdot), i^*(k_s, \cdot)) = (k_s, i^{\max}(k_s)),$$

specifically, we have

$$\lim_{k_0 \to k_s} i^*(k_0, 0) = i^{\max}(k_s).$$

In the following we assume w.l.o.g. $k_0 \geq k_s$. Two cases have to be distinguished,

$$i^*(k_0, 0) < i^{\max}(k_s), \quad k_0 \neq k_s,$$

or there exists a $\bar{k} > k_s$ such that

$$i^*(k_0, 0) \begin{cases} < i^{\max}(k_s) & \text{for} \quad k_0 > \bar{k}, \\ = i^{\max}(k_s) & \text{for} \quad k_s \leq k_0 \leq \bar{k}. \end{cases}$$

The first case can be excluded due to assumption (34). Since in that case the costate function $\lambda(k_0, \cdot)$ corresponding to the optimal solution for $k(0) = k_0$ satisfies

$$\lim_{k_0 \to k_s} i^*(k_0, 0) = \lim_{k_0 \to k_s} i^{\text{int}}(\lambda(k_0, 0)) = i^{\text{int}}(\lambda_s) = i^{\max}(k_s),$$

which violates assumption (34), $i^{\text{int}}(\lambda_s) > i^{\max}(k_s)$.

The second case implies that the costate function $\lambda(k_0, \cdot)$ satisfies the adjoint equation (25) and hence,

$$\lim_{k_0 \to k_s} \lambda(k_0, \cdot) = \lambda_s,$$

or using the relation of the costate and the optimal value function, $\frac{\partial}{\partial k} V^*(k) = \lambda(k, 0)$

$$\lim_{k \to k_s} \frac{\partial}{\partial k} V^*(k) = \lambda_s,$$

implying Eq. (35).

Moreover, we find that in the vicinity of k_s the non-negativity of the Lagrange multiplier (26) is fulfilled,

$$\nu(k_0, \lambda(k_0, 0), 1) \geq 0, \quad k_s < k_0 < \bar{k}$$

and hence

$$\lim_{k_0 \to k_s} \nu(k_0, \lambda(k_0, 0), 1) = \nu(k_s, \lambda_s, 1) \geq 0,$$

proving that (k_s, λ_s) satisfies the necessary optimality conditions for $\lambda_0 = 1$ and hence the problem is normal. This finishes the proof. ■

If the Sisyphus point and the related control are the optimal equilibrium solution but (k_s, λ_s) is a saddle the following proposition holds.

Proposition A3 *Let $(k^*(\cdot), i^*(\cdot)) \equiv (k_s, i^\circ(k_s))$ be the optimal solution for problem (1)–(3) with $k(0) = k_s$ and let (k_s, λ_s) be a saddle for $\lambda_0 = 1$. Then the problem is abnormal i.e. $\lambda_0 = 0$.*

Let $V^*(k)$ with $|k - k_s| < \varepsilon$ be the optimized value function, then $V^*(\cdot)$ is not

Lipschitz continuous in k_s. Specifically it satisfies

$$\lim_{k \to k_s, k \neq k_s} \frac{\partial}{\partial k} V^*(k) = \infty. \tag{36}$$

For $\lambda_0 = 0$ the point $(k_s, 0)$ is a saddle of the canonical system Eqs. (23)–(24).

Proof Repeating the first part of the proof of Prop. A2 we find the two cases an optimal solution in the vicinity of k_s can satisfy

$$i^*(k_0, 0) \begin{cases} < i^{\max}(k_s) & \text{for} \quad k_0 > k_s, \\ = i^{\max}(k_s) & \text{for} \quad k_0 \leq k_s. \end{cases}$$

The first case

$$i^*(k_0, 0) < i^{\max}(k_s), \quad k_0 \neq k_s, \tag{37}$$

can be excluded. Since in Prop. A1 we showed that if (k_s, λ_s) is a saddle, the Lagrange multiplier is negative. But from Eq. (37) and the slackness condition it follows that

$$\nu(k_0, \lambda(k_0, 0), 1) = 0$$

and hence

$$\lim_{k_0 \to k_s} \nu(k_0, \lambda(k_0, 0), 1) = \nu(k_s, \lambda_s, 1) = 0,$$

which violates

$$\nu(k_s, \lambda_s, 1) < 0.$$

For this argument we used the continuity of the Lagrange multiplier, which is a result of the uniqueness of the control value in the Hamilton maximizing condition,

$$i^*(t) = \operatorname{argmax} H(k^*(t), i, \lambda(t), \lambda_0) \quad \text{with} \quad 0 \leq i \leq mk^*(t)^2 + bk^*(t).$$

Thus, in the neighborhood of k_s the costate functions $\lambda(k_0, \cdot)$ satisfy the adjoint equation (24) and due to the properties of the saddle (k_s, λ_s) derived in Prop. A1 we find

$$\lim_{k_0 \to k_s} \lambda(k_0, 0) = \infty,$$

or using the relation of the costate and the optimal value function, $\frac{\partial}{\partial k} V^*(k) = \lambda(k, 0)$,

$$\lim_{k \to k_s} \frac{\partial}{\partial k} V^*(k) = \infty,$$

which implies that the optimal objective value is not Lipschitz continuous in k_s.
To prove the abnormality of the problem we first state that (k_s, λ_s) do not satisfy the necessary optimality conditions, since the Lagrange multiplier is negative. Assume that we choose some $\lambda(k_s, 0)$ such that

$$\nu(k_s, \lambda(k_s, 0), 1) \geq 0.$$

Since $\{(k_s, \lambda) \colon \lambda \in \mathbb{R}\}$ is the stable manifold of (k_s, λ_s) this implies the existence of some time $\tau(\lambda(0))$ such that for all $t > \tau(\lambda(0))$ the Lagrange multiplier evaluated at the costate function $\lambda(k_s, \cdot)$ fulfills

$$\nu(k_s, \lambda(k_s, t), 1) < 0$$

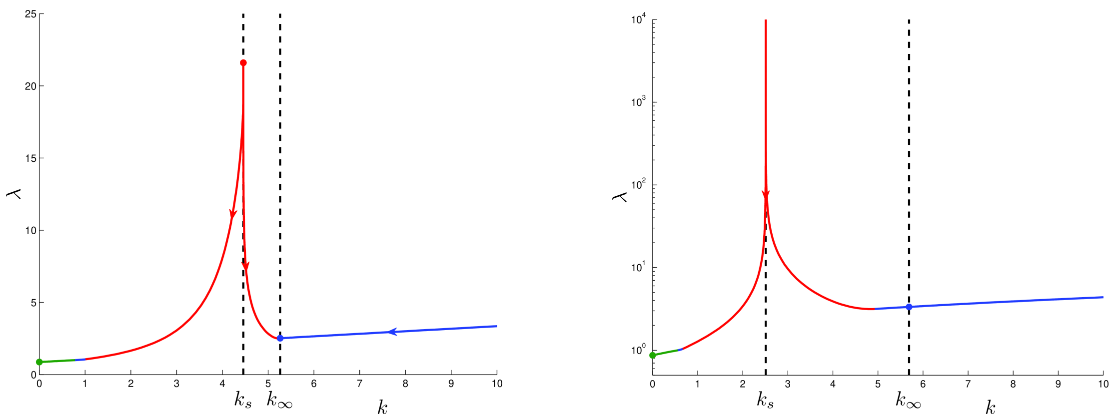
and hence violates the necessary optimality conditions. Consequently the necessary optimality conditions for the optimal solution $(k_s, i^{\max}(k_s))$ are only satisfied for $\lambda_0 = 0$ and therefore the problem is abnormal.
Setting $\lambda_0 = 0$ we find that $(k_s, 0)$ is a saddle of the canonical system and hence for every initial $\lambda(0) > 0$ the necessary optimality conditions are satisfied, specifically,

$$(\lambda(\cdot), \lambda_0) \neq 0.$$

This finishes the proof. ■

Remark An interpretation of abnormality of the problem in the Sisyphus point in economic terms can be the following. In the Sisyphus point the optimal control is on the edge. There is no other possibility than to choose $i^{\max}(k_s)$ and the slightest change in the state value yields a sharp (infinite) relative increase/decrease in the optimal profit.

In our numerical examples presented in Sec. 5.1 there is a small region $m \in (0.0352\ldots, 0.0353\ldots)$, where (k_s, λ_s) is an unstable node and Prop. A2 applies, see Fig. A.1a. For parameter values $m > 0.0353\ldots$ the equilibrium (k_s, λ_s) is a saddle and Prop. A3 applies, see Fig. A.1b.



(a) $m = 0.036$ (b) $m = 0.05$

Figure A.1: *In panel (a) the equilibrium* (k_s, λ_s) *is an unstable node (see Prop. A2) and the problem for* $k(0) = k_s$ *is normal. In panel (b) the equilibrium* (k_s, λ_s) *is a saddle (see Prop. A3) and the problem for* $k(0) = k_s$ *is abnormal. The parameter values are taken from* (16).