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Approximating open-loop and closed-loop optimal control by model predictive control*

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Abstract

This research report contains an extraction of some results from the report [6] and an additional material, as described in the next lines. We consider a finite-horizon continuous-time optimal control problem with nonlinear dynamics, an integral cost, control constraints and a time-varying parameter which represents perturbations or uncertainty. After time-discretization of the problem we employ a Model Predictive Control (MPC) algorithm, which uses a "prediction"/forecast for the uncertain parameter and (possibly inexact) measurements of the state vector, and generates a piecewise constant control signal by solving auxiliary open-loop control problems. In our main result we derive an estimate of the difference between the MPC-generated control and the optimal feedback control, both obtained for the same value of the perturbation parameter, in terms of the step-size of the discretization and the measurement error. We also estimate the distance from the MPC-generated control to the the optimal open-loop control in the problem with the "true" value of the uncertain parameter, depending on the prediction error.

1 Introduction

In this paper we consider an optimal control problem over a fixed finite time interval, which involves an integral cost functional, a control system described by a nonlinear ordinary differential equation, and control constraints given by a closed and convex set. Both the dynamics and the cost depend on a time-varying parameter which represents perturbations or uncertainty. We assume that only a reference trajectory of the time evolution of this parameter

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is available in advance, which is interpreted as a reference or a prediction for the true time-evolution of the parameter.

The optimal feedback synthesis is a basic problem in control theory. It assumes that the current state can be measured at each instance of time and consists in finding a device (mathematically, a mapping) to automatically generate an optimal control, which is then applied to the input of the system. The advantages of using optimal feedback control, when compared with open-loop control, are well known. However, finding an exact optimal feedback law for a nonlinear system with control constraints could be quite challenging.

In this paper we consider a Model Predictive Control (MPC) algorithm aiming at generating an approximate optimal feedback. It uses an initial time-discretization of the optimal control problem over a uniform mesh $\{t_i\}_0^N$, and assumes that measurements of the state are taken at every t_i from the mesh. Based on this, a control on the current time interval $[t_i, t_{i+1}]$ is computed by solving a corresponding discretized optimal control problem on the remaining time horizon $[t_i, t_N]$, which is then applied to the system. The main contribution of this paper is an estimate of the norm of the difference between the MPC-generated state and control trajectories and the state and control trajectories generated by the optimal feedback control law. We note that there are various ways to evaluate the loss of optimal performance; here we focus on the one represented by the difference between the respective solutions. To the best of authors' knowledge, this problem has not been treated in the literature in this setting.

The MPC has been extensively explored in the last decades; see e.g. the books [11] and [17] for a broad coverage of the basic aspects of it. MPC has also found numerous applications in various industries. Common variants of MPC involve receding or moving horizon implementation which leads to a time-invariant feedback law for stabilization and tracking. In other problems, such as problems involving spacecraft landing or docking, helicopter landing on ships, missile guidance, control of chemical batch processes etc., the control is performed over a finite time interval. The application of MPC in this setting leads to shrinking horizon formulations. Shrinking horizon MPC has been considered, for instance, in [18, 19], but compared to the receding horizon MPC, it has been less studied. As for the receding horizon MPC, the shrinking horizon MPC is expected to provide an approximation of the optimal finite horizon feedback control law that can be used to handle systems whose dimension is higher than those to which dynamic programming is applicable. At the same time, since shrinking horizon MPC is a form of a feedback law, it will improve robustness to uncertainty when compared with open-loop finite horizon control. The latter expectation is also supported by the results in this paper. We note that the effect of discretization addressed in the current paper has been considered for sampled-data MPC in a receding horizon setting, e.g. in [9, 10, 16]), but not for the finite horizon case studied here.

To put the stage, in this paper we consider the following optimal control problem, which we call problem $\mathcal{P}_p$:

$$\min \left\{ J_p(u) := g(x(T)) + \int_0^T \varphi(p(t), x(t), u(t))\,\mathrm{d}t \right\}, \tag{1}$$

subject to

$$\dot{x}(t) = f(p(t), x(t), u(t)), \quad u(t) \in U, \quad x(0) = x_0, \tag{2}$$

where the time $t \in [0, T]$, the state $x(t)$ is a vector in $I\!R^n$, the control u has values $u(t)$ that belong to a convex and closed set U in $I\!R^m$ for almost every (a.e.) $t \in [0, T]$, and $p(t) \in I\!R^l$ is the value of a parameter p which is a function of time on $[0, T]$ representing uncertainty. When we say a "value" of the parameter, we mean a specific function of time $t \in [0, T]$ representing the parameter evolution in time. The initial state x_0 and the final time $T > 0$ are fixed. The set of feasible control functions u, denoted in the sequel by $\mathcal{U}$, consists of all Lebesgue measurable and essentially bounded functions $u : [0, T] \to U$. The parameter p is a Lebesgue measurable and essentially bounded function on $[0, T]$ with values in $I\!R^l$. A state trajectory, denoted by $x[u, p]$, is a solution of (2) for a feasible control u and a value p of the parameter; accordingly, the state trajectories are Lipschitz continuous functions of time $t \in [0, T]$. For a specific choice of p, the state equation (2) is thought of as representing the "true" dynamics of a "real" system. Although the "true" representation of the parameter p is considered unknown, some reference (prediction) $\bar{p}$ for it is assumed to be available. In applications, $\bar{p}$ represents a preview/forecast of p which could be a disturbance (e.g., wind gust for an aircraft or road grade for a car) or a load in an electrical power system. In this paper we do not assume that the values of the parameter p are updated throughout the iterations of the MPC algorithm; taking into account such updates is of considerable interest and will be addressed in further research.

We consider the following MPC algorithm. First, the problem is discretized; we use for that the simplest Euler scheme over a uniform mesh. Given a natural number N, let $\{t_k\}_0^N$ be a grid on $[0, T]$ with equally spaced nodes t_k and a step-size $h = T/N$. To describe the MPC iteration, fix $k \in \{0, 1, \ldots, N-1\}$ and assume that a control u^N with $u^N(t) \in U$ is already determined on $[0, t_k)$. This control is applied to the real system (that with the value p of the parameter). Assume that the corresponding state at time t_k, $x[u^N, p](t_k)$, is measured (or estimated) with an additive error ξ_k, that is, the vector $x_k^0 := x[u^N, p](t_k) + \xi_k$ becomes available at time t_k. The next step is to solve the discrete-time optimal control problem

$$\min \left\{ g(x_N) + h \sum_{i=k}^{N-1} \varphi(\bar{p}(t_i), x_i, u_i) \right\},$$

subject to

$$\begin{aligned} x_{i+1} = x_i + hf(\bar{p}(t_i), x_i, u_i), \quad x_k := x_k^0, \\ u_i \in U, \quad i = k, \ldots, N-1. \end{aligned} \tag{3}$$

Note that this problem is solved for the reference value $\bar{p}$ of the parameter. For $k = 0$ we have $x_0^0 = x_0 + \xi_0$. Suppose that a locally optimal discrete-time control $(\tilde{u}_k, \ldots, \tilde{u}_{N-1})$ is obtained as a solution of this problem. Define the constant in time function

$$u^N(t) = \tilde{u}_k \quad \text{for } t \in [t_k, t_{k+1}),$$

change k to $k+1$ and continue the iterations as long as $k < N$, obtaining at the end a control u^N which is a piecewise constant function and which we call the *MPC-generated control*. Note that the MPC-generated control u^N may not be uniquely determined, e.g., because the discrete-time optimal control problem appearing at some stage does not have a unique solution. Also note that we keep the final time T fixed, so that the time horizon shrinks at each iteration.

Assume that there exists an (exact) optimal feedback $u^*(t,x)$ for problem $\mathcal{P}_{\bar{p}}$ with the reference value $\bar{p}$ of the parameter, provided that the state x entering the optimal feedback $u^*(t,x)$ is measured exactly. In the next section, we give an answer to the following question: what is the impact on state and control trajectories (and consequent loss of performance) if the MPC-generated control u^N (possibly in presence of measurement errors) is used in the system (2), with the value p of the parameter, instead of the exact optimal feedback u^*. The distance between the obtained solutions is estimated in appropriate norms as at most proportional to the sum of the discretization step and the averaged size of the measurement errors.

The first two sections are extracted from the research report [6], where detailed proofs are provided. Based on that, in Section 3 we compare the MPC-generated control u^N with the open-loop optimal control in problem (1) with the true value, $p(\cdot)$, of the uncertain parameter. This estimate involves, additionally, the L^1-norm of the prediction error, $\|p - \bar{p}\|_{L^1}$.

Section 4 contains some open problems for future research.

2 Main result

The notations used in this paper are fairly standard. The euclidean norm and the scalar product in $I\!R^n$ (the elements of which are regarded as vector-columns) are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. The transpose of a matrix (or vector) E is denoted as $E^\top$. For a function $\psi : I\!R^p \to I\!R^r$ of the variable z, we denote by $\mathrm{gph}\,(\psi)$ its graph and by $\psi_z(z)$ its derivative (Jacobian) at z, represented by an $(r \times p)$-matrix. If $r = 1$, $\nabla_z \psi(z) = \psi_z(z)^\top$ denotes its gradient (a vector-column of dimension p). For $r = 1$, $\psi_{zz}(z)$ denotes the second derivative (Hessian) at z, represented by a $(p \times p)$-matrix. For a function $\psi : I\!R^{p \times q} \to I\!R$ of the variables (z,v), $\psi_{zv}(z,v)$ denotes its mixed second derivative at (z,v), represented by a $(p \times q)$-matrix. The space L^k, with $k = 1, 2$ or $k = \infty$, consists of all (classes of equivalent) Lebesgue measurable vector-functions defined on an interval of real numbers, for which the standard norm $\|\cdot\|_k$ is finite (the dimension and the interval will be clear from the context). As usual, $W^{1,k}$ denotes the space of absolutely continuous functions on a scalar interval for which the first derivative belongs to L^k. In any metric space we denote by $I\!B_a(x)$ the closed ball of radius a centered at x.

We begin by stating the assumptions under which problem $\mathcal{P}_p$ is considered.

Assumption (A1). The set U is closed and convex, the functions $f : I\!R^l \times I\!R^n \times I\!R^m \to I\!R^n$, $\varphi : I\!R^l \times I\!R^n \times I\!R^m \to I\!R$ and $g : I\!R^n \to I\!R$ are two times continuously differentiable in (x,u) and these functions together with their derivatives in (x,u) up to second order are locally Lipschitz continuous in (p,x,u).

Assumption (A2). The time-varying parameter p representing uncertainty belongs to the following set of functions:

$$\Pi = \{p : [0,T] \to I\!R^l \ : \ p \in L^\infty(0,T), \ \|p\|_\infty \le \bar{M}, \ \|p - \bar{p}\|_1 \le \delta\},$$

where $\bar{M}$ and δ are positive constants. In addition, the reference parameter $\bar{p} \in \Pi$ is Lipschitz continuous in $[0,T]$. Moreover, problem $\mathcal{P}_{\bar{p}}$ has a locally optimal solution $(\bar{x}, \bar{u})$.

The local optimality is understood in the following (weak) sense: there exists a number $e_0 > 0$ such that for every $u \in \mathcal{U}$ with $\|u - \bar{u}\|_\infty \le e_0$, either the differential equation (2) has no solution on $[0,T]$ for u and $\bar{p}$, or $J_{\bar{p}}(u) \ge J_{\bar{p}}(\bar{u})$.

In terms of the Hamiltonian

$$H(t,x,u,\lambda) = \varphi(\bar{p}(t),x,u) + \lambda^\top f(\bar{p}(t),x,u)$$

for problem $\mathcal{P}_{\bar{p}}$, the Pontryagin maximum (here minimum) principle claims that there exists an absolutely continuous (here Lipschitz) function $\bar{\lambda} : [0,T] \to I\!R^n$ such that the triple $(\bar{x}, \bar{u}, \bar{\lambda})$ satisfies for a.e. $t \in [0,T]$ the following optimality system:

$$0 = -\dot{x}(t) + f(\bar{p}(t),x(t),u(t)), \quad x(0) - x_0 = 0, \tag{4}$$
$$0 = \dot{\lambda}(t) + \nabla_x H(t,x(t),u(t),\lambda(t)), \tag{5}$$
$$0 = \lambda(T) - \nabla_x g(x(T)), \tag{6}$$
$$0 \in \nabla_u H(t,x(t),u(t),\lambda(t)) + N_U(u(t)), \tag{7}$$

where the normal cone mapping N_U to the closed and convex set U is defined as

$$I\!R^m \ni u \mapsto N_U(u) = \begin{cases} \{y \in I\!R^n \mid \langle y, v - u\rangle \le 0 \ \forall v \in U\} & \text{if } u \in U, \\ \emptyset & \text{otherwise.} \end{cases}$$

To shorten the notations we skip arguments with "bar" in functions, shifting the "bar" to the function; that is, $\bar{H}(t) := H(t,\bar{x}(t),\bar{u}(t),\bar{\lambda}(t))$, $\bar{H}(t,u) := H(t,\bar{x}(t),u,\bar{\lambda}(t))$, $\bar{f}(t) := f(\bar{p}(t),\bar{x}(t),\bar{u}(t))$, etc. Define the matrices

$$\begin{aligned} &A(t) = \bar{f}_x(t),\ B(t) = \bar{f}_u(t),\ F = g_{xx}(\bar{x}(T)), \\ &Q(t) = \bar{H}_{xx}(t),\ S(t) = \bar{H}_{xu}(t),\ R(t) = \bar{H}_{uu}(t). \end{aligned}$$

Assumption (A3) – Coercivity. There exists a constant $\rho > 0$ such that

$$y(T)^\top F y(T) + \int_0^T \Big(y(t)^\top Q(t)y(t) + w(t)^\top R(t)w(t) + 2y(t)^\top S(t)w(t)\Big)\,\mathrm{d}t \ \ge\ \rho \int_0^T |w(t)|^2\,\mathrm{d}t$$

for all $w \in L^2$ with $w(t) \in U - U$ for a.e. $t \in [0,T]$, where $y \in W^{1,2}$ is the solution of the linearized equation $\dot{y}(t) = A(t)y(t) + B(t)w(t)$ with $y(0) = 0$.

Coercivity condition (A3) first appeared in [13] (if not earlier); after the publication of [2] it has been widely used in studies of regularity and approximations for problems like $\mathcal{P}_{\bar{p}}$. In particular, it plays a key role for Lipschitz continuity of the open-loop optimal control; also see Theorem 1 below. Later in the paper we show that Assumption (A3) implies a strong form of the Legendre condition regarding the positive definiteness of $R(t)$.

Next, we describe what we mean by optimal feedback control for problem $\mathcal{P}_{\bar{p}}$. For that we use the definition introduced in [7]. For any $\tau \in [0, T)$ and $y \in I\!R^n$ consider the following problem, denoted $\mathcal{P}_{\bar{p}}(\tau, y)$:

$$\min_{\mathcal{U}_\tau} \left\{ J_{\bar{p}}(\tau, y; u) := g(x(T)) + \int_\tau^T \varphi(\bar{p}(t), x(t), u(t))\,\mathrm{d}t \right\},$$

where x is the solution of the initial-value problem

$$\dot{x}(t) = f(\bar{p}(t), x(t), u(t)) \quad t \in [\tau, T], \quad x(\tau) = y, \tag{8}$$

and $\mathcal{U}_\tau$ is the set of feasible controls $u \in \mathcal{U}$ restricted to the interval $[\tau, T]$.

Definition 1. A function $u^* : [0, T] \times I\!R^n \to U$ is said to be a locally optimal feedback control around a reference optimal solution pair $(\bar{x}, \bar{u})$ of problem $\mathcal{P}_{\bar{p}}$ if there exist positive numbers η and a, and a set $\Gamma \subset [0, T] \times I\!R^n$ such that

(i) $\mathrm{gph}\,(\bar{x}) + \{0\} \times I\!B_\eta(0) \subset \Gamma$;

(ii) for every $(\tau, y) \in \Gamma$ the equation

$$\dot{x}(t) = f(\bar{p}(t), x(t), u^*(t, x(t))), \quad x(\tau) = y, \tag{9}$$

$t \in [\tau, T]$, has a unique absolutely continuous solution $\bar{x}[\tau, y]$ on $[\tau, T]$ which satisfies $\mathrm{gph}\,(\bar{x}[\tau, y]) \subset \Gamma$;

(iii) for every $(\tau, y) \in \Gamma$ the function
$\bar{u}[\tau, y](\cdot) := u^*(\cdot, \bar{x}[\tau, y](\cdot))$ is measurable, bounded, and satisfies

$$\|\bar{u}[\tau, y] - \bar{u}\|_\infty \le a \quad \text{and} \quad J(\tau, y; \bar{u}[\tau, y]) \le J(\tau, y; u),$$

where u is any admissible control on $[\tau, T]$ with $\|u - \bar{u}\|_\infty \le a$, for which a corresponding solution x of (9) exists on $[\tau, T]$ and is such that $\mathrm{gph}\,(x) \subset \Gamma$;

(iv) $u^*(\cdot, \bar{x}(\cdot)) = \bar{u}(\cdot)$.

In particular, property (iv) yields that $\bar{x}$ is a solution of (9) for $\tau = 0$ and $y = x_0$. Then the uniqueness requirement in (ii) implies that $\bar{x}[0, x_0] = \bar{x}$. In the sequel we call the function $t \mapsto \hat{u}(t) := u^*(t, x[u^*, p](t))$ a *realization* of the feedback control u^* when u^* is applied to (2) with value p of the parameter and with exact measurements. As before, here $x[u, p]$ denotes a solution of equation (2) for control u and parameter p. Note that, under our assumptions for the system, the solution is unique if u is feedback control which is a Lipschitz continuous function.

Recall that the admissible controls are elements of the space L^∞, that is, every admissible u is actually a class of functions $u : [0, T] \to U$ that differ from each other on a set of zero Lebesgue measure. Any of the members of this class (call it "representative") generates the same trajectory of (2) and the same value of the objective functional (1). Lemma 4.1 in [7] claims that under (A1)–(A3) there exists a "special" representative of the optimal control $\bar{u}$ which satisfies for *all* $t \in [0,T]$ the inclusion (7) in the maximum principle (with $\bar{x}$ and $\bar{\lambda}$ at the place of x and λ) and the pointwise coercivity condition

$$w^\top R(t) w \geq \rho |w|^2 \quad \text{for every } w \in U - U. \tag{10}$$

The following condition is introduced in [1] and used in [7] to prove existence of a Lipschitz continuous locally optimal feedback control.

Assumption (A4) – Isolatedness. The representative of the optimal control $\bar{u}$ described in the preceding lines is an isolated solution of the inclusion $\nabla_u \bar{H}(t,u) + N_U(u) \ni 0$ for all $t \in [0,T]$, meaning that there exists a (relatively) open set $\mathcal{O} \subset [0,T] \times I\!R^m$ such that

$$\{(t,u) \in [0,T] \times I\!R^m \;:\; \nabla_u \bar{H}(t,u) + N_U(u) \ni 0\} \cap \mathcal{O} = \mathrm{gph}\,(\bar{u}).$$

For example, the isolatedness assumption holds if for every $t \in [0,T]$ the inclusion $\nabla_u \bar{H}(t,u) + N_U(u) \ni 0$ has a unique solution (which has to be $\bar{u}(t)$). In this case, one can verify the isolatedness condition taking any (relatively) open set $\mathcal{O} \subset [0,T] \times I\!R^m$ containing $\mathrm{gph}\,(\bar{u})$. This will happen, for example, in the case when $\bar{H}$ is strongly convex in u for each t.

In the formulation (and in the proof) of our main theorem we use the following result:

Theorem 1 ([7, Theorem 5.2]). *Under conditions (A1)–(A4) there exists a locally optimal feedback control $u^* : [0,T] \times I\!R^n \to U$ around $(\bar{x}, \bar{u})$ which is Lipschitz continuous on a set Γ appearing (together with the positive numbers η and a) in Definition 1.*

Due to Theorem 1, if $\|p - \bar{p}\|_1$ is sufficiently small, the feedback control u^* when plugged in (2) generates a unique trajectory on $[0,T]$, denoted by $x[u^*, p]$.

Recall that for any $p \in \Pi$ and measurement errors $\xi_0, \dots, \xi_{N-1}$, the MPC method, as described in the introduction, generates a control u^N (possibly not uniquely). In order to indicate the dependence of u^N on p and ξ we sometimes use the extended notation $u^N[p, \xi]$.

The main theorem of this paper follows.

Theorem 2. Suppose that assumptions (A1)–(A4) hold with constants $\bar{M}$ and ρ. Then there exist constants N_0, c and $\delta > 0$ such that for every $N \geq N_0$, for every $p \in \Pi$, where the set Π defined in (2) depends on $\bar{M}$ and δ, and for every $\xi = (\xi_0, \dots \xi_{N-1})$ with $\max_{k=0,\dots,N-1} |\xi_k| \leq \delta$ there exists a control u^N generated by the MPC algorithm for the system (2) with disturbance parameter p and measurement error ξ such that,

$$|u^N(t_i) - \hat{u}(t_i)| \leq c\Big(h + |\xi_i| + h \sum_{k=0}^{N-1} |\xi_k|\Big), \quad i = 0, \dots, N-1.$$

where $\hat{u}(t) := u^*(t, x[u^*, p](t))$ *is the realization of the feedback control* u^**. Furthermore, if* $\hat{x} = x[\hat{u}, p] = x[u^*, p]$ *is the trajectory of (2) for* $\hat{u}$ *and* p *and* $x^N := x[u^N, p]$ *is the trajectory of (2) for* u^N *and* p*, then*

$$\|u^N - \hat{u}\|_1 + \|x^N - \hat{x}\|_{W^{1,1}} \leq c\Big(h + h\sum_{k=0}^{N-1} |\xi_k|\Big). \tag{11}$$

In the next lines we make some comments concerning the last theorem.

(i) As mentioned in the introduction, the MPC algorithm considered does not necessarily generate a unique control u^N, since the optimal control sequence $(\tilde{u}_k, \ldots, \tilde{u}_{N-1})$ appearing at each stage k of the MPC does not need to be unique. Of course, the MPC-solution u^N will be uniquely determined if each of the discrete optimal control problems which is solved at each stage of the MPC algorithm has a unique solution.

(ii) Clearly, the first term in the parentheses in the right-hand side of (11) comes from the discretization, while the second term is due to the measurement errors. Note that the bound (11) does not depend on the difference between the reference $\bar{p}$ and the real p; the only condition involved is that p must be sufficiently close to $\bar{p}$ for the estimate to hold. That is, a possible change of the parameter p affects in the same way (modulo $O(h)$) both the state-control pair corresponding to the optimal feedback control and the state-control pair obtained by applying the MPC algorithm.

(iii) The proof of Theorem 2 (given in detail in the research report [6]) uses results obtained in the recent papers [5] and [7]. In both papers the optimal control problem (1)–(2) is considered under basically the same assumptions. In [5] it is shown that the solution mapping of the discrete-time problem has a Lipschitz continuous single-valued localization with respect to the parameter, whose Lipschitz constant and the sizes of the neighborhoods do not depend on the number N of mesh points, for all sufficiently large N. Theorem 1 is the main result in [7].

(iv) The estimate (11) in Theorem 2 was numerically tested on an example for axisymmetric spacecraft spin stabilization [20]. The results ot the numerical simulations, presented in the research report [6], are consistent with the theoretically obtained estimate.

3 Comparison with the open-loop optimal solution

In this section we use Theorem 2 to estimate the distance from the MPC-generated control u^N and the optimal open-loop control for the problem $\mathcal{P}_p$ with the true value of the parameter p (although it is unknown). In this way we may also estimate the possible loss of performance resulting from the prediction error when the MPC algorithm is applied. To simplify the situation here we additionally assume the following.

Assumption (A5). Each of the problems $\mathcal{P}_p$, $p \in \Pi$, and each of the problems $\mathcal{P}_{\bar{p}}(t_k, x_k)$ that appears at the stages of the MPC algorithm has a unique open-loop optimal control, and the values of all these controls are contained in a bounded set.

Assumption (A5) implies, in particular, that for any N the MPC-generated control u^N exists and is unique.

Theorem 3. Suppose that assumptions (A1)–(A5) hold with constants $\bar{M}$ and ρ. Then there exist constants N_0, c and $\delta > 0$ such that for every $N \geq N_0$, every $p \in \Pi$ with $\|p - \bar{p}\|_\infty \leq \delta$ and every $\xi = (\xi_0, \ldots \xi_{N-1})$ with $\max_{k=0,\ldots,N-1} |\xi_k| \leq \delta$, the control u^N generated by the MPC algorithm for the system (2) with a parameter p and measurement error ξ satisfies

$$\|u^N - \hat{u}_p\|_1 \leq c\Big(h + h \sum_{k=0}^{N-1} |\xi_k| + \|p - \bar{p}\|_\infty\Big).$$

where $\hat{u}_p(t)$ is the open-loop optimal control in problem $\mathcal{P}_p$.

Similar estimates hold for the trajectories corresponding to u^N and $\hat{u}_p$ in the space $W^{1,1}$ and also for the difference of the values of the cost.

Proof of Theorem 3. We begin with some preliminaries. Introduce the spaces

$$\begin{array}{rcl} Y & := & \Big\{x \in W^{1,\infty} \ : \ x(0) = x_0\Big\} \times L^\infty \times W^{1,\infty}, \\ Z & := & L^\infty \times L^\infty \times I\!R^n \times L^\infty \end{array}$$

and the mappings $\psi^p : Y \to Z$ and $\Psi : Y \rightrightarrows Z$ (the second one is set-valued) as

$$\psi^p(y) := \begin{pmatrix} -\dot{x} + f(p,x,u) \\ \dot{\lambda} + \nabla_x H(p,x,u,\lambda) \\ \lambda(T) - \nabla_x g(x(T)) \\ \nabla_u H(p,x,u,\lambda) \end{pmatrix}, \quad \Psi(y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ N_U(u) \end{pmatrix},$$

where $y = (x,u,\lambda)$, $p \in \Pi$, and $H(p,y) = \varphi(p,x,u) + \lambda^\top f(p,x,u)$. Since $(\bar{u}, \bar{x})$ is optimal in problem $\mathcal{P}_{\bar{p}}$ and $(\hat{u}_p, \hat{x}_p)$ is optimal in problem $\mathcal{P}_p$, there exist adjoint functions $\bar{\lambda}$ and $\hat{\lambda}_p$ in $W^{1,\infty}$ such that $\bar{y} := (\bar{x}, \bar{u}, \bar{\lambda})$ and $\hat{y}_p := (\hat{x}_p, \hat{u}_p, \hat{\lambda}_p)$ satisfy

$$\psi^{\bar{p}}(\bar{y}) + \Psi(\bar{y}) \ni 0, \quad \psi^p(\hat{y}_p) + \Psi(\hat{y}_p) \ni 0.$$

The second inclusion implies that

$$\psi^{\bar{p}}(\hat{y}_p) + \Psi(\hat{y}_p) \ni d, \quad \text{where } \ d := \psi^{\bar{p}}(\hat{y}_p) - \psi^p(\hat{y}_p). \tag{12}$$

One can estimate

$$\begin{array}{rcl} |d(t)| & \leq & |f(\bar{p}(t), \hat{x}_p(t), \hat{u}_p(t)) - f(p(t), \hat{x}_p(t), \hat{u}_p(t))| \\ & & + |\nabla_x H(\bar{p}(t), \hat{y}_p(t)) - \nabla_x H(p(t), \hat{y}_p(t))| \\ & & + |\nabla_u H(\bar{p}(t), \hat{y}_p(t)) - \nabla_u H(p(t), \hat{y}_p(t))| \\ & \leq & \hat{L}|\bar{p}(t) - p(t)|, \end{array}$$

where $\hat{L}$ is a constant. To show that, one uses assumption (A1) and the last part of (A5). Hence,

$$\|d\|_\infty \leq \hat{L}\|p - \bar{p}\|_\infty. \tag{13}$$

Assumptions (A1) and (A3), together with the assumed in (A2) Lipschitz continuity of $\bar{p}$, imply that the mapping $\psi^{\bar{p}} + \Psi$ is strongly metrically regular (SMR) at $\bar{y}$ for zero (see e.g. [8] for the definition of SMR). This is proved (in a somewhat more restricted case and using a different terminology) in [2]. A proof that applies to the present setting and uses the contemporary terminology is given in [7] (see Proposition 1 there). The SMR property, together with (12) and the estimate (13) imply, in particular, that if the number $\hat{L}\|p - \bar{p}\|_\infty$ is sufficiently small (which can be ensured by choosing the number $\delta > 0$ sufficiently small), then

$$\|\bar{u} - \hat{u}_p\|_\infty \leq c_0\|p - \bar{p}\|_\infty, \tag{14}$$

where c_0 is an appropriate constant.

Having the estimate (14), we proceed with the proof of the theorem, using the notations introduced in Section 2. We estimate

$$\|u^N - \hat{u}_p\|_1 \leq \|u^N - \hat{u}\|_1 + \|\hat{u} - \bar{u}\|_1 + \|\bar{u} - \hat{u}_p\|_1.$$

From the identities $\bar{u}(t) = u^*(t, x[u^*, \bar{p}](t))$ (the realization of the optimal feedback control in problem $\mathcal{P}_{\bar{p}}$ when applied to (2) with $p = \bar{p}$ coincides with the unique optimal open-loop control for this problem) and $\hat{u}(t) = u^*(t, x[u^*, p](t))$ we obtain that

$$\|u^N - \hat{u}_p\|_1 \leq \|u^N - \hat{u}\|_1 + \|u^*(\cdot, x[u^*, p]) - u^*(\cdot, x[u^*, \bar{p}])\|_1 + \|\bar{u} - \hat{u}_p\|_1.$$

Since the first term in the right-hand side can be estimated using Theorem 2 and the last term can be estimated using (14) (both by const.$\|p - \bar{p}\|_\infty$), in order to complete the proof it remains to estimate the second term. Since u^* is a Lipschitz continuous function (see Theorem 1) we have

$$\|u^*(\cdot, x[u^*, p]) - u^*(\cdot, x[u^*, \bar{p}])\|_1 \leq L\|x[u^*, p] - x[u^*, \bar{p}]\|_1, \tag{15}$$

where L is the Lipschitz constant of u^*. Having in mind that $x[u^*, p]$ is the solution of the equation

$$\dot{x}(t) = f(p(t), x(t), u^*(t, x(t))), \quad x(0) = x_0$$

and $x[u^*, \bar{p}]$ is the solution of the same equation with $\bar{p}$ instead of p, and that the right-hand side is Lipschitz continuous in x, it is a standard exercise to estimate the right-hand side of (15) by constant times $\|p - \bar{p}\|_1 \leq T\|p - \bar{p}\|_\infty$. This completes the proof of the theorem.

4 Topics of further research

(i) In the present setting, a natural question to ask is whether one can obtain a better order of approximation with respect to the discretization step h than (11) by using higher order discretization schemes. The answer is, "yes, but conditionally". Second-order approximations to control-constrained optimal control problems in the form of $\mathcal{P}_p$ by Runge-Kutta discretizations are obtained in [4] under similar conditions as in the present paper. Results for even higher order approximations (essentially for problems without control constraints) are presented in [12]. Using these results, however, would improve only the first term $O(h)$ in the right-hand side of (11). Utilization of higher order schemes would be justified only if the total l_1-error in the measurements is consistent with the discretization error.

(ii) In many applications, MPC algorithms are used for problems with state constraints $x(t) \in X$, where the set $X \subset I\!\!R^n$ may have the form $l(x) \leq 0$ with a given function $l : I\!\!R^n \mapsto I\!\!R^k$. Extending Theorem 2 to this case appears to be difficult due to the possible discontinuity of the optimal feedback control (if it exists). However, an extension of Theorem 3 to the state constrained case seems to be tractable and is a subject of current work, based on an enhancement of the result in [3], where an error estimate of the Euler discretization is obtained for problems with state constraints.

(iii) The coercivity condition (A3) is of key importance for the results in this paper and the possible extensions mention in points (i) and (ii). However, in applications in electrical engineering and especially in power electronics, affine optimal control problems naturally appear, often in the form of tracking problems for switching systems. Generally, such problems can be represented in the form of (1)–(2), where the functions f and φ are affine with respect to the control u, e.g.,

$$\begin{aligned} f(p,x,u) &= a(p,x) + B(p,x)u, \\ \varphi(p,x,u) &= w(p,x) + \langle s(p,x), u \rangle. \end{aligned}$$

Clearly, the coercivity condition (A3) does not hold for such problems, since the matrix $R(t)$ vanishes (see (10), which follows from (A3)). Since the optimal feedback control for the reference problem $\mathcal{P}_{\bar{p}}$ (if it exists) is typically discontinuous in the affine case, the technique utilized in the proof of Theorem 2 (based on the Lipschitz continuity of the optimal feedback control) is not applicable. On the other hand, in the resent papers [14] and [15] conditions are obtained for strong metric sub-regularity (see [8, Chapter 3.9]) and strong bi-metric regularity of the mapping representing the right-hand side of the optimality conditions (4)–(7) (that is, the mapping $\psi^{\bar{p}} + \Psi$ in the proof of Theorem 3) under appropriate space specifications. These results may provide a ground for extending Theorem 3 to the case of affine problems, which is a subject of current research.

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