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Lipschitz Stability in Discretized Optimal Control

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Abstract: We consider a control constrained nonlinear optimal control problem under perturbations represented by changes of a vector parameter. Our main assumptions involve smoothness of the functions appearing in the integral functional and the state equations, an integral coercivity condition, and a condition that the reference optimal control is an isolated solution of the variational inequality for the control appearing in the maximum principle. We also consider a corresponding discrete-time optimal problem obtained from the continuous-time one by applying the Euler finite-difference scheme. Based on an enhanced version of Robinson's implicit function theorem, we establish that there exists a natural number $\bar{N}$ such that if the number N of the grid points is greater than $\bar{N}$, then the solution mapping of the discrete-time problem has a Lipschitz continuous single-valued localization with respect to the parameter whose Lipschitz constant and the sizes of the neighborhoods depend only on the ranges of values of the variables, the Lipschitz constants of the second derivatives of the functions involved, and the coercivity constant. As an application, we show that the Newton/SQP method converges uniformly with respect to the step-size of the discretization and small changes of the parameter. Numerical experiments with a satellite optimal control problem illustrate the results.

Key Words. optimal control, discrete approximation, uniform Lipschitz stability, Newton/SQP method.

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1 Introduction

In this paper, we consider the following continuous-time optimal control problem depending on a parameter p:

$$(1) \qquad \text{Minimize}_{(x,u)} \quad \left[J(p,x,u) = \int_0^T \varphi(p,x(t),u(t))dt \right]$$

subject to

$$(2) \qquad \dot{x}(t) = g(p,x(t),u(t)), \quad u(t) \in U \quad \text{for a.e. } t \in [0,T], \quad x(0) = 0,$$

with controls $u \in L^\infty$, the space of measurable and essentially bounded functions over $[0,T]$, whose values $u(t)$ are constrained to belong to a closed and convex set $U \subset \mathbb{R}^m$ for almost every (a.e.) $t \in [0,T]$. The state $x(t) \in \mathbb{R}^n$ and each state trajectory x is a function in $W_0^{1,\infty}$, the space of Lipschitz continuous functions x over $[0,T]$ such that $x(0) = 0$. In the sequel we also use the usual Lebesque space L^2 of square integrable functions over $[0,T]$ and the space $W^{1,2}$ of functions x over $[0,T]$ such that both x and its time-derivative $\dot{x}$ are in L^2. The parameter $p \in \mathbb{R}^d$; $\bar{p}$ is the reference value of p. The analysis presented in the paper also covers the case when, in addition to the integral functional, the cost includes a function dependent on the terminal state, say $\Phi(x(T))$. As well known, on the assumption that Φ is continuously differentiable, that case can be easily reduced to problem (1)–(2). The obtained results can be automatically extended to the case when the initial state is a function of the parameter, $x(0) = a(p)$, where $a : \mathbb{R}^d \to \mathbb{R}^n$ is Lipschitz continuous around $\bar{p}$; indeed, in that case a simple change of variables gives us an equivalent problem with zero initial state.

Our first assumption regarding problem (1)–(2) is the following:

– SMOOTHNESS: *for each $p \in I\!B(\bar{p})$ the functions $\varphi(p,\cdot,\cdot)$ and $g(p,\cdot,\cdot)$ are twice differentiable in $\mathbb{R}^n \times \mathbb{R}^m$ and their second derivatives are locally Lipschitz continuous. The first and second derivatives of φ and g with respect to (x,u) are Lipschitz continuous with respect to $p \in I\!B(\bar{p})$ uniformly in (x,u) belonging to bounded sets.*

We denote by $I\!B_a(x)$ the closed ball centered at x with radius a; then $I\!B_1(x)$ is denoted simply by $I\!B(x)$ and $I\!B(0)$ is denoted by $I\!B$. We use balls with radius one in order to avoid technicalities when dealing with superficial constants.

Let (x^*,u^*) be a locally optimal solution of problem (1)–(2) for $p = \bar{p}$. The Pontryagin maximum principle applied to (1)–(2) for $p = \bar{p}$ says that there exists a solution $q^* \in W^{1,\infty}$ of the adjoint equation

$$(3) \qquad \dot{q}(t) = -\nabla_x H(\bar{p},x^*(t),u^*(t),q(t)) \quad \text{for a.e. } t \in [0,T], \quad q(T) = 0,$$

where

$$(4) \qquad H(p,x,u,q) = \varphi(p,x,u) + q^T g(p,x,u)$$

is the associated Hamiltonian. Furthermore,

$$(5) \qquad \nabla_u H(\bar{p},x^*(t),u^*(t),q^*(t)) + N_U(u^*(t)) \ni 0 \quad \text{for a.e. } t \in [0,T],$$

where the normal cone mapping N_U to the set U is defined as

$$(6)\qquad v \mapsto N_U(v) = \begin{cases} \{y \in \mathbb{R}^n \mid y^T(w-v) \le 0 \text{ for all } \ w \in U\} & \text{if } \ v \in U, \\ \emptyset & \text{otherwise.} \end{cases}$$

Define the matrices

$$(7)\qquad \begin{array}{c} A(t) = \nabla_x g^*(t), \quad B(t) = \nabla_u g^*(t), \\ Q(t) = \nabla_{xx} H^*(t), \quad S(t) = \nabla_{xu} H^*(t), \quad R(t) = \nabla_{uu} H^*(t), \end{array}$$

where $g^*(t) = g(\bar{p}, x^*(t), u^*(t))$ and $H^*(t) = H(\bar{p}, x^*(t), u^*(t), q^*(t))$. Our next assumption is

– COERCIVITY: *there exists a constant* $\alpha > 0$ *such that*

$$(8)\qquad \int_0^T (y(t)^T Q(t) y(t) + w(t)^T R(t) w(t) + 2y(t)^T S(t) w(t)) dt \ge \alpha \int_0^T |w(t)|^2 dt$$

for all $y \in W^{1,2}, y(0) = 0, w \in L^2$ *such that* $\dot{y}(t) = A(t)y(t) + B(t)w(t),\ y(0) = 0,$ *and* $w(t) \in U - U$ *for a.e.* $t \in [0,T]$.

Here and further $U - U = \{w \mid w = v - v' \text{ for some } v, v' \in U\}$ and $|\cdot|$ is the Euclidean norm.

Clearly, the function u^* could be redefined on a subset of $[0,T]$ of measure zero without changing the optimality. As shown in [5, Lemma 4.1], see also the comment before [2, Theorem 5], under smoothness and coercivity, without loss of generality the optimal control u^* can be regarded as a function acting from $[0,T]$ to U which satisfies the variational inequality (5) for all $t \in [0,T]$; furthermore, the pointwise coercivity condition

$$(9)\qquad w^T R(t) w \ge \alpha |w|^2 \qquad \text{for all } \ w \in U - U$$

holds for all $t \in [0,T]$. In what follows we identify u^* with a representative with these properties.

–ISOLATEDNESS: *The function* u^* *is an isolated solution of the inclusion*

$$(10)\qquad \nabla_u H(\bar{p}, x^*(t), u, q^*(t)) + N_U(u) \ni 0 \qquad \textit{for all } t \in [0,T],$$

meaning that there exists a (relatively) open set $\mathcal{O} \subset [0,T] \times \mathbb{R}^m$ *such that*

$$\{(t,u) \in [0,T] \times \mathbb{R}^m \mid (10) \text{ holds } \} \cap \mathcal{O} = \operatorname{gph} u^*.$$

The coercivity condition is a rather standard assumption in optimal control; the form of it we use here goes back to the work of Hager [8]. In contrast, the isolatedness condition was introduced only recently in [1, Definition 3.6] in the context of the so-called differential variational inequalities with the aim to prevent different solution curves from crossing each other. Specifically, for the variational inequality (10), the isolatedness condition holds when the Hamiltonian has a unique minimizer for each $t \in [0,T]$. Also, from [1, Theorem 3.7] it follows that the isolatedness condition implies Lipschitz continuity of the optimal control u^*, provided that the variational inequality (10) is strongly regular at $u^*(t)$ for 0 uniformly in $t \in [0,T]$. Strong regularity is a property of mappings introduced by Robinson in [10]; we utilize it in Section 2 of the present paper. This line of research was further elaborated in the even more recent paper [5] devoted to the optimal control problem (1)–(2). Specifically, it was proved in [5, Theorem 4.4] that under the above smoothness, coercivity and isolatedness assumptions the representative u^* of the optimal control for which (5) and (9) hold for all $t \in [0,T]$ is a Lipschitz continuous function with respect to $t \in [0,T]$. This clearly implies that that there exists a closed set $\Delta \subset \mathbb{R}^{n+m}$ and a positive constant δ such that $(x^*(t), u^*(t))$ lies in Δ and the distance from $(x^*(t), u^*(t))$ to the boundary of Δ is at least δ for almost all $t \in [0,T]$. This is the standing assumption for [2, Theorem 5] which we use in Theorem 1.1 below.

For an arbitrary value p of the parameter, the optimality system for the perturbed problem (1)–(2) becomes

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & g(p, x(t), u(t)), \quad x_0 = 0, \\ \dot{q}(t) & = & -\nabla_x H(p, x(t), u(t), q(t)), \quad q(T) = 0, \\ 0 & \in & \nabla_u H(p, x(t), u(t), q(t)) + N_U(u(t)), \end{array}\right. \tag{11}$$

all for a.e. $t \in [0,T]$. Denote by $\mathcal{S}$ the associated solution mapping defined as $p \mapsto$ the set of solutions $(x(p), u(p), q(p))$ of (11) considered as elements of the function space $W_0^{1,\infty} \times L^\infty \times W^{1,\infty}$. The theorem below easily follows from [2, Theorem 5]:

Theorem 1.1. *Consider the optimal control problem* (1)–(2) *with a locally optimal solution* (x^*, u^*) *for* $\bar{p}$ *and a solution* q^* *of the associated adjoint equation* (3). *Suppose that the smoothness, coercivity and isolatedness conditions stated above are satisfied. Then there exist positive reals* $\bar{a}$ *and* $\bar{b}$ *such that the mapping*

$$\mathbb{B}_{\bar{b}}(\bar{p}) \ni p \mapsto \mathcal{S}(p) \cap \mathbb{B}_{\bar{a}}(x^*, u^*, q^*) \tag{12}$$

is a Lipschitz continuous function. Furthermore, for each $p \in \mathbb{B}_{\bar{b}}(\bar{p})$ *if* $(x(p), u(p), q(p)) \in \mathcal{S}(p) \cap \mathbb{B}_{\bar{a}}(x^*, u^*, q^*)$ *the pair* $(x(p), u(p))$ *is a strict local minimizer for problem* (1)–(2) *for* p *and* $q(p)$ *is an associated adjoint variable.*

Recall that the mapping (12) is said to be a *graphical localization* of $\mathcal{S}$ at $\bar{p}$ for (x^*, u^*, q^*) with constants $\bar{b}$ and $\bar{a}$. The aim of this paper is to extend Theorem 1.1 to a discrete approximation of problem (1)–(2). Choose a natural N and let $h = T/N$, $t_k = kh$, $k = 0, \dots, N$. The Euler finite-difference scheme with step-size h applied to problem (1)–(2) results in the following discrete-time optimal control problem:

$$\text{Minimize}_{(x,u)} \quad \left[J_N(p, x, u) = \sum_{i=0}^{N-1} h\varphi(p, x_i, u_i) \right] \tag{13}$$

subject to

$$x_{i+1} = x_i + hg(p, x_i, u_i), \quad u_i \in U \quad \text{for } i = 0, 1, \dots, N-1, \quad x_0 = 0, \tag{14}$$

where x_i and u_i correspond to the values of the state and control at the grid points t_i, $i = 0, \dots, N$. Our main result, presented in Theorem 1.2 below, shows that for all sufficiently large N the solution mapping of the optimality system associated with problem (13)–(14)

has a Lipschitz localization with respect to the parameter p which is *uniform* in the number N of the discretization steps (or the step size h); that is, the sizes of the neighborhoods and the Lipschitz constant involved do not depend on N.

We give next a necessary optimality condition for problem (13)–(14), by employing the Lagrangian

$$\mathcal{L}(p,x,u,q) = J_N(p,x,u) + \sum_{i=0}^{N-1} q_i^T(-x_{i+1} + x_i + hg(p,x_i,u_i)), \tag{15}$$

where q_i, $i = 0, \cdots, N-1$, are the Lagrange multipliers associated with the discrete-time dynamics, commonly called costates. The Lagrange multiplier rule for problem (13)–(14) is well known, see e.g. [11, Corollary 6.15], from which, adapted to the current setting, we obtain the following system involving the state equation, the adjoint equation determined by a backward recursion, and a variational inequality for the control:

$$\left\{ \begin{array}{lcl} x_{i+1} & = & x_i + hg(p,x_i,u_i), \quad x_0 = 0, \\ q_{i-1} & = & q_i + h\nabla_x H(p,x_i,u_i,q_i), \quad q_{N-1} = 0, \\ 0 & \in & \nabla_u H(p,x_i,u_i,q_i) + N_U(u_i), \end{array} \right. \tag{16}$$

where $i = 0, 1, \ldots, N-1$ in the first and in the last relations, $i = 1, 2, \ldots, N-1$ in the second equation, and H is the Hamiltonian defined in (4). Note that in the variational inequality we divide by h; this does not change the inclusion since N_U is a cone. Also note that the optimality system (16) can be also obtained by the Euler discretization of the continuous-time necessary optimality condition, with a minor adjustment in the indices. We consider system (16) for sequences $x = (x_1, \ldots, x_N)$, $u = (u_0, \ldots, u_{N-1})$ and $q = (q_0, \ldots, q_{N-1})$, where the initial state $x_0 = 0$ is considered as a datum and the final condition $q_{N-1} = 0$ for the costate is considered as an equation. Let $v = (x, u, q)$, let $C = (\mathbb{R}^n)^N \times \mathcal{U} \times (\mathbb{R}^n)^N$ where $\mathcal{U}$ is the product of N copies of the set U, and define

$$E(p,v) = \begin{pmatrix} \nabla_q \mathcal{L}(p,x,u,q) \\ \nabla_x \mathcal{L}(p,x,u,q) \\ \nabla_u \mathcal{L}(p,x,u,q) \end{pmatrix}.$$

Then the optimality system (16) can be conveniently written as a variational inequality of the form

$$E(p,v) + N_C(v) \ni 0. \tag{17}$$

Note that the function E depends not only on p and v but also on the number N of grid points. In Section 2 and later in the paper we use this compact description of (16) to simplify the notation.

In Section 3 we utilize a basic result in [2, Theorem 6], which gives conditions under which there exist a natural number N_0 and a constant $c > 0$ such that for all $N > N_0$ there exists a local minimizer $(\bar{x}^N, \bar{u}^N)$ of problem (13)–(14) for $p = \bar{p}$ and an associated costate $\bar{q}^N$ such that

$$\max_{0 \le i \le N-1} |\bar{u}_i^N - u^*(t_i)| + \max_{1 \le i \le N} |\bar{x}_i^N - x^*(t_i)| + \max_{0 \le i \le N-1} |\bar{q}_i^N - q^*(t_i)| \le ch. \tag{18}$$

We should note that the estimate (18) is obtained in [2] under conditions that look a bit stronger but are actually different from those in Theorem 1.2. First, it is assumed there that the optimal control u^* is of bounded variation while here we assume isolatedness for the optimal control. Also, [2] assumes an integral coercivity condition and a pointwise coercivity condition for the discretized problem. Furthermore, it is shown in [2, Lemma 11] that, under the additional assumption that the matrices defined in (7) are continuous in $[0, T]$, the integral discrete coercivity condition used in [2] follows from the continuous one (8), while here we prove this implication without any a priory assumption of continuity of the matrices. The isolatedness condition for u^*, however, does not follow from coercivity; this is demonstrated by the following example in [2, Remark 9]: Take $\varphi(p, x, u) = (u^2 - 1)^2, g = 0$ and $U = \mathbb{R}$; then for each measurable set $M \subset [0, T]$, the function u defined as $u(t) = 1$ for $t \in M$ and $u(t) = -1$ for $t \notin M$ is a discontinuous optimal control which does not satisfy the isolatedness condition.

Denote $\bar{v}^N := (\bar{x}^N, \bar{u}^N, \bar{q}^N)$. Our main result follows.

Theorem 1.2. *Consider the parameterized optimal control problem* (1)–(2) *with a reference solution* (x^*, u^*) *for* $\bar{p}$ *and suppose that the smoothness, coercivity and isolatedness conditions are satisfied. For a natural number* N*, consider also the discrete approximation* (13)–(14) *and let* $\mathcal{S}^N$ *be the solution mapping of the associated optimality system* (16)*. Then there exist a natural number* $\bar{N}$ *and positive reals* a*,* b *and* λ *such that for all* $N > \bar{N}$ *the graphical localization*

$$\mathbb{B}_b(\bar{p}) \ni p \mapsto \mathcal{S}^N(p) \cap \mathbb{B}_a(\bar{v}^N)$$

of the solution mapping $\mathcal{S}^N$ *is a Lipschitz continuous function with a Lipschitz constant* λ*. Furthermore, for each* $p \in \mathbb{B}_b(\bar{p})$ *if* $(x^N(p), u^N(p), q^N(p)) \in \mathcal{S}^N(p) \cap \mathbb{B}_a(\bar{v}^N)$*, then the pair* $(x^N(p), u^N(p))$ *is a strict local minimizer for problem* (13)–(14) *for* p*.*

A proof of this result is given on Section 3. It is based on an enhanced version of Robinson's implicit function theorem presented in Section 2.

As an application, in Section 4 we show quadratic convergence of the Newton/SQP method, applied to the optimality system (16), which is uniform with respect to both the number N of discretization points, for large N, and small changes of the parameter p around $\bar{p}$. Specifically, we prove that there exist a natural number N_4 and neighborhoods P of $\bar{p}$ and O of the reference solution-multiplier triple (x^*, u^*, q^*) such that for every sufficiently large $N > N_4$, every $p \in P$, and every $\varepsilon > 0$ there exists a constant k_ε, for which an estimate is given, such that the Newton/SQP method, applied to the discretized variational system with step-size $h = T/N$ and starting from a discrete-time representation of the solution-multiplier triple (x^0, u^0, q^0) which is in O, achieves accuracy ε, measured by the size of the residual of the optimality system, after not more than k_ε iterations.

In Section 5 we illustrate the result obtained in Section 4 by numerical experiments with an optimal control problem for spacecraft attitude.

This work is largely motivated by the Model Predictive Control (MPC) algorithm, in which discrete approximations are intrinsically involved when handling continuous-time optimal control problems. Roughly, at the first stage the MPC algorithm solves a discretized problem and then applies the optimal control computed to the continuous-time system on the first time-step, obtaining at the end of the first step a new initial state for the second stage. Repeating this procedure for each stage leads to an optimal feedback law which, as it turns out, may have a number of advantages that have been extensively explored in the last several decades; see e.g. the book [7] as a general reference. Thus, the MPC algorithm repeatedly solves discretized optimal control problems with varying parameters. If one applies the Newton/SQP method at each stage of the MPC, then it is beneficial if the computational performance does not vary too much from one time-step to another. As well known, the performance of the Newton/SQP method depends on the regularity properties of the associate solution mapping such as Lipschitz dependence on parameters. Hence, it is important to know how the regularity properties of the solution are affected by the discretization; this latter issue is what the current paper is about. We will not consider here applications of the results of this paper to MPC algorithm; we leave this for further research.

We use standard notation and terminology, basically from the book [4]. Specifically, we use small letters for vectors and capital letter for matrices. The transposition is indicated by ^{T} and then the scalar product in $\mathbb{R}^n$ is x^Ty. The derivative of a function f with respect to x at $\bar{x}$ is denoted by $\nabla_x f(\bar{x})$. A mapping F acting from, say, a Banach space X to a Banach space Y, allowed to be set-valued in general as indicated by the notation $F : X \rightrightarrows Y$, has graph $\operatorname{gph} F = \{\, (x, y) \in X \times Y \mid y \in F(x)\}$ and domain $\operatorname{dom} F = \{\, x \in X \mid F(x) \neq \emptyset\}$. It is *single-valued*, indicated by $F : X \to Y$, if $F(x)$ consists of only one y for each $x \in \operatorname{dom} F$; then $y \in F(x)$ is written as $F(x) = y$. The inverse of a mapping $F : X \rightrightarrows Y$ is the mapping $F^{-1} : Y \rightrightarrows X$ defined as $F^{-1}(y) = \{\, x \in X \mid y \in F(x)\}$, which of course may be set-valued even if F is single-valued.

In the paper we use a number of constants, such as the integers $\bar{N}$ and $N_i, i = 0, \ldots, 4$, as well as various reals. By tracing the proofs one may identify relations among some of them; we prefer not to do this here. For "long" vectors $z \in (\mathbb{R}^s)^N$ representing discrete approximations of functions on $[0, T]$ we utilize the 2-norm

$$\|z\|_2 = \left(\sum_{i=0}^{N-1} h|z_i|^2 \right)^{1/2},$$

and the ∞-norm

$$\|z\|_\infty = \max_{0 \le i \le N-1} |z_i|.$$

This corresponds to considering $\|z\|_2$ as the L^2 norm of a piecewise constant function $t \mapsto z(t)$ with values in $\mathbb{R}^s$ such that $z(t) = z_i$ for all $t \in [t_i, t_{i+1})$, $i = 0, 1, \ldots, N-1$. Correspondingly, $\|z\|_\infty$ is the L^∞ norm of a piecewise constant function $z(\cdot)$.

Recall that a mapping $F : X \rightrightarrows Y$ is said to be *strongly regular* at $\bar{x}$ for $\bar{y}$ if there exist positive constants a and b such that the graphical localization of the inverse $\mathbb{B}_b(\bar{y}) \ni y \mapsto F^{-1}(y) \cap \mathbb{B}_a(\bar{x})$ is a Lipschitz continuous function. This concept was introduced by S. M. Robinson in his seminal paper [10]; for a broad coverage of the major developments around this property as well as other related properties of mappings in variational analysis and optimization, see [4]. The next section is devoted to a basic result regarding strong regularity: the Robinson implicit function theorem.

2 The Robinson implicit function theorem revisited

In this section we present an enhanced version of the powerful Robinson's implicit function theorem [10] to handle the optimality system (16) written as the variational inequality (17). Recall that $\bar{p}$ is a fixed reference value of the parameter p and $\bar{v}^N = (\bar{x}^N, \bar{u}^N, \bar{q}^N) \in \mathbb{R}^l$, where $l = (n + m + n) \cdot N$, is a solution of (17) for $\bar{p}$. As it will be shown in Section 3 using [2, Theorem 6], a solution $\bar{v}^N$ exists and the estimation (18) holds.

Denote by x^{*N} the vector $(x^*(t_1), \cdots, x^*(t_N))$ of the values of the optimal state trajectory x^* for (1)–(2) at the grid points, and also $u^{*N} = (u^*(t_0), \cdots, u^*(t_{N-1}))$ for the control and $q^{*N} = (q^*(t_0), \cdots, q^*(t_{N-1}))$ for the costate. Let $v^{*N} = (x^{*N}, u^{*N}, q^{*N})$. Because of the uniform boundedness of v^{*N}, the assumed differentiability of the functions involved in the problem and the Lipschitz continuity of the derivatives implies that the supremum over $\mathbb{B}(\bar{p})$ of the Lipschitz constant of $\nabla^2 \mathcal{L}(p, \cdot)$ on $\mathbb{B}(v^{*N})$ is bounded by a constant independent of N. Further, if $v^N = (\bar{x}^N, \bar{u}^N, \bar{q}^N)$ is the optimal state–control–costate vector for the discrete-time problem (13)–(14), by using the estimate (18), we obtain that the supremum over $\mathbb{B}(\bar{p})$ of the Lipschitz constant of $\nabla^2 \mathcal{L}(p, \cdot)$ but now taken on $\mathbb{B}(v^N)$ is bounded by the same constant for all sufficiently large N. We denote that constant by L. In this section only, when we write $\|\cdot\|$ we mean the ∞-norm.

Introduce the parameterized linearization of (17) as follows:

$$Dv + N_C(v) \ni w, \tag{19}$$

where $w \in \mathbb{R}^l$ is again a parameter and $D := \nabla_v E(\bar{p}, \bar{v}^N)$. We employ the following assumption:

(20) *For each $w \in \mathbb{R}^l$ there is a unique solution $v(w)$ of* (19) *and the function $w \mapsto v(w)$ is Lipschitz continuous on $\mathbb{R}^l$ with a Lipschitz constant ℓ.*

In other words, the mapping

$$w \mapsto v(w) := (D + N_C)^{-1}(w)$$

is a Lipschitz continuous function over $\mathbb{R}^l$ with a Lipschitz constant ℓ.

Choose $a \in (0, 1/2]$ and $b \in (0, 1]$ such that

$$\ell L(a + b) < 1 \quad \text{and} \quad \ell L b \le a(1 - \ell L(a + b)). \tag{21}$$

Let $v, v' \in \mathbb{B}_a(\bar{v}^N)$ and $p \in \mathbb{B}_b(\bar{p})$. Utilizing the mean value theorem, we obtain

$$\begin{aligned} \|E(p, v) - E(p, v') - D(v - v')\| &\le \|\nabla_v E(p, \tilde{v}) - \nabla_v E(p, \bar{v}^N)\| \|v - v'\| \\ &\qquad + \|\nabla_v E(p, \bar{v}^N) - D\| \|v - v'\| \\ &\le L\|\tilde{v} - \bar{v}\| \|v - v'\| + L\|p - \bar{p}\| \|v - v'\| \end{aligned}$$

for some $\tilde{v} \in \mathrm{co}\{v, v'\} \subset \mathbb{B}_a(\bar{v}^N)$. Then

$$\|E(p, v) - E(p, v') - D(v - v')\| \le L(a + b)\|v - v'\|. \tag{22}$$

Fix $p \in I\!B_b(\bar{p})$ and consider the mapping

$$I\!B_a(\bar{v}^N) \ni v \mapsto \Phi_p(v) := (D + N_C)^{-1}(-E(p, v) + Dv).$$

Note that $v(p)$ is a solution of (17) in $I\!B_a(\bar{v}^N)$ for p if and only if $v(p) = \Phi_p(v(p))$. Using the mean value theorem and second inequality in (21), we have

$$\begin{aligned} \|\Phi_p(\bar{v}^N) - \bar{v}^N\| &= \|(D + N_C)^{-1}(-E(p, \bar{v}^N) + D\bar{v}) - (D + N_C)^{-1}(-E(\bar{p}, \bar{v}^N) + D\bar{v}^N)\| \\ &\leq \ell\|E(p, \bar{v}^N) - E(\bar{p}, \bar{v}^N)\| \leq \ell L\|p - \bar{p}\| \leq \ell L b \leq a(1 - \ell L(a + b)). \end{aligned}$$

Furthermore, for every $v, v' \in I\!B_a(\bar{v}^N)$, using (22), we have

$$\begin{aligned} \|\Phi_p(v) - \Phi_p(v')\| &\leq \|(D + N_C)^{-1}(-E(p, v) + Dv) - (D + N_C)^{-1}(-E(p, v')) + Dv'))\| \\ &\leq \ell\|E(p, v) - E(p, v') - D(v - v')\| \leq \ell L(a + b)\|v - v'\|. \end{aligned}$$

Hence, from the first inequality in (21), the mapping Φ_p is a contraction and then, by the basic contraction mapping principle, see e.g. [4, Theorem 1A.3], there exists a unique v in $I\!B_a(\bar{v}^N)$ which satisfies $v = \Phi_p(v)$; that is, $v = v(p)$ is the unique solution of (17) in $I\!B_a(\bar{v}^N)$ for p.

Let $p, p' \in I\!B_b(\bar{p})$. From (22) and again using the mean value theorem, we obtain

$$\begin{aligned} \|v(p) - v(p')\| &= \|(D + N_C)^{-1}(-E(p, v(p)) + Dv(p)) \\ &\qquad -(D + N_C)^{-1}(-E(p', v(p')) + Dv(p'))\| \\ &\leq \ell(\|E(p, v(p)) - E(p, v(p')) - D(v(p) - v(p'))\| \\ &\qquad +\ell\|E(p, v(p')) - E(p', v(p'))\| \\ &\leq \ell L(a + b)\|v(p) - v(p')\| + \ell L\|p - p'\|. \end{aligned}$$

This yields that $p \mapsto v(p)$ is Lipschitz continuous on $I\!B_b(\bar{p})$ with a Lipschitz constant any number λ such that $\lambda \geq \ell L/(1 - \ell L(a + b))$. We summarize the result obtained in the following version of Robinson's theorem:

Theorem 2.1. *Denote by S^N the solution mapping of the variational inequality* (17)*; that is,*

$$p \mapsto S^N(p) = \{v \mid E(p, v) + N_C(v) \ni 0\}.$$

Let $\bar{v}^N \in S^N(\bar{p})$ and assume that condition (20) *is satisfied. Then there exist positive numbers a, b and λ which depend on ℓ and L only such that the graphical localization*

$$I\!B_b(\bar{p}) \ni p \mapsto S^N(p) \cap I\!B_a(\bar{v}^N)$$

is a Lipschitz continuous function with Lipschitz constant λ.

Let us outline the differences between Theorem 2.1 and the original Robinson's theorem stated in [10], for a more recent version of it see [4, Theorem 2B.1]. In the original version, a condition weaker than (20) is used; namely, it is assumed that the mapping $(D + N_C)^{-1}$ has a Lipschitz localization at $\bar{t} := -E(\bar{p}, \bar{v}^N) + \nabla_v E(\bar{p}, \bar{v}^N)\bar{v}^N$ for 0. Here we utilize global Lipschitz continuity of the inverse of the linearization, which for the optimal control problem considered follows from the coercivity condition. Of course, we could use instead a weaker local version of this assumption; in the context of optimal control, however, this would require technically involved assumptions that are difficult to check. Still, such an extension of the present paper is potentially feasible and we put it aside for the future. Also note that, in the finite-dimensional setting of (17) we can still use a global invertibility condition if the set C is *polyhedral*. In that case we may use the argument in [4, Reduction Lemma 2E.4] reducing the considerations to the *critical cone* to C associated with the solution. This however again would leads to complications in translating the argument to the infinite-dimensional setting of the optimal control problem considered. Most important in the current situation is the fact that in Theorem 2.1 we identify the exact dependence of the sizes of the neighborhoods of the graphical localization and its Lipschitz constant on the data of the problem — as it turns out everything depends on a Lipschitz constant and on a coercivity constant. This is an important enhancement which is not present in the original Robinson's theorem.

3 Proof of Theorem 1.2

In the sequel we denote by c a generic positive constant which does not depend on the number N of grid points and may have different values in different relations.

A first step in the proof if to utilize a result from [2] regarding the coercivity condition. Let A_i, B_i, Q_i, S_i, R_i denote the matrices in (7) evaluated at $t = t_i$. Define the quadratic form

$$\mathcal{B}(y, w) = \sum_{i=0}^{N-1} (y_i^T Q_i y_i + 2y_i^T S_i w_i + w_i^T R_i w_i). \tag{23}$$

Thanks to the continuity of u^* we may apply [2, Lemma 11], which claims that the coercivity condition (8) implies the existence of a natural number N_1 and a positive constant $\alpha_1 \le \alpha$ such that for all $N > N_1$ the following discrete-time analogue of the coercivity condition holds:

$$\mathcal{B}(y, w) \ge \alpha_1 \sum_{i=0}^{N-1} |w_i|^2 \tag{24}$$

for all (y, w) from the set

$$\mathcal{C} = \{(y, w) \mid y_{i+1} = y_i + hA_i y_i + hB_i w_i,\ w_i \in U - U,\ i = 0, 1, \ldots, N-1,\ y_0 = 0\}.$$

According to (9),

$$w^T R_i w \ge \alpha |w|^2 \qquad \text{for all} \quad w \in U - U. \tag{25}$$

We should mention here that in [2, Lemma 11] condition (25) was *assumed*, along with (24). Now we know that, thanks to the isolatedness assumption, (25) follows from the coercivity assumption (8).

Define the matrices $\bar{A}_i = \nabla_x \bar{g}_i^N, \bar{B}_i = \nabla_u \bar{g}_i^N, \bar{Q}_i = \nabla_{xx} \bar{H}_i^N, \bar{S}_i = \nabla_{xu} \bar{H}_i^N, \bar{R}_i = \nabla_{uu} \bar{H}_i^N$ where $\bar{g}_i^N = g(\bar{p}, \bar{x}_i^N, \bar{u}_i^N)$ and $\bar{H}_i^N = H(\bar{p}, \bar{x}_i^N, \bar{u}_i^N, \bar{q}_i^N)$, $i = 0, 1, \ldots, N-2$. From (18), we obtain

$$\max_{0 \le i \le N-1} |\bar{u}_i^N| + \max_{1 \le i \le N} |\bar{x}_i^N| + \max_{0 \le i \le N-1} |\bar{q}_i^N| \le c, \tag{26}$$

as well as

$$\max_{0 \le i \le N-1} (|\bar{A}_i| + |\bar{B}_i| + |\bar{Q}_i| + |\bar{S}_i| + |\bar{R}_i|) \le c, \tag{27}$$

where in (27) we use the operator norms of the matrices.

According to Theorem 2.1, we need to consider a parameterized linearization of (16) at $(\bar{p}, \bar{x}^N, \bar{u}^N, \bar{q}^N)$. By differentiating the functions in (16) and then, for convenience, shifting by $(\bar{x}^N, \bar{u}^N, \bar{q}^N)$, that is, replacing $x - \bar{x}^N$ by x, $u - \bar{u}^N$ by u and $q - \bar{q}^N$ by q, we obtain the following system consisting of two linear difference equations running in opposite directions, and a linear variational inequality:

$$\left\{ \begin{array}{rcl} x_{i+1} & = & x_i + h\bar{A}_i x_i + h\bar{B}_i u_i + h r_i, \quad x_0 = 0, \\ q_{i-1} & = & q_i + h\bar{A}_i^T q_i + h\bar{Q}_i x_i + h\bar{S}_i u_i + h s_i, \\ q_{N-1} & = & h s_N, \\ 0 & \in & \nabla_u \bar{H}_i^N + \bar{R}_i u_i + \bar{S}_i^T x_i + \bar{B}_i^T q_i + e_i + N_U(u_i + \bar{u}_i), \end{array} \right. \tag{28}$$

for $i = 0, 1, \ldots, N-1$ in the first and the last relation, $i = 1, 2, \ldots, N-1$ in the second equation, where r_i, s_i, e_i are vector parameters of respective dimensions whose values are in neighborhoods of the zeros in respective spaces.

Define the matrices

(29)
$$\bar{A} = h^{-1} \begin{pmatrix} -I & 0 & \ldots & 0 & 0 \\ I + h\bar{A}_1 & -I & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & I + h\bar{A}_{N-1} & -I \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{B}_0 & 0 & \ldots & 0 \\ 0 & \bar{B}_1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \bar{B}_{N-1} \end{pmatrix},$$

$$\bar{Q} = \begin{pmatrix} \bar{Q}_1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \bar{Q}_{N-1} & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} \bar{S}_0 & 0 & \ldots & 0 \\ 0 & \bar{S}_1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \bar{S}_{N-1} \end{pmatrix}, \quad \bar{R} = \begin{pmatrix} \bar{R}_0 & 0 & \ldots & 0 \\ 0 & \bar{R}_1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \bar{R}_{N-1} \end{pmatrix}$$

and the vectors

$$r = \begin{pmatrix} r_0 \\ \vdots \\ r_{N-1} \end{pmatrix}, \quad s = \begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix}, \quad e = \begin{pmatrix} e_0 \\ \vdots \\ e_{N-1} \end{pmatrix} \quad \text{and} \quad \nabla_u \bar{H} = \begin{pmatrix} \nabla_u \bar{H}_0 \\ \vdots \\ \nabla_u \bar{H}_{N-1} \end{pmatrix}.$$

Note that $\bar{A} = \nabla_{qx} \mathcal{L}(\bar{p}, \bar{x}^N, \bar{u}^N, \bar{q}^N)$, $\bar{B} = \nabla_{qu} \mathcal{L}(\bar{p}, \bar{x}^N, \bar{u}^N, \bar{q})$, $\bar{Q} = \nabla_{xx} \mathcal{L}(\bar{p}, \bar{x}^N, \bar{u}^N, \bar{q}^N)$, $\bar{S} = \nabla_{xu} \mathcal{L}(\bar{p}, \bar{x}^N, \bar{u}^N, \bar{q}^N)$, $\bar{R} = \nabla_{uu} \mathcal{L}(\bar{p}, \bar{x}^N, \bar{u}^N, \bar{q}^N)$. Thus, (28) represents a linearization of the optimality system (16) at the optimal solution-multiplier $(\bar{x}^N, \bar{u}^N, \bar{q}^N)$ for $\bar{p}$. In terms of the aggregated matrices and vectors, it has the form

$$
\left\{
\begin{array}{l}
\bar{A}x + \bar{B}u + r = 0, \\
\bar{Q}x + \bar{S}u + \bar{A}^T q + s = 0, \\
\nabla_u \bar{H} + \bar{S}^T x + \bar{R}u + B^T q + e + N_{\mathcal{U}}(u + \bar{u}^N) \ni 0.
\end{array}
\right.
\tag{30}
$$

Clearly, the matrix $\bar{A}$ is nonsingular for any $h > 0$. In our further analysis we will need an estimate of the norm of the matrix $\bar{A}^{-1}$ regarded as an operator acting from l_2 to l_∞. To do that, we take an arbitrary $z = (z_0, \ldots, z_{N-1})$ and consider the equation $\bar{A}y + z = 0$, which reads as

$$y_{i+1} = y_i + h\bar{A}_i y_i + hz_i, \quad y_1 = hz_0, \quad i = 1, \ldots, N-1.$$

Then

$$y_i = \sum_{j=0}^{i-1} \prod_{k=j+1}^{i-1} (I + h\bar{A}_k) h z_j$$

(where by definition $\prod_{k=i}^{i-1}(I + h\bar{A}_k) = I$) from which, since $|\bar{A}_i|$ is uniformly bounded from (27), we obtain

$$|y_i| \le \sum_{j=0}^{i-1} ch|z_i|,$$

which yields, via the Cauchy-Schwarz inequality,

$$\|y\|_\infty \le ch\sqrt{N \sum_{j=0}^{N-1} |z_i|^2} = c\sqrt{T \sum_{j=0}^{N-1} h|z_i|^2} \le c\sqrt{T}\|z\|_2, \tag{31}$$

and hence the operator norm $|\bar{A}^{-1}|$ from l_2 to l_∞ is bounded by the constant which is independent of N. Define the matrix

$$\bar{P} = \begin{pmatrix} \bar{Q} & \bar{S} \\ \bar{S}^T & \bar{R} \end{pmatrix}. \tag{32}$$

Observe that (30) is the first-order optimality condition for the quadratic programming problem

$$\min \left[\frac{1}{2}(x^T \; u^T)\bar{P} \begin{pmatrix} x \\ u \end{pmatrix} + s^T x + (e + \nabla_u \bar{H})^T u \right] \tag{33}$$

subject to

$$\bar{A}x + \bar{B}u + r = 0, \qquad u \in \bar{u}^N + U. \tag{34}$$

We will now eliminate the variable x from this latter problem. Clearly, the equation in (34) has a unique solution $x = -\bar{A}^{-1}(\bar{B}u + r)$. Elementary transformations lead to

$$\frac{1}{2}(x^T \; u^T)\bar{P} \begin{pmatrix} x \\ u \end{pmatrix} + s^T x + (e + \nabla_u \bar{H})^T u = \frac{1}{2}u^T \bar{W} u + \varphi^T u + \varphi_0,$$

where

$$\bar{W} = (\bar{A}^{-1}\bar{B})^T\bar{Q}\bar{A}^{-1}\bar{B} - (\bar{A}^{-1}\bar{B})^T\bar{S} - \bar{S}^T\bar{A}^{-1}\bar{B} + \bar{R},$$

and

$$\varphi = ((\bar{A}^{-1}\bar{B})^T\bar{Q}\bar{A}^{-1} - \bar{S}^T\bar{A}^{-1})r - (\bar{A}^{-1}\bar{B})^T s + e + \nabla_u\bar{H}. \tag{35}$$

The remainder φ_0 is a vector independent of x and u which does not play any role in the minimization. Thus, problem (51)–(34) is equivalent to

$$\min \frac{1}{2}u^T\bar{W}u + \varphi^T u \quad \text{subject to} \quad u \in \bar{u}^N + \mathcal{U}. \tag{36}$$

Consider now the system (28) but with matrices $\bar{A}_i, \bar{B}_i, \bar{Q}_i, \bar{S}_i, \bar{R}_i$ replaced by A_i, B_i, Q_i, S_i, R_i that are the respective derivatives calculated at $(\bar{p}, x^*(t_i), u^*(t_i), q^*(t_i))$. In the same way as in (29) define matrices A, B, Q, S, R, as well as P and W. Then, making the transformations described in the preceding lines, the coercivity condition (24) implies that

$$u^T W u = \bar{\mathcal{B}}(x, u) \geq \alpha_1 \sum_{i=0}^{N-1} |u_i|^2 \quad \text{for all} \quad u_i \in U - U, \; i = 0, 1, \ldots, N-1, \tag{37}$$

where x is determined from the first group of equations in (28) with $r = 0$, as in the definition of the set $\mathcal{C}$ after (24).

From (18) it follows that for N sufficiently large the matrices A, B, Q, S, R become arbitrarily close to $\bar{A}_i, \bar{B}_i, \bar{Q}_i, \bar{S}_i, \bar{R}_i$. Since the matrix W is a continuous function of the matrices A, B, Q, S, and R, then W can be made arbitrarily close to $\bar{W}$ for N sufficiently large. Thus, for such large N we conclude that

$$u^T\bar{W}u = u^T W u - u^T(W - \bar{W})u \geq \frac{\alpha_1}{2}\sum_{i=0}^{N-1}|u_i|^2. \tag{38}$$

As a result, we obtain a coercivity condition defined for the solution $(\bar{x}^N, \bar{u}^N)$ of the discretized problem and the associated costate $\bar{q}^N$. Namely, let the quadratic form $\bar{\mathcal{B}}(y, w)$ be defined as in (23) but with the matrices $\bar{A}_i, \bar{B}_i, \bar{Q}_i, \bar{S}_i, \bar{R}_i$ instead of A_i, B_i, Q_i, S_i, R_i. Then, for all N sufficiently large we have

$$\bar{\mathcal{B}}(y, u) \geq \frac{\alpha_2}{2}\sum_{i=0}^{N-1}|u_i|^2 \tag{39}$$

for all (y, u) such that $y_{i+1} = y_i + h\bar{A}_i y_i + h\bar{B}_i u_i$, $u_i \in U - U$, $i = 0, 1, \ldots, N-1$, $y_0 = 0$. Furthermore, from (38) we obtain that (36) is a strongly convex quadratic problem. As well known, this problem has a unique global solution for every φ. Furthermore, the solution of this strongly convex quadratic problem is globally Lipschitz continuous with respect to a "tilt" perturbation, see e.g. [4, Theorem 2F.7] with a Lipschitz constant being the reciprocal of the modulus of strong convexity.

For completeness, we will present next a short proof of that fact. Let u and u' denote the solutions of (36) associated with φ and φ', respectively. The optimality of u and u' implies

$$u'^T \bar{W}(u-u') + \varphi'^T(u-u') \geq 0$$

and, symmetrically,

$$u^T \bar{W}(u'-u) + \varphi^T(u'-u) \geq 0.$$

Adding these inequalities we obtain

$$\begin{aligned}(u-u')^T \bar{W}(u-u') &\leq (\varphi'-\varphi)^T(u-u') \leq \|\varphi-\varphi'\|_\infty \sum_{i=0}^{N-1} |u_i - u_i'| \\ &\leq \|\varphi-\varphi'\|_\infty \sqrt{N \sum_{i=0}^{N-1} |u_i-u_i'|^2}.\end{aligned}$$

Combining this with (37) (note that $u_i - u_i' \in \bar{u}_i^N + U - \bar{u}_i^N - U = U - U$) and applying the Cauchy-Schwarz inequality, as in (31), gives us

$$\frac{\alpha_1}{2} \sum_{i=0}^{N-1} h|u_i-u_i'|^2 \leq \|\varphi-\varphi'\|_\infty \sqrt{T} \sqrt{\sum_{i=0}^{N-1} h|u_i-u_i'|^2}, \tag{40}$$

which implies

$$\|u-u'\|_2 \leq \frac{2\sqrt{T}}{\alpha_1} \|\varphi-\varphi'\|_\infty. \tag{41}$$

That is, the solution mapping $\varphi \mapsto u(\varphi)$, considered as a function acting from l_∞ to l_2, is globally Lipschitz continuous with a constant $2\sqrt{T}/\alpha_1$. Thus, the u part of the solution of (51), and hence of (30), is globally Lipschitz continuous with respect to the parameter $\rho = (r, s, d)$ with a Lipschitz constant, denoted ℓ, which, by (40) and the definition of φ in (35), is bounded by the expression

$$\frac{2\sqrt{T}}{\alpha_1} \max\left\{ |(\bar{A}^{-1}\bar{B})^T \bar{Q}\bar{A}^{-1} - \bar{S}^T \bar{A}^{-1}|, |\bar{A}^{-1}\bar{B}|, 1 \right\}. \tag{42}$$

Clearly, the Lipschitz constant ℓ is bounded by the matrix norms of $\bar{A}$, $\bar{B}$, $\bar{Q}$, $\bar{S}$ as in (27), which in turn are bounded by the Lipschitz constants of the derivatives of $\bar{g}$ and $\bar{H}$ over the unit balls centered at the reference solution, and hence does not depend on N for all N sufficiently large.

We will now give a sharper version of the last result. Let $\Delta r, \Delta s, \Delta e$ be increments of the parameters r, s, e, respectively, that is, $\Delta r = r - r'$ for r, r' close to the origin, etc., and denote the corresponding changes of the solution mapping of (28) by $\Delta x, \Delta q, \Delta u$. From the state equation in (28), we have

$$\Delta x_{i+1} = \Delta x_i + h\bar{A}_i \Delta x_i + h\bar{B}_i \Delta u_i + h\Delta r_i, \quad i = 0, 1, \ldots, N-1, \quad \Delta x_0 = 0,$$

hence, taking norms, from the discrete Gronwall lemma we get

$$|\Delta x_i| \le c \sum_{j=0}^{i} h(|\Delta u_j| + |\Delta r_j|), \quad i = 1, \dots, N.$$

Thus, by (41) and the definition of φ in (35), we obtain

$$\max_{1 \le i \le N} |\Delta x_i| \le c(\|\Delta r\|_\infty + \|\Delta s\|_\infty + \|\Delta e\|_\infty). \tag{43}$$

Analogously, by applying the discrete Gronwall lemma backward, for the change of the costate, we get

$$|\Delta q_{i-1}| \le c \left(\sum_{j=i}^{N-1} h(|\Delta x_j| + |\Delta u_j|) + \sum_{j=i}^{N-1} h|\Delta s_j| \right), \quad i = 1, \dots, N-1,$$

which leads to the estimate

$$\max_{0 \le i \le N-2} |\Delta q_i| \le c(\|\Delta r\|_\infty + \|\Delta s\|_\infty + \|\Delta e\|_\infty). \tag{44}$$

Utilizing the definition of the normal cone mapping (6), the variational inequality for the control in (28) gives us

$$\Delta u_i^T \bar{R}_i \Delta u_i \le -\Delta u_i^T (\bar{S}_i^T \Delta x_i + \bar{B}_i^T \Delta q_i + \Delta e_i), \quad i = 0, \dots, N-1.$$

By (25), since $\bar{R}_i$ can be made arbitrarily close to R_i, for each $i = 0, \dots, N-1$, for all sufficiently large N we have

$$w^T \bar{R}_i w \ge \frac{\alpha_1}{2} |w|^2 \qquad \text{for all} \quad w \in U - U.$$

Hence, we have that

$$|\Delta u_i| \le c(|\Delta x_i| + |\Delta q_i| + |\Delta e_i|), \quad i = 0, \dots, N-1. \tag{45}$$

Combining (43), (44) and (45), we obtain that the control part u of the solution of the linearized optimality system (28) satisfies

$$\max_{0 \le i \le N-1} |\Delta u_i| \le c(\|\Delta r\|_\infty + \|\Delta s\|_\infty + \|\Delta e\|_\infty).$$

That is, the control part is Lipschitz continuous with respect to $\rho = (r, s, e)$ as a function acting from l_∞ to l_∞ with a Lipschitz constant independent of N. Finally (43) and (44) give us Lipschitz continuity with respect to $\rho = (r, s, e)$ from l_∞ to l_∞ of the state trajectory x and the costate trajectory q.

We will now repeat the argument in the beginning of Section 2.1 regarding the bound on the Lipschitz constant of $\nabla^2 \mathcal{L}(p, \cdot)$, obtaining that Theorem 2.1 applies to the variational inequality (17) representing the first-order optimality condition of problem (13)–(14). As a result, we conclude that the solution mapping $\mathcal{S}^N$ of the optimality system (16) has a graphical localization with the properties claimed in the statement of Theorem 1.2.

The last step in the proof is to observe that, for the state-control pair $(\bar{x}^N, \bar{u}^N)$ obtained from solving the optimality system (16) for $p = \bar{p}$, the coercivity condition (39) is a sufficient condition for strict local minimum. Moreover, for all p sufficiently close to $\bar{p}$, this condition remains valid, in which case the state-control pair $(x^N(p), u^N(p))$ remains a strict local minimum of the perturbed problem. This statement can be derived from various (more general) results scattered in the literature the closest of which seems to be the one given in [2, Appendix 1], see also [11, Statement 13.25].

4 An application to Newton/SQP method

In this section we consider convergence properties of the Newton/SQP method applied to the optimality system (16). As in the Introduction, we write the optimality system (16) as the variational inequality (17), where $v = (x, u, q)$ represents the tripple state–control–costate. Recall that the Newton iteration for the variational inequality (17), as proposed originally by Josephy [9], has the following form

$$E(p, v^k) + \nabla_v E(p, v^k)(v^{k+1} - v^k) + N_C(v^{k+1}) \ni 0. \tag{46}$$

Roughly speaking, since (17) represents an optimality system coming from the first-order necessary optimality condition, then, in the numerical realization of the method, the linear variational inequality (46) can be regarded as an optimality system of a quadratic program to which a quadratic programming method can be applied. Therefore this form of the Newton method is commonly known as the Sequential Quadratic Programming (SQP), and this method is currently a method of choice for solving nonlinear programming problems.

Applied to the optimality system (16), the Newton/SQP iteration (46) is as follows: having $v^k = (x^k, u^k, q^k)$, the next iterate $v^{k+1} = (x^{k+1}, u^{k+1}, q^{k+1})$ is given by

$$x^{k+1} = x^k + \xi^k, \quad u^{k+1} = u^k + \nu^k, \quad q^{k+1} = q^k + \eta^k,$$

where (ξ^k, η^k, ν^k) is a solution of the linear variational inequality

$$\left\{ \begin{array}{l} \xi_{i+1} - \xi_i + r_i^k - hA_i^k \xi_i - hB_i^k \nu_i = 0, \quad \xi_0 = 0, \\ \eta_{i-1} - \eta_i + s^k + h(A_i^k)^T \eta_i + hQ_i^k \xi_i + hS_i^k \nu_i = 0, \quad \eta_{N-1} = 0, \\ w^k + R_i^k \nu_i + (S_i^k)^T \xi_i + (B_i^k)^T \eta_i + N_U(u_i^k + \nu_i^k) \ni 0, \end{array} \right. \tag{47}$$

where the index i runs from 0 to $N-1$ and

$$r^k = x_{i+1}^k - x_i^k - hg(p, x_i^k, u_i^k), \quad s^k = q_{i-1}^k - q_i^k + h\nabla_x H(p, x_i^k, u_i^k, q_i^k), \quad w^k = \nabla_u H(p, x_i^k, u_i^k, q_i^k)$$

$$A_i^k = \nabla_x g(p, x_i^k, u_i^k), \;\; B_i^k = \nabla_u g(p, x_i^k, u_i^k),$$

$$Q_i^k = \nabla_{xx} H(p, x_i^k, u_i^k, q_i^k), \;\; S_i = \nabla_{xu} H(p, x_i^k, u_i^k, q_i^k), \;\; R_i^k = \nabla_{uu} H(p, x_i^k, u_i^k, q_i^k).$$

From Theorem 1.2 we know that, under the smoothness, coercivity and isolatedness conditions stated in the Introduction, there exist a natural $\bar{N}$ and positive reals a, b and

λ such that for all $N > \bar{N}$, there is a single-valued graphical localization $\mathbb{B}_b(\bar{p}) \ni p \mapsto \mathcal{S}^N(p) \cap \mathbb{B}_a(v^N(p))$ of the solution mapping $\mathcal{S}^N$ of (16), associated with $\mathbb{B}_a(\bar{v}^N)$ and $\mathbb{B}_b(\bar{p})$, which is Lipschitz continuous on $\mathbb{B}_b(\bar{p})$ with Lipschitz constant λ; here $\bar{v}^N = (\bar{x}^N, \bar{u}^N, \bar{q}^N)$. Denote this localization by $\bar{v}^N(p) = (\bar{x}^N(p), \bar{u}^N(p), \bar{q}^N(p))$. It should be emphasized that a, b and λ do not depend on N for $N > \bar{N}$. But note that the function E in (17) and (46) depends on N. By the argument in the beginning of Section 2.1, we know that the supremum over $\mathbb{B}(\bar{p})$ of the Lipschitz constants of $\nabla E(p, \cdot)$ on $\|v - \bar{v}^N\|_\infty \leq 1$ is bounded by a constant L for all sufficiently large N.

Let $\varepsilon > 0$. Pick a starting point v^0 and apply the iteration (47), until the following stopping test is satisfied:

$$\operatorname{dist}(0, E(p, v^k) + N_C(v^k)) \leq \varepsilon. \tag{48}$$

where $\operatorname{dist}(z, Z)$ denotes the distance from a point z to a set Z. Let k_ε be the index of the first iteration where the stopping test (48) holds.

We wish to apply a combination of theorems 3.1 and 4.1 from [3], where, however, there is no dependence on the parameter p; that is, we have $E(v)$ as in (17) which depends on the number N of discretization points but not on p. The assumptions used in [3] involve smoothness of the function E and Lipschitz continuity of the derivative ∇E; furthermore, it is assumed in [3, Theorems 3.1] there that there exist constants α, β and γ such that for each N sufficiently large the localization

$$\mathbb{B}_\beta(0) \ni y \mapsto \big(E(\bar{v}^N) + \nabla E(\bar{v}^N)(\cdot - \bar{v}^N) + N_C\big)^{-1}(y) \cap \mathbb{B}_\alpha(\bar{v}^N) \tag{49}$$

is a Lipschitz continuous function with Lipschitz constant γ. In order to take into account the dependence on the parameter p we need the following enhanced form of (49): there exist a natural number N_2 and real constants κ, α', β', and γ' such that for every $p \in \mathbb{B}_\kappa(\bar{p})$ and for every for $N \geq N_2$ the mapping

$$\mathbb{B}_{\beta'}(0) \ni y \mapsto \big(E(p, v^N(p)) + \nabla_v E(p, v^N(p))(\cdot - v^N(p)) + N_C\big)^{-1}(y) \cap \mathbb{B}_{\alpha'}(v_h(p)) \tag{50}$$

is a Lipschitz continuous function with Lipschitz constant γ'. That this is true follows from the following theorem which is a part of [4, Theorem 5G.3][6]. Because of its importance, we give it here in full:

Theorem 4.1. *For Banach spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, consider a mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Let α, β, and γ be positive scalars such that the mapping $\mathbb{B}_\alpha(\bar{y}) \ni y \mapsto F^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x})$ is a Lipschitz continuous function with Lipschitz constant γ. Let $\mu > 0$ be such that $\gamma\mu < 1$ and let $\gamma' > \gamma/(1 - \gamma\mu)$. Then for every positive α' and β' such that*

$$\alpha' \leq \alpha/2, \quad 2\mu\alpha' + 2\beta' \leq \beta \quad \text{and} \quad 2\gamma'\beta' \leq \alpha',$$

and for every function $g : X \to Y$ satisfying

$$\|g(\bar{x})\|_Y \leq \beta' \quad \text{and} \quad \|g(x) - g(x')\|_Y \leq \mu\|x - x'\|_X \quad \text{for every } x, x' \in \mathbb{B}_{2\alpha'}(\bar{x}),$$

the mapping $\mathbb{B}_{\beta'}(\bar{y}) \ni y \mapsto (g+F)^{-1}(y) \cap \mathbb{B}_{\alpha'}(\bar{x})$ is a Lipschitz continuous function Lipschitz constant γ'.

[6] See Errata and Addenda at https://sites.google.com/site/adontchev/

By comparing (49) and (50), and utilizing the smoothness of E and Theorem 4.1 we obtain that there exist a natural N_3 and constant κ such that for every $p \in I\!B_\kappa(\bar{p})$ and for every for $N \geq N_3$ condition (50) is satisfied. Thus, [3, Theorems 3.1, 4.1], see also [4, Theorem 6D.2] can be applied to obtain the following theorem, where we use the notation defined in the beginning of Section 2. Namely, $v^{*N} = (x^{*N}, u^{*N}, q^{*N})$ is the long vector with the values of the optimal state, control and costate for the continuous-time problem (1)–(2) at the grid points.

Theorem 4.2. *On the assumptions of Theorem 1.2 there exist a natural N_4 and reals τ, σ, Θ, and c such that for every $p \in I\!B_\tau(\bar{p})$ and every $N \geq N_4$, if the method (47) is applied for solving the discretized variational system (16) with step-size $h = T/N$ and starting from a point $(x^0, u^0, q^0) \in I\!B_\sigma(v^{*N})$ there exists an infinite sequence $v^k := (x^k, u^k, q^k)$ satisfying (47) and such that $v^k \in I\!B_{2\sigma}(v^{*N})$ for $k = 1, 2, \ldots$; moreover, there is no other sequence with these properties. Furthermore, the sequence v^k is q-quadratically convergent to a locally unique solution $v^N(p) := (x^N(p), u^N(p), q^N(p))$ of (16) for p and the convergence is uniform with respect to $p \in I\!B_\tau(\bar{p})$ and $N \geq N_4$ in the sense that*

$$\|v_i^{k+1} - v_i^N(p)\|_\infty \leq \Theta \|v^k - v^N(p)\|_\infty^2. \tag{51}$$

In addition, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, every $p \in I\!B_\tau(\bar{p})$ and every $N \geq N_4$, if the iteration (47) is combined with the stopping criterion (48), then it terminates at iteration k_ε, where

$$k_\varepsilon \leq c(1 - \lg \varepsilon),$$

and the resulting ε-optimal solution $v^{k_\varepsilon}(p) := (x^{k_\varepsilon}(p), u^{k_\varepsilon}(p), q^{k_\varepsilon}(p))$ obtained at the k_ε iteration satisfies

$$\|v^{k_\varepsilon}(p) - v^{*N}\|_\infty \leq c(\varepsilon + h + |p - \bar{p}|).$$

5 Numerical experiments

In this section we will illustrate Theorem 4.2 by numerical experiments with the satellite control problem detailed in [6]. The state describing the dynamics of the attitude of a vehicle is represented by the Euler angles θ_1 (the roll angle), θ_2 (the pitch angle), and θ_3 (the yaw angle), as well as the vector of angular velocities $\omega \in \mathbb{R}^3$. The control vector $u \in \mathbb{R}^3$ describes the control torques acting on the body frame axes. The control system has the form

$$\begin{cases} \dot{\theta} = E(\theta)\omega \\ M\dot{\omega} = \hat{\omega} M \omega + u, \end{cases} \tag{52}$$

for a.e. $t \in [0, T]$, with given initial conditions $\theta(0) = \theta_0(p)$ and $\omega(0) = \omega_0(p)$, which depend on a scalar parameter p. Here[7]

$$E(\theta) = \frac{1}{c(\theta_2)} \begin{bmatrix} c(\theta_2) & s(\theta_1)s(\theta_2) & c(\theta_1)s(\theta_2) \\ 0 & c(\theta_1)c(\theta_2) & -s(\theta_1)c(\theta_2) \\ 0 & s(\theta_1) & c(\theta_1) \end{bmatrix},$$

[7]For short, $c(x)$, $s(x)$ denote $\cos(x)$, $\sin(x)$. We neglect here the issue of handling the "gimble lock" singularity at $\theta_2 = \pm\pi/2$ as it does not affect the numerical performance.

and

$$\hat{\omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

The components of the matrix M describe the moments of inertia of three reaction wheels along each of the body frame axes; for the numerical experiments we take $M = \mathrm{diag}([120\ 100\ 80])\ kg\cdot m^2$. The goal is to minimize the cost functional

$$J(p,x,u) = \frac{1}{2}\int_0^T \big(x(t)^T Q x(t) + u(t)^T R u(t)\big)\, dt,$$

where the state $x = \begin{pmatrix} \theta \\ \omega \end{pmatrix}$, subject to (52) and the control constraints

$$|u(t)| \le u_{\max} \qquad \text{for a.e. } t \in [0,T].$$

Let a natural N representing the discrete-time horizon and let $h = T/N$ be the step-size. Then the state equation (52) is discretized as

$$x_{i+1} = x_i + h\begin{bmatrix} E(\theta_i)\omega_i \\ M^{-1}(-\hat{\omega}_i M \omega_i + u_i) \end{bmatrix}, \quad i = 0, \cdots, N-1, \quad x_0 = \begin{pmatrix} \theta_0(p) \\ \omega_0(p) \end{pmatrix}, \tag{53}$$

and the discrete-time cost function is to minimize

$$J_N(p,x,u) = \frac{h}{2}\sum_{i=0}^{N-1} \big(x_i^T Q x_i + u_i^T R u_i\big)$$

subject to (53) and the control constraints

$$|u_i| \le u_{\max} \qquad \text{for} \quad i = 0, 1, \dots, N-1.$$

The numerical experiments are performed using the the cost matrices

$$Q = \mathrm{diag}[10\ 10\ 10\ 100\ 100\ 100] \qquad \text{and} \qquad R = \mathrm{diag}[0.1\ 0.1\ 0.1],$$

whereas the bound for the control values is $u_{\max} = 0.5\ Nm$. The optimal control problem is solved for the initial condition $\bar{\theta}(0) = [\pi/3\ 0\ -\pi/6]\ rad$, $\bar{\omega}(0) = 10^{-3}[0\ 5\ 1]\ rad/s$. The perturbation parameter $p \in \mathbb{R}$ enters the initial condition in the following way:

$$\theta(0) = \bar{\theta}(0) + p\frac{\bar{\theta}(0)}{10}, \qquad \omega(0) = \bar{\omega}(0) + p\frac{\bar{\omega}(0)}{10}.$$

Note that in our problem (1)–(2), the initial condition does not explicitly depend on p; however, a simple change of the variables moves the parameter p to the control system. The solution of the unperturbed problem, with $\bar{p} = 0$, is used as a starting point for the SQP iteration.

Figure 6.1 illustrates the behavior of the control error $\|u^k - u^N(p)\|_\infty$ for several values of the parameter p and of the number of discretization steps N. Most notably, for a fixed p one can see that for large N the behavior of the algorithm becomes independent of N.

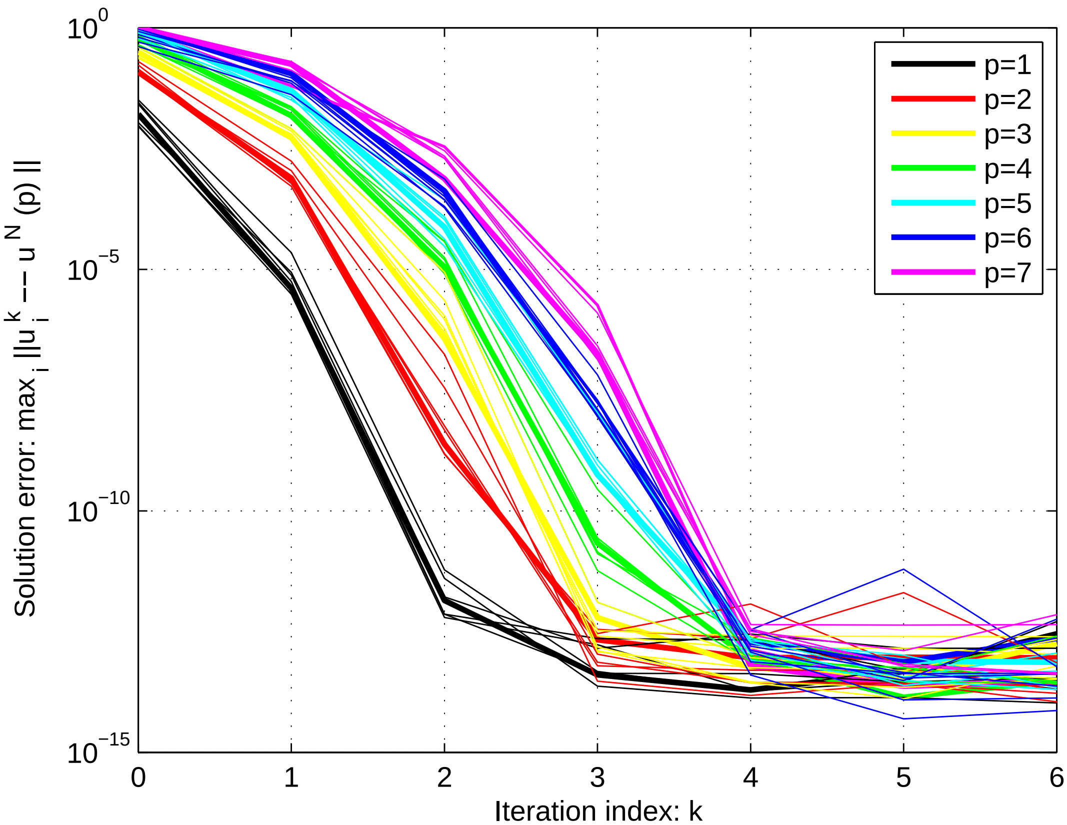

Figure 6.1: l_∞ control error at each SQP iteration for different values of p and N. For a given value of p, curves sharing the same pattern are obtained for $N = 20, 25, 30, 35, 40, 45, 50$.

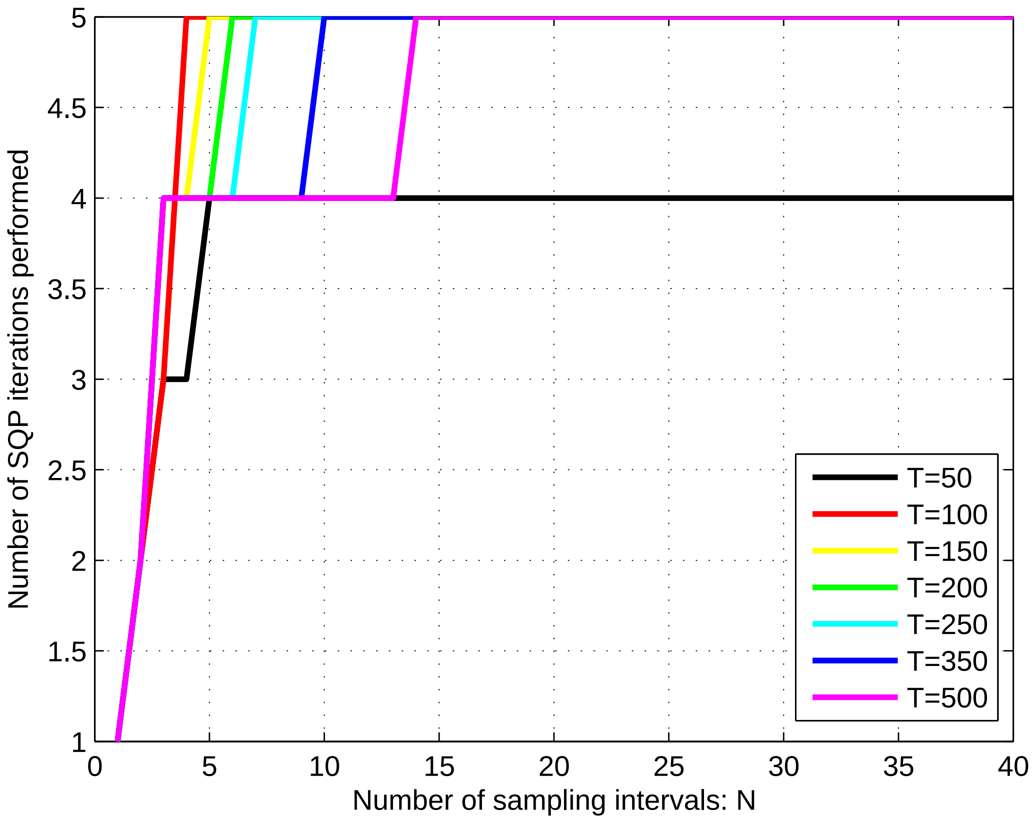

Figure 6.2: The number of iterations needed to reach the desired accuracy versus the number of discretization steps N. Each curve is obtained for a different value of the prediction horizon T.

The control error decreases quadratically in agreement with Theorem 4.2 and then stabilizes around the value 10^{-13} due to machine precision. The observed value for the coefficient in (51) is $\Theta = 0.32$.

Figure 6.2 shows the number of SQP iterations required to meet the termination criterion (48) with $\varepsilon = 10^{-10}$. All simulations are performed using $p = 4$ to compute the perturbed initial conditions. The figure clearly shows that the number of iterations (k_ε in Theorem 4.2) needed until the stopping criterion is satisfied remain constant when the number of discretization points increases. It is worth noting that, as shown in the figure, the number N_4 of grid points where saturation occurs monotonically increases with the horizon length T. Interestingly enough, the numerical experiments satisfy $T/N_4(T) \approx 35$ sec in all realizations, thus suggesting that the value of N_4 can be determined *a priori*.

As illustrated in Figure 6.3, when using a number of grid points for which the stopping test engages leads to a suboptimal solution which is still not quite close to the continuous-time solution. This is consistent with (18) since the sampling time $h = 35.71$ sec is relatively large with respect to the system dynamics.

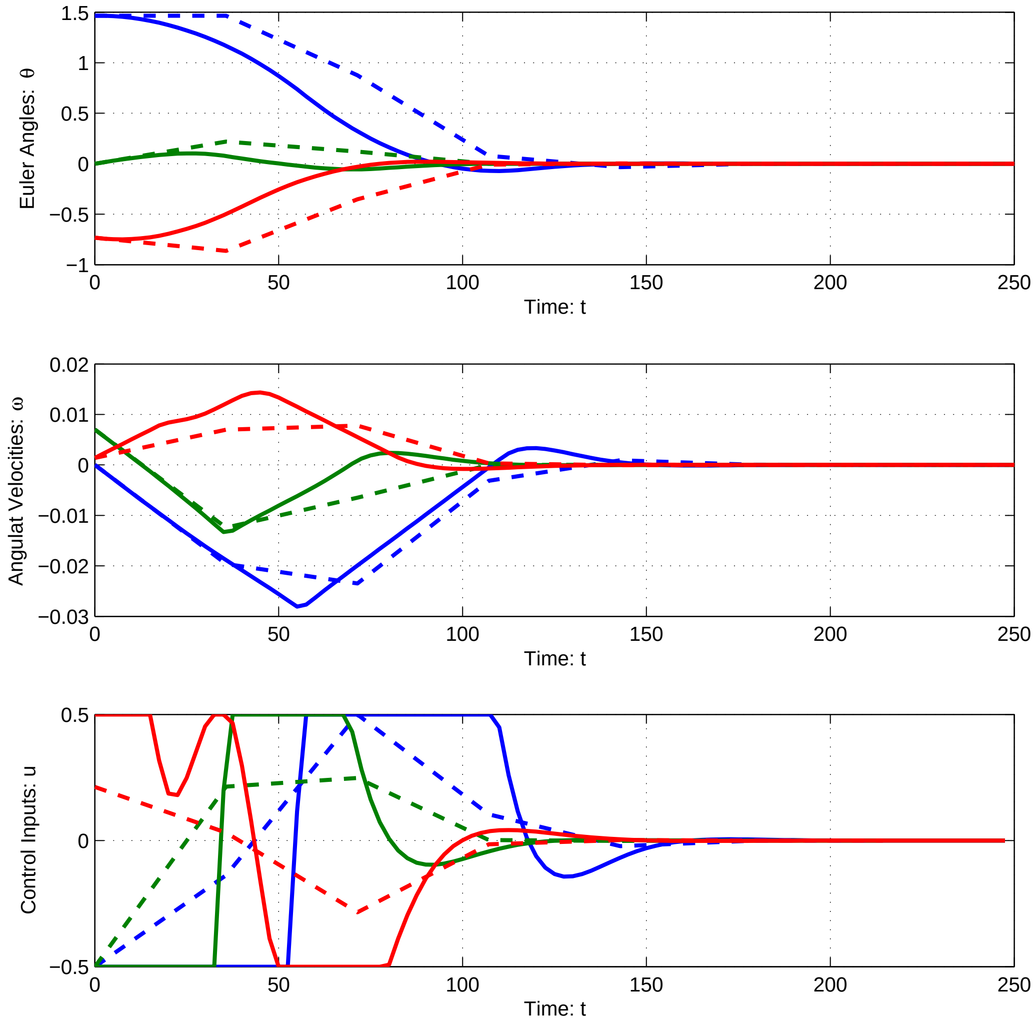

Figure 6.3: The dashed lines are the state and control functions obtained for $N = 7$ whereas the solid lines depicts those obtained for $N = 100$.

References

[1] R. Cibulka, A. L. Dontchev, M. I. Krastanov, V. M. Veliov, Metrically regular differential generalized equations. SIAM J. Control Optim. **56** (2018) 316–342.

[2] A. L. Dontchev, W. W. Hager, Lipschitzian stability in nonlinear control and optimization, SIAM J. Control Optim. **31** (1993), 569–603.

[3] A. L. Dontchev, W. W. Hager, V. M. Veliov, Uniform convergence and mesh independence of Newton's method for discretized variational problems, SIAM J. Control Optim. **39** (2000), 961–980.

[4] A. L. Dontchev, R. T. Rockafellar, *Implicit functions and solution mappings*, Second edition, Springer 2014.

[5] A. L. Dontchev, M. I. Krastanov, V. M. Veliov, On the existence of Lipschitz continuous optimal feedback control in ODE control problems, Research Report 2018-04, ORCOS, TU Wien, 2018.

[6] A. Giuggiani, I. V. Kolmanovsky, P. Patrinos, and A. Bemporad, Fixed point constrained Model Predictive Control of spacecraft attitude, *Proc. of the American Control Conference*, (2015) 2317–2322.

[7] L. Grüne, J. Pannek, Nonlinear model predictive control. Theory and algorithms. Second edition. Communications and Control Engineering Series. Springer, Cham, 2017.

[8] W. W. Hager, Multiplier methods for nonlinear optimal control, SIAM J. Numer. Anal. **27** (1990) 1061–1080.

[9] N. H. Josephy, Newton's method for generalized equations and the PIES energy model, Ph.D. Dissertation, Department of Industrial Engineering, University of Wisconsin-Madison, 1979.

[10] S. M. Robinson, Strongly regular generalized equations, Math. Oper. Res. **5** (1980) 43–62.

[11] R. T. Rockafellar, R. J-B Wets, Variational analysis, Springer 2009.