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An extragradient-type method for solving nonmonotone quasi-equilibrium problems

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Abstract

In this paper a quasi-equilibrium problem with a nonmonotone bifunction is considered in a finite dimensional space. The primary difficulty with this problem is related to the fact that one must simultaneously solve a nonmonotone equilibrium problem and calculate a fixed point of a multivalued mapping. An extragradient-type method is presented and analyzed for its solution. The convergence of the method is proved under the assumption that the solution set of an associated dual equilibrium problem is nonempty. Finally, some numerical experiments are reported.

Keywords: Extragradient-type method; Quasi-equilibrium problem; Nonmonotone operator; Convergence

1 Introduction

The aim of the paper is to present and analyze an extragradient-type method for solving the nonmonotone quasi-equilibrium problem, that is an equilibrium problem in the sense of Blum and Oettli [1] or Ky Fan's inequality [2], with a constraint set depending on the current point and without any monotonicity assumptions for the bifunction.

The equilibrium problem has been extensively studied in recent years; see, for example, [3, 4, 5, 6, 7, 8, 9, 10] and the references therein. Equilibrium problems

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include, as particular cases, scalar and vector optimization problems, saddle-point problems, variational inequalities, Nash equilibria, complementarity problems, fixed point problems, etc. It is well known that quasi-variational inequalities, where the constraint sets are moving sets, do not fall within the scope of equilibrium problems but under quasi-equilibrium problems. Perhaps the most important instance of such problems is the generalized Nash equilibrium problem [11, 12, 13] which models a large number of applications in engineering, economics, management science and other areas (see, e.g., [14, 15, 16]). Quasi-equilibrium problems address extensions of the well-known classical equilibrium concept, making it possible to model and study more general settings.

Many numerical methods have been proposed for solving equilibrium problems such as the projection method [17, 18, 19], the proximal point methods [20, 21], the extragradient methods with or without line searches [22, 23, 24, 25, 26, 27, 28], the methods using the Bregman distance [29, 30] and the gap function methods [31, 32, 33]. Each solution method is adapted to a class of equilibrium problems. The reader is referred to [34] and the references quoted therein for an excellent survey on the existing methods. In most of these methods it is assumed that the equilibrium bifunction is pseudomonotone with the consequence that the solution set of the equilibrium problem coincides with the solution set of the Minty equilibrium problem [35, 36]. This property is also satisfied for quasi-equilibrium problems when the equilibrium bifunction is pseudomonotone [37, 38].

Recently, in [39], an extragradient-type method has been proposed for solving the equilibrium problem when the equilibrium bifunction is not assumed to be pseudomonotone (see also [40]). In that case, contrary to the pseudomonotone case, at each iteration, all the hyperplanes separating the solution set from the previous iterates must be kept in memory to build the next iterate. Let us mention here that the authors of [41] have recently proposed a similar method for solving the equilibrium problem in the nonmonotone case.

Our aim in this paper is to solve a quasi-equilibrium problem having a constraint set $K(x)$ which depends on x and is contained in a fixed subset X of the finite dimensional space $\mathbb{R}^n$. This will be done without requiring any monotonicity assumption on the equilibrium bifunction. In this purpose, we associate with the quasi-equilibrium problem, the Minty quasi-equilibrium problem, and in order to guarantee nonempty solution sets for these two problems, we impose that there exists a fixed point of the multivalued mapping K that is also a solution of the classical Minty equilibrium problem. This condition can be considered as the dual version of a condition introduced in [38, 42] for solving pseudomonotone quasi-equilibrium problems.

Under this assumption, the strategy used for proving the convergence of the resulting algorithm to a solution of the quasi-equilibrium problem consists in considering successively the following two steps. First we prove that the sequence generated by the algorithm converges to a fixed point of the multivalued mapping K and after, we show that this sequence also converges to a solution of the quasi-equilibrium problem.

After presenting the nonmonotone quasi-equilibrium problem and recalling some useful results in Sect. 2, we study the convergence of the algorithm in Sect. 3. Finally, some numerical results are reported in Sect. 4 to show the validity of the proposed method.

2 Preliminaries and useful lemmas

Let X be a nonempty closed convex subset of the finite dimensional space $\mathbb{R}^n$ with the scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$, let $K(\cdot)$ be a multivalued mapping from X into itself such that for every $x \in X$, $K(x)$ is a nonempty closed convex subset of X and let $f : X \times X \to \mathbb{R}$ be a bifunction such that for every $x \in X$, $f(x, x) = 0$ and $f(x, \cdot)$ is a convex function on X. The quasi-equilibrium problem associated with f and K is denoted $QE(f, K)$ and consists in finding $x^* \in K(x^*)$, i.e., a fixed point x^* of $K(\cdot)$, such that

$$f(x^*, y) \geq 0 \quad \text{for all } y \in K(x^*).$$

Here we denote by $Fix(K)$ the set of fixed points of the multivalued mapping $K(\cdot)$.

The associated Minty quasi-equilibrium problem, denoted by $MQE(f, K)$, can be expressed as finding $x^* \in K(x^*)$ such that

$$f(y, x^*) \leq 0 \quad \text{for all } y \in K(x^*).$$

We denote by S_E^* and S_M^* the solution sets of problems $QE(f, K)$ and $MQE(f, K)$, respectively. When the function $f(\cdot, y)$ is in addition upper semicontinuous on X for every $y \in X$, the function f has the upper sign property ([43], Definition 2.1) and since the multivalued mapping $K(\cdot)$ has convex values, we obtain that $S_M^* \subseteq S_E^*$ ([43], Proposition 3.1).

When the constraint set $K(x)$ is equal to X for every $x \in X$, problem $QE(f, K)$ becomes an equilibrium problem and problem $MQE(f, K)$ its associated Minty equilibrium problem. They can be written under the form

$$E(f, X) \qquad \text{Find } x^* \in X \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in X$$

and

$$ME(f, X) \qquad \text{Find } x^* \in X \text{ such that } f(y, x^*) \leq 0 \text{ for all } y \in X.$$

We denote by S_E and S_M the solution sets of these problems, respectively.

In this paper we assume that there exists a fixed point of the multivalued mapping K that is a solution of the classical Minty equilibrium problem, i.e., that the set

$$S_* = \{x \in K(x) \mid f(y, x) \leq 0 \text{ for all } y \in X\}$$

is nonempty. Since $S_* \subseteq S_M^* \subseteq S_E^*$, this assumption guarantees that the solution sets S_M^* and S_E^* are nonempty.

When the function f is also pseudomonotone on X, i.e., when

$$\forall x, y \in X \quad f(x,y) \geq 0 \quad \Rightarrow \quad f(y,x) \leq 0,$$

we obtain that $S_E^* \subseteq S_M^*$ and thus that $S_M^* = S_E^*$. In this situation, the set S_* has been replaced in [38, 42] by the set

$$S^* = \{x \in \cap_{z \in X} K(z) \mid f(x,y) \geq 0 \text{ for all } y \in \cup_{z \in X} K(z)\}.$$

Here we do not assume that f is pseudomonotone on X. Hence we only have the relationships

$$\emptyset \neq S_* \subseteq S_M^* \subseteq S_E^*.$$

When $f(x,y) = \langle F(x), y - x \rangle$ for every $x, y \in X$ where F is a mapping from X to X, problem $QE(f,K)$ becomes a quasi-variational inequality problem that can be expressed as

$$\text{Find } x^* \in K(x^*) \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0 \text{ for all } y \in K(x^*).$$

This problem has been studied in [42] when F is supposed to be pseudomonotone.

From now on, we denote by $CCB(X)$ the family of nonempty convex closed bounded subsets of X and we recall that a multivalued mapping $T : X \to CCB(X)$ is said to be $\star$-nonexpansive ([44], p.4) if

$$\|P_{T(x)}x - P_{T(y)}y\| \leq \|x - y\| \text{ for all } x, y \in X, \tag{1}$$

where $P_{T(x)}x$ denotes the orthogonal projection of x onto the nonempty convex closed bounded subset $T(x)$.

Finally, let us also recall the *Demiclosedness Principle* (see for example [45], Theorem 3.4 and Corollary 3.5): Let $T : X \to CCB(X)$ be a $\star$-nonexpansive mapping. Then $Fix(T)$ is convex and closed and $I - T$ is demiclosed at 0, i.e., for every sequence $\{x^k\} \subset X$ such that $x^k \to x$ and $d(x^k, T(x^k)) \to 0$ as $k \to \infty$, we have $x \in T(x)$.

In this paper the following assumptions are supposed to be satisfied on the data of the problem.

Assumption (A)

(A1) f is defined on $\Delta \times \Delta$ where Δ is an open convex subset of $\mathbb{R}^n$ containing X; $f(x,x) = 0$ for all $x \in X$ and $f(x, \cdot)$ is convex on X for all $x \in X$.

(A2) f is jointly continuous on $\Delta \times \Delta$ in the sense that if $x, y \in \Delta$, and $\{x^k\}$ and $\{y^k\}$ are two sequences in Δ converging to x and y, respectively, then $f(x^k, y^k) \to f(x, y)$.

(A3) For all $x \in X$, $K(x)$ is a nonempty closed convex bounded subset of X.

(A4) The multivalued mapping $K(\cdot)$ is $\star$-nonexpansive on X and lower semicontinuous at each $\bar{x} \in X$, i.e., if $\{x^k\} \subset X$ and $x^k \to \bar{x}$, then for any $\bar{y} \in K(\bar{x})$, there exists a sequence $\{y^k\}$ with $y^k \in K(x^k)$ for all k, such that $y^k \to \bar{y} \quad (k \to \infty)$.

(A5) The set S_* is nonempty.

Let us observe that under Assumptions $(A1)$ and $(A2)$, it follows from [43] that the upper sign property holds and that $S_M^* \subseteq S_E^*$. Furthermore these sets are nonempty by Assumption $(A5)$.

The next lemmas are useful to prove the convergence of our algorithm.

Lemma 2.1 ([46], Theorem 24.5) (see also [47]) *Let $f : \Delta \times \Delta \to I\!R$ be a function satisfying conditions (A1) and (A2). Let $\bar{x}, \bar{z} \in \Delta$ and $\{x^k\}$, $\{z^k\}$ be two sequences in Δ converging to $\bar{x}$, $\bar{z}$, respectively. Then, for any $\varepsilon > 0$, there exist $\eta > 0$ and $k_\varepsilon \in I\!N$ such that*

$$\partial_2 f(z^k, x^k) \subseteq \partial_2 f(\bar{z}, \bar{x}) + \frac{\varepsilon}{\eta} B,$$

for every $k \geq k_\varepsilon$, where B denotes the closed unit ball in $I\!R^n$ and $\partial_2 f(z, x)$ the subdifferential of the convex function $f(z, \cdot)$ at x.

Lemma 2.2 ([48], Theorem 5.5) *Let $\{x^k\}$ be a sequence in $I\!R^n$ and let C be a nonempty subset of $I\!R^n$. Suppose that every limit point of $\{x^k\}$ belongs to C and that the sequence $\{x^k\}$ is Fejér monotone with respect to C in the sense that the following inequality*

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\| \quad \text{for all } x^* \in C$$

holds for all k. Then the whole sequence $\{x^k\}$ converges to a point in C.

3 A Convergent Algorithm

From now on we assume that Assumption (A) is satisfied, and we consider the following algorithm for solving the quasi-equilibrium problem $QE(f, K)$:

Algorithm 1

Step 0 Let $x^0 \in X$, $c \in]0,1[$ and $\alpha \in]0,1[$. Let also $\{\mu_k\} \subseteq [a,b]$ where $0 < a \leq b < 1$.
Set $k = 0$.

Step 1 Compute $y^k = \arg\min_{y \in K(x^k)} \{f(x^k, y) + \frac{1}{2} \|y - x^k\|^2\}$.
If $y^k = x^k$, then Stop. Otherwise go to Step 2.

Step 2 Find m the smallest nonnegative integer such that

$$\begin{cases} \langle g^{k,m}, x^k - y^k \rangle \geq c \|x^k - y^k\|^2 \\ \text{where } z^{k,m} = (1 - \alpha^m) x^k + \alpha^m y^k \text{ and } g^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}) \end{cases}$$

and set $\alpha_k = \alpha^m$, $z^k = z^{k,m}$, and $g^k = g^{k,m}$. Consider the half-space

$$H_k = \{x \in \mathbb{R}^n \mid \langle g^k, x - z^k \rangle \leq 0\}.$$

Step 3 Find $u^k = P_{C_k}(x^k)$ where C_k denotes the convex closed set

$$C_k = X \cap [\cap_{i=0}^k H_i].$$

Calculate $x^{k+1} = \mu_k x^k + (1 - \mu_k) v^k$ where $v^k = P_{K(u^k)} u^k$.
Set $k := k + 1$ and go back to Step 1.

Remark 1 *When $f(x,y) = \langle F(x), y - x \rangle$ for every $x \in X$, problem* QE(f,K) *becomes a quasi-variational inequality problem. In this case, y^k is the projection of $x^k - F(x^k)$ onto $K(x^k)$, and $g^k = F(z^k)$.*

First we prove that Algorithm 1 is well-defined.

Proposition 3.1 *Each subset $C_k \cap Fix(K)$ is convex, closed, and contains the nonempty set S_*.*

Proof. The set $C_k \cap Fix(K)$, being convex and closed, contains the nonempty set S_*. Indeed, let $x^* \in S_*$. Then $x^* \in K(x^*)$ and $f(y, x^*) \leq 0$ for every $y \in X$. So, for all $0 \leq i \leq k$, $x^* \in H_i \cap Fix(K)$ because by definition of $g^i \in \partial_2 f(z^i, z^i)$, we can write

$$\langle g^i, x^* - z^i \rangle \leq f(z^i, x^*) \leq 0.$$

Hence $x^* \in C_k \cap Fix(K)$ and $S_* \subseteq C_k \cap Fix(K)$.

In particular, the set $C_\infty = \cap_{k=0}^\infty C_k$ is such that $C_\infty \cap Fix(K) \supseteq S_*$. So, it follows from (A5) that

$$C_\infty \cap Fix(K) \text{ is nonempty, convex and closed.}$$

Furthermore, the projection u^k of x^k onto C_k and the projection v^k of u^k onto $K(u^k)$ are well-defined at Step 3 of Algorithm 1.

Next, we prove that x^k is a solution of problem $QE(f,K)$ when $y^k = x^k$ for some k.

Proposition 3.2 *For all k and for every $y \in K(x^k)$, we have the following inequality*

$$\|x^k - y^k\|^2 + f(x^k, y^k) \leq f(x^k, y) + \langle x^k - y^k, x^k - y \rangle. \tag{2}$$

In particular, when $x^k = y^k$ for some k, the vector x^k is a solution of problem QE(f, K).

Proof. Let $k \geq 0$ and $y \in K(x^k)$ be fixed. The vector y^k being a solution of a convex minimization problem, the optimality conditions associated with this problem imply that $y^k \in K(x^k)$ and that there exists $s^k \in \partial_2 f(x^k, y^k)$ such that

$$0 \in s^k + y^k - x^k + \mathcal{N}_{K(x^k)}(y^k)$$

where $\mathcal{N}_{K(x^k)}(y^k)$ denotes the normal cone to $K(x^k)$ at y^k. Hence, by definition of this cone, we obtain that $\langle x^k - y^k - s^k, y - y^k \rangle \leq 0$, and thus that

$$\langle s^k, y - y^k \rangle \geq \langle x^k - y^k, y - x^k \rangle + \|x^k - y^k\|^2. \tag{3}$$

On the other hand, since $s^k \in \partial_2 f(x^k, y^k)$, we can write

$$f(x^k, y) \geq f(x^k, y^k) + \langle s^k, y - y^k \rangle. \tag{4}$$

Combining (3) and (4), we obtain inequality (2) for all $y \in K(x^k)$. Consequently, when $y^k = x^k$ for some k, we obtain that $x^k \in K(x^k)$ and from (2) that $f(x^k, y) \geq 0$ for every $y \in K(x^k)$. So the vector x^k is a solution of problem $QE(f, K)$.

From now on, we assume that $y^k \neq x^k$ for all k. In this situation it was proven in [47, Section 5] that it is possible to find by bisection $\alpha_k \in]0, 1[$ in a finite number of steps in such a way that the point $z^k = (1 - \alpha_k)x^k + \alpha_k y^k$ satisfies the inequality

$$\langle g^k, x^k - y^k \rangle \geq c \, \|x^k - y^k\|^2$$

where $g^k \in \partial_2 f(z^k, z^k)$. Since $y^k \neq x^k$, we have that $g^k \neq 0$.

First we prove that the sequence $\{x^k\}$ generated by Algorithm 1 converges to some $x^* \in C_\infty \cap Fix(K)$. To derive this convergence result, we begin by proving that the sequence $\{x^k\}$ is Fejér monotone with respect to $C_\infty \cap Fix(K)$ in the sense that it satisfies for all k the inequalities

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\| \ \text{ for all } x^* \in C_\infty \cap Fix(K).$$

Proposition 3.3 *Let $x^* \in C_\infty \cap Fix(K)$. Then, for all k, the following inequalities hold:*

(i) $\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - x^k\|^2 \leq \|x^k - x^*\|^2$

(ii) $\|x^{k+1} - x^*\| \leq \|x^k - x^*\|$.

Furthermore, $\lim_{k\to\infty} \|x^k - x^*\|$ *exists and the sequences* $\{x^k\}$ *and* $\{u^k\}$ *are bounded.*

Proof. Let k be fixed.

(i) Since u^k is the projection of x^k onto C_k and $x^* \in C_\infty \subset C_k$, it follows that

$$\langle x^k - u^k, x^* - u^k \rangle \leq 0$$

and thus that

$$\begin{aligned}\|x^k - x^*\|^2 &= \|x^k - u^k\|^2 + \|u^k - x^*\|^2 + 2\langle x^k - u^k, u^k - x^* \rangle \\ &\geq \|x^k - u^k\|^2 + \|u^k - x^*\|^2.\end{aligned}$$

So, we obtain for all k that

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - x^k\|^2 \leq \|x^k - x^*\|^2. \tag{5}$$

(ii) By definition of x^{k+1}, we can write

$$x^{k+1} - x^* = \mu_k x^k + (1-\mu_k)v^k - x^* = \mu_k(x^k - x^*) + (1-\mu_k)(v^k - x^*) \tag{6}$$

where $v^k = P_{K(u^k)}u^k$. Therefore, since $x^* \in Fix(K)$ and $K(\cdot)$ is $\star$-nonexpansive on X, we obtain that

$$\begin{aligned}\|x^{k+1} - x^*\| &\leq \mu_k\|x^k - x^*\| + (1-\mu_k)\|v^k - x^*\| \\ &\leq \mu_k\|x^k - x^*\| + (1-\mu_k)\|u^k - x^*\| \\ &\leq \|x^k - x^*\| \text{ (by (5))}.\end{aligned}$$

Therefore $\lim_{k\to\infty} \|x^k - x^*\|$ exists and the sequence $\{x^k\}$ is bounded. Furthermore, by (5), the sequence $\{u^k\}$ is also bounded.

Proposition 3.4 *Let* $w_k = P_{K(x^k)}x^k$ *for all* k. *Then the sequences* $\{\|u^k - x^k\|\}$ *and* $\{\|w^k - x^k\|\}$ *tend to* 0 *as* $k \to \infty$.

Proof. The set $C_\infty \cap Fix(K)$ being nonempty, let $x^* \in C_\infty \cap Fix(K)$ and let k be fixed. Using successively (6), the $\star$-nonexpansiveness of $K(\cdot)$ and (5), we can write

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\|v^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2 \\ &\leq \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\|u^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2 \qquad (7) \\ &\leq \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\|x^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2 \\ &\leq \|x^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2.\end{aligned}$$

Since $\mu_k(1-\mu_k) \geq a(1-b) > 0$, we obtain from the last inequality that

$$0 \leq a(1-b)\|x^k - v^k\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.$$

Hence by Proposition 3.3 (ii), we can deduce that $\|x^k - v^k\| \to 0$ as $k \to \infty$. Furthermore, using (7) and Proposition 3.3 (i), we can also observe that for all k

$$\begin{array}{rcl} \|x^{k+1} - x^*\|^2 & \leq & \mu_k \|x^k - x^*\|^2 + (1-\mu_k)\|u^k - x^*\|^2 \\ & \leq & \mu_k \|x^k - x^*\|^2 + (1-\mu_k)\left[\|x^k - x^*\|^2 - \|u^k - x^k\|^2\right] \\ & = & \|x^k - x^*\|^2 - (1-\mu_k)\|x^k - u^k\|^2. \end{array}$$

and thus, since $1 - b \leq 1 - \mu_k$, that

$$0 \leq (1-b)\|x^k - u^k\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.$$

Therefore, we can deduce, from Proposition 3.3 (ii), that $\|x^k - u^k\| \to 0$ as $k \to \infty$. Consequently, since the multivalued mapping $K(\cdot)$ is $\star$-nonexpansive, $v_k = P_{K(u^k)}u^k$ and $w_k = P_{K(x^k)}x^k$ for all k, we have that $\|w^k - v^k\| \leq \|x^k - u^k\|$ for all k and we obtain that $\|w^k - v^k\| \to 0$ as $k \to \infty$. Moreover, since $\|w^k - x^k\| \leq \|w^k - v^k\| + \|v^k - x^k\|$, we can easily see that $\|w^k - x^k\| \to 0$ as $k \to \infty$.

Proposition 3.5 *Every limit point $\bar{x}$ of the sequence generated by Algorithm 1 belongs to $C_\infty \cap Fix(K)$.*

Proof. Let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ converging to $\bar{x}$.

(i) First we prove that $\bar{x} \in Fix(K)$. From Proposition 3.4 we have that $\|x^{k_j} - w^{k_j}\| \to 0$ as $j \to \infty$ where, for each j, w^{k_j} is the projection of x^{k_j} onto $K(x^{k_j})$. Then, we have $d(x^{k_j}, K(x^{k_j})) \to 0$ as $j \to \infty$ and, $K(\cdot)$ being $\star$-nonexpansive on X, we have that the operator $I - K(\cdot)$ is demi-closed and thus that $\bar{x} \in K(\bar{x})$. Hence $\bar{x}$ is a fixed point of $K(\cdot)$.

(ii) Now we prove that $\bar{x} \in C_\infty$. Since $C_\infty = \cap_{k=0}^{\infty} C_k$, it is sufficient to prove that $\bar{x} \in C_k$ for all integer k to obtain that $\bar{x} \in C_\infty$. Let N be a fixed integer. Since $k_j \to \infty$, there exists an integer j_0 such that $k_j \geq N$ for all $j \geq j_0$. The sequence $\{C_k\}$ being nonincreasing, we have that

$$x^{k_j} \in C_{k_j} \subset C_N \quad \text{for all } j \geq j_0. \tag{8}$$

Consequently, the set C_N being closed, contains $\bar{x}$, the limit of the sequence $\{x^{k_j}\}$. Since N is arbitrary, we obtain that

$$\bar{x} \in \cap_{k=0}^{\infty} C_k = C_\infty.$$

Finally, using Lemma 2.2, we obtain the following convergence result.

Theorem 3.6 *Assume that Assumption (A) holds. Then the whole sequence $\{x^k\}$ generated by Algorithm 1 converges to some x^* belonging to $C_\infty \cap Fix(K)$.*

Proof. By Propositions 3.5 we have that every limit point of the sequence $\{x^k\}$ belongs to $C_\infty \cap Fix(K)$ and by Proposition 3.3 we have that the sequence $\{x^k\}$ is Fejér monotone with respect to $C_\infty \cap Fix(K)$. Hence it follows from Lemma 2.2 that the whole sequence $\{x^k\}$ converges to some $x^* \in C_\infty \cap Fix(K)$.

It remains to prove that any limit point of the sequence $\{x^k\}$ generated by Algorithm 1 is a solution of problem $QE(f, K)$ to obtain that the sequence $\{x^k\}$ converges to a solution of problem $QE(f, K)$. In this purpose, let $\bar{x}$ be a limit point of the sequence $\{x^k\}$ and let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ converging to $\bar{x}$. First we prove that the associated subsequences $\{y^{k_j}\}$ and $\{z^{k_j}\}$ are bounded.

Proposition 3.7 *Assume that $x^{k_j} \to \bar{x}$. Then the two associated subsequences $\{y^{k_j}\}$ and $\{z^{k_j}\}$ are bounded.*

Proof. Let j be fixed and let $w^{k_j} = P_{K(x^{k_j})}(x^{k_j})$. From Proposition 3.2, we have that

$$\|x^{k_j} - y^{k_j}\|^2 \le -f(x^{k_j}, y^{k_j}) + f(x^{k_j}, w^{k_j}) + \langle x^{k_j} - y^{k_j}, x^{k_j} - w^{k_j} \rangle. \tag{9}$$

Let $s^{k_j} \in \partial_2 f(x^{k_j}, x^{k_j})$. It follows from the definition of the subgradient s^{k_j} that $f(x^{k_j}, y^{k_j}) \ge \langle s^{k_j}, y^{k_j} - x^{k_j} \rangle$ and thus that

$$-f(x^{k_j}, y^{k_j}) \le \|s^{k_j}\| \, \|y^{k_j} - x^{k_j}\|. \tag{10}$$

Let also $\bar{s}^{k_j} \in \partial_2 f(x^{k_j}, w^{k_j})$. Then $0 \ge f(x^{k_j}, w^{k_j}) + \langle \bar{s}^{k_j}, x^{k_j} - w^{k_j} \rangle$ and thus

$$f(x^{k_j}, w^{k_j}) \le \|\bar{s}^{k_j}\| \, \|x^{k_j} - w^{k_j}\|. \tag{11}$$

Since $\langle x^{k_j} - y^{k_j}, x^{k_j} - w^{k_j} \rangle \le \|x^{k_j} - y^{k_j}\| \, \|x^{k_j} - w^{k_j}\|$, we obtain, combining (9), (10), (11) and after division by $\|x^{k_j} - y^{k_j}\| \neq 0$, that

$$\|x^{k_j} - y^{k_j}\| \le \|s^{k_j}\| + \|\bar{s}^{k_j}\| \frac{\|x^{k_j} - w^{k_j}\|}{\|x^{k_j} - y^{k_j}\|} + \|x^{k_j} - w^{k_j}\|.$$

Hence, as $w^{k_j} = P_{K(x^{k_j})} x^{k_j}$, we have that $\frac{\|x^{k_j} - w^{k_j}\|}{\|x^{k_j} - y^{k_j}\|} \le 1$ and

$$\|x^{k_j} - y^{k_j}\| \le \|s^{k_j}\| + \|\bar{s}^{k_j}\| + \|x^{k_j} - w^{k_j}\|. \tag{12}$$

Since, by Proposition 3.4, $\|x^{k_j} - w^{k_j}\| \to 0$ as $j \to \infty$ and, since the sequence $\{x^{k_j}\}$ converges to $\bar{x}$ when $j \to \infty$, we can deduce that $w^{k_j} \to \bar{x}$. Consequently, using Lemma 2.1, there exists j_0 such that for all $j \ge j_0$

$$s^{k_j} \in \partial_2 f(x^{k_j}, x^{k_j}) \subseteq \partial_2 f(\bar{x}, \bar{x}) + B$$

and

$$\bar{s}^{k_j} \in \partial_2 f(x^{k_j}, w^{k_j}) \subseteq \partial_2 f(\bar{x}, \bar{x}) + \eta^{-1} B,$$

where $\eta > 0$ and B denotes the closed unit ball in $\mathbb{R}^n$. Since $\|x^{k_j} - w^{k_j}\| \to 0$ as $j \to \infty$, and the sets B and $\partial_2 f(\bar{x}, \bar{x})$ are bounded, the sequences $\{s^{k_j}\}$, $\{\bar{s}^{k_j}\}$ and $\{\|x^{k_j} - w^{k_j}\|\}$ are bounded. Hence the sequence $\{y^{k_j}\}$ is bounded thanks to (12). Finally the sequence $\{z^{k_j}\}$ is also bounded because, for all j, z^{k_j} belongs to the segment joining x^{k_j} to y^{k_j}.

Proposition 3.8 *Let $\bar{x}$ be a limit point of the bounded sequence $\{x^k\}$. Assume that $x^{k_j} \to \bar{x}$ and that a subsequence of $\{\|x^{k_j} - y^{k_j}\|\}$ converges to* 0*. Then $\bar{x}$ is a solution of problem* QE (f, K)*.*

Proof. Let $\bar{x}$ be a limit point of the bounded sequence $\{x^k\}$. By Theorem 3.6, it is immediate that $\bar{x}$ is a fixed point of $K(\cdot)$. Consequently, it remains to prove that if $\{x^{k_j}\}$ converges to $\bar{x}$, then $f(\bar{x}, y) \geq 0$ for every $y \in K(\bar{x})$. In this purpose, let $y \in K(\bar{x})$. Since $x^{k_j} \to \bar{x}$, $y \in K(\bar{x})$ and K is lower semi-continuous on X, there exists a sequence $\{\bar{y}^{k_j}\}$ such that

$$\bar{y}^{k_j} \in K(x^{k_j}) \text{ for all } j \text{ and } \bar{y}^{k_j} \to y \text{ as } j \to \infty.$$

Using Proposition 3.2, we have for all j that

$$f(x^{k_j}, \bar{y}^{k_j}) \geq f(x^{k_j}, y^{k_j}) + \|x^{k_j} - y^{k_j}\|^2 + \langle x^{k_j} - y^{k_j}, \bar{y}^{k_j} - x^{k_j} \rangle. \tag{13}$$

Taking the limit in (13) as $j \to \infty$, and observing that $f(\cdot, \cdot)$ is jointly continuous, $\|x^{k_j} - y^{k_j}\| \to 0$ and $\bar{y}^{k_j} - x^{k_j} \to y - \bar{x}$, we deduce that $f(\bar{x}, y) \geq 0$.

Now our aim is to prove that $\|x^{k_j} - y^{k_j}\| \to 0$ as $j \to \infty$. This result is obtained thanks to the line search used in Step 1 of Algorithm 1.

Proposition 3.9 *The following inequality holds*

$$\alpha_k c \|x^k - y^k\|^2 \leq \|g^k\| \, \|u^k - x^k\| \quad \textit{for all } k, \tag{14}$$

where $g^k \in \partial_2 f(z^k, z^k)$. Furthermore,

$$\frac{\alpha_k c}{\|g^k\|} \|x^k - y^k\|^2 \to 0 \quad \textit{as } k \to \infty.$$

Proof. Let k be fixed. Since $u^k \in H_k$, it follows that $\langle g^k, u^k - z^k \rangle \leq 0$. Hence, using the line search (Step 1), we obtain successively

$$\begin{aligned}
\alpha_k c \|x^k - y^k\|^2 &\leq \langle g^k, x^k - z^k \rangle \\
&= \langle g^k, x^k - u^k \rangle + \langle g^k, u^k - z^k \rangle \\
&\leq \langle g^k, x^k - u^k \rangle \\
&\leq \|g^k\| \, \|u^k - x^k\|. \qquad (15)
\end{aligned}$$

Since, by Proposition 3.4, $\|u^k - x^k\| \to 0$ and $x^k \neq y^k$ for all k, we have that each $g^k \neq 0$ and that

$$\frac{\alpha_k c}{\|g^k\|} \|x^k - y^k\|^2 \to 0 \quad \text{as } k \to \infty.$$

Proposition 3.10 *Let $x^{k_j} \to \bar{x}$ as $j \to \infty$. Then there exists a subsequence of $\{\|x^{k_j} - y^{k_j}\|\}$ which converges to 0 as $j \to \infty$. Furthermore, $\bar{x}$ is a solution of problem* QE(f, K).

Proof. Two cases are examined:

Case 1. $\inf_j \alpha_{k_j} > 0$. From Proposition 3.9, we can immediately deduce that

$$\frac{\|x^k - y^k\|^2}{\|g^k\|} \to 0.$$

On the other hand, since the sequence $\{z^{k_j}\}$ is bounded, there exists a subsequence of $\{z^{k_j}\}$, again denoted $\{z^{k_j}\}$, which converges to some $\bar{z}$. Then it follows from Lemma 2.1 that there exists j_0 such that for all $j \geq j_0$

$$\partial_2 f(z^{k_j}, z^{k_j}) \subseteq \partial_2 f(\bar{z}, \bar{z}) + B,$$

where B denotes the closed unit ball in $\mathbb{R}^n$. Since B and $\partial_2 f(\bar{z}, \bar{z})$ are bounded, the sequence $\{g^{k_j}\}$ is also bounded. As a consequence, we have that

$$\|x^{k_j} - y^{k_j}\|^2 = \frac{\|x^{k_j} - y^{k_j}\|^2}{\|g^{k_j}\|} \|g^{k_j}\| \to 0 \quad \text{as } j \to \infty.$$

Hence $\|x^{k_j} - y^{k_j}\| \to 0$ as $j \to \infty$.

Case 2. $\inf_j \alpha_{k_j} = 0$. Then $\alpha_{k_j} \to 0$ (in fact a subsequence) and $\alpha_{k_j} < 1$ for j large enough with the consequence that the line search condition in Step 2 of Algorithm 1 is not satisfied for $\frac{\alpha_{k_j}}{\alpha}$. Let us denote the corresponding step by

$$\bar{z}^{k_j} = \left(1 - \frac{\alpha_{k_j}}{\alpha}\right) x^{k_j} + \frac{\alpha_{k_j}}{\alpha} y^{k_j}.$$

Since $x^{k_j} \to \bar{x}$ and the sequence $\{y^{k_j}\}$ is bounded, it is immediate that $\bar{z}^{k_j} \to \bar{x}$. We have

$$\langle \bar{g}^{k_j}, x^{k_j} - y^{k_j} \rangle < c \, \|x^{k_j} - y^{k_j}\|^2, \tag{16}$$

where $\bar{g}^{k_j} \in \partial_2 f(\bar{z}^{k_j}, \bar{z}^{k_j})$. Since the sequences $\{\bar{g}^{k_j}\}$ and $\{y^{k_j}\}$ are bounded, we obtain (in fact for a subsequence) that $\bar{g}^{k_j} \to \bar{g}$ and $y^{k_j} \to \bar{y}$ for some $\bar{g}$ and $\bar{y}$, respectively.

On the other hand, by definition of $\bar{g}^{k_j}$, we have

$$f(\bar{z}^{k_j}, y^{k_j}) \geq \langle \bar{g}^{k_j}, y^{k_j} - \bar{z}^{k_j} \rangle = \left(1 - \frac{\alpha_{k_j}}{\alpha}\right) \langle \bar{g}^{k_j}, y^{k_j} - x^{k_j} \rangle. \tag{17}$$

It follows from Proposition 3.2 that

$$\|x^{k_j} - y^{k_j}\|^2 \leq -f(x^{k_j}, y^{k_j}) + \varepsilon_{k_j}, \tag{18}$$

where $\varepsilon_{k_j} = f(x^{k_j}, w^{k_j}) + \langle x^{k_j} - y^{k_j}, x^{k_j} - w^{k_j} \rangle$ and $w^{k_j} = P_{K(x^{k_j})} x^{k_j}$. Since $x^{k_j} \to x$ and $\|x^{k_j} - w^{k_j}\| \to 0$ (by Proposition 3.4), we have that $w_{k_j} \to x$. Consequently, the sequence $\{y^{k_j}\}$ being bounded and the bifunction f being jointly continuous, we have immediately that $\varepsilon_{k_j} \to 0$ as $j \to \infty$.

Using successively (17), (16) and (18), we obtain that

$$\begin{aligned} f(\bar{z}^{k_j}, y^{k_j}) &> -c\left(1 - \frac{\alpha_{k_j}}{\alpha}\right) \|y^{k_j} - x^{k_j}\|^2 \\ &\geq c\left(1 - \frac{\alpha_{k_j}}{\alpha}\right) (f(x^{k_j}, y^{k_j}) - \varepsilon_{k_j}). \end{aligned} \tag{19}$$

Since f is jointly continuous on X and since $c \in]0,1[$, inequalities (19) imply, at the limit, that $f(\bar{x}, \bar{y}) \geq c\, f(\bar{x}, \bar{y})$, and thus that $f(\bar{x}, \bar{y}) \geq 0$. Therefore from (18) we can deduce that $\|x^{k_j} - y^{k_j}\| \to 0$ as $j \to \infty$. Finally, it remains to use Proposition 3.8 to get that $\bar{x}$ is a solution of problem $QE(f, K)$.

Finally, thanks to Theorem 3.6, Proposition 3.10, and Proposition 3.8, we obtain the following convergence result.

Theorem 3.11 *Assume that Assumption (A) holds. Then the whole sequence $\{x^k\}$ generated by Algorithm 1 converges to a solution of problem* $\mathrm{QE}(f, K)$*. Furthermore, $x^* \in C_\infty$.*

Remark 2 *When the sets $K(x)$ are all equal to the fixed closed convex subset X of H, the quasi-equilibrium problem* $\mathrm{QE}(f, X)$ *coincides with the equilibrium problem* $\mathrm{E}(f, X)$*. In that case when $f(\cdot, \cdot)$ is pseudomonotone, we do not need to use Proposition 3.5 (ii) to get the convergence of the sequence $\{x^k\}$ to a solution of problem* $\mathrm{E}(f, X)$*. Indeed, from Propositions 3.8 and 3.10, we have that any limit point of the sequence $\{x^k\}$ belongs to S_E. Furthermore, since $S_M = S_E \subseteq C_\infty$, it follows from Proposition 3.3, that for all k*

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\| \quad \textit{for all } x^* \in S_E.$$

Hence, using Lemma 2.2, we obtain that the sequence $\{x^k\}$ converges to a point of S_E. Furthermore, since we do not use Proposition 3.5 (ii), we can take $C_k = X \cap H_k$ instead of $C_k = X \cap [\cap_{i=0}^k H_i]$ in Step 2 of Algorithm 1.

4 Numerical Results

In this section, we consider a numerical example for illustrating the behavior of the method.

Let $X = [-1,1] \times [-1,1]$ and consider the equilibrium bifunction $f : X \times X \to \mathbb{R}$ defined for each $x, y \in X$ by

$$f(x,y) = |x_1 + x_2| \left[(y_1^2 - x_1^2) + (y_2^2 - x_2^2) \right]$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The solution sets for problems $E(f,X)$ and $ME(f,X)$ are given by

$$S_E = \{(t,-t) \,|\, t \in [-1,1]\} \quad \text{and} \quad S_M = \{(0,0)\},$$

respectively. Since $S_M \neq S_E$, the equilibrium bifunction f is not pseudomonotone. In the example considered here, the multivalued mapping $K : X \to CCB(X)$ is defined for each $x = (x_1, x_2) \in X$ by

$$K(x_1, x_2) = B\left(0, \frac{\|x\|}{2}\right),$$

where $B(0,R)$ denotes the closed ball of radius R centred at 0. So $K(x_1, x_2)$ is a nonempty closed bounded convex subset of X for each $(x_1, x_2) \in X$. Let $x^* = (x_1^*, x_2^*) \in X$. Then $x^* \in K(x^*)$ if and only if $x^* = (0,0)$. Furthermore, the so-defined mapping $K(\cdot)$ is $\star$-nonexpansive since

$$P_{K(x)} x = \frac{x}{2} \quad \forall x \in X.$$

Moreover, K is lower semicontinuous at $\bar{x} \in X$. Indeed, let $x^k \to \bar{x}$ and let $\bar{y} \in K(\bar{x})$. Then $\|\bar{y}\| \leq \frac{\|\bar{x}\|}{2}$. When $\bar{x} = 0$, we have that $\bar{y} = 0$ and that the sequence $\{y^k\}$ defined for all k by $y^k = 0$, is such that $y^k \in K(x^k)$ for all k and converges to $\bar{y}$. When $\bar{x} \neq 0$ and $\bar{y} \in K(\bar{x})$, we can choose

$$y^k = \frac{\|x^k\| \, \bar{y}}{\|\bar{x}\|} \qquad \forall k$$

to obtain that for all k,

$$\|y^k\| = \frac{\|x^k\| \, \|\bar{y}\|}{\|\bar{x}\|} \leq \frac{\|x^k\|}{2}, \text{ i.e., } y^k \in K(x^k).$$

Since $x^k \to \bar{x}$, we have that $y^k \to \bar{y}$.

Now it is easy to see that

$$S_E^* = S_M^* = \{(0,0)\}.$$

Furthermore, the set S_* is also nonempty and equal to $\{(0,0)\}$.

For this example, the subproblems $\min_{y \in K(x^k)} \{ f(x^k, y) + \frac{1}{2} \|y - x^k\|^2 \}$ can be expressed as

$$(QP)_k \qquad \min_{y \in K(x^k)} \left\{ \frac{1}{2} \left(1 + 2\,|x_1^k + x_2^k|\right) \left(y_1^2 + y_2^2\right) - x_1^k\, y_1 - x_2^k\, y_2 \right\}$$

where $y = (y_1, y_2)$ and $x = (x_1, x_2)$. These subproblems are convex quadratic programming problems which can be explicitly solved. Indeed, first we observe that the solution of the corresponding unconstrained problem is given by

$$y^k = \frac{x^k}{1 + 2\,|x_1^k + x_2^k|}.$$

Consequently, when the constraint $y \in K(x^k)$, i.e., $\|y\| \le \frac{\|x^k\|}{2}$, is incorporated into the unconstrained problem, we obtain that the solution of problem $(QP)_k$ is equal to

$$y^k = \begin{cases} \frac{x^k}{2} & \text{if } |x_1^k + x_2^k| < \frac{1}{2} \\ \frac{x^k}{1+2\,|x_1^k + x_2^k|} & \text{if } |x_1^k + x_2^k| \ge \frac{1}{2}. \end{cases}$$

Several starting points have been considered for solving problem $QE(f, K)$ and the following stopping criterion has been retained at iteration k:

$$\|y^k - x^k\| \le 10^{-6}.$$

The obtained numerical results are displayed in the following table where the starting point, the obtained solution, the number of iterations and the cpu time have been reported.

Starting point	Solution	Iterations	Cputime (secs)
$(-1, -1)$ and $(1, 1)$	$(0, 0)$	12	0.984 and 0.078
$(-1, 1)$ and $(1, -1)$	$(0, 0)$	23	0.218 and 0.109
$(-0.1, -0.1)$ and $(0.1, 0.1)$	$(0, 0)$	10	0.047 and 0.094
$(-0.1, 0.1)$ and $(0.1, -0.1)$	$(0, 0)$	19	0.171 and 0.078

Finally, let us mention that when the starting point belongs to S_E^*, the solution point coincides with this point.

5 Conclusions

In this paper we have introduced a new algorithm for solving the nonmonotone quasi-equilibrium problem in the framework of a finite dimensional Euclidean space. The convergence of the proposed algorithm has been studied and a numerical example has been provided to illustrate the validity of the method.

The literature on solution methods for nonmonotone quasi-equilibrium problems is not too extensive. Developing implementable and efficient methods for solving this difficult class of problems is still a challenging task. To the best of our knowledge, this is among the first papers to deal at the same time with quasi-equilibrium problems and with nonmonotone bifunctions.

One of the difficulties with these methods is that projections have to be done onto intersections of half-spaces and that the number of these half-spaces increases at each iteration. This was already the case for solving equilibrium problems with nonmonotone bifunctions. So to avoid a huge number of constraints in the quadratic subproblems, a strategy would be to aggregate the constraints with the possibility of limiting their number to two. Such a technique, but for solving a similar problem, has been proposed in Sect. 7.4.4 of [49]. The use of such a method adapted to our situation could be the subject of a future research.

Finally, it is also worth mentioning that another natural approach is to show that the quasi-equilibrium problem is equivalent to a global optimization problem by using gap (merit) functions. This alternative equivalent formulation is used to propose some descent type methods to solve the monotone quasi-equilibrium problem in a very recent paper [31].

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