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On the Regularity of Linear-Quadratic Optimal Control Problems with Bang-Bang Solutions*

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Abstract. The paper investigates the stability of the solutions of linear-quadratic optimal control problems with bang-bang controls in terms of metric sub-regularity and bi-metric regularity. New sufficient conditions for these properties are obtained, which strengthen the known conditions for sub-regularity and extend the known conditions for bi-metric regularity to Bolza-type problems.

Keywords: optimal control, regularity, linear-quadratic problems, bang-bang controls

1 Introduction

In this paper we investigate the stability with respect to perturbations of the solutions of the following optimal control problem:

$$\begin{aligned} &\text{minimize} && J(x,u) \\ &\text{subject to} && \dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t), \quad t \in [0,T], \\ &&& u(t) \in U := [-1,1]^m, \\ &&& x(0) = x_0, \end{aligned} \tag{P}$$

where

$$J(x,u) := g(x(T)) + \int_0^T \left(\frac{1}{2} x(t)^\top W(t) x(t) + x(t)^\top S(t) u(t) \right) dt. \tag{1}$$

Here, admissible controls are all measurable functions $u : [0,T] \to [-1,1]^m$, while $x(t) \in \mathbb{R}^n$ denotes the state of the system at time $t \in [0,T]$. The initial state x_0, the final time T and the terminal function $g : \mathbb{R}^n \to \mathbb{R}$ are given, as well as the matrices $A(t), W(t) \in \mathbb{R}^{n\times n}$, $B(t), S(t) \in \mathbb{R}^{n\times m}$ and $d(t) \in \mathbb{R}^n$, $t \in [0,T]$.

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The stability of the solutions of this problem is investigated within the general framework of *metric regularity* (see e.g. [2, Section 3E]) of the associated Pontryagin system of necessary optimality conditions.

The issue is challenging due to the linearity of the problem with respect to the control, which may result in bang-bang solutions. Only a few results are known in the literature that deal with the regularity of this problem, among which we mention [4, 6, 1]. The paper [6] introduces the notion of bi-metric regularity as an appropriate extension of the established notion of metric regularity, which is more relevant to problems with discontinuous optimal controls. However, the result in [6] applies to Mayer-type problems only, where the integral term in the objective functional (1) is missing. The integral term brings a substantial difference, due the presence of the state and the control in the adjoint equation.

In this paper we obtain a strengthened version of the Hölder sub-regularity result obtained in [1, Theorem 8], which provides a basis for further investigations, including error analysis of approximation schemes. We also announce a result about strong bi-metric regularity of the Pontryagin system of necessary conditions associated with problem (P), extending [6] to Bolza problems with bang-bang solutions.

2 Preliminaries

We begin with formulation of assumptions.

Assumption (A1). The matrix-functions A, B, W, S and d are Lipschitz continuous. The matrix $W(t)$ is symmetric for every $t \in [0, T]$. The function g is differentiable with locally Lipschitz derivative.

Let $(\hat{x}, \hat{u})$ be a solution of problem (P), from now on fixed; a standard compactness argument implies existence.

Assumption (A2). For every admissible pair (x, u) of (P) it holds that

$$\langle \nabla g(x(T)) - \nabla g(\hat{x}(T)), \Delta x(T) \rangle + \int_0^T (\langle W(t)\Delta x, \Delta x \rangle + 2\langle S(t)\Delta u, \Delta x \rangle) dt \geq 0,$$

where $\Delta x := x(t) - \hat{x}(t)$ and $\Delta u := u(t) - \hat{u}(t)$, and $\langle \cdot, \cdot \rangle$ is the scalar product.

By the Pontryagin maximum (here minimum) principle, there exists an absolutely continuous function $\hat{p}$ such that the triple $(\hat{x}, \hat{p}, \hat{u})$ solves for a.e. $t \in [0, T]$ the system

$$\begin{aligned} 0 &= \dot{x}(t) - A(t)x(t) - B(t)u(t) - d(t), \\ 0 &= \dot{p}(t) + A(t)^\top p(t) + W(t)x(t) + S(t)u(t), \\ 0 &\in B(t)^\top p(t) + S(t)^\top x(t) + N_U(u(t)), \\ 0 &= p(T) - \nabla g(x(T)), \end{aligned} \tag{PMP}$$

where $N_U(u)$ is the normal cone to U at $u \in \mathbb{R}^m$:

$$N_U(u) := \begin{cases} \emptyset & \text{if } u \notin U \\ \{l \in \mathbb{R}^m : \langle l, v - u \rangle \leq 0 \ \forall v \in U\} & \text{if } u \in U. \end{cases}$$

We recall that $\hat\sigma := B^\top \hat p + S^\top \hat x$ is the so-called *switching function* corresponding to the triple $(\hat x, \hat p, \hat u)$. For every $j \in \{1, \dots, m\}$, denote by $\hat\sigma_j$ its j-th component.

The following assumption requires that the optimal control $\hat u$ is *strictly bang-bang*, with a finite number of switching times, and that the switching function exhibits a certain growth in a neighborhood of any zero. A similar assumption is introduced in [4] in the case $\kappa = 1$ and in [7] for $\kappa > 1$.

Assumption (A3) There exist real numbers $\kappa \ge 1$ and $\alpha, \tau > 0$ such that for each $j \in \{1, \dots, m\}$ and $s \in [0, T]$ with $\hat\sigma_j(s) = 0$ it holds that

$$|\hat\sigma_j(t)| \ge \alpha |t - s|^\kappa \quad \forall t \in [s - \tau, s + \tau] \cap [0, T].$$

The Pontryagin minimum principle (PMP) can be recast as

$$0 \in F(x, p, u), \tag{2}$$

where $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ is a set-valued map defined as

$$F(x, p, u) := \begin{pmatrix} \dot x - Ax - Bu - d \\ \dot p + A^\top p + Wx + Su \\ B^\top p + S^\top x + N_U(u) \\ p(T) - \nabla g(x(T)) \end{pmatrix}. \tag{3}$$

We will investigate the stability under perturbations of the solution of problem (P) by studying the stability of the generalized equation $y \in F(x, p, u)$ with respect to a perturbation y. The mapping F is considered as acting in the space

$$\mathcal{X} := W^{1,1}_{x_0}([0, T], \mathbb{R}^n) \times W^{1,1}([0, T], \mathbb{R}^n) \times L^1([0, T], \mathbb{R}^m)$$

with values in the space

$$\mathcal{Y} := L^1([0, T], \mathbb{R}^n) \times L^1([0, T], \mathbb{R}^n) \times L^\infty([0, T], \mathbb{R}^m) \times \mathbb{R}^n,$$

which restricts the set of considered selections of the mapping $t \mapsto N_U(u(t))$ to essentially bounded ones. Here $W^{1,1}_{x_0}([0, T], \mathbb{R}^n) := \{x \in W^{1,1}([0, T], \mathbb{R}^n) : x(0) = x_0\}$. The spaces $\mathcal{X}$ and $\mathcal{Y}$ are endowed with the usual norms for $(x, p, u) \in \mathcal{X}$ and $(\xi, \pi, \rho, \nu) \in \mathcal{Y}$:

$$\|(x, p, u)\|_{\mathcal{X}} := \|x\|_{1,1} + \|p\|_{1,1} + \|u\|_1, \quad \|(\xi, \pi, \rho, \nu)\| := \|\xi\|_1 + \|\pi\|_1 + \|\rho\|_\infty + |\nu|.$$

3 Metric sub-regularity

We begin with an auxiliary result that is similar in spirit to [7, Lemma 1.3] (also cf. [8, Theorem 2.1]) but is proved on slightly less restrictive assumptions.

Lemma 1. *Let $l : [0, T] \to \mathbb{R}^m$ be a continuous function satisfying assumption (A3) (with l at the place of $\hat\sigma$). Then there exists a constant $c > 0$ such that for any $v \in L^\infty([0, T], \mathbb{R}^m)$ the following inequality holds:*

$$\|v\|_\infty^k \int_0^T \sum_{j=1}^m |l_j(t) v_j(t)| \, dt \ge c \|v\|_1^{\kappa + 1}. \tag{4}$$

Proof. The claim of this lemma is trivial when $v = 0$. If $v \neq 0$ then due to the homogeneity with respect to v of order $\kappa + 1$ of the two sides of (4), it is enough to prove the lemma in the case $\|v\|_\infty = 1$, which will be assumed in the remaining part of the proof. For any $0 < \delta \le \tau$, we set

$$I_j(\delta) := \bigcup_{s\in[0,T]:\, l_j(s)=0} (s-\delta, s+\delta) \cap [0,T], \qquad I(\delta) := \bigcup_{1\le j\le m} I_j(\delta).$$

Since l is continuous and Assumption (A3) holds for l_j, we have that

$$l_{\min} := \min_{1\le j\le m} \min_{t\in[0,T]\setminus I_j(\tau)} |l_j(t)| > 0.$$

Now we choose $\bar\delta \in (0,\tau)$ such that $\alpha\bar\delta^\kappa < l_{\min}$. Then for all $\delta \in (0,\bar\delta)$ and $j \in \{1,\dots,m\}$ we have

$$|l_j(t)| \ge \alpha\delta^\kappa \quad \forall t \in [0,T] \setminus I(\delta). \tag{5}$$

Indeed, if $t \notin I_j(\tau)$ then $|l_j(t)| \ge l_{\min} > \alpha\bar\delta^\kappa \ge \alpha\delta^\kappa$. If $t \in I_j(\tau) \setminus I(\delta)$, then $t \in I_j(\tau) \setminus I_j(\delta)$. Thus there exists a zero s of l_j such that $\delta \le |t-s| < \tau$. According to Assumption (A3), $|l_j(t)| \ge \alpha|t-s|^\kappa \ge \alpha\delta^\kappa$. Hence,

$$\begin{aligned}
\phi(v) &:= \int_0^T \sum_{j=1}^m |l_j(t)v_j(t)|\,dt \ge \int_{[0,T]\setminus I(\delta)} \sum_{j=1}^m |l_j(t)v_j(t)|\,dt \\
&\ge \alpha\delta^\kappa \sum_{j=1}^m \int_{[0,T]\setminus I(\delta)} |v_j(t)|\,dt \ge \alpha\delta^\kappa \left(\|v\|_1 - \sum_{j=1}^m \int_{I(\delta)} |v_j(t)|\,dt \right) \\
&\ge \alpha\delta^\kappa(\|v\|_1 - 2\lambda\delta),
\end{aligned}$$

where λ is sum of the maximum of the number of zeros of l_j over all $j \in \{1,\dots m\}$ (notice that Assumption (A3) implies $\lambda \le mT/2\tau + m$). If $\|v\|_1 \ge 4\lambda\bar\delta$ then we choose $\delta := \bar\delta$ to get

$$\phi(v) \ge \frac{\alpha\bar\delta^\kappa}{2}\|v\|_1$$

and since $\|v\|_1 \le T\|v\|_\infty = T$ we have that $\phi(u) \ge \frac{\alpha\bar\delta^\kappa}{2T^\kappa}\|v\|_1^{\kappa+1}$. If, on the other hand, $\|v\|_1 \le 4\lambda\bar\delta$ then we choose $\delta := \frac{\|v\|_1}{4\lambda} \le \bar\delta$ to get

$$\phi(v) \ge \frac{\alpha}{2^{2\kappa+1}\lambda^\kappa}\|v\|_1^{\kappa+1}.$$

Hence choosing $c := \min\{\frac{\alpha\bar\delta^\kappa}{2T^\kappa}, \frac{\alpha}{2^{2\kappa+1}\lambda^\kappa}\}$ we obtain that

$$\phi(v) \ge c\|v\|_1^{\kappa+1}.$$

Q.E.D.

The following theorem establishes a property of the mapping F associated with system (PMP), which is a somewhat stronger form of the well known property of *metric sub-regularity*, [2, Section 3H]. It extends [1, Theorem 8] in several directions: Assumption (A3) is weaker than the corresponding assumption there, the norms are different, and the function g is not necessarily quadratic and convex.

Theorem 1. *Let* $(\hat{x}, \hat{p}, \hat{u})$ *be a solution of* (PMP) *such that (A1)–(A3) are fulfilled. Then for any* $b > 0$ *there exists* $c > 0$ *such that for any* $y \in \mathcal{Y}$ *with* $\|y\| \leq b$*, there exists a triple* $(x, p, u) \in \mathcal{X}$ *solving* $y \in F(x, p, u)$*, and any such triple satisfies*

$$\|(x, p, u) - (\hat{x}, \hat{p}, \hat{u})\|_{\mathcal{X}} \leq c\|y\|^{\frac{1}{\kappa}}.$$

Proof. Since the inclusion $y \in F(x, p, u)$ represents a system of necessary optimality conditions of a problem of the form of (P) with appropriate, bounded in L^1, perturbations defined by y (a simple and well known fact), the evident existence of an optimal solution of this perturbed version of (P) implies existence of a solution (x, p, u) of the inclusion $y \in F(x, p, u)$.

Now let $b > 0$ be arbitrarily chosen and let (x, p, u) be a solution of $y \in F(x, p, u)$, where $y = (\xi, \pi, \rho, \nu) \in \mathcal{Y}$ and $\|y\| \leq b$. The following notations will be used. As before, $\hat{\sigma}(t) := B(t)^\top \hat{p}(t) + S(t)^\top \hat{x}(t)$, while $\sigma(t) := B(t)^\top p(t) + S(t)^\top x(t) - \rho(t)$. Furthermore, we denote $\Delta x := x(t) - \hat{x}(t)$, $\Delta p = p(t) - \hat{p}(t)$, $\Delta u := u(t) - \hat{u}(t)$, $\Delta\sigma := \sigma(t) - \hat{\sigma}(t)$, and skip the argument t whenever clear.

Integrating by parts, we have

$$\int_0^T \langle \Delta\dot{p}, \Delta x \rangle \, dt = \langle \Delta p(T), \Delta x(T) \rangle - \int_0^T \langle \Delta p, \Delta\dot{x} \rangle \, dt.$$

Substituting here the expressions for Δx and Δp resulting from the inclusions $y \in F(x, p, u)$ and $0 \in F(\hat{x}, \hat{p}, \hat{u})$ in view of (3), we obtain that

$$\int_0^T \langle -A^\top \Delta p - W\Delta x - S\Delta u + \pi, \Delta x \rangle \, dt$$
$$= \langle \nabla g(x(T)) - \nabla g(\hat{x}(T)) + \nu, \Delta x(T) \rangle - \int_0^T \langle \Delta p, A\Delta x + B\Delta u + \xi \rangle \, dt.$$

Rearranging the terms in this equality and using (A2) we get

$$\int_0^T (\langle \Delta p, B\Delta u \rangle + \langle S\Delta u, \Delta x \rangle) \, dt + \int_0^T (\langle \pi, \Delta x \rangle + \langle \xi, \Delta p \rangle) \, dt - \langle \nu, \Delta x(T) \rangle$$
$$= \langle \nabla g(x(T)) - \nabla g(\hat{x}(T)), \Delta x(T) \rangle + \int_0^T (\langle W\Delta x, \Delta x \rangle + 2\langle S\Delta u, \Delta x \rangle) \, dt \geq 0.$$

Using this inequality and the definitions of the functions σ and $\hat{\sigma}$ we obtain

$$\int_0^T \langle \Delta\sigma, \Delta u \rangle \, dt = \int_0^T \langle B^\top \Delta p + S^\top \Delta x - \rho, \Delta u \rangle \, dt \geq$$
$$\geq \int_0^T (-\langle \pi, \Delta x \rangle - \langle \xi, \Delta p \rangle - \langle \rho, \Delta u \rangle) \, dt + \langle \nu, \Delta x(T) \rangle. \qquad (6)$$

The third component of the inclusion $y \in F(x,p,u)$ reads as $-\sigma(t) \in N_U(u(t))$, which implies $\langle -\sigma(t),\, \hat{u}(t) - u(t)\rangle \le 0$. Then

$$-\int_0^T \langle \Delta\sigma, \Delta u\rangle \, dt = \int_0^T [-\langle \sigma, \Delta u\rangle + \langle \hat{\sigma},\, \Delta u\rangle] \, dt \ge \int_0^T \langle \hat{\sigma},\, \Delta u\rangle \, dt.$$

From here, using that $-\hat{\sigma}_j(t) \in N_{[-1,1]}(\hat{u}_j(t))$, hence $\hat{\sigma}_j(t)\, \Delta u_j(t) \ge 0$ for each j, Lemma 1 implies that

$$-\int_0^T \langle \Delta\sigma, \Delta u\rangle \, dt \ge \int_0^T \sum_{j=1}^m |\hat{\sigma}_j \, \Delta u_j| \, dt \ge c_1 \|\Delta u\|_1^{\kappa+1},$$

where the constant c_1 is independent of y and (x,p,u). Then using (6) and the Hölder inequality we obtain

$$\|\pi\|_1 \, \|\Delta x\|_\infty + \|\xi\|_1 \, \|\Delta p\|_\infty + |\nu| \, |\Delta x(T)| + \|\rho\|_\infty \, \|\Delta u\|_1 \ge c_1 \|\Delta u\|_1^{\kappa+1}. \quad (7)$$

Using Assumption (A1) and the Cauchy formula for Δx and Δp we get

$$\|\Delta x\|_\infty \le c_2(\|\xi\|_1 + \|\Delta u\|_1) \quad (8)$$

and

$$\|\Delta p\|_\infty \le c_3(\|\xi\|_1 + \|\pi\|_1 + \|\Delta u\|_1 + |\nu|) \quad (9)$$

for some constants c_2 and c_3 that are independent of y and (x,p,u). Therefore, using (7) we obtain that

$$(\|y\|^2 + \|y\| \|\Delta u\|_1) \ge c_4 \|\Delta u\|_1^{\kappa+1} \quad (10)$$

for some constant c_4, also independent of y and (x,p,u).

Now we distinguish two cases. First, if $\||\Delta y\| \le \|u\|_1$ then

$$2\|y\| \|\Delta u\|_1 \ge c_4 \|\Delta u\|_1^{\kappa+1},$$

which implies

$$\|\Delta u\|_1 \le \left(\frac{2}{c_4}\|y\|\right)^{1/\kappa}. \quad (11)$$

Otherwise, if $\|\Delta u\|_1 \le \|y\| \le b$ then

$$\|\Delta u\|_1 \le \|y\|^{1/\kappa} \|y\|^{(\kappa-1)/\kappa} \le b^{(\kappa-1)/\kappa} \|y\|^{1/\kappa}. \quad (12)$$

Inequality (11) and (12) imply that for any $b > 0$ there exists $c_5 > 0$ such that for any and $\|y\| \le b$,

$$\|\Delta u\|_1 \le c_5 \|y\|^{1/\kappa}.$$

Then the claim of the theorem follows with a suitable constant c from the above estimate, (8) and (9). Q.E.D.

We mention that the property established in Theorem 1 is stronger than metric sub-regularity (as defined e.g. in [2, Section 3H]) in that it is global with respect to the solution $(x,p,u) \in \mathcal{X}$, and also with respect to the size b of the "disturbance" y, although the constant c in the theorem may depend on b.

4 Bi-metric regularity

We begin this section by introducing appropriate modifications of the spaces $\mathcal{X}$ and $\mathcal{Y}$ defined in Section 2. First, we consider the set $\mathcal{U} \subset L^\infty([0,T],\mathbb{R}^m)$ of admissible controls (that is, the set of all measurable functions $u : [0,T] \to U$) as a metric space with the metric

$$d^{\#}(u_1,u_2) = \text{meas}\ \{t \in [0,T] : u_1(t) \neq u_2(t)\},$$

in $L^\infty([0,T],\mathbb{R}^m)$, where "meas" stands for the Lebesgue measure in $[0,T]$. This metric is shift-invariant and we shall shorten $d^{\#}(u_1,u_2) = d^{\#}(u_1-u_2,0) =: d^{\#}(u_1-u_2)$. Moreover, $\mathcal{U}$ is a complete metric space with respect to $d^{\#}$ (see [3, Lemma 7.2]). Then the triple (x,p,u) is considered as an element of the space

$$\widetilde{\mathcal{X}} = W^{1,1}_{x_0}([0,T],\mathbb{R}^n) \times W^{1,1}([0,T],\mathbb{R}^n) \times \mathcal{U},$$

endowed with the (shift-invariant) metric

$$d_\sim(x,p,u) = \|x\|_{1,1} + \|p\|_{1,1} + d^{\#}(u). \tag{13}$$

Clearly $\widetilde{\mathcal{X}}$ is a complete metric space. We also define the space $\widetilde{\mathcal{Y}} \subset \mathcal{Y}$ as

$$\widetilde{\mathcal{Y}} := L^\infty([0,T],\mathbb{R}^n) \times L^\infty([0,T],\mathbb{R}^n) \times W^{1,\infty}([0,T],\mathbb{R}^m) \times \mathbb{R}^n$$

with the usual norm of $y = (\xi,\pi,\rho,\nu) \in \widetilde{\mathcal{Y}}$:

$$\|(\xi,\pi,\rho,\nu)\|_\sim := \|\xi\|_\infty + \|\pi\|_\infty + \|\rho\|_{1,\infty} + |\nu|. \tag{14}$$

The paper [6] introduces the notion of bi-metric regularity as a concept of regularity that is relevant to problems with bang-bang optimal controls. In the particular context of the present paper the definition of bi-metric regularity of the set-valued mapping $F : \widetilde{\mathcal{X}} \rightrightarrows \mathcal{Y}$ (see (3)) reads, in a somewhat more general form, as follows.

Definition 1. *The map $F : \widetilde{\mathcal{X}} \rightrightarrows \mathcal{Y}$ is strongly bi-metrically regular relative to (disturbance space) $\widetilde{\mathcal{Y}} \subset \mathcal{Y}$ at $\hat{z} \in \widetilde{\mathcal{X}}$ for $0 \in \widetilde{\mathcal{Y}}$ if $(\hat{z},0) \in \text{graph}(F)$ and there exist numbers $\varsigma \geq 0$, $\beta > 0$ and $a > 0$ such that the map $B_{\widetilde{\mathcal{Y}}}(0;\beta) \ni y \mapsto F^{-1}(y) \cap B_{\widetilde{\mathcal{X}}}(\hat{z};a)$ is single-valued and*

$$d_\sim(F^{-1}(y') \cap B_{\widetilde{\mathcal{X}}}(\hat{z};a),\ F^{-1}(y) \cap B_{\widetilde{\mathcal{X}}}(\hat{z};a)) \leq \varsigma\|y'-y\| \tag{15}$$

for all $y, y' \in B_{\widetilde{\mathcal{Y}}}(0;\beta)$. Here $B_{\widetilde{\mathcal{X}}}(\hat{z};a)$ is the ball of radius a centered at $\hat{z}$ in the space $\widetilde{\mathcal{X}}$, and $B_{\widetilde{\mathcal{Y}}}(0;\beta)$ is the ball of radius β (in the norm $\|\cdot\|_\sim$) centered at $0 \in \widetilde{\mathcal{Y}}$.

The following theorem extends the result for bi-metric regularity of F obtained in [6] for Mayer's problems for linear systems to the present Bolza problem. For that we need the following strengthened forms of assumptions (A1) and (A2).

Assumption (A1'). The functions A, W and d are Lipschitz continuous, B and S have first order Lipschitz derivatives. The matrices $W(t)$ and $S^\top(t)B(t)$ are symmetric for every $t \in [0,T]$. The function g is differentiable with locally Lipschitz derivative.

Assumption (A2'). The function J is convex on the set of admissible pairs (x, u).

Theorem 2 (Bi-metric regularity). *Let Assumptions (A1') and (A2') be fulfilled. Let* $(\hat{x}, \hat{p}, \hat{u})$ *be a solution to* (PMP) *such that (A3) is fulfilled with* $\kappa = 1$. *Then the mapping* $F : \widetilde{\mathcal{X}} \rightrightarrows \mathcal{Y}$ *introduced in* (3) *is strongly bi-metrically regular (relative to* $\widetilde{\mathcal{Y}} \subset \mathcal{Y}$*) at* $(\hat{x}, \hat{p}, \hat{u}) \in \widetilde{\mathcal{X}}$ *for* $0 \in \widetilde{\mathcal{Y}}$.

The proof of this theorem is too long to be placed here, therefore it will be presented as a part of a full size paper. This also applies to applications of Theorem 1 and Theorem 2 in qualitative analysis and error analysis of numerical approximations in the spirit of [5].

We mention, that the strong bi-metric regularity for Mayer's problems is proved in [6] for a general polyhedral set U and also in the case $\kappa > 1$. Extension of Theorem 2 to a general compact polyhedral U set is a matter of modification of Assumption (A3) and technicalities that we avoid in this paper, while the case $\kappa > 1$ is still open and challenging for the Bolza problem.

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