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Optimal cyclic harvesting of renewable resource*

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Abstract

The paper obtains existence of a solution and necessary optimality conditions for a problem of optimal (long run averaged) periodic extraction of a renewable resource distributed along a circle. The resource grows according to the logistic law, and is harvested by a single harvester periodically moving around the circle.

1 Introduction

The present paper extends the previous research [1] by the authors and studies the problem of optimal harvesting on a smooth closed curve, assumed to be (diffeomorphic to) the unit circle. At each point of the circle (further also called "position") the dynamics of the resource is described by the logistic equation with parameters that may depend on the position. The extraction of the resource is done by a harvester that moves around the curve with a position-dependent velocity and applies a certain (also position-dependent) harvesting effort. The instantaneously extracted quantity depends on the velocity of the harvester, on the applied harvesting effort, and on the available

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resource at the current position. The assumed law of extraction is consistent with the literature about search and acquisition (see e.g [5]). The velocity of the harvester and the harvesting effort are considered as control variables depending on the current position. The same control is applied in each round of the harvester on the circle. The goal is to find such a repeatedly applied control, that the long run averaged economic revenue of harvesting is maximal. The considered problem belongs to the class of averaged infinite-horizon problems, see e.g. [3] which, in the present context, agrees with the idea of sustainability of the resource utilization. The novelty in the present paper, compared with [1] and the other contributions quoted therein is, that the harvesting effort is introduced as an additional control, along with the velocity, which substantially changes the problem and makes it more realistic. The main results include existence of an optimal harvesting policy and a set of necessary optimality conditions which are convenient for qualitative and numerical analysis.

2 Statement of the problem

Let $p(t, x)$ be the available resource at time $t \geq 0$ and point x on the unit circle, further denoted by S^1. A single harvester moves clockwise around S^1, starting from a point $O \in S^1$, with a positive (tangential) velocity $v(x)$ and applying harvesting effort $u(x)$ at the current point x. Both u and v are regarded as position-dependent (and time-independent) control variables. The point O plays a special role, since the harvester can stop for a pause at O and then repeat the same control mode $(v(\cdot), u(\cdot))$ with the same pause at O in the next rounds.

When the harvester crosses a point $x \in S^1$ with a positive velocity $v(x)$ and applies effort $u(x)$ at this point, a fraction $1 - e^{-\gamma(x)u(x)/v(x)}$ of the available resource at x is harvested. Here $\gamma(x)$ is a resource acquisition parameter, which characterizes the ability to detect/extract resource at x. The actual efficiency of extraction is assumed proportional to the harvesting effort and inversely proportional to the velocity. The above expression for the harvested fraction of the resource has its foundation in retrieval theory (see e.g. [4, 5]). The harvesting effort and the velocity are constraint as follows:

$$0 \leq u(x) \leq 1, \qquad v_1(x) \leq v(x) \leq v_2(x), \quad x \in S^1, \tag{1}$$

where $v_1(x)$ and $v_2(x)$ are lower and upper bounds, respectively, and $v_1(x) \geq$

$v_0 > 0$.

All measurable pairs (u, v) satisfying (1) are considered as admissible controls. Given an admissible velocity control function v, the time needed to the harvester to make one round can be represented as $\int_{S^1} \frac{\mathrm{d}x}{v(x)}$. Since the harvester can make a pause at O, the total time, T, for one round including the pause, satisfies the constraint

$$\theta(v) := \int_{S^1} \frac{\mathrm{d}x}{v(x)} \le T. \tag{2}$$

Thus $T - \theta(v)$ is the duration of the pause. A triple (u, v, T) of an admissible control pair (u, v) and a number T satisfying (2) will be considered as *admissible harvesting policy.*

Now we describe the dynamics of the harvester and the resource, given an admissible harvesting policy (u, v, T). Let the position of the harvester is represented by the absolutely continuous function $x : [0, \infty) \to S^1$. Then starting from O at time zero, we have $\dot{x}(t) = v(x(t))$ on $[0, \theta(v)]$ and $x(t) = O$ on $(\theta(v), T]$. Then $x(\cdot)$ is extended periodically to $[0, +\infty)$.

For each $x \in S^1 \setminus O$ (the resource at the single point O is irrelevant) the dynamics of $p(\cdot, x)$ is described by the logistic equation

$$\dot{p}(t, x) = (a(x) - b(x)p(t, x))\, p(t, x), \quad p(0, x) = p_0(x), \tag{3}$$

where an upper dot "$\dot{}$" denotes differentiation with respect to t, and $p_0(x)$ is an initial resource distribution. The position-dependent parameter $a(x) \ge 0$ represents the recovery rate, and $b(x)p$ represents the mortality rate in a population of size p, where $b(x) \ge b_0 > 0$. Each time the harvester crosses the point x (that is $x(t) = x$), the resource at x jumps down:

$$p(t + 0, x) = e^{-\gamma(x)u(x)/v(x)}\, p(t - 0, x). \tag{4}$$

This determines the evolution of resource for a given admissible harvesting policy. The problem of maximization of the long run averaged revenue takes the form

$$\max_{u,v,T} \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau p(t, x(t)) \left(1 - e^{-\gamma(x(t))u(x(t))/v(x(t))}\right) v(x(t))\, \mathrm{d}t \tag{5}$$

on the set of all admissible harvesting policies.

Further we assume that a, b, p_0, γ, v_1, $v_2 : S^1 \to [0, \infty)$ are measurable and bounded functions, $p_0(x) > 0$, $b(x) \ge b_0 > 0$ for $x \in S^1$, and the set $S := \{x \in S^1 :\ a(x)\gamma(x) \ne 0\}$ has a positive measure.

3 Reformulation of the problem and existence of optimal harvesting policy

In a similar way as in [1], the problem (5) can be reformulated as

$$\max_{u,v,T} \frac{1}{T} \int_{S^1} p_\infty(x, u(x)/v(x), T) \left(1 - e^{-\gamma(x)u(x)/v(x)}\right) \mathrm{d}x, \tag{6}$$

where $p_\infty(x, \cdot, \cdot) \equiv 0$ if $a(x) = 0$, otherwise:

$$p_\infty(x, z, T) = \max\left\{0, \; \frac{a(x)}{b(x)} \frac{e^{a(x)T} - e^{\gamma(x)z}}{e^{a(x)T} - 1}\right\}. \tag{7}$$

It will be convenient to introduce the new decision variables $w = 1/T$ and $r(x) = 1/v(x)$, and $z(x) = u(x)/v(x) = r(x)u(x)$, so that the constraints (1) and (2) transform to $0 \le z(x) \le r(x)$ and

$$r_1(x) \le r(x) \le r_2(x), \qquad w \int_{S^1} r(x)\,\mathrm{d}x \le 1, \tag{8}$$

where $r_1(x) := 1/v_2(x)$ and $r_2(x) := 1/v_1(x) \le \bar{r} := 1/v_0$ for $x \in S^1$. Since obviously the maximization with respect to $u \in [0, 1]$ in (6) can be done separately for each x, we can reformulate problem (6) as

$$\max_{r,w} w \int_{S^1} f(x, r(x), w)\,\mathrm{d}x, \tag{9}$$

subject to (8), where $f(x, r, w) := \max_{z \in [0,r]} p_\infty(x, z, 1/w)\left(1 - e^{-\gamma(x)z}\right)$, for $x \in S^1 \backslash \{O\}$. Therefore, from now on we consider any pair (r, w) satisfying (8) as admissible harvesting policy (instead of the triple (u, v, T)). Since for any admissible policy we have

$$w \le 1/\int_{S^1} r(x)\,\mathrm{d}x \le 1/\int_{S^1} r_1(x)\,\mathrm{d}x =: w^0 > 0,$$

we consider the function f on the domain $S^1 \times [0, \bar{r}] \times (0, w^0]$.

Proposition 1 *Problem (8), (9) has a solution.*

The proof is similar to that of Proposition 1 in [1], where the following lemma has to be used instead of Lemma 1 in [1].

Lemma 1 *The function f is measurable in x, bounded from above by $\|a\|_{L_\infty}/b_0$, and is Lipschitz continuous in $(r,w) \in [0,\bar{r}] \times (0,w^0]$ uniformly with respect to $x \in S^1$. Moreover,*

(i) $f(\cdot,r,w) = 0$ on $S^1 \setminus S$; for any $x \in S$ and $w \in (0,w^0]$, the function $f(x,\cdot,w)$ is differentiable and concave on $[0,\bar{r}]$, strictly concave and strictly positive on $(0,r^0(x,w))$, and constant for $r \geq r^0(x,w)$, where $r^0(x,w) = \frac{a(x)}{2\gamma(x)w}$.

(ii) for every $(x,r) \in S^1 \times [0,\bar{r}]$, the function $w \mapsto w\, f(x,r,w)$ is differentiable and concave.

The concavity property in (i) is obtained due to the presence of the harvesting effort u in the present model. It does not hold in the simplified consideration in [1]. The advantage of changing the variable $T = 1/w$ is revealed by the concavity property in (ii), which has no counterpart in [1]. As a result, the present enhancement of the model in [1] even simplifies the proofs of Proposition 1 and of the optimality conditions in the next section. We also mention that the optimal harvesting policy is not unique for some data configurations.

4 Optimality conditions

The following theorem presents a necessary optimality condition with an explicit representation of the Lagrange multiplier associated with the second constraint in (8).

Theorem 1 *Let $(\hat{r}(\cdot),\hat{w})$ be an optimal harvesting policy in problem (8), (9) and let $\hat{w} < w^0$. Then with the number*

$$\hat{\lambda} = \hat{w} \int_{S^1} f(x,\hat{r}(x),\hat{w})\,\mathrm{d}x + \hat{w}^2 \int_{S^1} \frac{\partial f}{\partial w}(x,\hat{r}(x),\hat{w})\,\mathrm{d}x \geq 0, \tag{10}$$

that is Lagrange multiplier to the second inequality in (8) the following maximum condition *and* complementary slackness condition *are fulfilled:*

$$f(x,\hat{r}(x),\hat{w}) - \hat{\lambda}\,\hat{r}(x) = \max_{r \in [r_1(x),r_2(x)]} \left\{ f(x,r,\hat{w}) - \hat{\lambda}\, r \right\}, \quad \textit{for a.e. } x \in S^1; \tag{11}$$

$$\hat{\lambda} \left(\hat{w} \int_{S^1} \hat{r}(x)\,\mathrm{d}x - 1 \right) = 0. \tag{12}$$

Observe, that the case $\hat{w} = w^0$, which is excluded in the above theorem, is trivial, since the only control r for which the pair $(r, \hat{w})$ is admissible is $r = r_1$.

Taking into account the properties of the function $f(x, \cdot, w)$ in Lemma 1 we consider the following cases.

Case 1: $\hat{\lambda} > 0$. In this case for every $x \in S^1$ there is a unique maximizer $r^*(x, w; \hat{\lambda})$ in (11) and

$$r^*(x, w; \lambda) = \min\{r_2(x),\, \max\{r_1(x), \bar{r}(x, w; \lambda)\}\}\,, \tag{13}$$

where $\bar{r}(x, w; \lambda)$ is the unique maximizer of $f(x, r, w) - \lambda\, r$ on $r \in [0, \infty)$. If $a(x)\,\gamma(x) = 0$ for some x, then $\bar{r}(x, w; \lambda) = 0$, otherwise $\bar{r}(x, w; \lambda)$ can be determined by solving with respect to r the equation $\frac{\partial f}{\partial r}(x, r, w) - \lambda = 0$. Simple but somewhat lengthly calculations give the following expression:

$$\bar{r}(x, w; \lambda) = \frac{a(x)}{2\gamma(x)w} - \frac{1}{\gamma(x)} \operatorname{arsinh}\left(\lambda\, \frac{b(x)\sinh(\frac{a(x)}{2w})}{a(x)\gamma(x)}\right), \tag{14}$$

Case 2: $\hat{\lambda} = 0$, $a(x)\,\gamma(x) > 0$, and $r_2(x) \le r^0(x, w)$. In this case the objective function in (11) has a unique maximizer $r^*(x, w; \hat{\lambda}) = r_2(0)$ given by (13) and (14) as in Case 1.

Case 3: $\hat{\lambda} = 0$ and either $a(x)\,\gamma(x) = 0$ or $r_2(x) > r^0(x, w)$. In this case either $f(x, \cdot, w)$ is identically zero, or it is zero at least on $[\max\{r_1(x), r^0(x, w)\}, r_2(x)]$. Thus any element of $[\max\{r_1(x), r^0(x, w)\}, r_2(x)]$ solves (11). It is reasonable to define $r^*(x, w; \lambda) = \max\{r_1(x), r^0(x, w)\}$. Indeed, with this definition the function $r^*(\cdot, w; \lambda)$ is the minimal solution of (11) and hence, it is the most favorable for constraint (2).

Combining the above analysis with Theorem 1 we obtain the following corollary.

Corollary 1 *Let $(\hat{r}, \hat{w})$ be a solution of problem (8), (9) and let $\hat{\lambda}$ be defined as in (10). Then the complementary slackness condition (12) holds together with the following relations: (i) $\hat{r}(x) \ge r^*(x, w; \hat{\lambda})$ for every $x \in S^1$, and (ii) $\hat{r}(x) = r^*(x, w; \hat{\lambda})$, for all $x \in S^1$ for which Case 1 or Case 2 takes place.*

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