Autonomous and advertising-dependent "word of mouth" under costly dynamic pricing

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Abstract. Autonomous 'word of mouth', as a channel of social influence that is out of firms' direct control, has acquired particular importance with the development of the Internet. Depending on whether a given product or service is a good or a bad deal, this can significantly contribute to commercial success or failure. Yet the existing dynamic models of sales in marketing still assume that the influence of word of mouth on sales is at best advertising-dependent. This omission can produce ineffective management and therefore misleading marketing policies. This paper seeks to bridge the gap by introducing a contagion sales model of a monopolist firm's product where sales are affected by advertising-dependent as well as autonomous word of mouth. We assume that the firm's attraction rate of new customers is determined by the degree at which the current sales price is advantageous or not compared with the current customers' reservation price. A primary goal of the paper is to determine the optimal sales price and advertising effort. We show that, despite costly price adjustments, the interactions between sales price, advertising-dependent and autonomous word of mouth can result in complex dynamic pricing policies involving history-dependence or limit cycling consisting of alternating attraction of new customers and attrition of current customers.

Keywords. Dynamic pricing, Advertising effort, Word of mouth, Marketing strategy.

JEL Classification. D42, C62, M31.

1. Introduction

Word of mouth is an essential medium of marketing influence for products and services (Rosen, 2000). With the development of the Internet, *autonomous word of mouth*, which is beyond firms' direct control, has become much more salient (e.g., Moe and Trusov, 2011; Shrihari and Srinivasan, 2012; Yadav and Pavlou, 2014). By portraying a given product or service as either a good or a bad deal, autonomous word of mouth (WOM) can make the difference between commercial success and failure. This is why firms should be aware of the strength of interactions between their products' market value and autonomous WOM when designing pricing and communication strategies.

Nonetheless, the existing dynamic models of sales in marketing assume that the influence of WOM on sales is at best mediated by advertising effort (see Feichtinger *et al*., 1994, and Huang *et al*., 2012). As a result, they disregard the role of autonomous WOM and how it can be shaped over time by a product's market value. These omissions can produce ineffective WOM management and therefore misleading pricing and communication policies.

This paper seeks to bridge the gap by introducing a dynamic sales model of a monopolistic firm's product where, as in Gould (1970), advertising-dependent WOM and advertising allow the firm to attract new customers. We extend Gould (1970) by introducing autonomous WOM that does not depend on advertising effort. We further assume that the firm's attraction rate of new customers is determined not by the number of remaining potential customers, as in most marketing models, but by the degree at which the current sales price is advantageous or not, i.e., the price advantage. Price advantage is measured by the difference between the current sales price and a reservation price based on the current customers' maximum willingness to pay to repurchase the product.

The notion of price advantage is closely related to price fairness. Bolton *et al.* (2010) found that U.S. consumers assess comparatively higher prices as less fair and comparatively lower prices as more fair. Ferguson *et al.* (2014) showed that price fairness is highest when a price is advantageous and lowest when it is disadvantageous. In addition, greater price unfairness leads to greater intention to spread negative word-of-mouth (see also, for instance, Campbell, 1999; Zeelenberg and Pieters, 2004). In this context, "spreading negative word of mouth is a low-cost action that helps buyers cope with their negative feelings of disappointment or regret and prevents other customers in their social network from being exploited" (Xia *et al.*, 2004). An illustration of extreme reaction to perceptions of price unfairness is the public relations nightmare endured by Amazon.com when an customer discovered that he was charged a higher price for the same same-title DVDs on the basis of his purchasing history (Adamy 2000).

Conversely, when the price is advantageous, suspicion of the seller does not come into play and the price is accepted as fair (Ferguson *et al.*, 2014). This latter result is consistent with Grewal *et al.* (1998) according to which greater perceived transaction value brings about greater perceived acquisition value. Furthermore, "consumers are more likely to experience satisfaction with a shopping experience involving a rebate and, subsequently, are more likely to express intentions to engage word-of-mouth communication about the product" (Hunt *et al.*, 1995).

Through the latter mechanism, a price decrease generates positive WOM, as suggested in Dean (1976) and Nagle et *al.* (2011, p. 127) and conjectured in Ajorlou *et al.* (2014). An empirical evidence of a link between price reduction and WOM is provided by Godinho de Matos *et al.* (2015). Based on a randomized experiment using movies over Video-on-Demand (VoD) system of a large European telecommunications provider offering triple-play to more than half a million households, the authors show a positive effect of word-of-mouth on the

sales of movies in this VoD system, which is particularly large for the movies offered at discounted prices. In fact, 50% of the sales attributed to WOM in this VoD system are associated to movies offered at discounted prices. The authors conclude that firms that use social network information to shape promotion campaigns and determine target consumers are likely to perform better.

Overall, the above-mentioned literature suggests two important assumptions, that is, i) as price disadvantage decreases, the intention to spread negative WOM decreases, and ii) positive WOM increases with price advantage. Our model makes use of these two assumptions to characterize the optimal pattern of price-induced WOM in a context of costly price adjustments.

A primary goal of the paper is to analyze the optimal tradeoff by a monopolistic firm between sales price and advertising effort for a given product, and its implications for dynamic pricing and WOM effectiveness. To address this issue, an optimal control problem is formulated where the attraction of new customers depends both on autonomous and advertising-dependent WOM. Depending on the magnitude of price advantage, the attraction of new customers is reversible and may turn into attrition of current customers, and *vice versa*. Due to adjustment efforts in production capacity, sales price variations are costly. In this context, the firm has to determine the optimal intertemporal tradeoff between investing in WOM effectiveness and incurring sales price adjustment costs.

The contribution of our paper is twofold. First, we extend the contagion sales models by introducing two important levers that might affect the typical pattern of advertising policies, namely autonomous WOM and price advantage. Second, we show that, despite price rigidity, the interactions between sales price, advertising-dependent and autonomous WOM can result in complex dynamic pricing policies involving history-dependence or limit cycling consisting of alternating attraction of new customers and attrition of current customers.

The paper is organized as follows. In the next section, we review the relevant literature. In Section 3, an optimal control model is formulated where a monopolistic firm seeks to determine the dynamic optimal strategy in terms of advertising and sales price adjustment efforts. In Section 4, we characterize the solution of the problem by qualitative means. Section 5 investigates the model with numerical means, and Section 6 concludes the paper.

2. Literature review

Our research lies at the intersection of three literatures, those concerning contagion sales models, autonomous WOM and price advantage.

Dynamic models of sales in a monopoly are usually based on the assumption that the evolution of instantaneous sales over time results from a combination of two kinds of influences: an attraction rate of new customers and an abandon rate of current customers due to forgetting. Dynamic models of sales differ from diffusion models because they rely on the evolution of instantaneous rather than cumulative sales (see Jørgensen and Zaccour, 2004).

Some well-known specifications of these two influences, attraction rate of new customers and abandon rate of current customers, are presented in Table 1, where $s(t) = S(t)/M$ is the sales rate with $S(t)$ the stock of current customers, M the constant market potential, $u(t)$ the advertising effort, $p(t)$ the sales price, γ, φ and δ positive constants, and Θ, Ξ, Ω, Φ, and Ψ positive real functions.

	Vidale-Wolfe (1957) Sethi (1973)	Ozga (1960) Gould (1970) Sethi (1979)	Sasieni (1971)	Feichtinger (1982)	Mahajan-Muller (1986)	Feinberg (2001)
Attraction rate	$\gamma u(1-s)$	$\varphi u s(1-s)$	$\Theta(s,u)$	$\Xi(p,u)$	$\Omega(u)(1-s)$	$\Omega(u)\Phi(s)$
Abandon rate	δs	δs		δs	δs	$\Psi(s)$

Table 1. Specification of dynamic monopolistic models of sales

The models above assume that advertising effort is crucial in attracting new customers because it signals the existence of the product (Nelson, 1970). These models divide the existing literature into two streams, depending on whether the influence of WOM on new customers is nonexistent (Vidale-Wolfe, 1957; Sethi, 1973; Sasieni, 1971; Feichtinger, 1982; Mahajan and Muller, 1986; Feinberg, 2001) or at best mediated by advertising effort (Ozga, 1960; Gould, 1970; Sethi, 1979, Feinberg, 2001).

The latter stream of the literature, to which our paper contributes, assumes a contagion process toward new customers based on advertising-dependent WOM that gives rise to history-dependent advertising policies. In the context of a contagion sales model, Gould (1970) was the first to identify multiple long run equilibria, one of them being an unstable focus and the two others saddle points whose basins of attraction are separated by the so-called Skiba threshold (Skiba, 1978). Depending on whether the initial magnitude of WOM (i.e., the initial number of current customers) is above or below the Skiba threshold, it is optimal either to find a trade-off between an increase in sales through advertising effort and the related costs, or to gradually reduce the advertising effort and stop selling the product.

To date, the literature on contagion sales models have disregarded two important intertwined levers that might affect the typical pattern of advertising policies described above: autonomous WOM and price advantage.

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Autonomous WOM, as a channel of social influence that is not under the firm's direct control, has recently gained considerable attention in the marketing literature, notably because of the development of online product ratings. Previous research has mainly focused on identifying the consequences and antecedents of WOM. According to Chevalier and Mayzlin (2006), potential buyers are increasingly relying on the information provided by others in these forums which implies that online customer ratings have the potential to significantly affect product sales. Libai *et al*. (2013) show that customer WOM can generate value both through market expansion, i.e., it can gain customers who would not otherwise have bought the product, and market acceleration, i.e., it can accelerate the purchases of customers who would have purchased anyway. According to Shrihari and Srinivasan (2012), consumers who influence others are themselves influenced by other consumers and that this influence is contingent on their product experience. Moe and Trusov (2011) confirm that consumer online product ratings reflect both the customers' experience with the product and the influence of others' ratings. Trusov *et al*. (2009) show that WOM referrals have substantially longer carryover effects than traditional marketing actions and produce significantly higher response elasticities. Based on the idea that consumers spread WOM on brands as a result of social, emotional, and functional drivers, Lovett *et al*. (2013) find that the social and functional drivers are the most important for online WOM, while the emotional driver is the most important for offline WOM. Berger and Milkman (2012) indicate that positive content is more viral than negative content. Godes and Mayzlin (2009) confirm that firms can create WOM that drives sales, and suggest that it may be more impactful for the firm to target less loyal customers to participate in a WOM campaign. Berger and Schwartz (2011) suggest that promotional giveaways may help boost WOM.

In this stream of literature studies, the nature of interactions that prevails between autonomous and advertising-dependent WOM remains under-investigated. This issue is of significant concern for practitioners who debate whether WOM and advertising are complementary to or substitute for each other (Armelini and Villanueva, 2010). In addition, the relationship between autonomous WOM and pricing decisions is generally disregarded, so the price-induced WOM issue is not considered.

Recent research trends in price advantage investigate the reference effect, which represents the impact of reference price on demand. Reference price is the price customers have in mind and to which they compare the current price of a specific product. Customers compute the deviation of the sales price from the reference price to evaluate their gains (price advantage) and losses (price disadvantage). According to the prospect theory (Kahneman and Tversky,

1979), which received a vast empirical support in the context of reference prices (see Kalyanaram and Winer, 1995), there is an inherent asymmetry in customer's perception between price advantage and price disadvantage, in that price disadvantage loom larger than price advantage of the same magnitude due to loss aversion. In the setup of a dynamic model where the demand is a linear function of both the sales price and the price advantage with loss-averse customers, Fibich *et al*. (2003) show that the optimal convergence to the steady-state sales price is monotonic. This result, which generalizes those obtained by Sorger (1988) and Greenleaf (1995) is confirmed by Kopalle *et al*. (1996) and Popescu and Wu (2007). Kopalle *et al*. (1996) and Popescu and Wu (2007) also demonstrate that cyclical pricing policies are obtained in the case of loss-seeking customers. Though this type of behavioral asymmetry has found some empirical validation in the marketing literature (e.g., Krishnamurthi *et al*. 1992; Greenleaf, 1995), it remains debatable because it is inconsistent with prospect theory (Kahneman and Tversky, 1979).

An unexamined question is how price advantage can be shaped by WOM effectiveness over time. Another important issue is how the existence of price rigidity affects the evolution of price advantage under autonomous and advertising-dependent WOM.

In this paper, we extend the contagion sales model of Gould (Gould, 1970), introducing two distinguishing features. First, WOM can develop autonomously such that sales can still vary without advertising effort. Second, autonomous and advertising-dependent WOM affect the sales process in proportion with the price advantage. To account for the effect of consumption experience, price advantage is given as the difference between the current sales price and a reservation price, where the reservation price is based on current customers' maximum willingness to pay to repurchase the product. Current customers compare the actual sales price with their reservation price to evaluate their price advantage. WOM will cause these evaluations to result either in loyalty of current customers and attraction of new customers in the case of price advantage, or in attrition of current customers in the case of price disadvantage. We assume an asymmetric influence of WOM on the sales process, in that price advantage attracts more new customers through the joint influence of autonomous and advertising-dependent WOM, than a price disadvantage drives away current customers through autonomous WOM negative.

Doing so, we evaluate how a firm's autonomous and advertising-dependent WOM can be leveraged relative to price advantage. We also determine the nature of interactions between autonomous and advertising-dependent WOM, and characterize a broad spectrum of related pricing policies in the presence of costly adjustments.

3. Model

In general, when handling the available price information, customers compare it with a *reservation price*, i.e., the maximum price that customers consider reasonable to pay for a particular good or service (Steedman, 1987). In our model, this benchmark reflects the sales price level at which current customers have purchased the product. We consider that a reservation price is based on current customers' maximum willingness to pay to repurchase the product. The current customers' maximum willingness to pay to repurchase the product, $\tilde{p}(t)$, is the minimum value of the sales price $p(t) \geq 0$ at which the current customers have purchased the product and will continue to do so. We assume that it is linearly decreasing with the number of current customers, so that $\tilde{p}(t) = a - bS(t)$, where $a > 0$ is the upper value of the current customers' maximum willingness to pay, and $b > 0$ is the marginal reduction of maximum willingness to pay required by an additional current customer to repurchase the product. Note that this marginal reduction can be weakened by customer satisfaction (Homburg *et al.*, 2005) *via*, e.g., product quality.

We assume that new customers' inflows and current customers' outflows are determined by the price advantage, which is the difference between the reservation price and the current sales price, $\tilde{p}(t) - p(t)$. If the sales price is set above the reservation price, a fraction of current customers who have purchased the product at a sales price $\tilde{p}(t)$ lower than or equal to $p(t)$ will be lost. This loss of current customers will result in a subsequent update of the reservation price, which increases according to the new number of current customers. Conversely, a lower sales price than the reservation price will result in all the current customers' continuing to purchase the product, and new customers being attracted by the advantageous sales price. The corresponding increase in sales will then lead to a lower reservation price that decreases according to the gain of current customers.

The above arguments are reflected in the following dynamic equation for the sales:

$$\dot{S}(t) = \alpha\big(\kappa + u(t)\big)S(t)\big[\big(a - bS(t)\big) - p(t)\big] - \delta S(t), \qquad S(0) = S_0 > 0, \qquad (1)$$

where the sales are denoted by $S(t) \geq 0$, advertising effort by $u(t) \geq 0$, α, κ, a and b being positive constants, and $0 < \delta < 1$.

In the first term in the right-hand side of (1), a positive (negative) value of the difference $\big[\big(a - bS(t)\big) - p(t)\big]$ reflects the potential of attraction of new customers (attrition of current customers) whenever the sales price is smaller (greater) than the reservation price. This

difference is influenced twofold, that is, by autonomous WOM, $\alpha \kappa S(t)$, and advertising-dependent WOM, $\alpha u(t) S(t)$.

As in dynamic models of sales in marketing, the second term in the right-hand side of (1), $\delta S(t)$, is intended to model the abandon rate that reflects the number of current customers who forget about the product due to exogenous factors such as product obsolescence.

As noted in Gallego and Van Ryzin (1997), "in many cases, firms do not have complete flexibility in setting prices". The reason is that price adjustments require extra implementation and coordination efforts (Çelik *et al*., 2009; Chen and Hu, 2012), that frequently depend on the magnitude of the price change, e.g., managerial efforts (Blinder *et al*., 1998; Zbaracki *et al*., 2004). Such efforts include increasing capacity or producing below capacity. We assume that the price adjustment efforts are driven by modifications in the production capacity and impose that the sales price obeys the transition equation:

$$\dot{p}(t) = v(t), \qquad p(0) = p_0 \geq 0, \qquad (2)$$

where $v(t) \overset{>}{\underset{<}{-}} 0$ inversely reflect efforts of modification the production capacity, and the initial sales price inversely mirrors the initial production capacity.

Hence, a given sales price level in (2) can be seen as the assignment of a fixed production capacity to a customers' class (Gallego and Van Ryzin, 1997). A similar idea can be found in the screening model by Mussa and Rosen (1978) and in the cross-sectional model by Wernerfelt (2008).

Our representation of price stickiness (or rigidity) is simpler than that in the competitive model of Fershtman and Kamien (1987), where adjustments depend on the differences between the market demand and the sum of outputs. In our model, price and sales are assumed to gradually adjust to each other depending on the price advantage variations. The price advantage either increases or decreases depending on whether the production capacity increases while the sales decrease or increase more slowly than the production capacity (i.e., $-v(t) > b\dot{S}(t)$), or the sales increase while the production capacity decreases or increases more slowly than the sales (i.e., $-v(t) < b\dot{S}(t)$).

Remark. Without autonomous WOM and with infinitely costly price adjustment efforts, equation (2) reduces, as it should, to Gould's (1970) sales model.

We now turn to a definition of the firm's objective criterion.

At any time, the sales revenue is $p(t)S(t)$. Yet the unit production cost, $c \geq 0$, is assumed to be constant. For simplicity, it is normalized to zero, without loss of generality (e.g., Popescu and Wu, 2007).

The cost of advertising effort, $E(u(t))$, is described by the following quadratic function (e.g., Jørgensen and Zaccour, 2004):

$$E(u(t)) := eu(t)^2/2$$

where $e > 0$. Note that the results would remain qualitatively the same for general cost functions. Convex advertising costs can be defended as follows: to increase advertising over certain levels, a firm has to resort to additional and less efficient advertising media or more expensive advertising channels (i.e., decreasing returns to scale).

Finally, the costs of price adjustment efforts are assumed to be quadratic (e.g., Çelik *et al.*, 2009), that is:

$$F(v(t)) := fv(t)^2/2$$

where $f > 0$. The price adjustments costs being convex, makes sure that, as in the literature of sticky prices (see, for instance, Fershtman and Kamien, 1987), the price level behaves continuously over time. This implies that the price does not adjust instantaneously to the reservation price. Here also, the results would remain qualitatively the same for general functions.

Assuming an infinite time horizon and a positive discounting rate, $r > 0$, the firm's problem is:

$$\underset{u(.),v(.)}{\text{Max}} = \int_0^\infty e^{-rt} \left(p(t)S(t) - \frac{eu(t)^2}{2} - \frac{fv(t)^2}{2} \right) dt \tag{3a}$$

$$\dot{S}(t) = \left\{ \alpha(\kappa + u(t)) \left[(a - bS(t)) - p(t) \right] - \delta \right\} S(t) \tag{3b}$$

$$\dot{p}(t) = v(t) \tag{3c}$$

$$p(t) \geq 0 \tag{3d}$$

$$u(t) \geq 0 \tag{3e}$$

$$S(0) = S_0 > 0, \, p(0) = p_0 \geq 0 \tag{3f}$$

So we have an optimal control problem with two state variables, S and p; and two controls, u and v. Expression (3a) is the objective. The firm maximizes the discounted cash flow stream, where instantaneous cash flow consists of the difference between revenue and the costs of advertising and price adjustments.

Equation (3b) describes the evolution of sales influenced by autonomous and advertising-dependent WOM. Equation (3c) reflects the price dynamics, whereas expression (3d) is the state constraint making sure that price is non-negative. Advertising is non-negative, and this is assured by expression (3e), while the initial values of the state variables are given in expression (3f).

4. Analysis

Due to the state dynamics and constraints, it is obvious that the states remain in a bounded region and the controls are bounded as well. For admissible controls, the solution of the ordinary differential equations exists for all $t > 0$ and hence an optimal solution exists. Furthermore, we will consider every possible candidate for a long-run solution and calculate the corresponding basins of attraction. This guarantees that we will find the global maximizers.

The order of the state constraint in (3d) is one and it satisfies the full rank condition, since:

$$h(S,p) := p, \qquad \frac{\mathrm{d}h(.)}{\mathrm{d}t} = \dot{p}(t) = v(t), \qquad \frac{\mathrm{d}^2 h(.)}{\mathrm{d}v\mathrm{d}t} = 1. \qquad (4)$$

Hence, the extended Hamiltonian writes (Grass *et al.*, 2008):

$$H(S,p,\lambda_1,\lambda_2,u,v,\mu,\upsilon) := pS - \frac{eu^2}{2} - \frac{fv^2}{2} + \lambda_1 \left\{ \alpha(\kappa+u)\left[(a-bS)-p\right] - \delta \right\}S + \lambda_2 v + \mu u + \upsilon p \quad (5)$$

where $\lambda_i \equiv \lambda_i(t)$ are costate variables, $i = 1,2$, and $\mu \equiv \mu(t)$ and $\upsilon \equiv \upsilon(t)$ are Lagrange multipliers.

Let $\left(u(S,p,\lambda_1,\lambda_2)^\circ, v(S,p,\lambda_1,\lambda_2)^\circ\right)$ be a solution of the Hamiltonian maximizing condition, i.e.:

$$\left(u(S,p,\lambda_1,\lambda_2)^\circ, v(S,p,\lambda_1,\lambda_2)^\circ\right) := \underset{u,v}{\operatorname{argmax}} H(S,p,\lambda_1,\lambda_2,u,v,\mu,\upsilon) \qquad (6)$$

To keep notation short, we use $(u^\circ(t), v^\circ(t))$. Skipping the time index for convenience, the state and adjoint dynamics determine the canonical system:

$$\dot{S} = \alpha(\kappa+u^\circ)(a-bS-p)S - \delta S \qquad (7a)$$

$$\dot{p} = v^\circ \qquad (7b)$$

$$\dot{\lambda}_1 = \lambda_1 \left[r + \delta - \alpha(\kappa+u^\circ)(a-2bS-p) \right] - p \qquad (7c)$$

$$\dot{\lambda}_2 = r\lambda_2 + \left[\lambda_1 \alpha(\kappa+u^\circ) - 1 \right]S \qquad (7d)$$

A solution $(u^\circ, v^\circ, \mu, \upsilon)$ of (6) has to satisfy the following first order necessary conditions, and the complementary slackness conditions resulting from the non-negativity conditions (3d-e):

$$\frac{d}{du} H\left(S, p, \lambda_1, \lambda_2, u^\circ, v^\circ, \mu, \upsilon\right) = 0 \tag{8}$$

$$\frac{d}{dv} H\left(S, p, \lambda_1, \lambda_2, u^\circ, v^\circ, \mu, \upsilon\right) = 0 \tag{9}$$

$$\mu u = 0 \tag{10}$$

$$\upsilon p = 0 \tag{11}$$

We now investigate the possible arcs separately and derive some theoretical results.

4.1 Interior solution

For an interior solution with $u > 0$ and $p > 0$, the multipliers vanish:

$$\upsilon = \mu = 0 \tag{12}$$

and the conditions (8) and (9) reduce to:

$$H_u = -eu + \lambda_1 \alpha (a - bS - p) S = 0 \tag{13}$$

$$H_v = -fv + \lambda_2 = 0 \tag{14}$$

yielding:

$$u^\circ = \frac{\lambda \alpha S (a - bS - p)}{e} \tag{15}$$

$$v^\circ = \lambda / f \tag{16}$$

It is easily verified that the Legendre-Clebsch condition is satisfied for (13)-(14), because the Hessian is negative definite. This implies that the Hamiltonian is concave with respect to the control vector (u, v) and guarantees a (local) maximum of the Hamiltonian.

From (13), if $(a - bS - p) < 0$, it holds that $u = 0$. Therefore, advertising-dependent WOM should not be activated under a disadvantageous price. This rule is supported empirically by Jones *et al*. (2009) who show that negative electronic WOM has a negative impact on advertising credibility. This implies that WOM has an asymmetric influence on the sales process because an advantageous sales price attracts new customers through the joint influence of autonomous and advertising-dependent WOM, while a disadvantageous sales price makes autonomous WOM negative which drives away current customers.

Proposition 1. For a given value of $\lambda_1 > 0$, advertising effort decreases with the sales price, and either increases or decreases depending on whether sales are low or high.

Proof. From (15), we compute the partial derivative of advertising effort with respect to sales price, which gives $u_p^\circ = -\lambda_1 \alpha S / e$. In our context, the positive influence of sales on the

objective function suggests that its corresponding costate variable λ_1 should be positive. Therefore, $u_p^o > 0$ for $S > 0$. However, the partial derivative of advertising effort with respect to sales is $u_S^o = \lambda_1 \alpha (a - 2bS - p)/e$, which suggests that $u_S^o \gtrless 0$ depending on whether

$$a - 2bS \underset{<}{\overset{>}{=}} p . \square$$

This result suggests that advertising-dependent WOM should be activated more intensively when the sales price is lower, and *vice versa*. This implies a negative relationship between sales price and advertising-dependent WOM. From (15), we see that the firm should not advertise under a disadvantageous price, that is, $a - bS - p \leq 0$. Further, it can be easily seen that the partial derivative of u^o with respective to price advantage, $a - bS - p$, is positive whenever $S > 0$, which implies that the firm should advertise more because the price becomes more advantageous whenever WOM is strictly positive.

In contrast, from (15), the firm should not advertise when sales are either zero or too large, i.e., $S \geq (a - p)/b$. This is intuitive given that WOM is ineffective in the first case and useless in the second case. However, the maximum level of advertising effort should occur for $S = (a - p)/2b$, which means that the relation between advertising effort and WOM has a parabolic shape. This result, which is consistent with Gould (1970), suggests a negative relationship between advertising-dependent WOM and autonomous WOM because large enough autonomous WOM can gradually take over the sales increase process from the advertising effort.

4.2 Boundary solution with zero price, p = 0

We first establish that giving the product away for free can only occur in combination with positive advertising.

Proposition 2. Provided the reservation price $a - bS$ is non-negative, a solution with both zero sales price and advertising effort is not optimal.

Proof. Assume on the contrary that a boundary solution $p = 0$ and $u = 0$ exists on an interval of positive length. Then, (8) reduces to:

$$H_u = -eu + \lambda_1 \alpha (a - bS - p) S + \mu = 0 \tag{17}$$

and inserting $p = 0$ and $u = 0$ yields:

$$\mu = -\lambda_1 \alpha (a - bS) S \tag{18}$$

From (18), it is obvious that $\mu \geq 0$ requires that $S > a/b$, which will never happen because it corresponds to a negative reservation price. □

Actually, unless the reservation price is negative, a zero sales price can be advantageous and should lead to maximal attraction of new customers by setting a positive advertising effort. In such conditions, a zero advertising effort would correspond to a suboptimal policy.

While a solution piece with $p = 0$ and $u = 0$ can be optimal in some transient phases, it is not possible to have it in the long run.

Proposition 3. A solution with zero sales price, $p = 0$, and positive advertising, $u > 0$, cannot be a long run optimal equilibrium.

Proof. Assume on the contrary that an equilibrium with zero price, $p^\infty = 0$, and positive advertising, $u^\infty > 0$, exists. Then the profit rate in (3a) is negative, that is:

$$pS - \frac{eu^2}{2} - \frac{fv^2}{2} = -\frac{e(u^\infty)^2}{2} < 0 \tag{19}$$

Hence, rather than staying in this steady state with constant and negative profit, it would be better to choose zero price, $p = 0$, and zero advertising, $u = 0$, which gives zero profit. Hence, an optimal solution can never end up in a steady state with $p^\infty = 0$ and $u^\infty > 0$. □

Note that the solution with zero sales price, which is an extreme case of market penetration strategy, can be part of an optimal trajectory.

4.3 The possibility of limit cycles

Proposition 4. In an appropriate parameter region, an optimal limit cycle may occur.

Proof. The equilibria for system (7a)-(7d) along with (13)-(14) with $\delta = 0$ can be computed analytically, where we only consider the solution with $S > 0$. This yields:

$$\begin{pmatrix} S^\infty & p^\infty & \lambda_1^\infty & \lambda_2^\infty \end{pmatrix}^T = \begin{pmatrix} \dfrac{\alpha a - r}{2\alpha b} & \dfrac{\alpha a + r}{2\alpha} & \dfrac{1}{\alpha} & 0 \end{pmatrix}^T \tag{20}$$

where $\alpha a - r > 0$, with the corresponding Jacobian:

$$J = \begin{bmatrix} \dfrac{r - \alpha a}{2} & \dfrac{r - \alpha a}{2b} & 0 & 0 \\ 0 & 0 & 0 & 1/f \\ \dfrac{(r - \alpha a)^2}{4\alpha^2} - 2b & -\dfrac{(\alpha a - r)^2}{4\alpha^2 b} & \dfrac{r + \alpha a}{2} & 0 \\ -\dfrac{(\alpha a - r)^2}{4\alpha^2 b} & -\dfrac{(\alpha a - r)^2}{4\alpha^2 b^2} & \dfrac{\alpha a - r}{2b} & r \end{bmatrix}$$

In Dockner and Feichtinger (1991), a necessary and sufficient condition for the possible occurrence of limit cycles for the canonical system (7a)-(7d) along with (13)-(14) with a two dimensional stable manifold are stated. To check this condition, we have to calculate the sum of all principal minors (denoted as M_2) of the Jacobin matrix of order two, which yields:

$$K = M_2 - r^2 = \frac{(r - \alpha a)\left[\alpha^2 b^2 f (r + \alpha a) + (r - \alpha a)\right]}{4\alpha^2 b^2 f}$$

The following conditions are defined:

- The conditions

$$K < 0 \tag{21}$$

$$0 < \det J \le K^2/4 \tag{22}$$

are necessary and sufficient for all eigenvalues to be real, two being positive and two being negative.

- The conditions:

$$\det J > K^2/4 \tag{23}$$

$$\det J > K^2/4 - r^2 K/4 > 0 \tag{24}$$

are necessary and sufficient for all eigenvalues to be complex, two having negative real parts and two having positive real parts.

These terms satisfy the following properties for varying f.

$$\det J(f) = \frac{(\alpha a - r)^2}{2bf} > 0 \qquad \text{for all } f > 0 \tag{25}$$

$$\lim_{f \to \infty} \det J(f) = 0 \tag{26}$$

$$\lim_{f \to \infty} K(f) = (r - \alpha a)(r + \alpha a)/4 < 0 \tag{27}$$

$$\lim_{f \to 0} K(f) = \frac{(\alpha a - r)^2}{4\alpha^2 b^2} \lim_{f \to 0} \frac{1}{f} = \infty \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}f} K(f) = -\frac{(\alpha a - r)^2}{f^2 \alpha^2 b^2} < 0 \tag{29}$$

Equations (26) and (27) thus yield for f large enough:

$$K(f_\infty) < 0$$

$$\det J(f_\infty) < \left(\frac{K(f_\infty)}{2}\right)^2$$

Equations (27) to (29) guarantee that there exists a unique f_0 with $K(f_0) = 0$ and hence:

$$\det J(f_0) > \left(\frac{K(f_0)}{2} \right)^2 = 0$$

$$\det J(f_0) > \left(\frac{K(f_0)}{2} \right)^2 - \frac{r^2 K(f_0)}{2} = 0$$

with $f_0 = \dfrac{\alpha a - r}{\alpha^2 b^2 \left(\alpha a + r \right)^2}$.

We proved that for f_∞ large enough, the equilibrium in (20) satisfies (21)-(22) and therefore is a saddle point with two-dimensional stable manifold. For f_0, conditions in (23)-(24) are satisfied, hence there exists f_h with $f_0 < f_h < f_\infty$, where (20) undergoes a Hopf-bifurcation and in the neighborhood of f_h, a limit cycle with two-dimensional stable manifold exists. These results were derived for $\delta = 0$. Thus we can guarantee that this also holds true for δ small enough. $\square$

One main implication of limit cycling is that the current solution is reversible in the sense that the firm can alternate successive phases involving improvement and exploitation of the price advantage. A phase of *improvement of the price advantage* should lead to reactivating the advertising-dependent WOM to raise sales and then take advantage of the development of autonomous WOM. This latter phase is that of *exploitation of the price advantage*.

Therefore, the presence of costly price adjustments does not preclude the possibility of cyclical pricing. In spite of costly production capacity adjustments, the firm will apply dynamic pricing discrimination on a continuous basis, oscillating between lower and upper sales price levels respectively with and without advertising effort and inducing identical sales levels at regular time intervals. Put differently, the occurrence of limit cycles imply that one sales price level should result in two different realizations of sales at different periods.

Note that cyclical optimal solutions are not uncommon in marketing models. In the ADPULS model (Simon, 1982, for a time discrete version and Luhmer *et al.*, 1988, for a continuous time version), the source of the cycle is the delay in the advertising response. An important question that arises from the above result is whether the cyclical policy in our model is driven by the asymmetric influence of WOM on the sales process. In other words, is cyclical pricing caused by the fact that price advantage attracts more new customers than a price disadvantage drives away current customers? In the next section, we show that the asymmetric influence of

WOM on the sales process is compatible with a broad spectrum of pricing policies, including cyclical patterns.

5. Numerical analysis

To better characterize the possible patterns of optimal pricing and advertising effort policies, we conduct a numerical analysis. For the numerical resolution, a boundary value approach with an adaptive time mesh is used (see Grass *et al*., 2008 and Grass, 2012). The infinite time horizon was truncated to a sufficiently large time horizon that guaranteed that the last point of the solution path lied on the (linear) stable manifold of the equilibrium and was in a small epsilon neighborhood. The computations are made with OCMat (MATLAB package for the computation of optimal control problems).

Description	R	e	f	α	κ	a	b	δ
Unique equilibrium (Fig. 1)	0.03	1	1	0.01	1	1000	10	0.05
Skiba with two equilibria (Fig. 2 and 3)	*	*	*	0.00001	*	*	*	0.75
Skiba with a single equilibrium (Fig. 4 and 5)	*	*	*	0.0002	*	*	*	0.05
Limit cycle (Fig. 6.a and 7)	*	*	*	0.0001	*	*	*	0.05
Larger limit cycle (Fig. 6.b)	*	*	*	0.000005	*	*	*	0.05

Table 2. Parameter values and related solutions

We select the values for the parameters as shown in Table 2. By varying parameters α and δ, we show that various structurally different solutions are possible, including stable equilibria, different Skiba behaviors, and limit cycles.

5.1. Unique equilibrium

In some situations (e.g., the parameter set from Table 2), a unique equilibrium occurs that is also a saddle point (Figure 1).
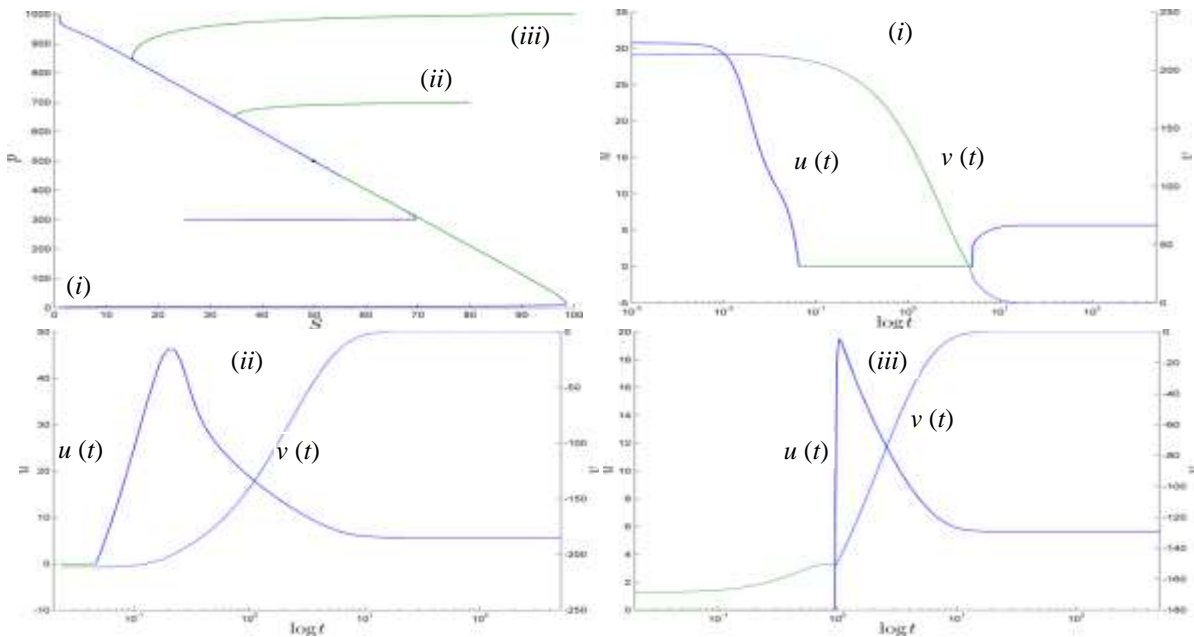

Fig. 1. Phase plane and time paths in the case of unique equilibrium (green parts refer to zero advertising effort)

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Let us first start with interpreting the situation (*i*) where the product has to be built up from the start. Initially the product has no reputation, i.e. the firm cannot offer it at a positive sales price while sales are negligible. The firm begins with keeping the sales price lower than the reservation price to make the product as attractive as possible for customers. It thus increases sales by activating advertising-dependent WOM with heavy advertising effort. In this way the firm enhances WOM effectiveness. During this initial stage the firm reduces advertising effort because, as time passes, autonomous WOM becomes stronger so that it can gradually take over the sales increase process from the advertising effort. During this initial phase of exploitation of the price advantage, the reservation price decreases rapidly as sales increase, and sales price increases very slightly due to limited production capacity reductions. This goes on until a kink appears where the sales price is equal to the reservation price (Putler, 1992). From this point, the firm stops advertising because the firm price becomes higher than the reservation price, to keep advertising-dependent WOM activated; see path (*i*) of Figure 1. Another implication is that autonomous WOM has a negative effect on sales. A new phase then starts where the firm substantially adjusts its production capacity to increase the sales price while sales are decreasing, thereby inflating the reservation price. This phase consists in *riding up the demand curve*. Due to this increase, at some point the sales price becomes lower than its reservation level. From that moment, the firm reactivates advertising-dependent WOM but more lightly than during the previous phase. Sales price keeps on increasing while sales decrease until the firm ends up in steady state where price and sales are constant. Therefore, the optimal policy under low initial price and sales alternate a phase of price advantage exploitation along with heavy advertising effort with a phase of riding up the demand curve supplemented by light advertising effort until a pricing equilibrium is reached.

Another initial situation to consider is when the product is very mature in the beginning, i.e., sales price and sales are initially high; see paths (*ii*) and (*iii*) in Figure 1. Because sales price is so high and reservation price is so low (due to high sales), negative autonomous WOM leads to attrition of current customers, which decreases sales. This is the reason why the firm initially refrains from activating advertising-dependent WOM. However, the counterpart of the sales decrease due to autonomous WOM is an increase in the reservation price. To complement the increasing trend of the reservation price, a decrease in the sales price (resulting from a production capacity increase) is the only possible option at this point but at a slow pace, to improve the price advantage. Once the sales price has decreased enough that it falls below the kink, the firm starts to advertise heavily to activate advertising-dependent WOM. Sales then increase again, which is also possible because sales price simultaneously

rapidly decreases. This phase consists in *riding down the demand curve* (or *skimming*). Along the way, advertising-dependent WOM decreases because it is gradually replaced by the growing effectiveness of autonomous WOM. This continues until the steady state is reached. Overall, the optimal policy under high initial price and sales alternates a phase of improvement of the price advantage marked by heavy advertising effort, then a phase of riding down the demand curve supplemented by light advertising effort.

		Initial price	
		Low	High
Initial sales	Low	Price advantage exploitation, then Riding up the demand curve	Riding down the demand curve
	High	Riding up the demand curve	Price advantage improvement, then Riding down the demand curve

Table 3. Initial configurations for sales and price and related strategies

Both solutions have in common a relationship between sales price and sales that is positive in the short run and negative in the long run. This can be explained by the fact that large initial sales imply both a large autonomous WOM effect and a low reservation price. Therefore, a high initial sales price first requires the firm to let the sales decrease more rapidly than the sales price to turn the autonomous WOM from negative to positive. The sales price then decreases more rapidly than the reservation price to increase sales. The converse scenario applies for both low initial sales and sales price.

Overall, the results can be summarized in the matrix below (Table 3). In all cases, the optimal policy is either monotonic or it involves at most one kink in the demand function.

5.2. Skiba point with two equilibria

The core result of Gould's contagion model (Gould, 1970) is the existence of three long run steady states in the state-control phase portrait, one of them being an unstable focus and two saddle points whose basins of attraction are separated by a Skiba threshold (for an introduction to multiple equilibria and Skiba thresholds, cf. Grass *et al*., 2008, chap.5). In Gould's model, the optimal behavior of a firm depends on the initial state such that if the sales rate is *above* a certain threshold, it is optimal to find a trade-off between an increase in the sales rate by advertising and the related expenditures. Conversely, for sales *below* this threshold it is better to gradually reduce the advertising rate and to stop selling the product.

Under certain conditions, the history-dependence property remains valid for the extension of Gould's model including autonomous WOM and sales price inertia. More precisely, there exists a Skiba curve separating two regions in the (S, p)-state space. Starting in one region, it

is optimal to converge to an interior steady state, while for initial states in the other part of the state space one is better off to choose a boundary equilibrium $S = 0$.

Figure 2.a presents the Skiba curve (in black) in the (S, p)-phase space with a multitude of solution trajectories (in blue for the unrestricted case, and in green for $u = 0$) emanating from the Skiba curve. Starting there, the firm is indifferent either to converge to the interior steady state (shown as a green point in the S-axis) or to go to a boundary point on the p-axis. Note that the trajectories leading to a boundary equilibrium are depicted as solid curves, while those converging to the interior steady state are dashed.
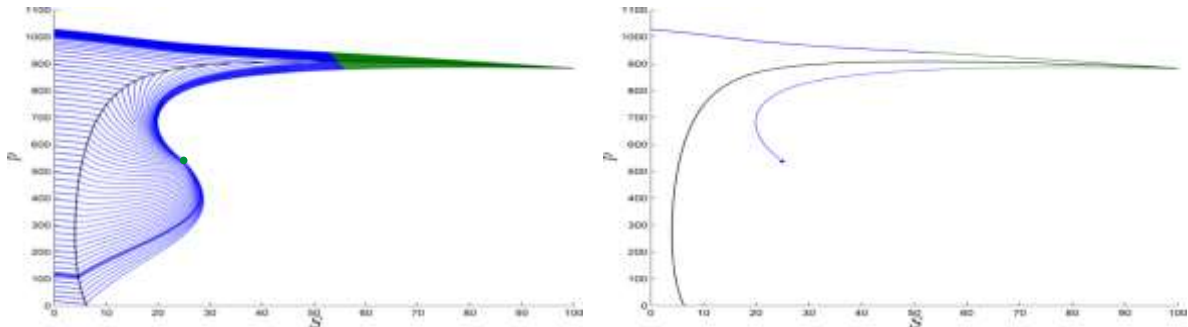

| 2.a. Multitude of optimal trajectories | 2.b. Two optimal trajectories |

Fig. 2. Skiba curve in the state space with various solution trajectories (in green for zero advertising effort)

There is a continuum of such boundary steady states that are reached within finite time. This Skiba-behavior occurs for very low WOM effectiveness ($\alpha = 0.00001$) and a high current customer abandon rate ($\delta = 0.75$). For initially limited WOM or excessively high price levels, it is therefore optimal to stop selling the product, while for all other cases gradual convergence to a steady state is possible. For very low initial WOM, it is always optimal to close the business, and no sales price policy can save the product. Overall, the product ceases to exist in finite time when either initial WOM is excessively low or when sales price is excessively high at the beginning of the planning period.

Only when the sales are strong enough and the sales price is not too high compared with the reservation price will an interior long-run steady state be optimal. In this case, depending on the initial state, the path to the interior equilibrium is somewhat similar to the case with unique equilibrium. Note that this steady state is characterized by a tradeoff between the costs of advertising and sales price adjustment and their impact on sales. To discuss the indifference mechanism in more detail, we observe that the Hamiltonian maximizing condition $H_u = 0$ in (23) causes the optimal advertising rate u to 'follow' both the costate of S and the logistic term $S(a - bS - p)$. The shape of the time path of the optimal advertising rate depends on which of these two factors dominates.

In Figure 2.b, a pair of trajectories starting from the extreme right of the Skiba curve is illustrated (bold for the closing-down solution and dashed for the path leading to the interior equilibrium). The Skiba curve here clearly delimits the area of initial values for which the sales price is sufficiently advantageous or not excessively disadvantageous to reach the interior equilibrium, providing there is a sufficiently high level of WOM effectiveness.
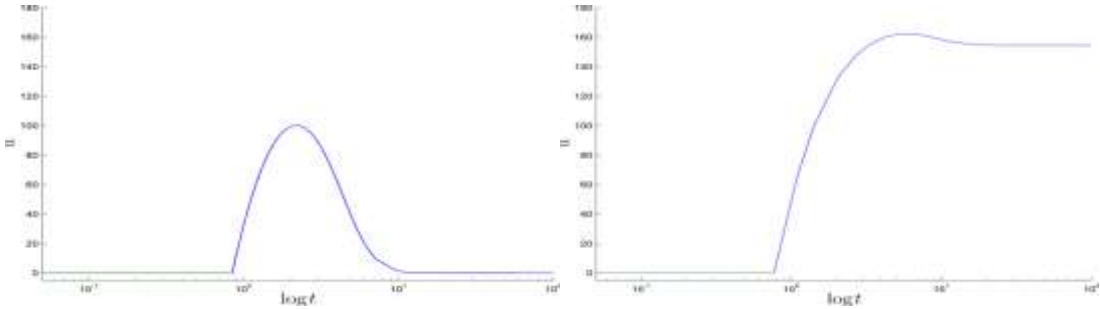

3.a. Initial state above the Skiba threshold   3.b. Initial state below the Skiba threshold
Fig. 3. Time paths of advertising effort above and below the Skiba threshold

Figure 3 shows the time path of advertising effort. For the case where the business finally closes down (bold trajectories) the optimal advertising rate follows $S(a-bS-p)$. Note that the costate is more or less constant, and u decreases monotonically to zero. Considering the dashed case, it turns out that it is λ_1 which predominantly determines the behavior of u.

5.3. Skiba threshold with unique equilibrium

In addition to the case in the previous section where – as usual – the Skiba curve separates separates the basins of attraction of two different equilibria. Here we have a solution where the Skiba curve separates regions where significantly different behavior is optimal while the long run equilibrium is the same. This happens in situations with both low (though not excessively) WOM effectiveness ($\alpha = 0.0002$) and low values of the current customers' abandon rate ($\delta = 0.05$).
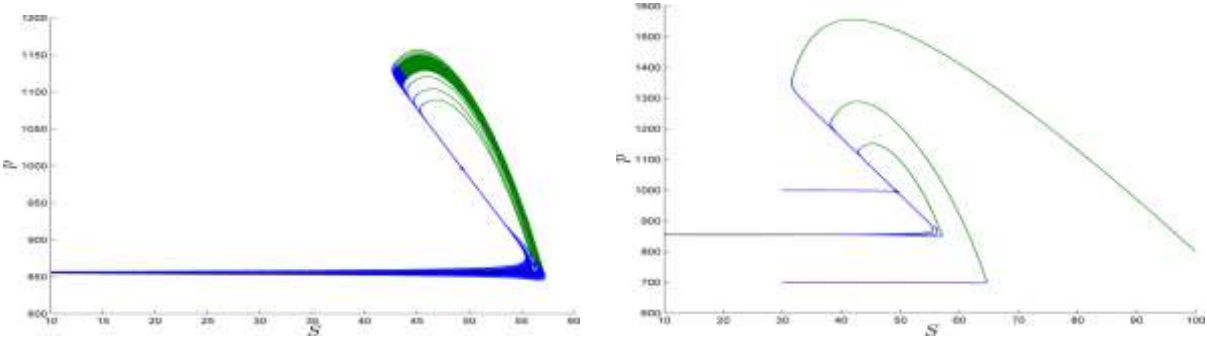

4.a. Multitude of initial states   4.b. A few initial states
Fig. 4. Phase plane for a Skiba threshold with a single equilibrium

In Figure 4.a, starting from both low initial sales price and sales, we observe the existence of a more or less horizontal Skiba threshold above which the optimal path is quite similar to the case with a unique equilibrium. The existence of the Skiba threshold either results in a straight

path to the steady state, or allows for optimal overshooting. An initial sales price below the Skiba threshold will lead both to a slight decrease in the sales price and a dramatic increase in sales. The phase corresponding to this path is that of market penetration; see Figure 4.b.

The increase in sales is caused by both the advantageous sales price and the heavy advertising effort that serves as a word-of-mouth activator; see Figure 5. As for the unique equilibrium case, the market penetration phase is supported by heavy advertising effort. Further, the sales evolve inversely to the unit profit margin. At the maximum value of sales, a first kink appears where the sales price, like the profit margin, is at its lowest value and equal to the reservation price. At this level, it becomes more profitable to sacrifice high sales to get a higher profit margin very rapidly. This phase consists in riding up the demand curve and requires no advertising effort because, during a finite time interval, WOM is negative due to the disadvantageous sales price. During this phase, the customers that purchased the product earlier at a lower price are lost at an increasing rate, and the reservation price rises. The path that corresponds to this phase is steeper compared with that of the subsequent final phase. At the maximum value of the sales price, a second kink is reached after which both sales price and reservation price start to decrease during a short time interval to allow for a recovery of the price advantage. At the final sequence, it becomes more profitable to sacrifice part of the profit margin to increase sales. During this phase, advertising effort becomes positive to reactivate WOM. Finally, the inertial price equilibrium is reached that reflects an optimal long run balance between profit margin and sales.
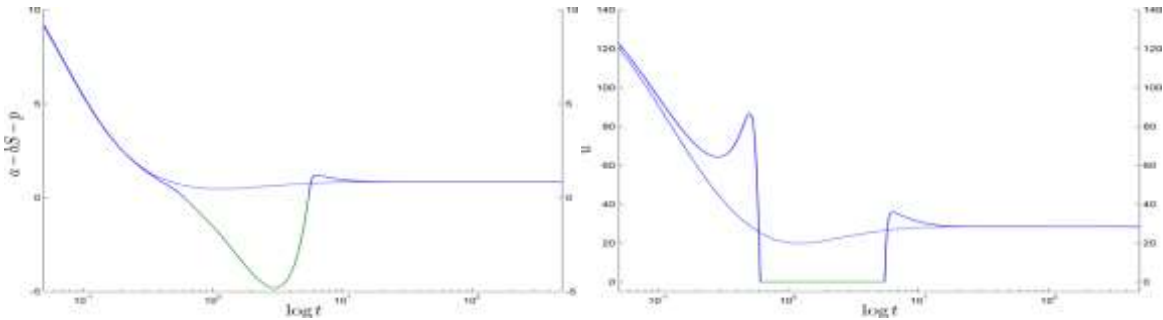
5.a. Price advantage   5.b. Advertising effort time paths
Fig. 5. Time paths for a Skiba threshold with a single equilibrium

Therefore, the optimal sequence here is first a market penetration phase, then a phase of riding up the demand curve supplemented by price advantage improvement, and finally a phase of riding down the demand curve. Both the penetration phase before the first kink and the riding down the demand curve phase after the second kink lead to leverage advertising-dependent WOM because advertising and WOM are complementary rather than substitutes, while the riding up the demand curve phase creates negative WOM as a corollary of a disadvantageous sales price.

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5.4. Limit cycle

For lower values of WOM effectiveness ($\alpha = 0.0001$), an interesting cyclical optimal solution with continuous price adjustments occurs in which a counter-clockwise movement can be observed (Figure 6).

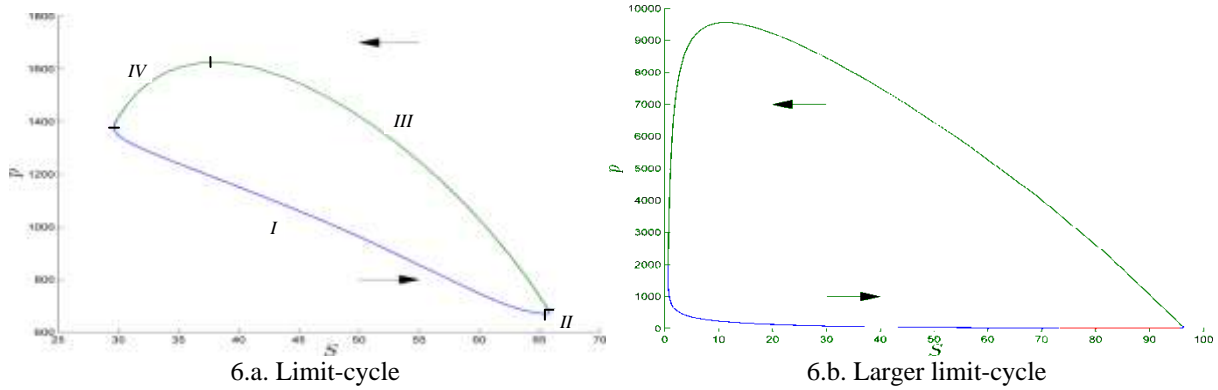6.a. Limit-cycle   6.b. Larger limit-cycle
Fig. 6. Phase-portrait diagram in the state space with limit-cycle

We start with a situation with a low sales value and intermediate sales price level. In such a situation the product has recovered from a previous phase of riding up the demand curve or has to initially establish itself in the market. Hence it starts with a riding down the demand curve phase (see regime *I* in Figure 6.a), where the sales price decreases quickly and advertising is positive (blue part) and increasing to support the rapid growth of sales by WOM.

When the sales have reached the first kink and the resulting reservation price is very low, the firm prepares for the subsequent phase of riding up the demand curve (regime *II*) by increasing the price while sales still increase slightly. Remarkably, this intermediate phase during which sales price and sales are complementary rather than substitutes is very short because it is not sustainable to maintain this phase for a long time. Doing so would lead both to an increase in the sales price and a further decrease in the reservation price, which is already at a low value. However, in this short time, an extremely high advertising pulse is applied to inform new customers of the very advantageous price through WOM (Figure 7.a).
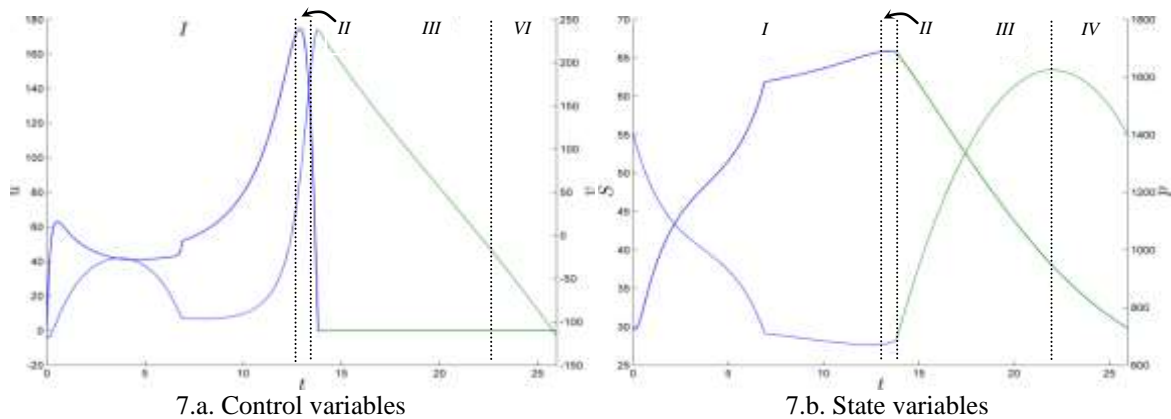

7.a. Control variables   7.b. State variables
Fig. 7. Time paths of the control and state variables for the limit cycle

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The increasing sales price quickly makes the (very high) sales volume decrease sharply and the firm enters regime *III*, corresponding to the riding up the demand curve phase in which the firm exploits its high sales and sluggish demand by charging an increasing sales price. Given that in a situation with a higher sales price, advertising would only add to the decreasing sales *via* negative word-of-mouth, the firm refrains from doing advertising in this phase (green part). When sales have dropped below a certain level, sales price cannot increase any further because the reservation price is already far below the sales price. Hence, in some intermediate regime *IV*, the firm has to prepare for recovery, where the sales price already drops but is still high so that also sales continue to decrease and reservation price to increase. This phase of price advantage improvement, where price and sales are once again complementary rather than substitutes, ends when sales have reached the lowest level at the second kink and start to increase again, i.e. entering the riding down the demand curve phase of regime *I* to start the next cycle. During this phase, the reservation price is at a maximum level before the firm cuts the sales price.

A few things are remarkable when observing the general pattern of the cycle. First, while all four phases occur on intervals of positive length, some are very short. As mentioned, intermediate regime *II* is not very sustainable for the firm, so the firm quickly moves to the riding up the demand curve phase (regime *III*). In addition, regime *IV* is not very profitable, but is necessary to maximize the subsequent attraction of new customers (Figs. 7.a and 7.b). The main phases are the riding down the demand curve phase (regime *I*) and the riding up the demand curve phase (regime *III*), in which the firm spends the most time. The latter is the most profitable in that, for a given price level, this phase results in higher sales at no advertising costs. As in the case of unique equilibrium, the negative relation between sales price and sales in the long run alternates with a positive relation between them. This means that the cycle is not a circle in the phase plane but is a flat object oriented from Northwest to Southeast (Figure 6.a).

Second, depending on the magnitude of WOM effectiveness, the cycle can be smaller or larger. Figure 6.b depicts a situation where WOM effectiveness is extremely low (α=0.000005) so that the cycle is very pronounced and the phases are significantly different than previously. That is, both phases of price advantage improvement and riding up the demand curve are now steeper, and result in a market penetration phase at the end of which the firm even gives away the product for free ($p = 0$, red part) before the subsequent riding up the demand curve phase to minimize the corresponding attrition of current customers due to an increasingly disadvantageous sales price. The transition from a riding up the demand curve phase to a

riding down the demand curve phase requires that the reservation price be increased as much as possible, while the transition from a market penetration phase to a riding up the demand curve phase compels the firm to increase sales as much as possible. Compared to the situation in Figure 6.a, now the firm can successively reach a high sales level and high level of sales price more easily and exploit them for a longer time.

6. Conclusions

The paper considers dynamic pricing and advertising policies in a sales model where WOM is prevalent in two forms: advertising-dependent WOM that depends on a firm's advertising effort, and autonomous WOM that develops beyond the firm's control. Both autonomous and advertising-dependent WOM affect the sales process proportionately with the price advantage, which is the difference between the sales price and a reservation price based on the current customer's willingness to pay to repurchase the product.

Two situations are possible. First, the current sales price may be low compared with the reservation price. Through autonomous and advertising-dependent WOM, potential customers become aware of the advantageous sales price with the implication that a number of them get attracted to the product and sales increase. Second, the current sales price may be high compared with the current customers' reservation price. In such a case, some current customers get dissatisfied because of the disadvantageous price and stop purchasing the product, causing sales to decline. Customers react slowly to changes in price in the sense that sales and price gradually converge to a level where they correspond to each other.

The main qualitative results point to a negative relationship between price advantage and advertising-dependent WOM in the sense that the firm should always (never) advertise when the sales price is advantageous (disadvantageous). There is also a negative relationship between advertising-dependent WOM and autonomous WOM because large enough autonomous WOM can gradually take over the sales increase process from the advertising effort. Finally, the solution patterns include limit cycles.

Using WOM effectiveness and current customers' abandon rate as bifurcation parameters, an extensive numerical analysis led to the conclusion that for different scenarios very different solution patterns can occur.

First, when WOM effectiveness is very low and current customers' abandon rate is very high, we obtain history dependent (Skiba) trajectories that converge to a steady state when initial sales are high enough while the initial price level is low enough. Otherwise, the firm ceases to exist in finite time. Skiba threshold with two equilibria implies that the firm can only stay in

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business if the reservation price lies sufficiently above the actual price. Otherwise, bankruptcy is unavoidable, which explains the existence of the two equilibria.

Second, if the abandon rate is much lower, a range of qualitatively different solutions is possible, all having in common that the firm remains active on this market. Depending on WOM effectiveness, history-dependent trajectories can result where the steady state is reached either straightforwardly or by an overshooting trajectory. When WOM is very effective, it is optimal for the firm to approach a unique equilibrium in a relatively straightforward way. WOM being effective implies that it is relatively easy for the firm to approach and then stay in a long term ideal situation where price and sales stay at their optimal level. This solution entails linear pricing policies for initial configurations with high (low) price and low (high) sales, and non-linear pricing policies for initial configurations with both high (low) price and sales. The case with both high price and sales shows that the commonly criticized practice of artificially inflating the current customers' reservation price to take advantage of price advantage improvement is part of an optimal pricing policy.

With less efficient WOM it is more difficult to reach such an ideal situation with optimal price and sales level. In one scenario, it can still be reached in a straightforward way provided the initial reservation price is a little larger than the actual price. Otherwise an overshooting pattern occurs where sales first exceed its equilibrium level, but then folds back to converge to this equilibrium.

Finally, if WOM effectiveness is very low and current customers' abandon rate is sufficiently low, the ideal situation is never reached, where instead sales and price level keep on cycling around the steady state. That is, an optimal trajectory can be a limit cycle, where an intertemporal price discrimination (or differential pricing) policy is applied continuously despite costly adjustment efforts. This differential pricing policy involves various phases, the most profitable of which is characterized by attrition of current customers due to price advantage decrease.

This paper is new in that it combines in a common framework autonomous WOM, advertising-dependent WOM, and optimal price setting with adjustment costs. Our results are relevant from the viewpoint of communication strategy because we show that WOM management needs to account for the sign and magnitude of a price advantage. Our results are also important from the perspective of revenue management, because we suggest how a dynamic pricing policy with costly adjustments can be shaped by WOM effectiveness. In spite of costly price adjustments, our framework allows for the emergence of complex pricing policies involving either Skiba thresholds, which is a novelty in regard of the existing

literature, or limit cycles. In other words, kinked demand and cyclical pricing are both compatible with price stickiness. While cyclical pricing policies are usually treated as a consequence of the counter-intuitive behavioral asymmetry involved by loss seeking consumers (Kopalle *et al.*, 1996; Popescu and Wu, 2007), we show that they simply result in our model from a plausible combination of very low WOM effectiveness and sufficiently low current customers' abandon rate. More generally, our results suggest that cyclical pricing cannot be driven by the asymmetric influence of WOM on the sales process alone.

Examples of monopolistic competition markets to which our framework applies are characterized by repeat-purchase consumption behaviors, e.g., hospitality industry, airline industry, car rental industry, etc. Different pricing contexts, such as fines for traffic offenses and telephone charges, fall outside the scope of our model.

We would expect our results to be modified by changes in product quality and competition. Such factors would notably affect the patterns of building and leveraging price advantage via the reservation price. Their inclusion would both extend the scope of our model and allow for a thorough characterization of building and leveraging brand attitudes (Hanssens *et al.*, 2014). Important extensions therefore include investment in product quality and considering frameworks with competition like duopolistic and oligopolistic markets. The latter extension would require the application of differential games. These items are high on our research agenda.

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