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**Research Report 2015-15**

November, 2015

**Operations Research and Control Systems**
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# Optimal language policy for the preservation of a minority language

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**Abstract**

We develop a dynamic language competition model with dynamic state intervention. Parents choose the language(s) to raise their children in based on the communicational value of each language as well as on their emotional attachment to the languages at hand. Languages are thus conceptualized as tools for communication as well as carriers of cultural identity. The model includes a high and a low status language, and children can be brought up as monolinguals or bilinguals. Through investment into language policies, the status of the minority language can be increased. The aim of the intervention is to obtain the minority language in a bilingual subpopulation at low costs. We investigate the dynamic structure of the optimally controlled system as well as the optimal policy, identify stable equilibria and provide numerical case studies.

**Keywords:** Language Competition; Language Dynamics; Intergenerational language transmission; Optimal control;

## 1 Introduction

In many of the states in this world, one can find two or more larger language groups, often in form of a majority language and one or several minority languages. By no means this is a static situation, since "[a]ll over the world, people are stopping speaking minority languages and shifting to languages of

*Part of the research of Bengt-Arne Wickström and Torsten Templin leading to this paper has received funding from the European Community's Seventh Framework Program under grant agreement No. 613344 (Project MIME).

†Andrea Seidl and Gustav Feichtinger were supported by the Austrian Science Fund (FWF) under Grant P25979-N25.

wider communication" (Sallabank, 2012, p. 104). This often results in the displacement of the minority languages by the majority language. To some extend such processes are inevitable and can be observed throughout human history. Nevertheless, in the modern world the decline of minority languages appears to occur much faster than ever before. It is predicted that 90 percent of the currently 7000 spoken languages will not survive the end of the century (Krauss, 1992).

## Language shift and maintenance

In response to this accelerated process of (minority) language decline, revitalizing and maintaining (endangered) minority languages is not only on the agenda of many of their speakers. Governments, non-governmental organizations as well as international organizations such as the European Union "are actively working to save and stabilize endangered languages" (Fernando *et al.*, 2010, p. 49). In scientific discourses a large variety of arguments to support (minority) language rights or to save endangered languages were put forward over the past decades. In this paper we will not assess such arguments in detail or develop new ones[1], but rather investigate the possibilities, effects and costs of language policies which aim at saving endangered languages in a formal model setting. To do so, we first have to clarify three questions: 1) what are the causes of language shift, and 2) which measures are available to reverse language shift. Here again, we will not go into all the details and mostly refer to the extensive literature on this topics, see e.g. Fishman (1991), Crystal (2000), Nettle & Romaine (2000) and May (2011). The third question is concerned with the target function: 3) what is the desired state of affairs that language policies should aim at?

Referring to Nettle & Romaine (2000) and Crystal (2000), Sallabank groups causes for language shift in four often overlapping main categories: a) natural catastrophes, famine, disease, b) war and genocide, c) overt repression and d) cultural/political/economic dominance, where the last one is the most common, cf. Sallabank (2012, pp.103f). Since we are interested in such cases, where individuals *voluntarily choose* to change to the majority language or not to pass the minority language to the next generation, we concentrate on the last category. Especially in nation states with one official/national language (which often but not necessarily is the language of the majority), this language is dominant in education, politics, media and public life. In modern democratic states the result is "that the majority culture [...] is endemic and omnipresent; and minority cultures, having very little, if any, public legitimization and private space, thereby constantly decline in survival potential, the more their members participate in the 'greater general good'" (Fishman, 1991, p. 63). Here, uneven power relations between the *national majority* and minorities play a major role, when minorities are not only underrepresented in politics and in the public, but, furthermore, are often socially disadvantaged, cf. May (2011). This, in turn, can yield negative attitudes towards the minority language, which are also internalized by its speakers (Sallabank, 2012, p. 104). When the two main aspects of language are considered - language as a tool for communication and language as a carrier of cultural identity - it is no surprise, that a language that can not be used in the majority of societal domains and that is furthermore stigmatized to some degree will not be learned, spoken or passed to the next generation.[2]

Language shift usually passes through three phases. In a first phase, called diglossia, formal language domains are dominated by the majority language which implies a loss of official and public functions of the minority language. This *forces* the speakers of the minority language to use the dominant one. In a second phase more and more speakers of the minority language become bilingual, while both languages are still used, at least in some domains. Especially among the younger generation one can observe a decreasing number of speakers and a decline of domains where the minority language can be or is used, cf. May (2011). The third phase finally is the replacement of the minority language: "For a generation

[1] For an overview over current discussions concerning language rights see e.g. May or Sallabank. See also Fishman 1991 for a popular work on reversing language shift

[2] "The communicative value of languages is largely determined by the number of speakers it gives access to and by the status or social positions of these speakers" (Spolsky, 2012, p. 127).

or two, some bilingual arrangements may be observed, but often [...] these prove to be way-stations on the road to a new monolingualism in the larger language" (Edwards, 2010, p. 6).

The process of language shift can be contested by language policies aiming at the survival of the minority language. Language planning can be divided into three categories: status planning, corpus planning and acquisition planning. All three can have a positive impact on the chances of survival of minority languages. Through status planning, e.g. giving some official status to the minority language, the prestige of the language can be increased for its speakers as well as for the other members of the society. Corpus planning, which aims at standardizing the orthography and grammar of a language, can increase its prestige and at the same time can reduce learning costs. Teaching the minority language at school, which belongs to the category of acquisition planning, enables students to learn the language properly or in the first place and can also have a positive impact on its status and identity value. In general, (re)introducing and/or strengthening the minority language in at least some domains can enhance the chances that the minority language stays vital.

Although Fishman and other authors underline, that a strong incentive from the minority language group itself is needed to reverse language shift, we concentrate on the role of the state in such a revival process. We presuppose that the state is basically interested in supporting the minority language or to warrant minority language rights.[3]

At the same time, we assume that the state aims at ensuring social cohesion by enabling wide communication possibilities. The existence of two linguistically segregated language groups can threaten the solidarity between the society members and hence social cohesion. Even without referring to a necessity of a shared national identity for solidarity and cohesion one can at least say that "a shared language contributes to democracy" (Spolsky, 2012, p. 135). Enabling wide communication possibilities while guarantying minority rights can be achieved through widespread bilingualism. If the minority language can be preserved in form of a relatively large number of bilingual individuals, the language minority is able to pass cultural values linked to the minority language to the next generations while communication possibilities throughout the society are assured. As mentioned above, bilingualism is usually a second step in the decline of a minority language. Thus, it is modeled that the preservation of a vital bilingual community requires a continuous effort by the state. We assume - and this is translated into the target function - that the state tries to maximize the number of bilingual speakers at minimal expenditures.

## Language competition models

In the past two decade a wide variety of language competition models were developed. One important point of departure for this new research on language competition was the work by Abrams & Strogatz (2003). There, a simple population model with two monolingual subpopulations is developed. The fraction of speakers of each language evolves according to a differential equation, which takes into account the size of the subpopulations and the prestige of both languages. Although the authors could fit their model to aggregated empirical data of endangered languages, it shows some weaknesses. In Abrams & Strogatz (2003) neither bilingual speakers nor the social structure of the population are considered. Moreover, it is predicted that always one of the two competing languages will extinct in the long run. Due to such limitations, the model was revised and extended by many authors, especially from the field of (statistical) physics. Patriarca & Leppänen (2004) introduced a spatial dependence and could show that both languages can survive. Mira & Paredes (2005), Minett & Wang (2008) and others extended the

[3] As mentioned above, there are many arguments supporting such policies:
"Indeed, the dynamics of ethnic tension involving language, leading in some cases to political conflict, occur most often *not* when language compromises are made or language right are recognized, but where they have been historically avoided, suppressed or ignored" (May, 2011, p. 161).
"So people's self-respect and dignity are often affected by the esteem their language gets from others or from the state. We might then justify different language policies by appealing to the importance of language recognition for individuals' dignity" (Spolsky, 2012, p. 136).

A-S model by additionally considering bilinguals. Stauffer *et al.* (2007) or Schulze *et al.* (2007) propose microscopic or individual based versions of the AS model and apply simulation techniques instead of averaging over the whole population. A good review of the different approaches is given in Patriarca *et al.* (2012).

In the model of Abrams and Stogatz (A-S model) speakers of two language A and B are assumed. Speakers of A can switch/convert/change to speakers of language B and vis versa, while the population size remains constant. Minett and Wang pointed out that "in practice, [...] typically a speaker does not suddenly give up one language completely in favor of an other" (Minett & Wang, 2008, p. 23). Therefore, they include bilingual speakers in their adoption of the A-S model. Furthermore, Abrams and Strogatz implicitly consider language transmission from one generation to the other when fitting their mathematical model to empirical data from more than a hundred years without theorizing this fact. Minett and Wang therefore consider two modes of language transmission: 1) vertical, i.e. transmission from parents to their children and 2) horizontal, i.e. (adults) learning the second language and becoming bilingual. For the vertical mode, a uniparental model of transmission is applied. In contrast, Wickström (2005) only considers vertical transmission, but explicitly models family formation. It is assumed that adults mate due to a random search and matching process with a success probability that is smaller for couples with an A-monolingual and a B-monolingual partner than for all the other possible couples. In the so formed families offspring is produced and socialized in one - or in some cases both - of the parents' languages, depending on the communicational value of each language and their status/prestige. As Wickström (2005) we only consider the vertical mode, i.e. intergenerational language transmission[4].

In Wickström (2014) it is illustrated that the A-S model and its extension by Minett & Wang (2008) can be reformulated in terms of the general model presented in Wickström (2005), adapting probabilities for family formation and probabilities for mono- or bilingual socialization in each family type. Furthermore the spatial model in Patriarca & Leppänen (2004) can be interpreted a version of the Wickström framework with two subpopulations I and II, which value language A differently. It is shown that under some general assumptions on the nexus between transition probabilities and the size of the subpopulations stable steady states of the system are the same as derived by Patriarca & Leppänen (2004) in spatial terms.

For this paper we build on the general model formulation presented in Wickström (2005) and Wickström (2014). Hence we consider speakers of the majority language $A$, speakers of the minority language $B$ and bilingual speakers $C$.

Most of the formal language dynamic models so far concentrated on the description of language competition depending on some external parameters. They did not deal with any kind of linguistic intervention. To maintain a bilingual equilibrium Minett & Wang (2008) suggested a rather technical kind of intervention: whenever the amount of speakers of the minority language drops below some threshold value, then the status of the minority language or some other model parameters have to be increased. Here, increasing the status of the minority language automatically implies decreasing the status of the majority language. That such a "dramatic intervention" (Fernando *et al.*, 2010, p. 51) is quite unrealistic, was already mentioned by the authors of Minett & Wang (2008) themselves. It can be seen as a theoretical approximation of a more sophisticated intervention, which starts to increase the minority language status when the numbers come close the threshold. A greater effort to model language planning was done in Fernando *et al.* (2010). They consider intergenerational transmission as well as horizontal transmission. It is taken into account to what extend languages are heard and used outside of the home and if they are tough at school. Adults randomly form families, while couples with monolinguals of different languages are excluded. In contrast to Wickström (2005) parents then do not choose one or two languages to socialize their children in. Instead, in Fernando *et al.* (2010) the probability that a child speaks a language $L$ depends on the amount of $L$-conversations it is exposed to. In the private domain this amount just depends on the family constellation. Hence, on the one hand, in this approach parental decisions on the

[4]Transmission in the family is the 'gold standard' of language vitality and the most important factor in language survival (Fishman, 1991, p. 113).

potential future of their children or their emotional attachments to their language do not play any roll. On the other hand, the model does not only focus on the family sphere. It furthermore considers the influence of the community by taking into account languages heard in the public sphere and languages taught at school. This is also reflected in the three different kind of interventions contemplated here: 1) increasing the status of the minority language, 2) increase the amount of the minority language heard in society and 3) formal language teaching. Unlike to most of the models mentioned above, in the model developed in Fernando *et al.* (2010) the status is included indirectly, namely in the parameters $\alpha_L$ and $\alpha_H$, where $H$ denotes the high-status language and $L$ the low-status language. "$\alpha_L$ measures the effectiveness of hearing language $L$ in motivating its learning (i.e. the receptiveness of the child to $L$)" in a $HH$ or $HB$ family (Fernando *et al.*, 2010, p. 60), where $B$ denotes bilinguals. Here one could ask why especially very young children should have a higher response to the high-status language only because it has a higher (social) status. Furthermore, in their simulations Fernando et al. illustrate the effect of different kind of governmental interventions. After 100 years the learning of the low-status language at home is encouraged. In the model this is done by increasing $\alpha_L$ from 1 to 1.5 at year 100. Citing Fernando et al. (Fernando *et al.* (2010)) when reviewing the paper by Minett and Wang one can state: "How such a dramatic intervention could be achieved is not explained" (Minett & Wang, 2008, p. 51).

It is this last critical comment that motivated us to try to combine existing language dynamic models with optimal control theory. To increase or even stabilize the status of a (minority) language a continuous effort over some amount of time is necessary, and such efforts will always raise costs. In the model proposed here governmental intervention is not just a switch of model parameters. Instead, state intervention is a continuous investment into language policies that aim at maintaining the minority language, and hence a process $(s_t)_{t\geq 0}$. We assume that there is a maximal amount of investment, i.e. a limited budget on language maintenance. Thus, we can normalize the investment such that $s_t \in [0,1]$. Throughout the paper $S \in [0,1]$ will denote the relative status of minority language $B$. Respectively, the relative status of $A$ is $1-S$. In Fernando *et al.* (2010) the authors criticize such an assumption in the model of Minett and Wang because it implies "that it is impossible to make one language more attractive without making the other less so" (Minett & Wang, 2008, p. 50). However, in a language competition situation, where individuals have to decide for one language, the other or both, this assumption makes sense when we think of relative attractiveness instead of absolute attractiveness. So instead of statements as 'language $A$ has an attractiveness value of 3.5' the model here only allows statements like 'language $A$ is three times as attractive as language $B$'. We assume that without any governmental intervention the relative status $S$ will tend to zero in the long run, but can be increased and stabilized due investments into proper language policies.

The dynamic control model proposed here is a three-state model. The three states are: the fraction of speakers of language $A$ (denoted by $p_A$), the fraction of speakers of language $B$ (denoted by $p_B$) and the relative status of language $B$ (denoted by $S$). The fraction of bilingual speakers is simply given by $p_C = 1 - p_A - p_B$. In contrast to Wickström (2005) we follow the definition in Fernando *et al.* (2010, p. 53) by assuming that "bilingualism [is] the ability to function confidently in two languages, that is, the ability to have communicative competence in two languages". Hence, we do not require bilinguals to be fully balanced speakers of two languages.

The evolution of the system is described by three differential equations. The evolution of the status can be controlled directly though state intervention $s$, i.e. $\dot{S} = g(s,S)$, where $g$ is some function increasing in $s$. How the fractions of speakers evolve depends on the current distribution of speakers as well as on the status $S$. Hence, these fractions can be influenced by state intervention, but only indirectly through the controlled status. The purpose of state intervention is achieve a maximal number of bilingual speakers with lowest possible costs.

This paper is organized as follows. In Section 2 the general language dynamic model is introduced. In Section 3 we suggest specific functional forms for the general model described in the previous section. Section 4 aims at identifying the optimal public investment strategy. Furthermore, some general statements on steady states of the optimally controlled system are derived. In Section 5 we consider some

| F | $\phi_F$ |
|---|---|
| $AA$ | $p_A^2 + p_A p_B$ |
| $AB$ | 0 |
| $AC$ | $2p_A p_C$ |
| $BB$ | $p_B^2 + p_A p_B$ |
| $BC$ | $2p_B p_C$ |
| $CC$ | $p_C^2$ |

Table 1: Distribution of families for a given distribution of adult speakers.

case studies to illustrate our results numerically. Section 6 provides conclusions and some remarks for future research.

# 2 Model

We consider a (large) population consisting of individuals equipped with one of three different language repertoires $L$: monolingual speakers of the dominant language $A$, monolingual speakers of the minority language $B$ and bilinguals speakers $C$. The relative sizes (fractions of the population) of the respective language repertoire groups are denoted by $p_A$, $p_B$ and $p_C$. The fractions add up to 1, hence $p_C = 1 - p_A - p_B$. The variable $S$ represents the relative status of the minority language $B$ in the society.

## 2.1 Family formation

In every generation individuals form families. There are six family types $F$: $AA$ (two $A$ monolinguals), $AB$, $AC$, $BB$, $BC$ and $CC$. Family formation is assumed to be random but restricted by the condition that both adults should share a common language, i.e. they should be able to communicate with each other. Hence, couples with an A-monoglot and a B-monoglot are precluded. Given any distribution of speakers $p_A, p_B, p_C$, the distribution of family types is given in Table 1, where $\phi_F$ denotes the fraction of $F$-type families.

## 2.2 Family behavior

Families bring up their children either as monolinguals in $A$ or $B$, or as bilinguals. The fraction of $F$-type families bringing up children with language repertoire $L$ is denoted by $\alpha_L(F;\cdot) \in [0,1]$. Naturally, the $\alpha$'s add up to one: for every family type $F$

$$\sum_L \alpha_L(F;\cdot) = 1.$$

The $\alpha-$functions are one main ingredient of the model proposed here. Parents choose a language repertoire depending on their own languages, on their emotional attachment to those languages as well as on the communication values of all the languages at hand. Therefore, the fraction of families of type $F$ raising their children as $L$'s varies with the current distribution of speakers in the society as well as with the statuses of languages $A$ and $B$. Hence, $\alpha_L(F;\cdot) = \alpha_L(F;p_A,p_B,S)$. The dependence on the variables $p_L$ captures the practical advantage of belonging to a certain language group, since they measure the frequency with which an individual encounters another individual in group A, B and C, respectively, and hence measure how many people one can communicate with. Following the individual utility maximization approach developed in Wickström (2005), we assume that $\alpha_A$ is non-decreasing in

$p_A$ and $p_C$, and non-increasing in $p_B$, and vice versa for $\alpha_B$:

$$\frac{\partial \alpha_A(F; p_A, p_B, S)}{\partial p_A}, \frac{\partial \alpha_B(F; p_A, p_B, S)}{\partial p_B} \geq 0$$
$$\frac{\partial \alpha_C(F; p_A, p_B, S)}{\partial p_A}, \frac{\partial \alpha_C(F; p_A, p_B, S)}{\partial p_B} \geq 0$$
$$\frac{\partial \alpha_A(F; p_A, p_B, S)}{\partial p_B}, \frac{\partial \alpha_B(F; p_A, p_B, S)}{\partial p_A} \leq 0$$

This reflects the first aspect of language mentioned in the introduction: language as a tool for communication. The second aspect - language as a carrier for cultural identity - is reflected in the dependence of the $\alpha'$s on the family type $F$ and the relative status of the minority language $S$. It is hypothesized that the emotional attachment in the family to a certain language, and hence the frequency of its transmission to the next generation, depends on its strength in the family. The stronger the position of a language $L$ in the family, the higher is the fraction $\alpha_L$:

$$1 \geq \alpha_A(AA; \cdot) \geq \alpha_A(AC; \cdot) \geq \alpha_A(CC; \cdot) \geq \alpha_A(BC; \cdot) \geq \alpha_A(BB; \cdot) \geq 0$$
$$0 \leq \alpha_B(AA; \cdot) \leq \alpha_B(AC; \cdot) \leq \alpha_B(CC; \cdot) \leq \alpha_B(BC; \cdot) \leq \alpha_B(BB; \cdot) \leq 1$$

It is furthermore assumed that both parents shall be able to communicate with their children, cf. Fernando *et al.* (2010). Hence,

$$\alpha_A(BC; \cdot) = \alpha_A(BB; \cdot) = 0$$
$$\alpha_B(AC; \cdot) = \alpha_B(AA; \cdot) = 0$$

The average emotional attachment to a language $L$ also depends on the general prestige or cultural status of the language in the society. The higher the status, the higher is the willingness of its speakers to pass their language to the next generation. We therefore assume that $\alpha_A$ is non-increasing in $S$, while $\alpha_B$ is non-decreasing in $S$:

$$\frac{\partial \alpha_A(F; p_A, p_B, S)}{\partial S} \leq 0$$
$$\frac{\partial \alpha_B(F; p_A, p_B, S)}{\partial S} \geq 0.$$

From the assumptions made above two properties of the $\alpha'$s can be concluded. Since $\alpha_B(AA) = \alpha_A(BB) = 0$ we get

$$\frac{\partial \alpha_A(AA; p_A, p_B, S)}{\partial p_A} = \frac{\partial \alpha_B(BB; p_A, p_B, S)}{\partial p_B} = 0.$$

Furthermore, $\alpha_B(AC; \cdot) = \alpha_A(BC; \cdot) = 0$ yield

$$\frac{\partial \alpha_A(AC; p_A, p_B, S)}{\partial p_A} = \frac{\partial \alpha_B(BC; p_A, p_B, S)}{\partial p_B} = 0.$$

## 2.3 Dynamics

Throughout the paper we make the simplifying assumptions that the size of the population is constant and that there are two children per family. Hence, the dynamics of the system is described by:

$$\dot{p} = \sum_F \alpha(F; p_A, p_B, S) \cdot \phi_F - p$$

For languages $A$ and $B$ this reads as

$$\dot{p_A} = (p_A^2 + p_A p_B)\alpha_A(AA) + 2p_A p_C \alpha_A(AC) + p_C^2 \alpha_A(CC) - p_A \tag{2.1}$$

$$\dot{p_B} = (p_B^2 + p_A p_B)\alpha_B(BB) + 2p_B p_C \alpha_B(BC) + p_C^2 \alpha_B(CC) - p_B, \tag{2.2}$$

where $\alpha_L(F) = \alpha_L(F; p_A, p_B, S)$.

If families with two monolingual speakers, that is AA or BB, will always socialize their children in their respective language, i.e. $\alpha_A(AA) = \alpha_B(BB) = 1$, then the dynamics simplify to

$$\dot{p_A} = p_C \left[2p_A \alpha_A(AC; p_A, p_B, S) + (1 - p_A - p_B)\alpha_A(CC; p_A, p_B, S) - p_A\right]$$
$$\dot{p_B} = p_C \left[2p_B \alpha_B(BC; p_A, p_B, S) + (1 - p_A - p_B)\alpha_B(CC; p_A, p_B, S) - p_B\right].$$

### 2.3.1 The status variable

The status of the minority language $B$ is expressed in the variable $S$, $0 \leq S \leq 1$. The status can be influenced by investments in language policies supporting the minority language $B$, denoted by $s$

$$\dot{S} = f(S, s) - \mu S \tag{2.3}$$

It is assumed that the function $f$ is non-increasing in $S$ and non-decreasing in $s$. Furthermore, for $s = 0$ the function $f$ should be zero. This implies, that without any state intervention the minority language B will die out at rate $\mu$.

## 2.4 The objective function

The aim of state intervention is a preferably large bilingual subpopulation. At the same time, state interventions to increase the status of the minority language are costly. Hence, the decision maker is looking for an investment policy $(s(t))_{t\geq 0}$, $s_t \in [0, 1]$, that yields a high level of individual bilingualism (benefit) at low costs. By $w(p_A(t), p_B(t), s(t))$ we denote the value of the system at time $t$, i.e. benefits minus costs at time $t$. We require $w$ to be increasing in $p_C = 1 - p_A - p_B$, non-increasing in $p_A$ and $p_B$, and decreasing in $s$. The total discounted value is given by

$$\int_0^\infty e^{-rt} w(p_A(t), p_B(t), s(t)) dt,$$

where $r \in (0, 1)$ denotes the discount rate. The problem of finding the best investment strategy for language maintenance can now be formulated as a maximization problem:

$$\max_{(s_t)_{t\geq 0}} \int_0^\infty e^{-rt} w(p_A(t), p_B(t), s(t)) dt.$$

Note, $S(t)$ and therefore $p_A(t)$ and $p_B(t)$ depend on the size of $s$ prior to time $t$, cf. (2.3), (2.1) and (2.2).

# 3 Specific functional forms

In view of the theoretical assumptions and properties stated in the previous section, we will now provide specifications of the $\alpha$-functions, of the dynamics of the status as well as of the objective function.

For parameters $0 \leq \eta < \beta < \delta$ and $\varepsilon + \gamma < \zeta < 1$ let

$$\begin{aligned}
\alpha_A(AA; p_A, p_B, S) &= 1 - \eta S p_B \\
\alpha_A(AC; p_A, p_B, S) &= \max\{0, \zeta(1-S) - \beta S p_B\} \\
\alpha_A(CC; p_A, p_B, S) &= \max\{0, \varepsilon(1-S) + \gamma(1-S)p_A - \delta S p_B\}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_B(BB; p_A, p_B, S) &= 1 - \eta(1-S)p_A \\
\alpha_B(BC; p_A, p_B, S) &= \max\{0, \zeta S - \beta(1-S)p_A\} \\
\alpha_B(CC; p_A, p_B, S) &= \max\{0, \varepsilon S + \gamma S p_B - \delta(1-S)p_A\}.
\end{aligned}$$

These constructions imply, that given a sufficiently high fraction of $A$ speakers in the society and a sufficiently low status of the minority language $B$, bilingual or even mixed couples ($BC$) will not raise their children as monolinguals in $B$, since in this scenario neither $B$ is a much useful communicational tool in this society nor the prestige of this language can really compensate the communicational disadvantage.

Throughout the paper we will assume $\eta$ to be zero. In this case the system dynamics simplify to

$$\dot{p_A} = p_C\left[2p_A\alpha_A(AC; p_A, p_B, S) + p_C\alpha_A(CC; p_A, p_B, S) - p_A\right] \tag{3.4}$$

$$\dot{p_B} = p_C\left[2p_B\alpha_B(BC; p_A, p_B, S) + p_C\alpha_B(CC; p_A, p_B, S) - p_B\right]. \tag{3.5}$$

### 3.1 Dynamics for fixed status

For the moment let $S$ be fixed. The essential dynamics of $p_A$ and $p_B$ can each be described by two parameters, cf. Wickström (2005). These parameters are introduced in the following. Let $p_B^{\Delta}(S)$ denote the fraction of $B$ speakers where $p_A = 0$ and $\dot{p}_A = 0$. Hence,

$$\alpha_A(CC; p_A, p_B, S) = 0 \Rightarrow \varepsilon(1-S) - \delta S p_B = 0 \Leftrightarrow p_B^{\Delta}(S) = \frac{\varepsilon}{\delta}\frac{1-S}{S}.$$

For $p_A^{\Delta}$ respectively we get

$$p_A^{\Delta}(S) = \frac{\varepsilon}{\delta}\frac{S}{1-S}.$$

Next we look for $p_A^*$ and $p_B^*$. $p_A^*$ is the fraction when $\dot{p}_A = 0$ given $p_B = 0$. Hence, $p_A^*$ is a solution to

$$0 = 2p_A\alpha_A(AC; p_A, p_B, S) + (1-p_A)\alpha_A(CC; p_A, p_B, S) - p_A,$$

or, with the above specifications, $p_A^*$ is the unique positive solution to the quadratic equation

$$0 = \gamma p_A^2 - \left[2\zeta + \gamma - \varepsilon - \frac{1}{1-S}\right] p_A - \varepsilon. \tag{3.6}$$

Note, $p_A^* < 1$ iff $S > 1/2\zeta$. From this, we easily conclude that $p_A^*$ is increasing in $\zeta, \varepsilon$ and $\gamma$, and decreases with an increase of $S$. On the other hand, $p_A^{\Delta}$ increases in $\varepsilon$ and $S$ and decreases with an increase in $\gamma$. It is unaffected by a change of $\zeta$.

From the relations between $p_A^{\Delta}$, $p_B^{\Delta}$ and $p_A^*$, $p_B^*$ we can identify possible bilingual equilibria for the fixed status $S$:

**Lemma 3.1.** *Let* $\eta = 0$.

*(a) If* $p_A^{\Delta} \leq p_A^* < 1$ *there exists a stable equilibrium with* $0 < p_A < 1$ *and* $p_B = 0$*; the fraction of A-speakers equals* $p_A^*$

*(b) If $p_B^\Delta \leq p_B^* < 1$ there exists a stable equilibrium with $0 < p_B < 1$ and $p_A = 0$; the fraction of B-speakers equals $p_B^*$*

*(c) If $1 \geq p_A^\Delta > p_A^*$ and $1 \geq p_B^\Delta > p_B^*$, we have a stable equilibrium with bilinguals and monolinguals in both languages ($p_A, p_B, p_C > 0$).*

**Lemma 3.2.** *Let $\eta = 0$. For monolingual stable equilibria the following statements hold true*

*(a) $p_A = 1$ is a stable equilibrium if and only if $S \leq 1 - 1/2\zeta$ .*

*(b) $p_B = 1$ is no stable equilibrium*

*(c) $p_A, p_B \in (0, 1)$ with $p_A + p_B = 1$ is stable iff*

$$p_A \alpha_A(AC; p_A, p_B, S) + p_B \alpha_B(BC; p_A, p_B, S) \geq \frac{1}{2}. \tag{3.7}$$

*A necessary condition for this last inequality is $S \leq 1 - 1/2\zeta$.*

Lemma 3.1 can be proven by picture, see Wickström (2005). The proof of Lemma 3.2 can be found in the Appendix.

### 3.2 Variable status and status control

Now we specify the dynamics of the minority language status $S$, which is increasing as a result of investments into language policies and decreasing due to a general negative trend. We assume the following functional form:

$$\dot{S} = f(S, s) - \mu S = \nu(1 - 2S)\sqrt{s} - \mu S, \tag{3.8}$$

where $\nu > 0$ is a model parameter correlated to the effectiveness of intervention. Here two assumptions are made: a) for a low status language the necessary effort to increase its status is low, while for a high status language it takes more effort. b) Language $B$ stays the minority language. This assumptions is expressed in the term $(1-2S)$. The status can not exceed 1/2, while the $(1-S)$, which can be interpreted as the status of $A$, does not fall below 1/2. $A$ can be thought as the first official language.

The control variable $s$ is bounded ($s \leq 1$). Thus, any steady state status $S$ ($\dot{S} = 0$) has an upper bound:

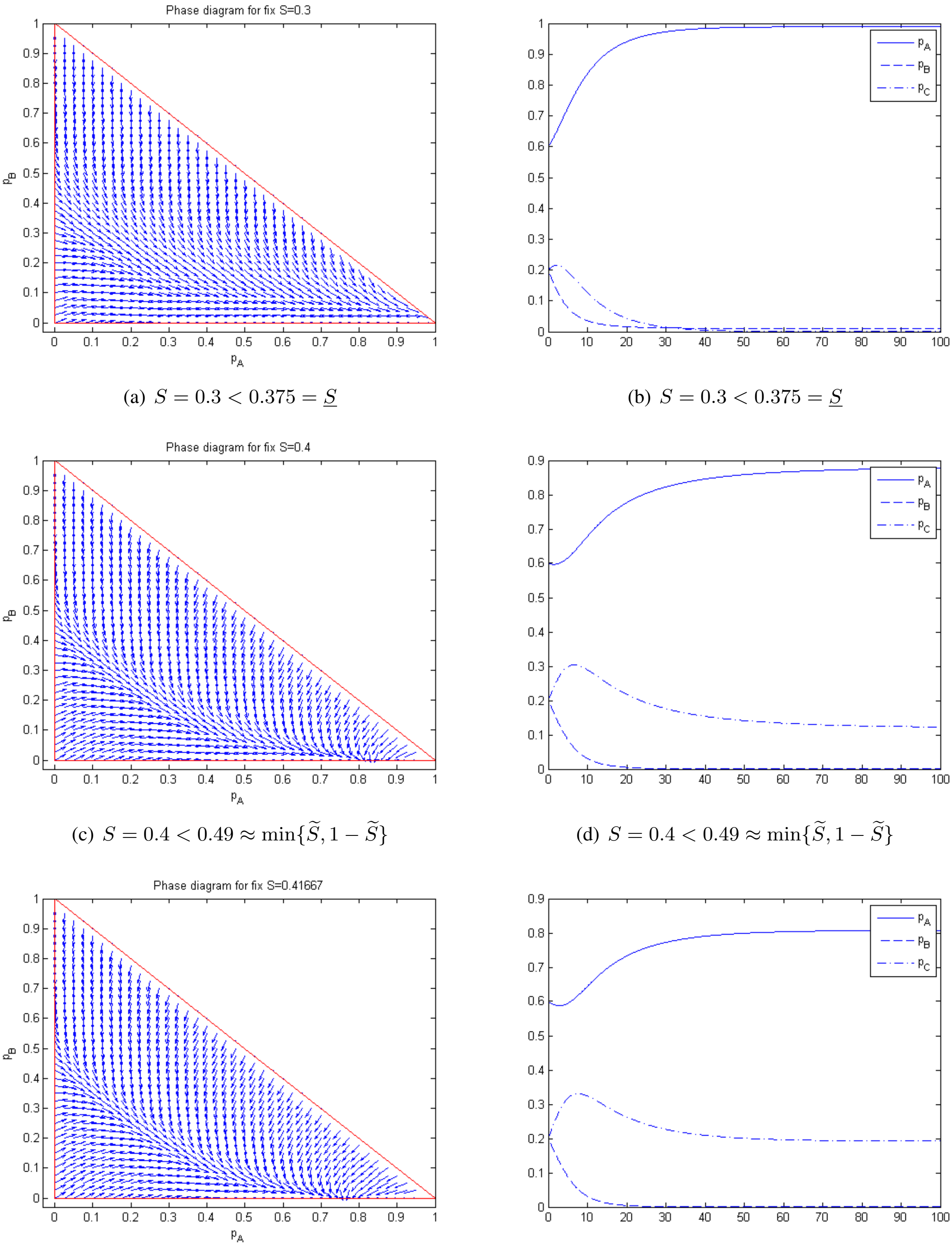
$$S \leq \frac{\nu}{2\nu + \mu}.$$

Since $p_A^*$ is decreasing in $S$, while $p_A^\Delta$ increases in $S$, Lemma 3.1 (a) yields a second upper bound for $S$, which is relevant for equilibria with $0 < p_A < 1$ and $p_B = 0$. A third one results from Lemma 3.1 (b), see below. A minimal value for this kind of equilibrium is given by $p_A^*(S) < 1$, where $p_A^*$ is the unique positive solution to (3.6).

We therefore introduce the following status thresholds

$$\begin{aligned}
\overline{S} &:= \frac{\nu}{2\nu + \mu} \\
\widetilde{S} &: p_A^*(\widetilde{S}) = p_A^\Delta(\widetilde{S}) \\
\underline{S} &:= 1 - \frac{1}{2\zeta}.
\end{aligned}$$

Note, that due to symmetry it holds $p_B^*(1 - \widetilde{S}) = p_B^\Delta(1 - \widetilde{S})$. Table 3.2 shows possible stable equilibria for the fixed status problem corresponding to these threshold values. Figure 1 illustrates some of the cases listed in 3.2.




(a) $S = 0.3 < 0.375 = \underline{S}$

(b) $S = 0.3 < 0.375 = \underline{S}$

(c) $S = 0.4 < 0.49 \approx \min\{\widetilde{S}, 1 - \widetilde{S}\}$

(d) $S = 0.4 < 0.49 \approx \min\{\widetilde{S}, 1 - \widetilde{S}\}$

(e) $S = \overline{S} \approx 0.42$

(f) $S = \overline{S} \approx 0.42$

Figure 1: Panals (a),(c) and (e) show phase diagrams for fixed $S$ for different values of $S$. Panals (b),(d) and (f) show trajectories for fixed $S$ for different values of $S$. For the trajectories the initial distribution is $p_A = 0.6$ and $p_B = 0.2$. Parameters are as in Example 5.1 in Section 5.

| $S \in$ | $[0\,,\,\underline{S}]$ | $(\underline{S}\,,\,\widetilde{S} \wedge 1-\widetilde{S})$ | $(\underline{S} \vee 1-\widetilde{S}\,,\,\widetilde{S}]$ | $(\underline{S} \vee \widetilde{S}\,,\,1-\widetilde{S})$ |
|---|---|---|---|---|
| steady state | A, AB | AC | AC, BC | ABC |

Table 2: Possible stable equilibria for the fixed status problem for different values of $S$. The first line contains intervals for $S$, while the second one shows the corresponding potential stable equilibria. "A, AB" means that a pure $A$-monolingual steady state as well as a steady state with monolingual speakers of $A$ and $B$ is possible.

To find optimal state intervention strategies we need to consider the derivatives of the function $f(S,s) = \nu(1-2S)\sqrt{s}$:

$$\frac{\partial f}{\partial s}(S,s) = \frac{\nu}{2}\frac{1-2S}{\sqrt{s}}, \tag{3.9}$$

$$\frac{\partial f}{\partial S}(S,s) = -2\nu\sqrt{s}. \tag{3.10}$$

### 3.3 Objective

Departing at the initial state $p_A(0), p_B(0)$ and $S(0)$ the aim of the optimization problem is to find the best investment policy $(s(t))_{t\geq 0}$ such that, $r \in (0,1)$, $k > 0$, $\xi \in [0,1]$,

$$\int_0^\infty e^{-rt}\left(k \cdot p_C(t) - [p_B(t) + p_C(t)]^\xi s(t)\right) dt \tag{3.11}$$

is maximized, while the system is developing according to (3.4), (3.5) and (3.8). For $\xi = 0$ the costs for the state intervention do not depend on the numbers of speakers of language B. Here one can think of adding language B to (street-)signs. For $\xi = 1$ the costs linearly increase with the number of speakers - one could think of bilingual education in schools.

## 4 Optimal control and optimal steady states

Substituting $p_B + p_C$ by $1 - p_A$ in the objective function, the Hamiltonian can be expressed as

$$H(p_A, p_B, S, s) = k \cdot p_C - (1-p_A)^\xi s + \lambda_A \dot{p}_A + \lambda_B \dot{p}_B + \lambda_S\left(f(S,s) - \mu S\right), \tag{4.12}$$

where $\lambda_A$, $\lambda_B$ and $\lambda_S$ are the costate variables measuring the marginal value of the corresponding state variables $p_A$, $p_B$ and $S$, respectively.

We assumed that the control variable is bounded, i.e. that the budget for language policies fostering bilingualism is limited. This budget constraint is formalized by the inequality $s \leq 1$. To include the constraint in the formal model we define the Lagrangian $L := H + \omega(1-s)$, where $\omega$ is the Lagrange multiplier. For the identification of the optimal intervention at a given state we consider the derivative of $L$ with respect to the control variable $s$:

$$L_s = -(1-p_A)^\xi + \lambda_S \frac{\partial f(S,s)}{\partial s} - \omega. \tag{4.13}$$

To identify optimal intervention, we are looking for $s$ and $\omega$ such that $L_s = 0$ and $\omega(1-s) = 0$. We have

$$L_s = 0 \;\Leftrightarrow\; (1-p_A)^\xi + \omega = \lambda_S \cdot \underbrace{\frac{\partial f(S,s)}{\partial s}}_{\geq 0} \quad\Rightarrow\quad \lambda_S \geq 0.$$

Note, if $p_A < 1$ then we even have $\lambda_S > 0$. For the explicit form of the function $f$ defined in (3.8) we get

$$L_s = 0 \Leftrightarrow (1-p_A)^\xi + \omega = \lambda_S \cdot \frac{\nu}{2}\frac{1-2S}{\sqrt{s^*}}$$

$$\Leftrightarrow s^* = \left(\lambda_S \frac{\nu}{2}\frac{1-2S}{(1-p_A)^\xi + \omega}\right)^2. \tag{4.14}$$

The second derivative of $L$ with respect to the control variable $s$ is non-positive if $\lambda_S > 0$ in which case the Legendre Clebsch condition is satisfied. Whenever $p_A = 1$ - in which case $\lambda_S = 0$ could be possible - $s = 0$ is obviously optimal. Applying the optimal control we have

$$\dot{S} \;=\; f(S, s^*) - \mu S \;=\; \lambda_S \frac{\nu^2}{2}\frac{(1-2S)^2}{(1-p_A)^\xi + \omega} - \mu S. \tag{4.15}$$

If the constraint is inactive, i.e. $s < 1$, then $\omega = 0$. If, in contrast, the constraint is active ($s = 1$), then

$$\omega = \lambda_S \frac{\nu}{2}(1-2S) - (1-p_A)^\xi \geq 0. \tag{4.16}$$

## 4.1 Stationary points

To state the co-state equations we first introduce some functions. For $L = A, B$ set

$$g_L(p_A, p_B, S) := 2p_L\alpha_L(LC; p_A, p_B, S) + p_C\alpha_L(CC; p_A, p_B, S) - p_L,$$

which equals $\dot{p}_L/p_C$ whenever $p_C > 0$. Then,

$$H = p_C(k + \lambda_A g_A + \lambda_B g_B) - (1-p_A)^\xi s + \lambda_S(f(S,s) - \mu S).$$

Using this notation we have

$$H_{p_A} = -(k + \lambda_A g_A + \lambda_B g_B) \;+\; \lambda_A p_C \frac{\partial g_A}{\partial p_A} \;+\; \lambda_B p_C \frac{\partial g_B}{\partial p_A} + \frac{\xi}{(1-p_A)^{1-\xi}} s, \tag{4.17}$$

$$H_{p_B} = -(k + \lambda_A g_A + \lambda_B g_B) \;+\; \lambda_A p_C \frac{\partial g_A}{\partial p_B} \;+\; \lambda_B p_C \frac{\partial g_B}{\partial p_B}, \tag{4.18}$$

$$H_S = p_C\left(\lambda_A \frac{\partial g_A}{\partial S} + \lambda_B \frac{\partial g_B}{\partial S}\right) + \lambda_S\left(\frac{\partial f(S,s)}{\partial S} - \mu\right). \tag{4.19}$$

The co-state equations are then given by

$$\dot{\lambda}_A = r\lambda_A - H_{p_A},$$
$$\dot{\lambda}_B = r\lambda_B - H_{p_B},$$
$$\dot{\lambda}_S = r\lambda_S - H_S.$$

To find inner stationary points we try to identify solutions $(\widehat{p}_A, \widehat{p}_B, \widehat{S}, \widehat{\lambda}_A, \widehat{\lambda}_B, \widehat{\lambda}_S)$ to

$$0 = \dot{p}_A = \dot{p}_B = \dot{S} = \dot{\lambda}_A = \dot{\lambda}_B = \dot{\lambda}_S.$$

For $p_A$ and $p_B$ to be stationary we need either $\widehat{p}_C = 0$ or $g_A(\widehat{p}_A, \widehat{p}_B, \widehat{S}) = g_B(\widehat{p}_A, \widehat{p}_B, \widehat{S}) = 0$.

Note, any steady state status $0 < \widehat{S} < \overline{S}$ corresponds to a steady state control variable $0 < \widehat{s}^* < 1$ and to hence to some $\widehat{\omega} = 0$. In this case, the stationarity of the status ($\dot{S}(\widehat{S}, \widehat{\lambda}_S) = 0$) yields an explicit relation between $\widehat{S}$ and $\widehat{\lambda}_S$, cf. (4.15):

$$\widehat{\lambda}_S = \frac{2\mu}{\nu^2}\frac{\widehat{S}}{(1-2\widehat{S})^2}(1-\widehat{p}_A)^\xi. \tag{4.20}$$

Plugging this into (4.14) we get for the stationary optimal intervention

$$\widehat{s}^* = \left(\frac{\mu}{\nu}\frac{\widehat{S}}{1-2\widehat{S}}\right)^2 < 1. \tag{4.21}$$

If $\widehat{S} = \overline{S}$, then $\widehat{s}$ has to be equal to one and thus $\widehat{\lambda}_S \geq 2\frac{2\nu+\mu}{\nu\mu}(1-p_A^*(\overline{S}))^\xi$ has to hold true, cf. (4.16).

Using the explicit expression for the function $f$ introduced in Section 3, the equation $\dot{\lambda}_S = 0$ yields

$$0 = -\,\widehat{p}_C\left(\widehat{\lambda}_A\frac{\partial g_A}{\partial S} + \widehat{\lambda}_B\frac{\partial g_B}{\partial S}\right) + \widehat{\lambda}_S\left(r+\mu+2\nu\left(\left[\widehat{\lambda}_S\cdot\frac{\nu}{2}\frac{1-2\widehat{S}}{(1-\widehat{p}_A)^\xi}\right]\wedge 1\right)\right). \tag{4.22}$$

### 4.1.1 Monolingual stationary points

First we want to consider stationary points with $\widehat{p}_C = 0$. Obviously, if $p_C = 0$, then the linguistic composition wont change anymore, since families of type $AB$ are impossible, while no bilinguals, which function as a kind of language transmitters, are part of the population. In the steady state all families are of types $AA$ and $BB$ and children of such families are raised monolingual in the respective language. Hence, both monolingual language groups reproduce themselves independent of the statuses of both languages. Thus, the state does not invest any money to support the status minority language, which would produce costs without having any positive effect, i.e. $\widehat{S} = \widehat{s}^* = 0$.

### 4.1.2 Bilingual stationary points

Now we want to consider stationary points with a bilingual sub-population, i.e. $p_C > 0$. Using the notation introduced above this yields that whenever $\widehat{p}_L > 0$ the stationarity implies $g_L(\widehat{p}, \widehat{S}) = 0$.

### 4.1.3 Bilingual stationary points with $p_B = 0$

The most interesting case is when monolingualism in the minority language $B$ vanishes and only monolinguals in $A$ and bilinguals remain. Such a state is desirable, since all society members are able to communicate with each other, while speakers of $B$ can still preserve their cultural identity. If $p_B = 0$ we need $\dot{p}_B \leq 0$. This is equivalent to $\widehat{p}_A(S) \geq \widehat{p}_A^\Delta$.

Let

$$\underline{S} < S \leq \min\{\overline{S}, \widetilde{S}\}$$

and $p_A = p_A^*(S)$. The co-state equation $\dot{\lambda}_A = r\lambda_A - H_{p_A} = 0$ is independent of $\lambda_B$, since $g_B = 0$ and $\partial g_B/\partial p_A = 0$, see 7.1. Hence, we can derive $\lambda_A(S) = \lambda_A(p_A, S)$. Given this $\lambda_A$ we can choose some $\lambda_B$ such that $\dot{\lambda}_B = 0$. In 7.1 it is also shown that $\partial g_B/\partial S = 0$.

To identify optimal steady states we have to distinguish two possibilities. First we can check if there is a steady state at $\overline{S}$. To do so, it has to be investigated if there exists a $\widehat{\lambda}_S > 2\frac{2\nu+\mu}{\nu\mu}(1-p_A^*(\overline{S}))^\xi$ which solves

$$0 = \dot{\lambda}_S(\widehat{\lambda}_S) = \dot{\lambda}_S(\overline{S}, p_A(\overline{S}), \lambda_A(\overline{S}), \widehat{\lambda}_S).$$

The second case covers $\underline{S} < S < \overline{S}$. Here, let $\lambda_S(S)$ be defined by (4.20). In this case steady states can be found by identifying statuses $S$ which solve

$$0 = \dot{\lambda}_S(S) = \dot{\lambda}_S(S, p_A(S), \lambda_A(S), \lambda_S(S)).$$

Depending on the parameter constellation and especially depending on $k, \nu$ and $\mu$ such a solution exists. If $k$ is too small, then no such solution exists, that means it is not profitable to maintain the minority language $B$.

**Lemma 4.1.** *For $k$ sufficiently large there exists at least one solution $\widehat{S}^* \in (\underline{S}, \min\{\overline{S}, \widetilde{S}\}]$ such that*

$$0 = \dot{\lambda}_S(\widehat{S}^*) = \dot{\lambda}_S(\widehat{S}^*, p_A(\widehat{S}^*), \lambda_A(\widehat{S}^*), \lambda_S),$$

*where $\lambda_S = \lambda_S(\widehat{S}^*)$ if $\widehat{S}^* < \overline{S}$, and $\lambda_S > 2\frac{2\nu+\mu}{\nu\mu}(1 - p^*_A(\overline{S}))^\xi$ if $\widehat{S}^* = \overline{S}$.*

For a proof see the Appendix.

#### 4.1.4 Bilingual stationary points with $p_B > 0$

For an optimal steady state with $p_A, p_B, p_C > 0$ we need

$$\underline{S} \vee \widetilde{S} < \widehat{S} \leq (1 - \widetilde{S}) \wedge \overline{S}.$$

This is only possible if $\widetilde{S} < \overline{S} < 1/2$, which does not hold true for all parameter constellations, cf. Example 5.1.

For fix $S$ we need the following for any steady state: $\alpha_A(AC), \alpha_B(BC), \alpha_B(CC) > 0$. The last inequality is due to $S < 1/2$ and $\zeta < 1$. If $\alpha_A(CC) = 0$, then $\zeta(1 - S) > 1/2$ has to hold true, else $\alpha_A(CC) > 0$.

As before, for suitable $S$ (here $\max\{\underline{S}, \widetilde{S}\} < S < \overline{S}$), we have can find $p_A(S)$ and $p_B(S)$ such that $\dot{p}_A = \dot{p}_B = \dot{S} = 0$. For some parameter constellations there can be more than one stable solution $p_A(S)$ and $p_B(S)$ such that $\dot{p}_A = \dot{p}_B = 0$. Furthermore we get a unique $\lambda_S(S)$. The co-state equations yield a linear system in $\lambda_A, \lambda_B$ with 3 equations and coefficients depending on $S$. To identify the optimal status, one has to check if this linear system has a solution for some suitable $S$.This also holds true at the left boundary. At the right boundary one has to check if the linear system in $\lambda_A$, $\lambda_B$ and $\lambda_S$ has a solution with a sufficiently large $\lambda_S$, see above.

## 5 Numerical calculations

In this section we numerically investigate the linguistic behavior of the population under the optimal policy. We show the existence of different stable and optimal steady states. Moreover, we illustrate the dependence of the selected steady state on the initial distribution of speakers as well as on how much bilingualism is valued with respect to expenditures by the decision maker (parameter $k$). To analyze the evolution towards the steady states we plot exemplary trajectories.

Two examples are considered. For both of them we set $\eta = 0$. In Example 5.1 we choose $\mu$, the rate of decline of the minority language status $S$, to be $0.2$, which is relatively high. In contrast, Example 5.2 depicts a case where the status of the minority language declines rather slowly over time ($\mu = 0.01$). Furthermore, the parameter $\zeta$, which measures the aggregated weight that is put on the status in the decision of $LC$ families, $L = A, B$, to socialize their children as monolinguals in $L$, is slightly higher in Example 5.1. In both example we chose the discount rate $r$ to be $0.5$.

**Example 5.1.** $\beta = 0.4$, $\delta = 0.7$; $\gamma = 0.1$, $\varepsilon = 0.4$, $\zeta = 0.8$; $\nu = 0.5$, $\mu = 0.2$ *and* $\xi = 0$

**Example 5.2.** $\beta = 0.4$, $\delta = 0.7$; $\gamma = 0.1$, $\varepsilon = 0.4$, $\zeta = 0.7$; $\nu = 0.5$, $\mu = 0.01$ *and* $\xi = 0$

First we calculate the $S$- thresholds, cf. Subsection 3.2. In Example 5.1 we have $\underline{S} = 0.375$, $\overline{S} = 0.417$ and $\widetilde{S} = 0.492$, while in Example 5.2, $\underline{S} = 0.286$, $\overline{S} = 0.495$ and $\widetilde{S} = 0.463$. According to these numbers and the statements made in Subsection 3.2, stable equilibria with $p_A, p_C > 0$ and $p_B = 0$ are possible for both examples. In Example 5.2 furthermore equilibria with $p_A, p_C > 0$ and $p_B > 0$ are possible, since $\widetilde{S} < \overline{S}$. This is not the case for Example 5.1, since there $\overline{S} < \widetilde{S}$. The actual stable

| | $k$ | $\xi$ | $\widehat{S}$ | $\widehat{s}^*$ | $\widehat{p}_A$ | $\widehat{p}_B$ | $k\widehat{p}_C - \widehat{s}^*$ |
|---|---|---|---|---|---|---|---|
| Example 5.1 | 60 | 0 | - | - | - | - | - |
| | 75 | 0 | 0.41 | 0.74 | 0.85 | 0 | 10.3 |
| | | 1 | $\overline{S} \approx 4.2$ | 1 | 0.81 | 0 | 13.4 |
| | 90 | 0 | $\overline{S} \approx 4.2$ | 1 | 0.81 | 0 | 16.3 |
| Example 5.2 | 20 | 0 | 0.47 | 0.03 | 0.44 | 0.03 | 10.6 |

Table 3: This table contains stable bilingual steady states - if such exist - for Examples 5.1 and 5.2 for different values of $k$. The steady state values of the status $\widehat{S}$, the optimal control $\widehat{s}^*$, the fraction of speakers $\widehat{p}_A$ and $\widehat{p}_B$ as well as the steady state objective $k\widehat{p}_C - \widehat{s}^*$ are listed. Here $r = 0.5$ and $\xi = 0$.

bilingual equilibria are displayed in Table 3. For Example 5.1 we investigate the influence of different values of $k$, namely $k = 60$, $k = 75$ and $k = 90$. For Example 5.2 we concentrate on the case of $k = 20$. For any parameter constellation there also is a manifold of steady states at $(\widehat{p}_A, \widehat{p}_B, \widehat{S}) = (\widehat{p}_A, 1-\widehat{p}_A, 0)$, where $\widehat{p}_A$ can take any value between zero and one. In these steady states it is optimal to have $\widehat{s} = 0$. Note, however, that not every point on this manifold is a candidate for the optimal long run solution due to its stability properties, cf. Lemma 3.2. Next, we analyze the two examples in greater detail.

**Example 5.1,** $k = 60$

If $k$ is small the decision maker does not have a particularly high incentive to support the status of the minority language B in the long run. As can be seen in the first row of Table 3 there is no bilingual steady steady. The following happens. Let us consider a situation where the fraction of $A$ speakers, $p_A$, is relatively high, while $p_B$ and $p_C$ and the status variable $S$ are small. Because of the dominance of $A$ speakers, most families are of type $AA$. Thus, $p_A$ increases. Initially $p_C$ decreases due to the low status of $B$ and the low chances of $A$ speakers of meeting a bilingual partner. This development is challenged by the decision maker who invests much into raising the status of $B$. Under such a policy the incentive to raise their children bilingual increases for $AC$ and $CC$ couples. This yields an increase in the number of bilinguals. An other effect of is that $BC$ couples have a stronger incentive to raise their children as $B$-monoglots. However, since the fraction of $B$ and $C$ speakers is small, the policy does not have a strong effect on the overall development of the language and over all $p_B$ decreases even further. As a result, it soon does not pay of anymore to invest into the status of the language as these measures affect less and less people. Thus, the status of $B$ decreases again. Consequently, the incentive to raise the kids bilingual and therefore the fraction of bilinguals decreases as well. In the long-run the majority of the population only speaks $A$ and bilingual speakers disappear completely in the long run. This behavior is illustrated in Figures 2 and 3.

**Example 5.1,** $k = 75$**,** $\xi = 0$

Table 3 shows that for $k = 75$ there exists a steady state with 15% bilinguals and no monolingual speakers of the minority language $B$. To obtain this fraction of bilingual speakers in the long run, 75% of the budget has to be used. If this bilingual steady is reached or not depends on the initial state values. For the initial states considered in Figures 4 and 5 the system converges to that steady states. If the initial $p_B$, $p_C$ and $S$ would be even smaller than in Figure 4, the system is likely to converge to a steady state with almost only $A$-monolingual speakers, few $B$-monoglots and no bilinguals.

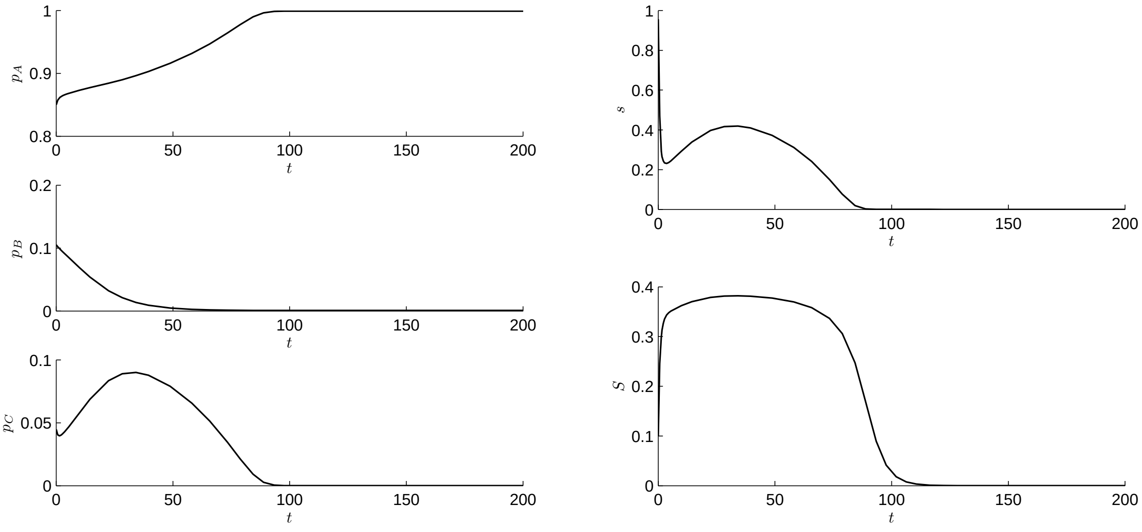



Figure 2: Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.1, $k = 60$).

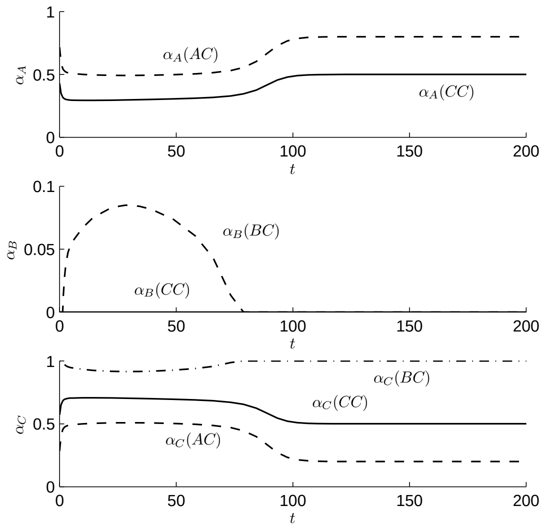



Figure 3: $\alpha$-functions for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.1, $k = 60$).

For the base case ($p_A = 0.85$, $p_B = 0.05$, $S = 0.1$), see Figure 4 and the left panel of Figure 6, the fraction of the bilingual population first decreases, since the status of language $B$ is low, as are the fractions of $B$ and $C$ speakers, so the majority of couples consists of $A$ speakers. Due to the dominance of $AA$ couples and the high likelihood that $AC$ and $CC$ couples raise their children as $A$-monoglots, $p_A$ first increases. Initially one would invest as much as possible into the status to increase it. As a first result of this policy $BC$ couples get a stronger incentive to raise their children as $B$-monoglots. Furthermore, $AC$ and $CC$ couples become less likely to raise their kids just as speakers of language $A$ and instead are more likely to raise the children bilingual than before. Consequently, $p_A$ now decreases while $p_C$ increases see Figures 4. Hence, the negative term in $\alpha_B(BC)$ decreases and even more $BC$ families raise their children as $B$'s. This is a problem as long as $p_B$, which is continuously decreasing, is still to big. To avoid this effect, the increase of $S$ is slowed down for a while, until $p_B$ is small enough

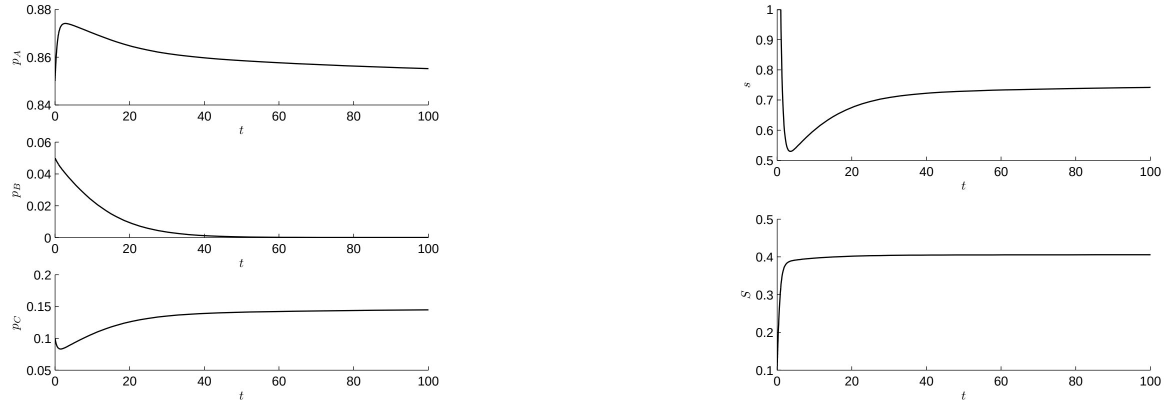


Figure 4: Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Ex. 5.1, $k = 75$, $\xi = 0$).

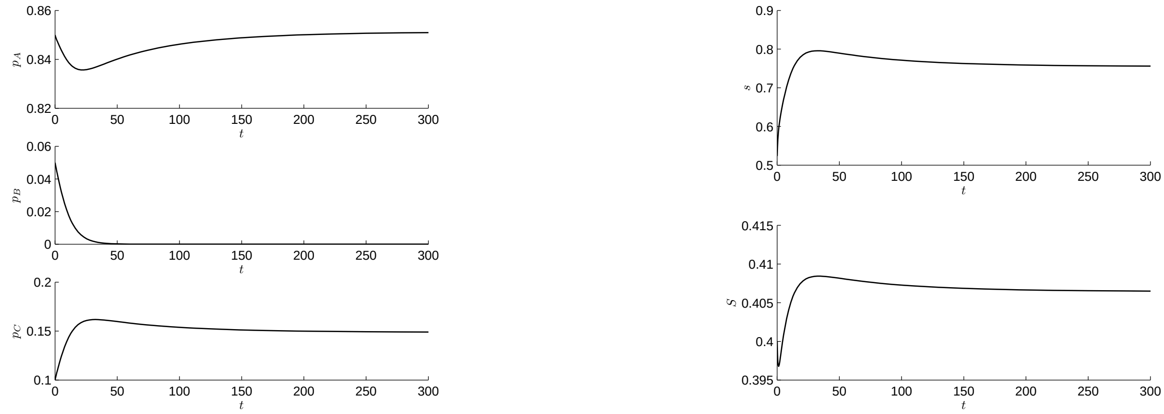


Figure 5: Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.4$ (Ex. 5.1, $k = 75$, $\xi = 0$).

and then increased again to obtain the steady state status.

If, in contrast to the base case, the initial status is high, see Figure 5 and the right panel of Figure 6, then initially the state does not have to invest as much into increasing the status of the minority language. Due to the high status of $B$, many $AC$ couples will raise their kids bilingual. As a result, at the beginning $p_A$ decreases while $p_C$ increases. Furthermore, the fraction of language $B$ speakers is so low that $BB$ and $BC$ couples are rather unlikely and $p_B$ decreases. To further support the growth of $p_C$ it is optimal to increase $s$ for some time. Due to the smaller fraction of $B$ speakers, $AA$ and $AC$ couples are more likely than $BC$ or $CC$ couples, thus, $p_A$ recovers after some time and even grows. At some point of time the status $S$ and the fraction of bilingual speakers $p_C$ is high enough while $p_B$ is very low, such that $s$ can be lowered again until it reaches its steady state.

**Example 5.1,** $k = 75$**,** $\xi = 1$

For $\xi = 1$ the costs for state intervention increase with the number of speakers of $B$, i.e. $B$-monoglots as well as bilinguals. Thus, the higher $p_A$, the lower are the costs for state intervention. In Figure 7 we can see that for the base case, the system behaves quite similar to the case of $\xi = 0$. The major difference is that state intervention is not just maximal in the beginning, but the entire budget is used over the entire time horizon. Due to the large amount of $A$-monolinguals the intervention is much cheaper compared to

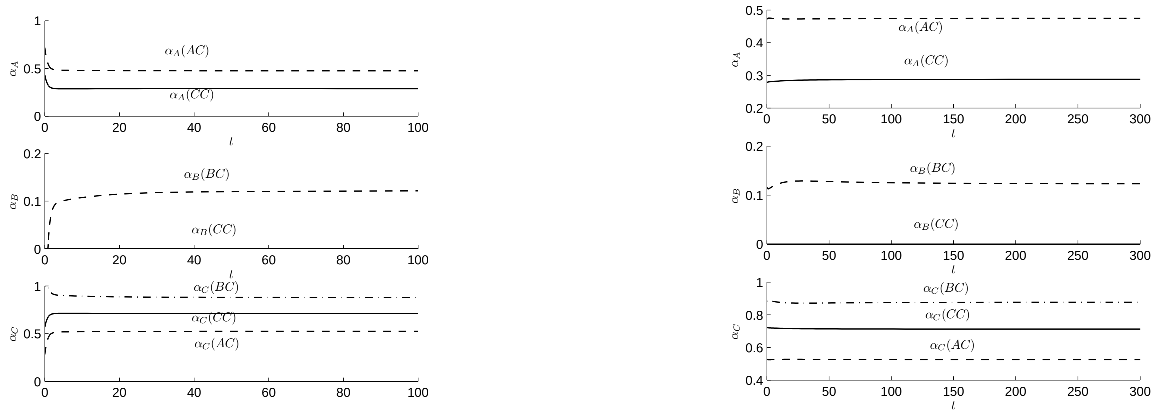



Figure 6: $\alpha$-functions. In the both panels $p_A(0) = 0.85$ and $p_B(0) = 0.05$. In the left panel $S(0) = 0.1$, while in the right one $S(0) = 0.4$ (Ex. 5.1, $k = 75$, $\xi = 0$).

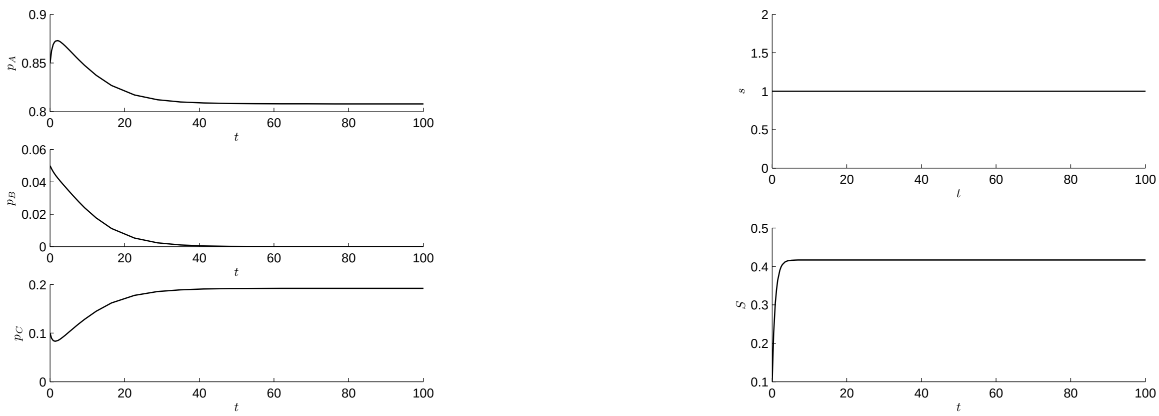


Figure 7: Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.4$ (Ex. 5.1, $k = 75$, $\xi = 1$).

the case where $\xi = 0$ (more than 80% cheaper). Therefore, in the long run the status and $p_C$ are higher while the $p_A$ is smaller, cf. Table 3.

**Example 5.1,** $k = 90$

If $k$ is large, then it is optimal to approach a steady state where the state invests the entire budget to reach the maximal possible status for minority language $B$, see Table 3. This yields a maximal amount of bilingual speakers while no $B$-monolinguals remain within the population. For the base case, see Figure 8, initially the state spends as much as possible for improving the status of $B$. For similar reasons as before, $p_A$ first increases while $p_B$ and $p_C$ first decrease. This changes after some time. Once $p_B$ has become small enough, the state can afford to decrease efforts. However, to ensure a growth in the number of bilingual speakers, it is necessary to increase expenditures after some time again. This is the main difference to the case with a low k; where one would first decrease, then increase, and then decrease the expenditures $s$. I.e. the later increase is apparently necessary reach a steady state with a proper bilingual population.

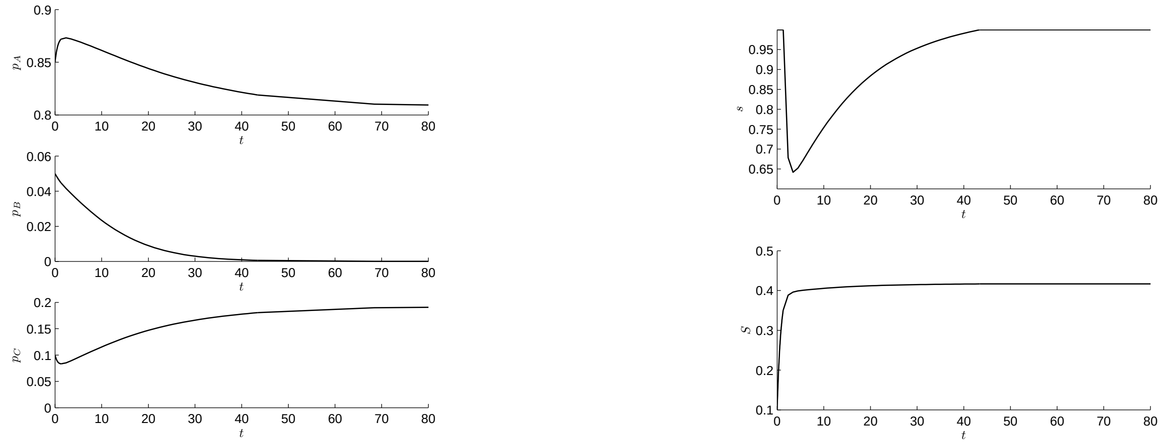



Figure 8: Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.1, $k = 90$).

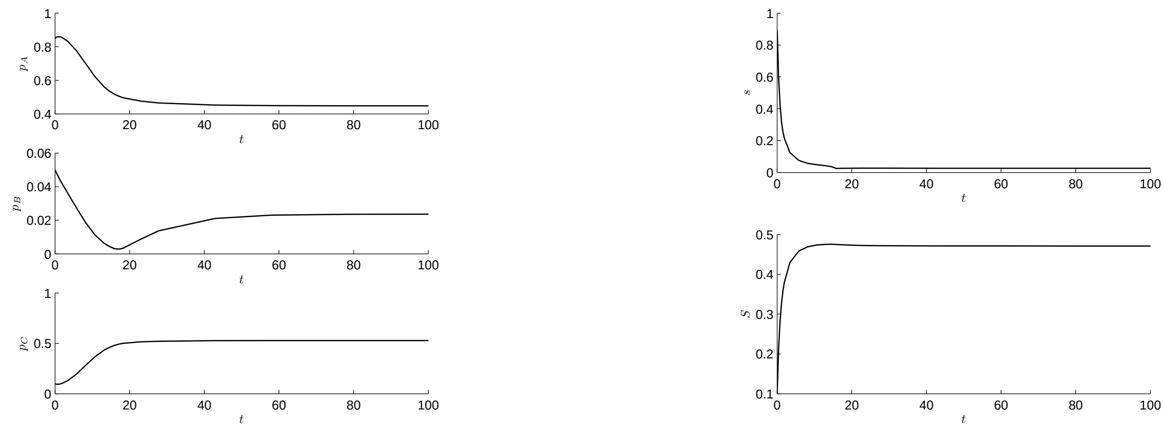



Figure 9: Time path for initial state $p_A(0) = 0.85$, $p_B(0) = 0.05$, $S(0) = 0.1$ (Example 5.2, $k = 20$).

**Example 5.2,** $k = 20$

Table 3 shows that in the bilingual steady state for the parameter constellation considered in Example 5.2 all three linguistic repertoires remain intact in the long run. This is the major difference to Example 5.1 and is mainly due to the much lower value of $\mu$ ($\mu = 0.2$ in Example 5.1 and $\mu = 0.01$ in Example 5.2). Here with the low $\mu$ it is much less costly to keep the status at a high level. The development of the population groups is similar to before, however, $p_B$ only decreases for a certain time, then the status of language $B$ is so high that even $CC$ couples have a small incentive to teach their kids only language $B$. Due to the small depreciation of $S$ it is not necessary to spend much for keeping the status high, so one would only invest much into the status in the beginning to get it to a high level and then decrease control efforts over time. Example 5.2 with $k = 20$ is visualized in Figure 9. Note, in the long run only 3% of the budget is used to guaranty that more than half of the population is bilingual.

## 6 Conclusions

The state aims at ensuring wide communication possibilities, while recognizing and supporting - if this is not *too costly* - minority language rights. This trade-off between a commonly spoken language and the preservation of a minority language is approached through bilingualism. To investigate language policies can be used to preserve a minority language in a bilingual subpopulation we developed an abstract language dynamics model. The point of departure is individual utility maximization, while here only intergenerational language transmission is considered. Families decide to bring up their children either as monolinguals in the majority or the minority language, or as bilinguals. This decision is based on how they value the communicational value of each language and their emotional attachment to the languages at hand. Through a continuous investment into language policies the state can increase the status of the minority language and thereby foster bilingual parenting in families with one or two bilingual parents. It is assumed that the state wants to maximize the number of bilingual speakers at minimal costs.

In Wickström (2005) it was already proven that for a constant status and proper parameter constellations stable bilingual steady states are possible. Here we could furthermore show that such bilingual steady states can even be optimal when costs for language policies are taken into account. It was illustrated that for some cases there are steady states only with monolingual speakers of the majority language and bilinguals but without any monolingual speakers of the minority language. In such a state all individuals within the population can - in principle - communicate with each other while the minority can preserve its language. For other cases we could see that small subpopulation with monolingual speakers of the minority language survives in the long run optimal state. As one would expect, bilingual steady states are only optimal, if bilingualism is valued high enough with respect to expenditures.

Whether or not a bilingual steady states is not only possible but really targeted by the decision maker, depends on the initial distribution of speakers as well as the initial status of the minority language. If both the status and number of speakers of the minority language are too low, then it is not worthwhile to invest in language maintenance in the long run, which results in a purely monolingual population. In most of the examples considered in the numerical analysis, the initial values were high enough and it was illustrated how expenditures change over time to achieve an optimal bilingual steady state in the long run.

For future research the current model will be extended. To get closer to the real-world complexity of language acquisition and transmission within a large population, we will add to the model language learning in formal education as well as adult language learning. Furthermore, language policies will be investigated in greater detail. We also intend to adjust the model to cases of *new* minorities, that means minorities which are based on temporary or permanent migration.

# 7 Appendix

## 7.1 Partial derivatives of $g_A$ and $g_B$ when $p_B = 0$

Given the definition of $g_A$ and $g_B$, their partial derivatives are given by

$$\begin{aligned}
\frac{\partial g_A}{\partial p_A} &= (1-S)\left[2\zeta - (\varepsilon + \gamma p_A) + (1-p_A)\gamma\right] - 1 \\
\frac{\partial g_B}{\partial p_A} &= -\alpha_B(CC) - \delta(1-S)(1-p_A)1_{\{\alpha_B(CC)>0\}} \\
\frac{\partial g_A}{\partial p_B} &= -\left[S(2\beta p_A + \delta p_C) + (1-S)(\varepsilon + \gamma p_A)\right] \\
\frac{\partial g_B}{\partial p_B} &= 2\alpha_B(BC) - \alpha_B(CC) + \gamma S(1-p_A)1_{\{\alpha_B(CC)>0\}} - 1 \\
\frac{\partial g_A}{\partial S} &= -2\zeta p_A - (\varepsilon + \gamma p_A)(1-p_A) \\
\frac{\partial g_B}{\partial S} &= (1-p_A)(\varepsilon + \delta p_A)1_{\{\alpha_B(CC>0)\}}.
\end{aligned}$$

Note, if $p_A \geq p_A^\Delta$, then $\alpha_B(CC) = 0$ and hence $\partial g_B/\partial p_A = \partial g_B/\partial S = 0$.

## 7.2 Proof of Lemma 3.2

Since $\eta = 0$, every constellation with $p_A + p_B = 1$, which implies $p_C = 0$, is a steady state ($\dot{p}_A = \dot{p}_B = 0$). In the following we investigate their stability. Let $f_{LL}$ denote the matrix

$$f_{LL} = \begin{pmatrix} \frac{\partial \dot{p}_A}{\partial p_A} & \frac{\partial \dot{p}_A}{\partial p_B} \\ \\ \frac{\partial \dot{p}_B}{\partial p_A} & \frac{\partial \dot{p}_B}{\partial p_B} \end{pmatrix}$$

and define $a := p_A(1 - 2\alpha_A(AC))$ and $b := p_B(1 - 2\alpha_B(BC))$. For $p_C = 0$ the matrix $f_{LL}$ equals $\left(\begin{smallmatrix} a & a \\ b & b \end{smallmatrix}\right)$ and has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a + b$. If $p_A = 1$ and hence $p_B = 0$ the non positivity of $a + b = a$ is equivalent to $S \leq 1 - 1/2\zeta$. If in contrast $p_A = 0$ and $p_B = 1$ we need for stability that $a + b = b \leq 0$. This is equivalent to $S \geq 1/2\zeta$ and can not be true since $S < 1/2$ and $\zeta \leq 1$. If $p_A, p_B > 0$ we have

$$a + b = 1 - 2(p_A\alpha_A(AC) + p_B\alpha_B(BC)).$$

Consider the function

$$h(p) = p\alpha_A(AC; p, 1-p, S) + (1-p)\alpha_B(BC; p, 1-p, S).$$

Then stability, i.e. $a + b <= 0$, is euqivalent to $h(p_A) \geq 1/2$. We will investigate the four possible cases seperately. If $\alpha_A(AC) = \alpha_B(BC) = 0$, then $h = 0$. So we can except this first case. As a second case let $\alpha_A(AC) = 0$ and $\alpha_B(BC) > 0$. Then,

$$f(p) = (1-p)(\zeta S - \beta(1-S)p) \leq (1-p)\zeta S < 1/2, \tag{7.23}$$

since $S < 1/2$ and $(1-p), \zeta < 1$. Thus, we can exclude this case as well. As a third case let $\alpha_A(AC) > 0$ and $\alpha_B(BC) = 0$. Here,

$$f(p) = (1-p)(\zeta(1-S) - \beta S(1-p) = p\zeta - S(p\zeta + \beta p(1-p)).$$

To get $f(p_A) \geq 1/2$ we need $p_A \geq 1/2$. Then, $f(p_A) \geq 1/2$ yields

$$S \leq \frac{p_A\zeta - 1/2}{p_A\zeta + \beta P - A(1-p_A)}.$$

The right hand side of the last inequality is increasing in $p_A$ for $p_A \geq 1/2$. Hence, to achieve $f(p_A) \geq 1/2$ we need at least

$$S \leq \frac{\zeta - 1/2}{\zeta} = 1 - \frac{1}{2\zeta}.$$

In case 4 we have $\alpha_A(AC), \alpha_B(BC) > 0$. Here, $f$ is a convex function in $p$:

$$f(p) = \zeta S + (\zeta - 2\zeta S - \beta)p + \beta p^2.$$

Hence, for all $0 < p < 1$, $f(p) \leq \max\{f(0), f(1)\}$. We have $f(0) = \zeta S < 1/2$ and $f(1) = \zeta(1-S)$. For $S > 1 - 1/2\zeta$, $f(1) < 1/2$. Summarizing we can see that in the first two cases no stable steady state exists, while in the last two cases a necessary condition for stability is given by $S \leq 1 - 1/2\zeta$. $\square$

$$\begin{aligned} a + b &= ... \\ &= \zeta p_A + \zeta(1-2p_A)S - \beta p_A(1-p_A) \\ &< \zeta p_A + \zeta(1 - 2(1 - \frac{\zeta}{\beta}\frac{1-S}{S}))S - \beta p_A(1-p_A) \\ &= \zeta p_A + \zeta(-1 + 2\frac{\zeta}{\beta}\frac{1-S}{S})S - \beta p_A(1-p_A) \\ &= \zeta p_A + \zeta(2\frac{\zeta}{\beta}(1-S) - S) - \beta p_A(1-p_A) \end{aligned}$$

### 7.3 Proof of Lemma 4.1

For $S \in [\underline{S}, \overline{S}]$ let $p_A = p_A^*(S)$, while $p_B = 0$. We will consider both cases separately.

**Case 1**: $S = \overline{S}_1$

The stationarity of $\lambda_A$ yields

$$0 = \left(r - p_C\frac{\partial g_A}{\partial p_A}\right)\lambda_A + k - \frac{\xi}{(1-p_A)^{1-\xi}}.$$

To achieve stationarity of $\lambda_S$, we have to find a $\lambda_S \geq 2\frac{2\nu+\mu}{\nu\mu}(1-p_A)^\xi$ such that

$$0 = \dot{\lambda}_S = -p_C\lambda_A\frac{\partial g_A}{\partial S} + \lambda_S(r + \mu + 2\nu).$$

Since $\lambda_A < 0$ increases in $k$ and $\frac{\partial g_A}{\partial S} < 0$, the solution to the above linear equation is sufficiently large, if $k$ is sufficiently large.

**Case 2**: $\underline{S} < S < \overline{S}_1$

Here the stationarity of $\lambda_A$ yields

$$0 = \left(r - p_C\frac{\partial g_A}{\partial p_A}\right)\lambda_A + k - \frac{\xi}{(1-p_A)^{1-\xi}}s^*(S),$$

and $\lambda_S = \lambda_S(S)$ is given by (4.20). We seek for a proper $S$ such that $\dot{\lambda}_S = 0$ holds, cf. (4.22), where $\partial g_B/\partial S = 0$. If the first summand of (4.22) is denoted by $f_1(S)$ and the second one by $f_2(S)$, then we aim to solve $-f_1(S) = f_2(S)$. It is easy to check that at $\underline{S}$ (note, $p_A^*(\underline{S}) = 1$) we have $f_1(\underline{S}) = 0$. Depending on $\xi$ it holds $f_2(\underline{S}) > 0$ (for $\xi = 0$) or $f_2(\underline{S}) = 0$ (for $\xi > 0$). Furthermore, $f_2(S) \to \infty$ for $S \to 1/2$, while $-f_1$ is bounded. Since $f_2$ is independent of the parameter $k$ while $-f1$ is growing linearly in $k$, we get for sufficiently large $k$ that $-f_1(S) > f_2(S)$ for some relevant $S$. Summarizing we have for sufficiently large $k$: $-f_1(\underline{S}) \leq f_2(\underline{S})$, $-f_1(S) > f_2(S)$ for some $S \in (\underline{S}, 1/2)$, $f_2(1/2) = \infty$, $-f_1(1/2) < \infty$ and $f_1, f_2$ are continuous functions on $(\underline{S}, 1/2)$. Hence, there exists at least one intersection between the two functions in the interval $(\underline{S}, 1/2)$. □

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