TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Operations Research and Control Systems

SWM ORCOS

On Optimal Harvesting in Age-Structured Populations

Anton O. Belyakov and Vladimir M. Veliov

Research Report 2015-08

March, 2015

Operations Research and Control Systems
Institute of Statistics and Mathematical Methods in Economics
Vienna University of Technology

Research Unit ORCOS
Wiedner Hauptstraße 8 / E105-4
1040 Vienna, Austria
E-mail: orcos@tuwien.ac.at

On Optimal Harvesting in Age-Structured Populations *

Anton O. Belyakov[†] and Vladimir M. Veliov [‡]

Abstract

The problem of optimal harvesting (in a fish population as a benchmark) is stated within a model that takes into account the age-structure of the population. In contrast to models disregarding the age structure, it is shown that in case of selective harvesting mode (where only fish of certain sizes are harvested) the optimal harvesting effort may be periodic. It is also proved that the periodicity is caused by the selectivity of the harvesting. Mathematically, the model comprises an optimal control problem on infinite horizon for a McKendrick-type PDE with endogenous and non-local dynamics and boundary conditions.

1 Introduction

This paper contributes to the understanding of the controversial issue of the pattern of exploitation of renewable resources: is it optimal to extract a renewable resource continually (which leads to a constant rate of extraction in the long run) or it is better to implement a periodic extraction pattern, in which periods of intense extraction are followed by recovery periods of no extraction. This issue arises in several contexts, including agricultural land-use and fishing. The later case is brought to our attention by Ulf Dieckmann, who suggested that a selective fishing mode (harvesting only sufficiently large fishes) may lead to detrimental evolutionary changes, due to which a periodic harvesting may be advantageous, since it gives time for the fish population to recover (see the more sophisticated analysis in [11] and the bibliography therein).

In the recent paper [4] we investigated the problem of optimal industrial fishing in a closed basin (in the sense that migration of the considered fish species in and out of the basin is negligible). It was shown there that in the problem of maximization of the averaged net revenue from the fishing activity on the infinite time-horizon $[0, \infty)$

*This research was supported by the Austrian Science Fund (FWF) under Grant P26640-N25
[†]Institute of Mechanics, Lomonosov Moscow State University, Russia
[‡]Institute of Statistics and Mathematical Methods in Economics, Vienna University of Technology, Austria

a periodic fishing effort may give a better objective value than every constant one. It is also shown that the selective fishing mode plays a key role for this result. Namely, we proved that in case of non-selective fishing and stationary vital rates the optimal fishing effort is constant.

The averaged objective functional takes into account only the asymptotics of the net revenue. In contrast, in the present paper we consider the total discounted net revenue as an objective of maximization. This is an essentially different problem, the solution of which depends on the discount rate, and may have a qualitatively different asymptotic behavior than the solution of the problem with averaged objective functional. However, in the analysis and in the numerical investigation in the present paper we substantially use the results obtained in [4]. Namely, on certain assumptions we show that for sufficiently small discount rates the asymptotic behavior of the solution of the discounted problem inherits the qualitative properties of the solution of the averaged problem. As a benchmark case for the numerical study we use the same data specifications for which we obtain in [4] superiority of a proper periodic fishing effort (compared with any constant ones). In this case study we establish periodicity of the optimal fishing effort in the discounted problem for sufficiently small discount rates.

In short, in the present paper we show that a selective fishing mode may lead to an optimal fishing effort which is asymptotically proper periodic. This may happen at least for sufficiently small discount rates. On the other hand, we show that if the fishing mode is not selective, then the optimal fishing effort is asymptotically constant. That is, the possible periodicity of the optimal fishing effort is caused by the selectivity of fishing.

In order to enable involvement of selective fishing modes in the mathematical analysis, we need to consider a distributed optimal control model, that takes into account the size- (or age-) structure of the fish population. The model we use employs a standard age-structured controlled differential equation (see e.g. [1, 7, 10, 15]). The main difficulty is, that the theory of establishing (asymptotically) periodic behavior of the optimal control for problems involving such equations is not developed (in contrast to the ODE case). For this reason we make use of the results in [4], as explained above.

The plan of the paper is as follows. In the next section we formulate the model and the main assumptions. In Section 3 we present a result suggesting that a periodic fishing can be superior compared with any (asymptotically) constant one. In Section 4 we give numerical results that exhibit periodicity of the optimal fishing effort in a case study with a selective fishing mode. In Section 5 we show that in case of non-selective fishing the optimal fishing effort is asymptotically constant. Some concluding remarks are given in Section 6.

2 Model formulation and preliminaries

The dynamics of the age-structured fish population is described by the same model as in [4], which we reproduce for readers' convenience.

Everywhere below $t \geq 0$ is the time, a is the fish age, assumed to be restricted in a finite interval $[0, A]$, where A is a proxy for the maximal age that a fish may achieve (meaning, that fishes of age higher than A can be disregarded, since they are not fertile and their total biomass is negligible). By $n(t, \cdot)$ we denote the (non-normalized) age-density of fish at time $t \in [0, +\infty)$. Informally, one says that $n(t, a)$ is the number of fish of age a at time t. Moreover, $z(t)$ will be the total biomass of fish, and $u(t)$ will be the harvesting intensity (effort) at time t. The latter can be interpreted as the number of ships or nets involved in harvesting at t.

The model involves the following age-specific data:
$\mu(a)$ – natural (without competition and harvesting) mortality rate at age a;
$\beta(a)$ – fertility rate;
$\gamma(a)$ – (average) biomass of a single fish of age a;
$\chi(a)$: – age-profile of the selective harvesting.

The last function needs an explanation. Essentially, the function $\chi(a)$ indicates at which ages (hence, indirectly, at which size) a fish is a subject to harvesting. Typically, $\chi(a)$ is zero for small ages and equals 1 after a certain age. Generally, $\chi(a)u(t)n(t, a)$ is the harvested flow of fish of age a resulting from fishing effort $u(t)$ in a population with density $n(t, \cdot)$.

Moreover, there is an additional mortality rate $M(z)$ depending on the total biomass, z.

The dynamics of the fish population is described by the age-structured system

$$\mathcal{D}n(t,a) = -\left(\mu(a) + M(z(t)) + u(t)\,\chi(a)\right) n(t,a), \quad (t,a) \in [0,\infty) \times [0,A], \tag{1}$$

$$n(0,a) = n^0(a), \tag{2}$$

$$n(t,0) = \int_0^A \beta(a)\, n(t,a)\, \mathrm{d}a, \tag{3}$$

$$z(t) = \int_0^A \gamma(a)\, n(t,a)\, \mathrm{d}a, \tag{4}$$

where $n^0(a)$ is a given initial density, and

$$\mathcal{D}n(t,a) := \lim_{\varepsilon \to 0+} \frac{n(t+\varepsilon, a+\varepsilon) - n(t,a)}{\varepsilon}$$

is the directional derivative of n in the direction $(1, 1)$. (Traditionally, the partial differentiation $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)$ is used instead of $\mathcal{D}$, meaning the same.) Since n is assumed absolutely continuous on almost every characteristic line $t - a =$ const., the traces $n(t, 0)$ and $n(0, a)$ make sense (see e.g. [1, 7, 10, 15] for the precise meaning of a solution of (1)–(4)).

The following assumptions are standing all over the paper.

Assumption (A1). The functions μ, β, γ, χ : $[0, A] \to [0, \infty)$ are measurable and bounded, $\gamma(a) \geq \gamma_0 > 0$ for every $a \in [0, A]$.

Assumption (A2). The following inequality holds:

$$\int_0^A \beta(a)\, e^{-\int_0^a \mu(\eta)\,\mathrm{d}\eta}\,\mathrm{d}a > 1.$$

Assumption (A3). $M : [0, +\infty) \to [0, +\infty)$ is continuously differentiable, $M(0) = 0$, $\lim\limits_{z \to +\infty} M(z) = +\infty$, and $M'(z) > 0$ for $z > 0$.

Assumption (A4). The set $\{a \in [0, A] : \; n^0(a)\beta(a) > 0\}$ has a positive measure.

Assumption (A2) is known as *above replacement fertility* (if the initial n^0 contains fertile fishes and there is no harvesting, then the population grows as long as its density $M(z(t))$ is sufficiently small. Assumption (A3) ensures that the population cannot grow infinitely even without harvesting. Assumption (A4) ensures that the population does not extinct in a finite time.

It is well known that on the above assumptions for every measurable and bounded function u a solution of system (1)–(4) exists on $[0, +\infty)$ and is unique (see e.g. [1, Theorem 2.2.3]). Moreover (A3) implies that the solution n is bounded.

Within the above model the harvested biomass, $x(t)$, per unit of harvesting effort $u(t)$ is

$$x(t) = \int_0^A \gamma(a)\,\chi(a)\,n(t, a)\,\mathrm{d}a. \tag{5}$$

If $c \geq 0$ is the cost of a unit of harvesting effort, then a reasonable optimization problem is to maximize the aggregated net discounted revenue:

$$J^\delta(u) := \int_0^\infty e^{-\delta t}\,(x(t) - c)\,u(t)\,\mathrm{d}t \;\longrightarrow \max, \tag{6}$$

with respect to the control function (harvesting effort) $u(\cdot)$. Admissible control functions are all measurable functions $u : [0, \infty) \to [0, U]$, where U is the maximal feasible harvesting effort. Denote this set of admissible controls by $\mathcal{U}$. The discount rate δ is assumed strictly positive.

Since $\delta > 0$, the integral in (6) is finite for every $u \in \mathcal{U}$ due to the boundedness of x and assumption (A1). Then optimality has the classical meaning, namely, $\hat{u} \in \mathcal{U}$ is optimal if $J^\delta(\hat{u}) \geq J^\delta(u)$ for every $u \in \mathcal{U}$.

As explained in the introduction, the main goal of this paper is to show that an optimal fishing control $u(t)$ is not necessarily tending to a constant in the long run.

In order to formalize this claim we introduce the following notations:

$$\mathcal{U}_c^\infty := \{u \in \mathcal{U} :\ \exists\, u^\infty \in [0, U] \text{ such that } \lim_{\tau \to +\infty} \|u(\cdot) - u^\infty\|_{L\infty(\tau,+\infty)} = 0\}, \tag{7}$$

$$\mathcal{U}_p := \{u \in \mathcal{U} :\ u \text{ is a proper periodic function}\}.$$

The first set consists of all asymptotically constant admissible controls, while $\mathcal{U}_p$ consists of all (essentially) periodic admissible controls for which a minimal positive period exists. In this notations, our goal is to show that for appropriate configurations of data $(\mu, M, \chi, \beta, \gamma, c, \delta > 0)$ there exists $\hat{u}_p \in \mathcal{U}_p$ such that

$$\sup_{u \in \mathcal{U}_c^\infty} J^\delta(u) < J^\delta(\hat{u}_p). \tag{8}$$

In other words, there exists a proper periodic admissible control u_p^δ which gives a strictly better performance value of J^δ than any asymptotically constant admissible control.

3 An analytic approach

This section deals with the case of positive discounting, $\delta > 0$. A direct formal verification of the key inequality (8) for a given data configuration seems to be difficult. Therefore, we propose an indirect way that uses the results in the recent paper [4], where a similar as (8) relation is obtained for the problem of maximization of the long run averaged net revenue. Below we extract the needed here information from [4].

For a given $u \in \mathcal{U}$ define

$$J_\tau(u) := \frac{1}{\tau} \int_0^\tau (x[u](t) - c)\, u(t)\, \mathrm{d}t$$

and

$$J(u) := \liminf_{\tau \to +\infty} J_\tau(u), \tag{9}$$

where $x[u]$ is defined as in (5) for the solution n of (1)–(4) corresponding to u. The existence of the above limit follows from the boundedness of $x[u]$.

In [4] we have shown that for appropriate configurations of data $(\mu, M, \chi, \beta, \gamma, c)$ there exists a proper periodic control $\hat{u}_p \in \mathcal{U}_p$ and a number $\alpha > 0$ such that

$$J(u) \le J(\hat{u}_p) - \alpha \quad \text{for every constant control } u \in [0, U]. \tag{10}$$

This is done by using a combination of analytic and reliable numerical arguments (the latter resulting from an enhancement of the so-called *properness test* originally developed in [6]), and under certain additional conditions that we do not mention here. Instead, we merely assume that for a given particular configuration of data

$(\mu, M, \chi, \beta, \gamma, c)$ inequality (10) is fulfilled. The existence of such data is shown in [4] and the used arguments clearly indicate that such data configurations are not exceptional.

Our goal in this and in the next section is to show that (10) implies (8) for all (or at least for some) sufficiently small discount rates $\delta > 0$. In this section we propose an analytic approach for achieving this goal, which is based on results in [3] and [8], which we briefly present for readers convenience. We begin with the following consequence of [3, Theorem 2.2].

Corollary 1 (of Theorem 2.2 in [3]) *For every $u \in \mathcal{U}_p$ with a period $\omega > 0$ there exists a measurable and bounded function $\bar{n} : [0, +\infty) \times [0, A] \to \mathbf{R}$ such that the mapping $t \mapsto \bar{n}(t, \cdot)$ is ω-periodic and the corresponding solution n of system (1)–(4) satisfies*

$$\lim_{t \to +\infty} \| n(t, \cdot) - \bar{n}(t, \cdot) \|_{L_\infty(0,A)} = 0.$$

We stress that Theorem 2.2 in [3] is proved under certain additional conditions denoted there by (H1)–(H5) (most of them implied by our assumptions (A1)–(A3), but not all). Since these conditions are not restrictive in our context, we do not recall them. Instead, we just assume that the claim of Corollary 1 holds true.

Lemma 2 *Let $u \in \mathcal{U}_p$. Then $\lim_{\tau \to +\infty} J_\tau(u)$ does exist.*

Proof. With the notations from Corollary 1 we can easily show that due to the convergence of $n(t, \cdot)$ we have

$$\lim_{\tau \to +\infty} \left| J\tau(u) - \frac{1}{\tau} \int_0^\tau (\bar{x}(t) - c) u(t) \, \mathrm{d}t \right| = 0,$$

where

$$\bar{x}(t) = \int_0^A \gamma(a) \, \chi(a) \, \bar{n}(t, a) \, \mathrm{d}a.$$

Since $f(t) := (\bar{x}(t) - c) u(t)$ is ω-periodic, we have

$$\frac{1}{\tau} \int_0^\tau f(t) \, \mathrm{d}t = \frac{1}{\tau} \sum_{i=0}^{k-1} \left(\int_{i\omega}^{(i+1)\omega} f(t) \, \mathrm{d}t + \int_{k\omega}^\tau f(t) \, \mathrm{d}t \right),$$

where k is the maximal natural number for which $k\omega \le \tau$, so that $\alpha_\tau := \tau - k\omega \in [0, \omega)$. The above exposed expression can be written as

$$\frac{1}{k\omega + \alpha_\tau} \left(k \int_0^\omega f(t) \, \mathrm{d}t + \int_{k\omega}^\tau f(t) \, \mathrm{d}t \right) \longrightarrow \frac{1}{\omega} \int_0^\omega f(t) \, \mathrm{d}t,$$

where the convergence is with respect to $\tau \to +\infty$, that is, $k \to +\infty$. ■

The next result that we use below is taken from [8].

Lemma 3 *Let $q : \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying $|q(t)| \le N$ for almost every $t \in [0, \infty)$. Assume that there exist $T > 0$ and $\sigma \in \mathbf{R}$ such that*

$$\frac{1}{\tau} \int_0^\tau q(t)\, \mathrm{d}t < \sigma \quad \textit{for all } \tau > T.$$

Then for any $\varepsilon > 0$ and all $\delta \in (0, \varepsilon / T(N + |\sigma + \varepsilon|)$ the following inequality holds:

$$\delta \int_0^\infty e^{-\delta t} q(t)\, \mathrm{d}t \le \sigma + \varepsilon.$$

We remind that for $u \in \mathcal{U}_c$ we denoted by $u^\infty \in [0, U]$ the limit value of $u(t)$ at infinity (see (7)). Thus for $u_\delta \in \mathcal{U}_c$ (that appears below), u_δ^∞ will denote the corresponding limit value.

Proposition 4 *Assume that (10) is fulfilled. Let $u_\delta \in \mathcal{U}_c^\infty$ be any function such that*

$$J^\delta(u_\delta) \ge \sup_{u \in \mathcal{U}_c^\infty} J^\delta(u) - \frac{\alpha}{16\delta}. \tag{11}$$

Assume also that there exist $T_0 > 0$ and $\delta_0 > 0$ such that

$$J_\tau(u_\delta) < J(u_\delta^\infty) + \frac{\alpha}{16} \qquad \textit{for all } \delta \in (0, \delta_0) \textit{ and all } \tau \ge T_0. \tag{12}$$

Then there exists $\bar{\delta} > 0$ such that

$$\sup_{u \in \mathcal{U}_c^\infty} J^\delta(u) < J^\delta(\hat{u}_p) - \frac{\alpha}{2\delta} \qquad \textit{for all} \quad \delta \in (0, \bar{\delta}).$$

Proof. By the definition of $J(\hat{u}_p)$ and Lemma 2 there exists $\bar{T} \ge T_0$ such that

$$|J_\tau(\hat{u}_p) - J(\hat{u}_p)| < \frac{\alpha}{16} \quad \text{for all } \tau \ge \bar{T}.$$

Hence,

$$-\frac{1}{\tau} \int_0^\tau (x[\hat{u}_p](t) - c)\hat{u}_p(t)\, \mathrm{d}t < -J(\hat{u}_p) + \frac{\alpha}{16}.$$

Then we apply Lemma 3 with $q(t) = -(x[\hat{u}_p](t) - c)\hat{u}_p(t)$, $\sigma = -J(\hat{u}_p) + \alpha/16$, and $\varepsilon = \alpha/16$. It gives that

$$-\delta J^\delta(\hat{u}_p) = \delta \int_0^\infty -e^{-\delta t}(x[\hat{u}_p](t) - c)\hat{u}_p(t)\, \mathrm{d}t < \sigma + \varepsilon = -J(\hat{u}_p) + \frac{\alpha}{8},$$

for every

$$\delta \in \left(0, \frac{\varepsilon}{(N + |\sigma + \varepsilon|)\bar{T}}\right) = \left(0, \frac{\alpha}{16(N + |-J(\hat{u}_p) + \alpha/8|)\bar{T}}\right),$$

where N is an upper bound for $|(x[\hat{u}_p](t) - c)\hat{u}_p(t)|$, $t \in [0, \infty)$. Since obviously also $J(\hat{u}_p) \le N$, we obtain that

$$\delta J^\delta(\hat{u}_p) > J(\hat{u}_p) - \frac{\alpha}{8} \quad \text{for all} \;\; \delta \in \left(0, \frac{\alpha}{2(16N + \alpha)\bar{T}}\right).$$

On the other hand, from (12) we have for every $\delta \in (0, \delta_0)$ that

$$\frac{1}{\tau} \int_0^\tau (x[u_\delta](t) - c)u_\delta(t)\,\mathrm{d}t < J(u_\delta) + \frac{\alpha}{16}. \tag{13}$$

Then we can apply Lemma 3 with $q(t) = (x[u_\delta](t) - c)u_\delta(t)\,\mathrm{d}t$, $\sigma = -J(u_\delta) + \alpha/16$, and $\varepsilon = \alpha/16$, and obtain in the same way as above that

$$\delta J^\delta(u_\delta) < J(u_\delta) + \frac{\alpha}{8} \quad \text{provided that} \;\; \delta \in \left(0, \frac{\alpha}{2(16N + \alpha)\bar{T}}\right).$$

Denoting $\bar{\delta} = \min\{\delta_0, \alpha/(2(16N+\alpha))\bar{T})\}$ and combining the last inequality with (13) and (11) we obtain that for every $\delta \in (0, \bar{\delta})$ and $\tau \ge \bar{T}$

$$\begin{aligned} \delta \sup_{u \in \mathcal{U}_c^\infty} J^\delta(u) \;\; &\le \;\; \delta J^\delta(u_\delta) + \frac{\alpha}{16} < J(u_\delta) + \frac{\alpha}{8} + \frac{\alpha}{16} < J(\hat{u}_p) - \frac{13\alpha}{16} \\ &< \;\; \delta J^\delta(\hat{v}_p) - \frac{13\alpha}{16} + \frac{\alpha}{8} < \delta J^\delta(\hat{v}_p) - \frac{\alpha}{2}. \end{aligned}$$

■

Clearly, the claim of the above proposition implies the desired inequality (8), since $\hat{u}_p \in \mathcal{U}_p$. However, the assumptions of the proposition need a discussion. The existence of the controls u_δ in (11) is evident. However, (12) is problematic. On one hand, since $u_\delta(t) \to u_\delta^\infty$, it is possible to prove that the "liminf" in (9) is in fact "lim", and then $J_\tau(u_\delta^\infty) \to J(u_\delta^\infty)$ by definition. Thus the convergence $J_\tau(u_\delta) \to J(u_\delta^\infty)$ can be verified. However, the assumption in Proposition 4 requires this convergence to be uniform with respect to δ in a sufficiently small interval $(0, \delta_0)$. One can further elaborate this assumption, but it still requires a number of qualitative properties of system (1)–(4) which are not available in the literature and are not simple to obtain in general. Therefore, we combine the argument in the above proposition with a numerical study done in the next section.

4 Periodic optimal fishing – numerical results

In this section we present a numerical results for a particular problem of the type (1)–(6), where the optimal fishing control $v(t)$ is clearly periodic. The specifications for this case study are taken from [4]. Namely, here $A = 30$, $\mu(a) = 0.005$, $M(z) = 0.001z$, the fertility function $\beta(a)$ equals 0.16 for ages $a \in [10, 20]$ and is zero for all

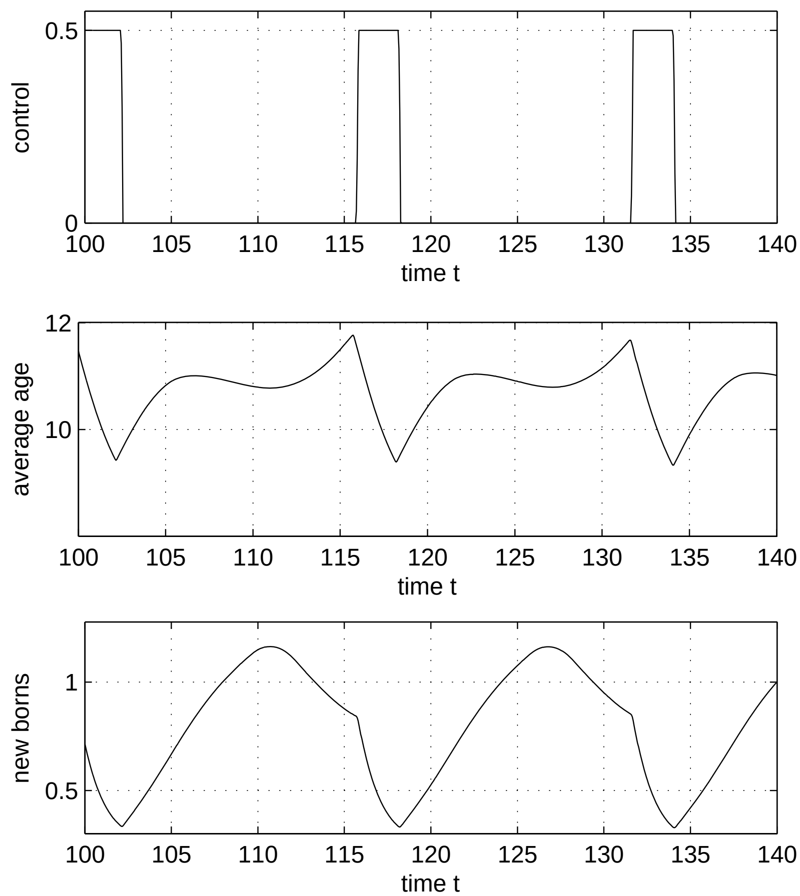

Figure 1: $r = 0.01$

other ages, the biomass of a fish of age a is $\gamma(a) = \frac{2a}{10+a}$. Moreover, the fishing is selective with $\chi(a) = 0$ for $a \leq 10$ and $\chi(a) = 1$ for $a > 10$. The cost of a unit of fishing effort is $c = 2$.

The choice of the above specifications has a good reason. It was shown in [4] that there exists a proper periodic admissible control $\hat{u}_p(t)$ that gives averaged objective value $J(\hat{u}_p)$ which is larger than the averaged objective value $J(u)$ resulting from any constant admissible control u. Then Proposition 4 suggests that $J^\delta(\hat{u}_p) > \sup_{u \in \mathcal{U}_c^\infty} J^\delta(u)$ for all sufficiently small $\delta > 0$. This turns out to be the case. The numerical results below are obtained for discount rate $\delta = 0.01$ and maximal fishing effort $U = 0.5$. The solution is obtained by using the optimality conditions in [7] (applied for a finite, but sufficiently large horizon) and utilization of a gradient projection method. A detailed description of the numerical approximation scheme, including error analysis, is presented in the forthcoming paper [14].

Figures 1 present the optimal control, $v(t)$, together with the average age of fish and the number of newborns, all restricted to the time-interval $[100, 140]$. After an initial transitional time-interval the optimal control stabilizes to a periodic pattern, which is of bang-bang type: either the harvesting has maximal intensity ($v(t) = U$), or the fish stock is left to recover ($v(t) = 0$).

Figure 1 helps to understand the reason for which selective fishing may lead to a periodic optimal control. We see that during an intense fishing period both the average age of fish and the inflow of newborns steeply decrease. The former, due to the fact that young fish is not harvested, the later, due to the harvesting of fish at fertile ages. A stock of young fish with maturation still to come stays unaffected. If the harvesting would continue longer, then a part of this stock would fail to give offspring. A period of no-harvesting enables the still unaffected wave of young fish to reproduce. In our case study this period takes about 14 units of time (call them years). As the third plot in Figure 1 shows, in this period the number of newborns steeply increases for awhile (some 10 years), then decreases again since the preserved stock of young fish becomes infertile (this also explains the increase of the average age some 10 years after the harvesting period).

We should mention that the above explanations are conditional: as suggested by Proposition 4, they apply to sufficiently small discount rates. If the discount rate δ is too large, then the concern about the regeneration potential is small. Indeed, further numerical analysis of the case study shows that for $r = 0.1$ the optimal harvesting effort converges to a constant level, that is, it is not asymptotically proper periodic.

5 The case of non-selective fishing

In this section we show that the selectivity of fishing (defined by the age-dependent fishing pattern $\chi(a)$) is responsible for the asymptotically non-constant/periodic behavior of the optimal fishing. To do this we consider the case of non-selective fishing: $\chi(a) \equiv \chi \in (0, \infty)$. Without any restriction we assume that $\chi = 1$.

Let us fix an arbitrary admissible control $u \in \mathcal{U}$. Following Section 3.2, Chapter 3 in [1] (see also [10, 13]) and making use of the stationarity of the vital rates in our model, we represent the unique solution $n(t,a)$ of system (1)–(4) in the form

$$n(t,a) = y(t)\, \tilde{n}(t,a),$$

where y is absolutely continuous and $\tilde{n}$ is measurable and absolutely continuous along the characteristic lines of $\mathcal{D}$. To do this, for an arbitrary real number α (viewed as a parameter) we consider the following two systems:

$$\mathcal{D}\tilde{n}(t,a) = -\,(\mu(a) + \alpha)\, \tilde{n}(t,a), \tag{14}$$

$$\tilde{n}(0,a) = n^0(a), \tag{15}$$

$$\tilde{n}(t,0) = \int_0^A \beta(a)\, \tilde{n}(t,a)\, \mathrm{d}a, \tag{16}$$

and

$$\dot{y}(t) = -\,(M(\tilde{z}(t)\, y(t)) - \alpha + u(t))\, y(t), \quad y(0) = 1, \tag{17}$$

where $\tilde{z}(t)$ is defined as

$$\tilde{z}(t) = \int_0^A \gamma(a)\, \tilde{n}(t,a)\, \mathrm{d}a. \tag{18}$$

We chose α to be the *Malthusian parameter*, that is, the value for which the unique solution of system (14)–(16), $\tilde{n}(t,\cdot)$, converges to a steady state $\bar{n}$ in the space $L_\infty(0,A)$. The Malthusian parameter α is determined from the condition that the *net reproduction rate* is equal to 1:

$$\int_0^A \beta(a)\, e^{-\int_0^a \mu(\eta)\,\mathrm{d}\eta - \alpha a}\, \mathrm{d}a = 1.$$

Due to Assumption (A2), for $\alpha = 0$ the left-hand side is strictly bigger than 1. Since the left-hand side is strictly decreasing and continuous in α, and converges to zero when $\alpha \to +\infty$, the above equation has a unique solution $\alpha > 0$.

It is a matter of direct substitution to check that $n = y\tilde{n}$ is the (unique) solution of system (1)–(4). Notice that $\tilde{n}$, thus also $\tilde{z}$, is independent of the control u. Then the optimal control problem (1)–(6) can be equivalently reformulated as

$$\int_0^\infty e^{-\delta t}\, (\tilde{z}(t)\, y(t) - c)\, u(t)\, \mathrm{d}t \to \max_{u(\cdot)}, \qquad u(t) \in [0, U], \tag{19}$$

subject to (17).

Due to Assumption (A3) for M it is straightforward to prove that there is a constant K such that the interval $[0,K]$ is invariant with respect to (17) with any admissible control u, and any trajectory enters $[0,K]$ in finite time.

We mention that by a standard argument problem (17), (19) has a solution due to the linearity with respect to u, the compactness of the admissible control values, $[0,U]$, the above boundedness property, and the inequality $\delta > 0$ (take an L_2-weakly convergent maximizing sequence and pass to the limit).

Our aim in the rest of the section is to prove (on minor additional assumptions) that any solution of problem (17), (19) is asymptotically constant. Then the same will be true for the optimal fishing problem (1)–(6).

We begin with some preliminary considerations and notations. The steady state $\bar{n}$ is nontrivial due to assumption (A4). This implies that $\tilde{n}(t,\cdot)$ is nontrivial for every $t \geq 0$. From (18) and $\gamma(a) \geq \gamma_0 > 0$ we have that $\tilde{z}(t) > 0$. Passing to the limit in (18) with $t \to +\infty$ and using the non-triviality of $\bar{n}(\cdot)$ we obtain that $\tilde{z}(t)$ converges to $\bar{z} := \int_0^A \gamma(a)\bar{n}(a)\,\mathrm{d}a > 0$. Since there exists $\bar{N}$ such that $\tilde{n}(t,a) \leq \bar{N}$ for all (t,a) and is Lipschitz continuous along the characteristic lines of (14), it is easy to prove that $\tilde{z}(\cdot)$ is (Lipschitz) continuous. Together with the already established properties of $\tilde{z}$, this implies that there exist $z_0 > 0$ and $z_1 > z_0$ such that $z(t) \in [z_0, z_1]$ for all $t \geq 0$.

Denote by y^* and y^0 the unique numbers for which

$$\bar{z}y^* = c \quad \text{and} \quad M(\bar{z}y^0) = \alpha.$$

Proposition 5 *Let* $(\tilde{y}, \tilde{u})$ *be a solution of problem (17), (19). Then the following claims hold:*

(i) If $M(c) = \alpha$ *then* $\tilde{y}(t)$ *converges to* y^0 *when* $t \to +\infty$*;*

(ii) if $M(c) > \alpha$ *then* $\tilde{u}(t) = 0$ *for all sufficiently large* t *and* $\tilde{y}(t)$ *converges to* y^0 *when* $t \to +\infty$*;*

(iii) if $M(c) < \alpha$ *and if, additionally,* $c > 0$*,* μ *is continuous,* β *is continuously differentiable,* M *is twice continuously differentiable,* $M(c) - \alpha + U \geq 0$*, and*

$$2M'(\xi) + (\xi - c)M''(\xi) > 0 \quad \text{for every} \quad \xi \in [c, M^{-1}(\alpha)], \tag{20}$$

then $\tilde{y}(t)$ *and* $\tilde{u}(t)$ *converge when* $t \to +\infty$*.*

Before proving the proposition we make some comments. First, in case (i) we prove only convergence of $\tilde{y}(t)$. It might be true that also $\tilde{u}(t)$ converges (to zero), however, this requires further analysis. We pay no attention to this case, since it is non-generic: notice that α is determined solely by the population parameters while c has purely economic meaning, thus the two are not correlated. Second, obviously assumption (20) is fulfilled if M is convex (a plausible assumption), but is even weaker than that. The other additional assumptions in (iii) are also not restrictive, except $M(c) - \alpha + U \geq 0$. The latter requires that the upper bound U is sufficiently large. We believe that the last assumption can be substantially relaxed, but the proof would require some additional work.

Proof of Proposition 5. We introduce the notations $\tilde{y}^*(t) = c/\tilde{z}(t)$ and $\tilde{y}^0(t) = M^{-1}(\alpha)/\tilde{z}(t)$. Observe that the optimal solution $(\tilde{y}, \tilde{u})$ has the following property: if $\tilde{y}(t) < \tilde{y}^*(t)$ then $\tilde{u}(t) = 0$. (Since harvesting decreases the stock of fish, the above statement has the obvious meaning that fish is harvested only if the net revenue is positive. The formal proof is easy.)

Part 1. Let $M(c) \geq \alpha$. Due to the monotonicity of M this implies that $y^0 \leq y^*$. Fix an arbitrary positive number $\rho < y^0$. Due to the convergence of $\tilde{z}(t)$ to $\bar{z}$, we have that $\tilde{y}^*(t) \to y^*$ and $\tilde{y}^0(t) \to y^*$. Thus there exists a number $\tau(\rho)$ such that $|\tilde{y}^0(t) - y^0| \leq \rho/2$ and $|\tilde{y}^*(t) - y^*| \leq \rho/2$ for every $t \geq \tau(\rho)$. Since $M'(\xi) > 0$ for $\xi > 0$, there exists $\varepsilon > 0$ such that $M(\tilde{z}(t)y) \geq \alpha + \varepsilon$ for $y \geq y^0 + \rho$, $M(\tilde{z}(t)y) \leq \alpha - \varepsilon$ for $y \leq y^0 - \rho$, and $\tilde{z}(t)y < c$ for $y \leq y^* - \rho$.

Assume that for some $t \geq \tau(\rho)$ it holds that $\tilde{y}(t) \geq y^0 + \rho$. Then

$$\dot{\tilde{y}}(t) \leq -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha)\tilde{y}(t) \leq -\varepsilon\tilde{y}(t).$$

This implies that $\tilde{y}(t) \leq y^0 + \rho$ for all sufficiently large t.

Now assume that for some $t \geq \tau(\rho)$ it holds that $\tilde{y}(t) \leq y^0 - \rho$. Then

$$\dot{\tilde{y}}(t) = -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha)\tilde{y}(t) \geq \varepsilon\tilde{y}(t),$$

where we use that $\tilde{z}(t)\tilde{y}(t) < c$, hence $\tilde{u}(t) = 0$. This implies that $\tilde{y}(t) \geq y^0 - \rho$ for all sufficiently large t. Since $\rho > 0$ was arbitrarily chosen (sufficiently small), we obtain that $\tilde{y}(t)$ converges to y^0, thus claim (i).

Part 2. Under the condition in claim (ii) we have, in addition, that $y^* > y^0$, thus $\tilde{y}(t) < y^* - \rho$ for some $\rho > 0$ and all sufficiently large t. Then $\tilde{z}(t)\tilde{y}(t) < c$ for all sufficiently large t, hence $\tilde{u}(t) = 0$. Claim (ii) is proved.

Part 3.1. The non-trivial case is $M(c) < \alpha$, which implies $y^* < y^0$. First, we shall establish some smoothness and convergence properties of $\tilde{z}$. Consider the function

$$B(t) := \int_0^A \beta(a)\tilde{n}(t,a)\,\mathrm{d}a.$$

Similarly as for $\tilde{z}$, one can argue that B is (Lipschitz) continuous. Moreover, from (14) $\tilde{n}$ can be expressed as

$$\tilde{n}(t,a) = e^{-\int_0^a (\mu(\theta)+\alpha)\,\mathrm{d}\theta} B(t-a), \qquad \text{for} \quad t > A. \tag{21}$$

Hence, B satisfies for $t > A$ the equation

$$B(t) = \int_0^A k(a)B(t-a)\,\mathrm{d}a, \qquad k(a) := \beta(a)e^{-\int_0^a (\mu(\theta)+\alpha)\,\mathrm{d}\theta}.$$

Using the continuity of B and the above equality, it is a routine task to prove that B is continuously differentiable and

$$\dot{B}(t) = \int_0^A k'(a)B(t-a)\,\mathrm{d}a + \beta(0)B(t) - k(A)B(t-A). \tag{22}$$

Due to the convergence of $\tilde{n}(t,\cdot)$ we have that $B(t)$ also converges when $t \to +\infty$ to the limit $\bar{B} := \int_0^A \beta(a)\bar{n}(a)\,\mathrm{d}a$. Then the right-hand side in (22) converges to

$$\int_0^A k'(a)\bar{B}\,\mathrm{d}a + \beta(0)\bar{B} - k(A)\bar{B} = 0.$$

Thus $\dot{B}(t)$ converges to zero. Then from (22) we recursively conclude that B is two-times differentiable for all $t > 2A$, three-times differentiable for all $t > 3A$, and so on. Each derivative converges to zero. Then from (21) it follows, in particular, that $\tilde{n}$ is two-times differentiable with respect to t for all $t > 3A$ and each partial derivative in t converges to zero uniformly in a. From (18) we obtain that $\dot{\tilde{z}}(t)$ and $\ddot{\tilde{z}}(t)$ exist for all $t > 3A$ and both converge to zero when $t \to +\infty$.

Part 3.2. Inequality (20) is equivalent to

$$2M'(\bar{z}y) + (\bar{z}y - c)M''(\bar{z}y) > 0 \quad \text{for every} \quad y \in [y^*, y^0]. \tag{23}$$

Then there exist positive numbers σ and $\rho < y^*$ such that

$$\bar{z}^2[2M'(\bar{z}y) + (\bar{z}y - c)M''(\bar{z}y)] > 2\sigma \quad \text{for every} \quad y \in [y^* - \rho, y^0 + \rho]. \tag{24}$$

Since $\tilde{z}(t)$, $\tilde{y}^*(t)$, $\tilde{y}^0(t)$ converge to $\bar{z}$, y^*, y^0, respectively, there exist $\tau > 0$ and $\nu > 0$ such that[1] for $t \geq \tau$ we have $\tilde{y}^*(t) > y^* - \rho$ and

$$M(\tilde{z}(t)y) - \alpha < -\nu \quad \text{for } y \leq y^* - \rho, \quad M(\tilde{z}(t)y) - \alpha > \nu \quad \text{for } y \geq y^0 + \rho.$$

This allows to prove that the interval $[y^0 - \rho, y^* + \rho]$ is invariant with respect to equation (17) with $u = \tilde{u}$, and $\tilde{y}(t)$ reaches it in a finite time. Indeed, if for some $t \geq \tau$ it holds that $\tilde{y}(t) > y^0 + \rho$ then

$$\dot{\tilde{y}}(t) = -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha + \tilde{u}(t))\tilde{y}(t) \leq -\nu\tilde{y}(t).$$

If it holds that $\tilde{y}(t) < y^* - \rho$, then $\tilde{y}(t) < \tilde{y}^*(t)$, thus $\tilde{u}(t) = 0$ and

$$\dot{\tilde{y}}(t) = -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha)\tilde{y}(t) \geq \nu\tilde{y}(t).$$

These two inequalities prove both the invariance of $[y^0 - \rho, y^* + \rho]$ and the fact that it is reached in a finite time. Thanks to this and the dynamic programming principle, for all sufficiently large τ the restriction of $(\tilde{y}, \tilde{u})$ to $[\tau, \infty)$ is a solution of the problem

$$\int_\tau^\infty e^{-\delta t} \, (\tilde{z}(t) \, y(t) - c) \, u(t) \, \mathrm{d}t \to \max_{u(\cdot)}, \qquad u(t) \in [0, U], \tag{25}$$

$$\dot{y}(t) = - \left(M(\tilde{z}(t) \, y(t)) - \alpha + u(t)\right) y(t), \quad y(\tau) = \tilde{y}(\tau), \tag{26}$$

$$y(t) \in [y^0 - \rho, y^* + \rho]. \tag{27}$$

Part 3.3. In order to investigate the solution $(\tilde{y}, \tilde{u})$ for $t > \tau$ we apply to problem (25)–(27) the most rapid approach path theorem, [9, Theorem 3.1], which deals with non-stationary problems.

Solving (26) for u and substituting u in (25) we come up with the problem

$$\max_y \int_{\tau_0}^\infty e^{-\delta t}[P(t, y(t)) + Q(t, y(t))\dot{y}(t)] \, \mathrm{d}t, \tag{28}$$

where,

$$P(y) = -(\tilde{z}(t)y - c)(M(\tilde{z}(t)y) - \alpha), \quad Q(y) = -\frac{\tilde{z}(t)y - c}{y}.$$

The initial condition is $y(\tau) = \tilde{y}(\tau) > 0$ and due to the constraint $u \in [0, U]$ the function y has to satisfy the inclusion

$$\dot{y}(t) \in \Omega(t, y(t)) \quad \text{with} \quad \Omega(t, y) := [-(M(\tilde{z}(t)y) - \alpha + U)y, -(M(\tilde{z}(t)y) - \alpha)y], \tag{29}$$

along with the state constraints (27). The restriction of $(\tilde{y}, \tilde{u})$ to $[\tau, \infty)$ is a solution of this problem.

[1] The number ρ may be eventually decreased, and τ increased, later on in a correct way, that is, this can be made right here, but we postpone it to a later point for more clarity.

In order to apply [9, Theorem 3.1] we have to investigate for $y \in [y^* - \rho, y^0 + \rho]$ the equation $I(t,y) = 0$ with

$$\begin{aligned} I(t,y) &:= -\delta Q(t,y) + N_t(t,y) - P_y(t,y) \\ &= -\dot{\tilde{z}}(t) + \delta\left(\tilde{z}(t) - \frac{c}{y}\right) + \tilde{z}(t)[M(\tilde{z}(t)y) - \alpha + (\tilde{z}(t)y - c)M'(\tilde{z}(t)y)]. \end{aligned}$$

Observe that

$$I(t,y) = \delta\left(\bar{z} - \frac{c}{y}\right) + \bar{z}[M(\bar{z}y) - \alpha + (\bar{z}y - c)M'(\bar{z}y)] + \varepsilon(t,y) =: \bar{I}(y) + \varepsilon(t,y),$$

where due to the convergence of $\tilde{z}(t)$ to $\bar{z}$ and of the first two derivatives of $\tilde{z}$ to zero, also the function $\varepsilon(t,y)$ as well as the derivatives $\varepsilon_t(t,y)$ and $\varepsilon_y(t,y)$ converge to zero when $t \to +\infty$, uniformly in $y \in [y^*, y^0]$. Since

$$\bar{I}(y^*) = \bar{z}[M(\bar{z}y^*) - \alpha] < 0, \qquad \bar{I}(y^0) = \delta\frac{\bar{z}y^0 - c}{y^0} + \bar{z}(\bar{z}y^0 - c)M'(\bar{z}y^0) > 0. \quad (30)$$

and (see (24))

$$\bar{I}'(y) = \delta\frac{c}{y^2} + \bar{z}^2[2M(\bar{z}y) + M''(\bar{z}y)(\bar{z}y - c)] \geq 2\sigma,$$

the function $\bar{I}$ has a unique zero $\bar{y} \in (y^*, y^0)$, which is the unique zero also in $[y^* - \rho, y^0 + \rho]$ if $\rho > 0$ is fixed as sufficiently small. Then the inequalities

$$I(t, y^* - \rho) < 0, \quad I(t, y^0 + \rho) > 0, \quad I_y(t,y) \geq \sigma$$

hold for every $y \in [y^* - \rho, y^0 + \rho]$ and $t \geq \tau$, provided that $\rho > 0$ and τ are appropriately fixed (see Footnote 1). Consequently, $I(\cdot, t)$ has a unique single zero $\hat{y}(t) \in ([y^* - \rho, y^0 + \rho)$. The implicit function theorem also claims that $\hat{y}$ is differentiable and

$$\dot{\hat{y}}(t) = \frac{I_t(t, \hat{y}(t))}{I_y(t, \hat{y}(t))} = \frac{\varepsilon_t(t, \hat{y}(t))}{\bar{I}_y(\hat{y}(t)) + \varepsilon_y(t, \hat{y}(t))} \longrightarrow 0.$$

Since every condensation point of $\hat{y}(t)$ at $+\infty$ satisfies $\bar{I}(y) = 0$ and belongs to $[y^* - \rho, y^0 + \rho]$, it must coincide with $\bar{y}$. Thus $\lim_{t \to +\infty} \hat{y}(t) = \bar{y}$.

As a recapitulation, in the paragraph above we show in particular that $I_y(t,y) > 0$, in $[y^* - \rho, y^* + \rho]$ and the equation $I(t,y) = 0$ has a unique solution, $\hat{y}(t)$, in this interval. This is one of the suppositions in [9, Theorem 3.1].

A second assumption in [9, Theorem 3.1] is that

$$\lim_{t \to +\infty} e^{-\delta t} \int_{y(t)}^{\hat{y}(t)} N(t,\xi)\, \mathrm{d}\xi \geq 0$$

for every admissible $y(\cdot)$ in problem (28), (29), (27). Obviously the above limit equals zero due to $y(t)$, $\hat{y}(t) \in [y^* - \rho, y^0 + \rho]$, where $N(t, \cdot)$ is bounded uniformly in t.

The third assumption in [9, Theorem 3.1] is that $\hat{y}(t)$ satisfies the inclusion in (29), which reads as

$$\dot{\hat{y}}(t) \in [-(M(\tilde{z}(t)\hat{y}(t)) - \alpha + U)\hat{y}(t),\, -(M(\tilde{z}(t)\hat{y}(t)) - \alpha)\hat{y}(t)].$$

Since we proved that $\hat{y}(t)$ converges to $\bar{y}$ and $\dot{\hat{y}}(t)$ converges to zero, and since $-(M(\bar{z}\bar{y}) - \alpha)\bar{y} > 0$ (due to $\bar{y} < y^0$) the upper constraint is satisfied by $\dot{\hat{y}}(t)$ for $t \geq \tau$, provided that τ is fixed sufficiently large. If we assume that the lower constraint is not satisfied for arbitrarily large t, then $\dot{\hat{y}}(t_k) < -(M(\tilde{z}(t_k)\hat{y}(t_k)) - \alpha + U)\hat{y}(t_k)$ for some sequence $t_k \to +\infty$. Passing to the limit we obtain

$$0 \leq -(M(\bar{z}\bar{y}) - \alpha + U)\bar{y}, \quad \text{hence,} \quad 0 \geq M(\bar{z}\bar{y}) - \alpha + U > M(c) - \alpha + U,$$

which contradicts the assumption in part (iii) of the theorem.

Now, we can apply Theorem 3 in [9], which climes that $\tilde{y}(t)$ reaches $\hat{y}(t)$ in a finite time and then coincides with it, hence it converges to $\bar{y}$. This completes the proof of the proposition. ■

6 Final discussion

Below we indicate some open questions related to subject of the present paper.

1. The proof of superiority of proper periodic controls given in Section 3 is based on the assumption (12) that is not directly checkable. Proving that it is fulfilled would make the conclusion for non-optimality of asymptotically constant controls ultimate (at least for small discount rates).

2. The harvesting age-profile $\chi(a)$ is assumed fixed in this paper. However, it makes sense to consider it also as a decision function, as in [1, Chapter 3, Section 3.3] or [12] for example, where control depends also on age, $u(t, a)$. It is an open question if the optimal harvesting effort will still be asymptotically non-constant in this case.

References

[1] S. Aniţa. *Analysis and Control of Age-Dependent Population Dynamics*. Mathematical Modelling Series. Springer, 2000.

[2] S. Aniţa, V. Arnăutu, and R. Ştefănescu. Numerical optimal harvesting for a periodic age-structured population dynamics with logistic term. *Numerical Functional Analysis and Optimization*, 30(3–4):183–198, 2009.

[3] L.-I. Aniţa, S. Aniţa, and V. Arnăutu. Global behavior for an age-structured population model with logistic term and periodic vital rates. *Applied Mathematics and Computation*, 206:368–379, 2008.

[4] A. Belyakov and V. Veliov. Constant versus periodic fishing: Age structured optimal control approach. *Mathematical Modelling of Natural Phenomena*, 9:20–37, 1 2014.

[5] D. A. Carlson, A. B. Haurie, A. Leizarowitz, *Infinite horizon optimal control. Deterministic and stochastic systems*, Springer, Berlin, 1991.

[6] F. Colonius, W. Kliemann. Infinite time optimal control and periodicity, *Applied Mathematics and Optimization.* 20(1):113–130, 1989

[7] G. Feichtinger, G. Tragler, and V. Veliov. Optimality conditions for age-structured control systems. *J. Math. Anal. Appl.*, 288(1):47–68, 2003.

[8] L. Grüne. On the relation between discounted and average optimal value functions. *Journal of Differential Equations*, 148(1):65–99, 1998.

[9] R. Hartl and G. Feichtinger. A new sufficient condition for most rapid approach paths. *Journal of Optimization Theory and Applications*, 54(2):403–411, 1987.

[10] M. Iannelli. *Mathematical theory of age-structured population dynamics.* Applied mathematics monographs. C.N.R., Giardini editori e stampatori, Pisa, 1995.

[11] P. Landi, C. Hui, and U. Dieckmann. Fisheries-induced disruptive selection. *Journal of Theoretical Biology*, 365:204–216, 2015.

[12] E.J. Park, M. Iannelli, M.Y. Kim, and S. Anita. Optimal Harvesting for Periodic Age-Dependent Population Dynamics. *SIAM J. Appl. Math.*, 58(5):1648–1666, 1998.

[13] H. Thieme. *Mathematics in Population Biology.* Mathematical Biology Series, Princeton University Press, 2003.

[14] V.M. Veliov. Numerical Approximations in Optimal Control of a Class of Heterogeneous Systems. Submitted. Preprint: Research Report 2015-01, ORCOS, TU Wien, 2015, http://orcos.tuwien.ac.at/research/research_reports/

[15] G. Webb. *Theory of Nonlinear Age-Dependent Population Dynamics.* Marcel Dekker, New York, 1985.