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On the Optimal Control of Heterogeneous Systems

Bernhard Skritek

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Operations Research and Control Systems
Institute of Statistics and Mathematical Methods in Economics
Vienna University of Technology

Research Unit ORCOS
Wiedner Hauptstraße 8 / E105-4
1040 Vienna, Austria
E-mail: orcos@tuwien.ac.at

Abstract

Heterogeneity plays an important role in modeling demographic, epidemic, biological and economical processes. The mathematical formulation of such systems can vary widely: age-structured systems, trait-structured systems, or systems with endogenously changing domains are some of the most common. Controls in such systems may be non-distributed (targeting the whole system), or distributed (targeting one particular part of the system); some have to further satisfy constraints, such as integral constraints. This thesis investigates necessary optimality conditions of Pontryagin's type involving a Hamiltonian functional.

At first, infinite horizon age-structured systems are analyzed. They are governed by partial differential equations with boundary conditions, coupled by non-local integral states. Despite the numerous applications in population dynamics and economics, which are naturally formulated on an infinite horizon, a complete set of optimality conditions is missing, because of the difficult task of defining appropriate transversality conditions.

For problems on the infinite horizon, the objective value may become infinite. Therefore, the necessary optimality conditions are derived for controls that are weakly overtaking optimal. To prove the result, a new approach (recently developed for ordinary differential equations by S. Aseev and V. Veliov) has been used for a system affine in the states, but non-linear in the controls and with a non-linear objective function.

Furthermore, a problem which arises in demography is studied. Due to a low birth rate, many countries need immigration to sustain their population size. The age of the immigrants has severe implications on the stability of social security systems, therefore, the optimal age-pattern of immigrants is studied. The problem is on the infinite time horizon with a rather specific equality constraint. It is shown that there exists an optimal solution (although the problem is non-concave), and that this optimal control is time-invariant. A numerical case study is carried out for the Austrian population.

A second focus lies on heterogeneous systems with a fixed domain of heterogeneity, which are

used in epidemiology to describe the spreading of contagious diseases, but are also employed in economics. While necessary optimality conditions for problems on the finite horizon are known, a Hamiltonian formulation was missing. A Hamiltonian functional is introduced and its constancy shown for autonomous problems. This functional also allows to reproduce the primal and the adjoint system. With explicitly defined solutions of the adjoint system (using the above mentioned approach), it is proved that for a problem on the infinite horizon, any weakly overtaking optimal control maximizes this Hamiltonian. The model is non-linear, and the non-local integral states (which do not depend explicitly on the control) may enter the objective function and the differential equation of the distributed states.

The third type of heterogeneous systems considered in this thesis deals with models in which the domain of heterogeneity changes endogenously. Such systems arise, for example, for a profit-maximizing company which can invest to improve existing products, or invest in research to increase the variety of products. A maximum principle for such systems was proved by A. Belyakov, Ts. Tsachev, and V. Veliov. However, the strong form, in which the differential inclusion for the adjoint variable collapses to a differential equation, holds only under an a priori regularity assumption on the optimal control. It is shown for a certain optimal control problem arising in economics, that this regularity assumption is fulfilled. Additionally, in case of stationary data, it is proved that the Hamiltonian is constant along the optimal control.

Contents

Chapter 1

Motivation

Heterogeneous dynamic systems play an important role in modeling biological, demographic, or economic phenomena. They can be used, for example, to describe populations, epidemics, or evolution of production capital. Heterogeneity occurs in such systems due to differences with respect to preferences, technologies, abilities, wealth, space, age, size, etc. In most of these models, there exists a variable that can be influenced by a decision maker. Therefore, it is natural to consider optimal control theory for such systems, where the task is to find optimal solutions, maximizing some objective function.

This thesis deals with necessary optimality conditions of Pontraygin's type for different types of heterogeneous models: age-structured models on an infinite horizon arising, for example, in biology and demography; trait-structured models on an infinite horizon occurring in, for example, economics and epidemiology; and models with endogenously changing domain of heterogeneity with economic applications.

Heterogeneous control systems can be considered as extensions of dynamic control systems governed by ordinary differential equations (ODEs). The theory for optimal control of ODE-systems has been developed further a lot since the second half of the 20th century. In particular the work by Lev Semenovich Pontryagin and his collaborators, and one of their publications [37], need to be mentioned. There are a lot of results available: existence of solutions; necessary and sufficient optimality conditions of first and higher order; controllability; Hamiltonian representations (satisfying in many respects the properties of a Hamiltonian as in classical mechanics) and Hamilton-Jacobi theory; stability results, etc. Still, some tasks remain difficult, for example, optimal control on an infinite horizon and the treatment of state constraints.

In contrast to ODE problems, in heterogeneous systems the state variable is infinite-dimensional. Stability results and necessary optimality conditions of Pontryagin's type have been proved for certain systems, especially for age-structured systems on a finite horizon. At the same time, the number of successful applications of heterogeneous control systems in biology, demography, economics, epidemiology, and even physical systems, has drastically increased. Exemplary references will be given in the following paragraphs.

Many heterogeneous models can (under the right regularity assumptions) be integrated in one formal framework, as it is shown in [54]. However, this generality increases the complexity of the analysis, therefore, we stick in this thesis to specialized notations for each type of problem. This makes it easier to underline the important points. Although we will use different notations, some analogies in the results will appear, which can be explained by the existence of such general framework.

The following paragraphs shortly motivate the different types of problems studied in this thesis, more detailed introductions to the models and presentation of the results obtained follow in the corresponding chapters. The structure of the thesis will be presented thereafter.

Age-structured Control Models

Age-structured models provide a basic tool for modeling populations. Therefore, applications can be found in biology (e.g., for competitive and non-competitive species, [32, 33]), epidemiology (e.g. in HIV treatment [26]), social sciences (e.g., in marketing [29] or drug treatment [2]). In economy, recently age has been used to differentiate between machines of different vintages (dates of production), see, for example, [10, 21, 22, 23], but also in overlapping generation models (OLG) in labor and population economics, [24, 38, 39].

Mathematically, we deal with infinitely many ordinary differential equations (the McKendrick equations), where a single ODE describes the evolution of the population born at a certain time. They are coupled together by non-local dynamics and endogenous boundary conditions (see, e.g., [3]). The number of newborns is given by some boundary condition, which may depend on the state of the remaining population. The differential equations may be coupled by non-local integral states or some non-distributed control.

For the finite horizon, there are results on necessary conditions and stability of such systems available in the literature (see, e.g., [14, 25, 3, 33]). Some problems remain challenging, for example, if the time horizon is infinite, in which case results exist only for special cases (e.g. in [33, 27]).

We discuss necessary optimality conditions for a general age-structured system on an infinite horizon, and a model arising in demography. For the latter, properties of the optimal control such as existence, uniqueness, and stationary are proved additionally.

Trait-structured Control Models

Trait-structured control models are also referred to as parametric parameter models. In this thesis, the term "trait" is chosen, because such models are, for example, used in epidemiology to differentiate a population by a given trait. The heterogeneity of the population can describe different susceptibilities or the contact rates, which strongly influence the probability of an infection, [19]. In economy, the trait can be some indicator for a continuum of different production technologies, or the trait describes different abilities or initial resources of people, [46, 50].

Mathematically, we have infinitely many ordinary differential equations, coupled by aggregate states, which may enter the dynamics, as well as the objective function.

Such systems have received yet less attention in the literature. As in the age-structured case, one can find some results on the finite horizon (e.g. necessary optimality conditions of Pontryagin's type for a wide class of trait-structured systems are proved in [52]). However, the case of an infinite has not yet been investigated more closely. Also, a Hamiltonian formulation has received little attention so far.

We analyze necessary optimality conditions of Pontryagin's type for a trait-structured control problem on an infinite horizon. Additionally, a Hamiltonian functional is presented, which is for autonomous problems constant along the optimal control and the optimal trajectories.

Models on a Controlled Domain

Models with endogenously changing domain can be considered as extension of trait-structured systems. Consider, for example, a variety of different production technologies. A company can invest to improve the production capacity, but it can also invent new technologies, which increases the domain of available traits. This occurs in particular when modeling endogenous economic growth, but other applications of the general system are also possible.

In economic literature, often a variety of goods is present. However, in many cases (e.g. [20, 30, 48]), these are viewed as identical. In some economic growth literature (such as [40]), the problem is disentangled by having different decision makers for increasing the variety of goods and investments in production technologies. Others (such as [11, 47]) allow for heterogeneous products but neglect the dynamics of production capital.

The mathematical theory for such problems is not yet far developed. A maximum principle has been presented recently in [12], although it takes a non-standard form because the objective value is non-differentiable as a function of the controls. The second version presented in the same paper, which is easier to apply, only holds true under additional “regularity assumptions” for the optimal control, which are hard to prove a priori.

Therefore, we prove for a certain control problem arising in economics that the “regularity assumptions” hold for the optimal control. Further, the right formulation of a Hamiltonian for this problem is discussed.

Structure of the Thesis

The thesis is structured as follows:

In Chapter 2 some notations are introduced, and some well-known facts about differential equations and integral equations are repeated. Furthermore, different notions of optimality are discussed and some recent results from the literature on infinite-horizon optimal control problems for ordinary differential equations are presented.

Chapter 3 deals with age-structured systems on an infinite horizon. The distributed state is governed by a partial differential equation and complemented by an initial condition and a boundary condition depending on a non-local integral state. Two different models are analyzed: the first (which is based on a joint publication with V. Veliov, [45]) is more general in many respects, while the latter stems from an application in demography (and is based on the joint work with C. Simon and V. Veliov, [42]).

The model considered in Chapter 4 is motivated by applications in epidemiology, but is also interesting in economics. It is a trait-structured model on an infinite horizon, for which necessary optimality conditions of Pontryagin’s type are proved. Additionally, it is proved that the maximized Hamiltonian functional is constant along the optimal trajectory for autonomous problems. It is based on joint work with V. Veliov, a publication is in preparation.

In Chapter 5, an economic problem is analyzed on an endogenously changing domain. A company can increase it’s variety of products by investments in research and development. Furthermore, it is proved that the maximized Hamiltonian is constant for autonomous problems. It is based on a joint work with Ts. Tsachev and V. Veliov, [44].

The results and potential future work are discussed in Section 6.

Chapter 2

Preliminaries

In this chapter we state some facts that will be used throughout the thesis. In Section 2.1, some notational conventions are introduced. Section 2.2 deals with fundamental matrix solutions for ordinary differential equations (ODEs). In Section 2.3, different notions of optimality are presented, while Section 2.4 summarizes recent literature on the Pontryagin maximum principle for ODEs. Some facts about Volterra integral equations are the topic of Section 2.5.

2.1 Notational Conventions

In the whole thesis, square brackets denote closed intervals, round ones open intervals. By t we denote time, which takes ranges in $[0,T]$ with $0 < T < \infty$ or $T = \infty$. The parameter of heterogeneity will vary and will be a in the age-distributed systems of Chapter 3, and σ in Chapter 4 and 5, and takes values in varying domains, which will be stated in the corresponding sections.

Let (P,Π,μ) be a measurable space. In this thesis, P will be a subset of $\mathbb{R}$ or $\mathbb{R}^2$ with Π being the Lebesgue-Sigma-Algebra and μ the Lebesgue-Borel-measure. Consider the set of all measurable functions from P to $\mathbb{R}$, whose absolute value is integrable, that is: $f \in L_1(P)$ if and only if

$$\|f\|_{L_1(P)} := \int_P |f|\,\mathrm{d}\mu < \infty,$$

and two functions f and g are identified with each other if they are equal except on a set of measure zero. By $L_\infty(P)$, the space of all functions $f : P \to \mathbb{R}$ such that

$$\|f\|_{L_\infty(P)} := \inf\{C > 0 : |f(x)| \leq C \text{ for a.e. } x \in P\} < \infty,$$

where here and in the rest of the thesis *a.e.* stands for almost everywhere and means except on a set of measure zero. By $L_\infty^{\mathrm{loc}}(P)$ we denote the set of locally bounded functions, that is, all functions that are in $L_\infty(Q)$ for every compact subset $Q \subset P$.

The measure of a set P is denoted by $\mathrm{meas}(P)$, while $\mathrm{co}\{a,b\}$ denotes the convex hull of a and b, that is, all points $c := ax + b(1-x)$ with $x \in [0,1]$.

The variables (x,y,z) denote the states of the systems: $x(t)$ is a non-distributed state, which follows an ordinary differential equation; $y(t,\cdot)$ is a distributed state, which obeys a partial differential equation; $z(t)$ denotes a non-local integral state. The states $(x(t),y(t,a),z(t))$ (Chapter 3) or $(x(t),y(t,\sigma),z(t))$ (Chapter 4 and 5) may take values in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_z}$ with n_x, n_y, and n_z some natural numbers.

As a convention, we use capital letters to denote sets and matrices, lower-case Latin letters to denote numbers and column vectors (thus, all state vectors are column vectors), and Greek lower-case letters to denote numbers and row-vectors. There will be a few exceptions (such as δ and Δ) which will not lead to confusions.

Adjoint variables are row vectors and therefore denoted by greek letters (ψ,ξ,ζ) in the corresponding dimensions. They satisfy certain adjoint equations, which will always be introduced in the corresponding section.

Distributed controls will be denoted by $u(t,\cdot)$ and the non-distributed by $v(t)$. In general, we consider measurable and locally bounded control functions $(u,v) \in \mathscr{U} \times \mathscr{V} := L_\infty^{loc}(D;U) \times L_\infty^{loc}([0,\infty);V)$ that take values in some subsets U and V of finite dimensional euclidean spaces. Deviations from this notion will be stated where needed.

An optimal control is denoted by $(\hat{u},\hat{v})$ and the corresponding states (the solutions of the dynamic system corresponding to these controls) by $(\hat{x},\hat{y},\hat{z})$. The solution of the adjoint system with the optimal control and trajectories inserted is denoted by $(\hat{\psi},\hat{\xi},\hat{\zeta})$.

In order to emphasize the dependence of $x(t)$ on a certain control u, the notation $x[u](t)$ is used. The dependence on an initial condition $x(0) = x_0$ is emphasized by the notation $x(t;x_0)$.

We will make use the following notational convention: Given a function $f(t,a,y(t,a),z(t),u(t,a),v(t))$, the dependency on (t,a) is omitted, $f(t,a,y,z,u,v)$. Furthermore, we skip variables with a "hat" when they appear as variables of other functions, e.g. $f(t,a,u) := f(t,a,\hat{y}(t,a),\hat{z}(t),u(t,a),\hat{v}(t))$.

Partial derivatives of the function $f(t,a,y,z,u,v)$ are denoted by an index, e.g. $f_y(t,a,y,z,u,v) := \frac{\partial}{\partial y} f(t,a,y,z,u,v)$. Derivation with respect to time will be denoted by a dot, $\dot{x}(t) := \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, or

$\dot{y}(t,\sigma) := \frac{\mathrm{d}}{\mathrm{d}t} y(t,\sigma)$. By $\mathscr{D}$ we denote the directional derivative

$$\mathscr{D}y(t,a) := \lim_{\varepsilon \to 0} \frac{y(t+\varepsilon, a+\varepsilon) - y(t,a)}{\varepsilon}.$$

The integrand of the objective function is denoted by L, while the value of the objective function on time horizon $[0,T]$ for given controls (u,v) is denoted by $J_T(u,v)$. In case $T = \infty$, we write just $J(u,v)$.

2.2 Fundamental Matrix Solutions

Consider on $[0,\infty)$ the differential equation

$$\dot{x}(t) = f(t)x(t), \qquad x(0) = x_0, \tag{2.1}$$

with some measurable and locally bounded matrix-valued function $f(t)$ and some arbitrary $x_0 \in \mathbb{R}^{n_x}$. Then there exists a unique solution $\bar{x}(\cdot)$, defined on the whole interval $[0,\infty)$, and this solution is locally bounded. Furthermore, there exists a fundamental matrix $X(t)$ such that $\bar{x}(t) = X(t)x_0$. That is, the columns $X^i(t)$, $i = 1,\dots,n$, are solutions to (2.1) with initial condition $X^i_j(0) = \delta_{i,j}$, where $\delta_{i,j} = 1$ if $i = j$ and zero otherwise.

Define the state transition matrix $X(t,\tau) := X(t)X(\tau)^{-1}$, then it satisfies $\bar{x}(t) = X(t,\tau)\bar{x}(\tau)$, where $\bar{x}$ is a solution of (2.1). The slight overload of notation will not lead to a confusion.

It is easy to verify, that the inverse $X(t)^{-1}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(X(t)^{-1}\right) = -X(t)^{-1} f(t), \qquad X^{-1}(0) = I.$$

A more detailed presentation can be found, e.g., in [15, Appendix A.2].

2.3 Notion of Optimality

We now discuss different notions of optimality.

Consider any optimal control problem on the infinite horizon $[0,\infty)$. We call a pair of controls $(u,v) \in \mathscr{U} \times \mathscr{V}$ admissible, if the corresponding state trajectories $(x, y(\cdot), z)$ exist on $[0,\infty)$ and the objective functional is locally integrable. Then for every $T > 0$, and every admissible pair, the value $J_T(u,v)$ is finite.

The notion of *strong optimality* requires for an admissible control $(\hat{u},\hat{v})$ to be optimal, that the value $J(\hat{u},\hat{v})$ is finite, and that for any other admissible pair (u,v) it holds that

$$J(\hat{u},\hat{v}) \geq \limsup_{T\to\infty} J_T(u,v),$$

The assumption of a finite objective value is not always appropriate when considering problems on an infinite horizon. It can be relaxed by considering other notions of optimality. The following notion proves to be more useful:

Definition 2.1 (Weak Overtaking Optimality (WOO))**.** A set of admissible controls $(\hat{u},\hat{v})$ is weakly overtaking optimal, if for any other admissible pair of controls (u,v) and for every $\varepsilon > 0$ and $T > 0$, there exists a $T' > T$ such that

$$J_{T'}(\hat{u},\hat{v}) \geq J_{T'}(u,v) - \varepsilon.$$

The WOO is weaker than the strong optimality, but stronger than, for example, finite optimality (see, e.g., [31]). If for all (u,v) the integral $J(u,v)$ is convergent, then weak overtaking optimality implies strong optimality. For a more complete presentation of the different optimality conditions, see, e.g., [16, Section 1.5].

In presenting some recent results in Section 2.4, the following local version is used. It is a slight generalization of WOO of mainly academic interest:

Definition 2.2 (Locally Weak Overtaking Optimality (LWOO))**.** An admissible pair $(\hat{v},\hat{x})$ is locally weakly overtaking optimal if there exists $\delta > 0$ such that for any other admissible pair (v,x) satisfying

$$\text{meas}\{t \geq 0 : v(t) \neq \hat{v}(t)\} \leq \delta,$$

and for arbitrary $\varepsilon > 0$, $T > 0$ there exists a $T' > T$ such that

$$J_{T'}(\hat{v}) \geq J_{T'}(v) - \varepsilon.$$

2.4 The Pontryagin Maximum Principle

In order to demonstrate the underlying ideas that will be used in Chapter 3, 4, and 5, we present in this subsection a Pontryagin maximum principle and a Hamiltonian for a non-distributed optimal control problem. For detailed proofs see the original publications: [7] for the maximum principle; and [37] for the assertions about the Hamiltonian.

Consider the optimal control problem

$$\max_{v} \int_0^\infty L(t,x(t),v(t))\,\mathrm{d}t, \tag{2.2}$$

subject to

$$\dot{x}(t) = f(t,x(t),v(t)), \qquad x(0) = x_0, \tag{2.3}$$

$$v(t) \in V, \tag{2.4}$$

where $x_0 \in G$, with G being a nonempty open convex subset of $\mathbb{R}^{n_x}$, and V an arbitrary nonempty set in $\mathbb{R}^m$. Let the following assumptions hold:

Assumption 2.1. *[7, Assumption (A1)] The functions $f : [0,\infty) \times G \times V \to \mathbb{R}^n$ and $L : [0,\infty) \times G \times V \to \mathbb{R}$, together with their partial derivatives $f_x(\cdot,\cdot,\cdot)$ and $L_x(\cdot,\cdot,\cdot)$ are defined and locally bounded; measurable in t for every $(x,v) \in G \times V$ for almost every $t \in [0,\infty)$, and continuous in (x,u) for almost every $t \in [0,\infty)$.*

Denote by $\hat{v} \in \mathscr{V}$ an optimal control, and let $\hat{x}$ be the corresponding trajectory.

Assumption 2.2. *[7, Assumption (A2)] There exist a number $\gamma > 0$ and a non-negative integrable function $\lambda : [0,\infty) \to \mathbb{R}$, such that for every $\tilde{x}_0 \in G$ with $|\tilde{x}_0 - x_0| < \gamma$, equation (2.3) with $v(\cdot) = \hat{v}(\cdot)$ and initial condition $x(0) = \tilde{x}_0$ instead of $x(0) = x_0$ has a solution $\tilde{x}(\cdot;\tilde{x}_0)$ on $[0,\infty)$ in G, and*

$$\max_{\theta \in \mathrm{co}\{\tilde{x}(t;\tilde{x}_0),\hat{x}(t)\}} |L_x(t,\theta,\hat{v}(t))(\tilde{x}(t;\tilde{x}_0) - \hat{x}(t))| \le |\tilde{x}_0 - x_0|\lambda(t).$$

With X being the fundamental matrix solution of the equation $\dot{x}(t) = f_x(t,\hat{x},\hat{u})x$ (compare with Section 2.2), define

$$\hat{\psi}(t) := \int_t^\infty L_x(s)X(s)\,\mathrm{d}s\,X(t)^{-1}. \tag{2.5}$$

Due to Assumption 2.2, the integral is absolutely convergent, therefore, it is well defined.

Define the Hamilton-Pontryagin function

$$\mathscr{H}(t,x,v,\psi) := L(t,x,v) + \psi f(t,x,v).$$

Theorem 2.3. *[7, Theorem 4.1] Let Assumption 2.1 be fulfilled and $\hat{v}$ be an admissible LWOO control and $\hat{x}$ the corresponding trajectory and let Assumption 2.2 hold for $(\hat{v},\hat{x})$. Then function $\hat{\psi}$ defined in (2.5) is locally bounded, locally absolutely continuous, and satisfies the core conditions of the normal form maximum principle, i.e.*

(i) $\hat{\psi}$ is a solution of the (present value) adjoint system

$$\dot{\psi}(t) = -\mathscr{H}_x(t, \hat{x}(t), \hat{v}(t), \psi(t)),$$

(ii) the maximum condition holds for almost every $t \in [0, \infty)$,

$$\mathscr{H}(t, \hat{x}(t), \hat{v}(t), \hat{\psi}(t)) = \max_{v \in V} \mathscr{H}(t, \hat{x}(t), v, \hat{\psi}(t)).$$

Note that this maximum principle is in "normal form", that is, the Lagrange multiplier of the objective functional is equal to one. Furthermore, there is no transversality condition involved, because the adjoint variable is defined explicitly. The adjoint variable does not have to be bounded either, but only the integral has to be absolutely convergent – the term $X(t)^{-1}$ can be unbounded.

From (2.5), one can derive (because of the absolute convergence of the integral) a condition which can be seen as a transversality condition:

$$\lim_{t \to \infty} \hat{\psi}(t) X(t) = 0. \tag{2.6}$$

It is now easy to relate this "transversality condition" to the "classical" transversality conditions

$$\lim_{t \to \infty} \hat{\psi}(t) = 0, \qquad \lim_{t \to \infty} \hat{\psi}(t) \hat{x}(t) = 0.$$

For example, if the fundamental matrix solution X behaves asymptotically like the state x (that is, it describes the behavior of the optimal state $\hat{x}$ sufficiently well), then the latter is fulfilled. The first one holds, for example, if the fundamental matrix solution is strictly positiv, $|X| \geq c_0 > 0$.

Remark 2.4. In the famous example by Halkin (see [31] or [7, Section 5]), $L(t,x,v) = f(t,x,v) = (1-x)v$. Thus, the fundamental matrix solution is $X(t) = e^{-\int_0^t \hat{v}(s)\,\mathrm{d}s}$ while the state is $\hat{x}(t) = 1 - X(t)$. The adjoint variable itself is constant over time, $\hat{\psi}(t) \equiv -1$. Therefore, the adjoint itself does not converge to zero, but also the adjoint times the state does not converge to zero. However, condition (2.6) is fulfilled, and the maximum principle in Theorem 2.3 is fulfilled.

The Hamilton function, as defined above, contains also the information about the primal and adjoint system:

$$\begin{aligned} \dot{x}(t) &= \mathscr{H}_\psi(t, \hat{x}(t), \hat{v}(t), \hat{\psi}(t)), \\ -\dot{\psi}(t) &= \mathscr{H}_x(t, \hat{x}(t), \hat{v}(t), \hat{\psi}(t)). \end{aligned}$$

For an autonomous problem along the optimal trajectory, the Hamiltonian is constant:

Theorem 2.5. *[37, Lemma 7] Let L and f do not depend explicitly on t. Denote by $\hat{v}$ the optimal control and the corresponding trajectory by $\hat{x}$. Then the maximized Hamiltonian*

$$\max_{v\in V} \mathscr{H}(\hat{x}(t), v, \hat{\psi}(t)) \tag{2.7}$$

is constant on every interval $[0,T]$, $T > 0$.

The following properties of the Hamiltonian are of particular importance in the context of optimal control:

(i) the primal and the dual system can be reproduced,

(ii) the maximum principle is fulfilled with a solution of the adjoint system,

(iii) for autonomous problems, the functional is constant along optimal trajectories.

2.5 Volterra Equation

Consider for $t \in [0,T]$, with $0 < T \leq \infty$ the Volterra integral equation of the second kind,

$$y(t) = f(t) + \int_0^t K(t,s)y(s)\,\mathrm{d}s, \tag{2.8}$$

for a measurable and locally bounded $(m \times m)$-matrix function $K(t,s)$ and a locally bounded m-dimensional function f.

The locally bounded kernel $K(t,s)$ defines a resolvent $R(t,s)$ which is a priori also defined for $t \in [0,T]$ and $s \in [0,t]$. The resolvent R is itself a measurable and locally bounded $(m \times m)$-matrix function.

According to Definition 3.2 in [28, Chapter 9], the resolvent satisfies

$$R(t,s) = K(t,s) + \int_s^t K(t,x)R(x,s)\,\mathrm{d}x = K(t,s) + \int_s^t R(t,x)K(x,s)\,\mathrm{d}x, \tag{2.9}$$

and a solution of the integral equation (2.8) is given by [28, Chapter 9, Theorem 3.6]

$$y(t) = f(t) + \int_0^t R(t,s)f(s)\,\mathrm{d}s.$$

For convenience, we extend the definition of $R(t,s)$ and $K(t,s)$ to $[0,T] \times [0,T]$ by setting $R(t,s) = K(t,s) = 0$ for $s > t$.

The following two statements can be easily checked by using (2.9). If equation (2.8) is considered on the restricted interval $[\tau, T]$, then the resolvent does not change. Thus, for

$$y(t) = f(t) + \int_{\tau}^{t} K(t,s)y(s)\,\mathrm{d}s, \tag{2.10}$$

the unique locally bounded solution $y(t)$ is given by

$$y(t) = f(t) + \int_{\tau}^{t} R(t,s)y(s)\,\mathrm{d}s.$$

Neither does the resolvent change, when the equation is considered inverse in time. Let $\tau > 0$ and $t \in [0, \tau]$, then a solution of

$$\zeta(t) = \beta(t) + \int_{t}^{\tau} \zeta(s)K(s,t)\,\mathrm{d}s, \tag{2.11}$$

is given by

$$\zeta(t) = \beta(t) + \int_{t}^{\tau} \beta(s)R(s,t)\,\mathrm{d}s. \tag{2.12}$$

The converse implication is also true: For $\tau < \infty$ define ζ by (2.12), then it solves equation (2.11). If $\tau = \infty$ the assertion is true if the integral in (2.12) is convergent when $\tau \to \infty$ and locally bounded in t.

Denote by $L^r_\infty(0,T)$ the space of functions that are uniformly bounded when multiplied by e^{-rt}. That is, a function $\beta(t)$ belongs to $L^r_\infty(0,T)$ if $\|\beta(t)e^{-rt}\|_{L_\infty(0,T)} < \infty$. For example, $e^{rt} \in L^r_\infty(0,T)$.

The norm of the operator $(\mathscr{K}\beta)(t) := \int_t^\infty \beta(s)K(s,t)\,\mathrm{d}s$ as an operator from $L^r_\infty(0,\infty)$ into itself is given by (cf. [28, Chapter 9, Definition 2.2 and Theorem 2.7])

$$\|\mathscr{K}\| = \operatorname{ess\,sup}_{t\in[0,\infty)} \int_{t}^{\infty} |K(s,t)|e^{r(s-t)}\,\mathrm{d}s. \tag{2.13}$$

If

$$\|\mathscr{K}\|_{L^r_\infty(0,\infty)\to L^r_\infty(0,\infty)} < 1,$$

then the operator $\mathscr{R}$ defined as $(\mathscr{R}\beta)(t) := \int_t^\infty \beta(s)R(s,t)\,\mathrm{d}s$ maps the L^r_∞ into itself and for $\beta \in L^r_\infty(0,\infty)$, $\gamma(t) := \beta(t) + (\mathscr{R}\beta)(t)$ is an element of $L^r_\infty(0,\infty)$.

Chapter 3

Age-Structured Optimal Control Problems on an Infinite Horizon[1]

Age-structured dynamic systems provide a main tool in demography and biology for modeling populations, see, for example, [55, 33]. Recently, they are also employed in economics, where age is involved in order to distinguish machines or technologies of different vintages (dates of production). Optimal control problems for such systems are also widely investigated (see, e.g., [10, 13, 25] and the bibliography therein). Most of these problems are naturally formulated on an infinite time-horizon.

Infinite-horizon optimal control problems are still challenging even for systems of ordinary differential equations (see, e.g., the recent contributions [5, 7] or the summary in Section 2.4). The key issue is to define appropriate *transversality conditions*, which allow to select the "right" solution of the adjoint system for which the Pontryagin maximum principle holds. In the infinite dimensional case (including age-structured systems), this issue is open, especially in the case of non-local dynamics or boundary conditions as considered here.[2] This is one reason for which often optimal control problems are considered on a truncated time-horizon (see, e.g., [25, 3, 23, 38, 39] and the examples in Section 3.2.6), although the natural formulation is on infinite horizon.

In this chapter, one rather general and one specific demographic age-structured optimal control model are investigated. They differ by the dependence of the functions on the controls and the non-local integral states. In both cases, necessary optimality conditions of Pontryagin's type are derived. The set of conditions is "complete", that is, the solution of the adjoint system, for which the maximization condition of the Pontryagin maximum principle holds is defined in a unique way.

[1] This chapter is based on the joint publications [45] with V. Veliov, and [43] with C. Simon and V. Veliov.

[2] We do not mention some publications, where transversality conditions are introduced ad hoc or based on non-sound arguments (see the recent paper [35] for more information). There are some exceptions, out of which we mention [10], where, however, the dynamics and the boundary conditions are local.

The approach implemented was recently developed for ODEs in [5, 7, 6], and defines the solution of the adjoint system explicitly, omitting the necessity of transversality conditions. The extension to age-structured models is, however, not straightforward and requires substantial additional work. Note that the obtained maximum principle is in normal form (in contrast to many known results for ODEs), that is, with the Lagrange multiplier of the objective functional equal to one.

Before presenting the models, in Section 3.1 some important results from the literature for age-structured systems are summarized.

A rather general model is considered in Section 3.2. The non-local integral states enter the boundary condition (which depends also on a control) and the objective function. The dynamics of the control system is affine in the states, but nonlinear in the controls. The objective function may be nonlinear in both the states and the controls. The objective value may be unbounded for some admissible controls, therefore the notion of strong optimality is not appropriate, and the notion of weak overtaking optimality is used, which compares the intertemporal behavior of controls (cf. Section 2.3). Overall, this approach is also implementable in the non-linear case, where known additional arguments from the stability theory for (non-local) age-structured systems have to be involved.

In Section 3.3, a human population is considered, which needs immigration in order to keep its size constant. The control is the age-profile of immigrants which can vary only within certain bounds. The optimal control maximizes some functional, which corresponds to the stability of the social system and depends linearly on the population and immigration rate. From a mathematical point of view, in addition to the distributed system on an infinite horizon, the challenges are (i) that the problem involves a state constraint, although in a rather specific form, and (ii) that we deal with the maximization of a non-concave functional where the existence of a solution and the well-posedness are problematic. The main contributions are (i) the proof of existence of an optimal solution, (ii) the derivation of a Pontryagin maximum principle is derived, (iii) under an additional generic well-posedness condition, the proof of time-invariance of the optimal solution, and its independence of the initial distribution. Additionally, a case study for the Austrian population is carried out.

3.1 Preliminaries for Age-Structured Systems

In this section, we remind some known facts and provide some auxiliary material for age-structured systems, which will be used in the sequel of the chapter.

This chapter deals with optimal control problems of the following form

$$\max_{u,v} J(u,v) := \int_0^\infty \int_0^\omega L(t,a,y,z,u,v)\,\mathrm{d}a\,\mathrm{d}t, \tag{3.1}$$

subject to the state dynamics,

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) y(t,a) = f(t,a,y,z,u,v), \qquad y(0,a) = y_0(a), \tag{3.2}$$

$$y(t,0) = \varphi(t,z,v), \tag{3.3}$$

$$z(t) = \int_0^\omega h(t,a,y,u,v)\,\mathrm{d}a, \tag{3.4}$$

and the control restrictions

$$u(t,a) \in U, \qquad v(t) \in V. \tag{3.5}$$

The following assumptions are standing in this section.

The sets U and V are subsets of finite dimensional Euclidean spaces; V is convex. The functions L, f, φ, h, y_0, together with the partial derivatives L_y and L_z, and the partial derivatives of all the functions above with respect to v, are locally bounded, measurable in (t,a) for every (y,z,u,v), and locally Lipschitz continuous in (y,z,u,v). The maximal age ω will be finite in Section 3.2, and infinite in Section 3.3. Denote the complete domain by $D := [0,\infty)\times[0,\omega]$ and let D_T be the truncated horizon, $D_T = [0,T]\times[0,\omega]$.

We use the classical partial differential equation (PDE) representation of the transport-reaction equation (3.2), although the left-hand side should be interpreted as the directional derivative of y in the direction $(1,1)$:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) y(t,a) = \mathscr{D}y(t,a) := \lim_{\varepsilon\to 0+} \frac{y(t+\varepsilon,a+\varepsilon)-y(t,a)}{\varepsilon}. \tag{3.6}$$

Further on we use the notation $\mathscr{D}$ for this directional derivative.

The notion of admissible controls and of solutions will differ (due to the different maximal age and the problem statements) in the following sections and introduced in Section 3.2.1 and 3.3.1, respectively. They have in common that for admissible controls the state trajectories exist for $[0,\infty)$ and are locally bounded which will be needed in the considerations of the following subsections.

Let $(\hat{u},\hat{v})$ be an optimal solution and denote by $(\hat{y},\hat{z})$ the corresponding trajectories. We remind of the notational convention, that variables which are fixed at their optimal value are suppressed.

3.1.1 A Pontryagin Maximum Principle on the Finite Horizon

In [25], an optimal control system is considered on the finite horizon, which is similar to the one considered here. It is more general than (3.1)–(3.5): there exist additional state variables and an initial control, but in the following presentation, we simplify the results to meet our needs.

Define on $[0,T]$ the adjoint system

$$\begin{aligned} -\mathscr{D}\xi(t,a) =& \xi(t,a) f_y(t,a) + \zeta(t) h_y(t,a) + L_y(t,a), \qquad \xi(t,\omega) = \xi(T,a) = 0, \\ \zeta(t) =& \xi(t,0)\varphi_z(t) + \int_0^\omega [\xi(t,a) f_z(t,a) + L_z(t,a)]\,\mathrm{d}a, \end{aligned}$$

and the pre-Hamiltonian functional

$$\mathscr{H}(t,a,y,z,u,v,\xi,\zeta) := L(t,a,y,z,u,v) + \xi f(t,a,y,z,u,v) + \zeta h(t,a,y,z,u,v).$$

Theorem 3.1. *[25, Theorem 2] Let $(\hat{u},\hat{v})$ be a pair of optimal controls and $(\hat{y},\hat{z})$ the corresponding trajectories. Denote by $(\hat{\xi},\hat{\zeta})$ the unique solution of the adjoint system. Then the following holds true:*

$$\begin{aligned} &0 \leq \mathscr{H}(t,a,\hat{u}(t,a)) - \mathscr{H}(t,a,u), \qquad u \in U \\ &0 \leq \left[\int_0^\omega \mathscr{H}_v(t,a,\hat{v}(t))\,\mathrm{d}a + \hat{\xi}(t,0)\varphi_v(t,\hat{v}(t))\right](\hat{v}(t) - v), \qquad v \in V. \end{aligned}$$

3.1.2 Lipschitz Stability of the Primal System

Below we present a simplified and adapted version of the stability result in [25, Proposition 1], splitting it in two parts.

For a number $\tau \geq 0$ and a function $\delta \in L_\infty(0,\omega)$ we consider system (3.2)–(3.4) (with $(u,v) = (\hat{u},\hat{v})$) on $[\tau,\infty)$, with a "disturbed initial condition"

$$\tilde{y}(\tau,a) = \hat{y}(\tau,a) + \delta(a).$$

Denote by $(\tilde{y},\tilde{z})$ the corresponding solution on $[\tau,\infty)$.

Proposition 3.2. *For every $T > 0$ there exists a constant $c_0(T)$, such that for every $\tau \in [0,T)$,*

$\delta \in L_\infty(0,\omega)$, and $(\tilde{y},\tilde{z})$ defined as above, it holds that

$$\|\tilde{y}-\hat{y}\|_{L_k(D_T\setminus D_\tau)} + \|\tilde{z}-\hat{z}\|_{L_k([\tau,T])} \le c_0(T)\,\|\delta\|_{L_k(0,\omega)}, \quad k \in \{1,\infty\}.$$

Now, let $B(\tau,b;\alpha)$ denote the box $[\tau-\alpha,\tau]\times[b-\alpha,b]$, where $\tau>0$, $b\in(0,\omega]$. If $0<\alpha\le\alpha_0:=\min\{\tau,b\}$, then $B(\tau,b;\alpha)\subset D$. Let $\bar{u}: B(\tau,b;\alpha_0)\to U$ and $\bar{v}:[\tau-\alpha_0,\tau]\to V$ be two measurable and bounded functions. Consider again system (3.2)–(3.4) for the new pair of admissible controls:

$$u_\alpha(t,a) = \begin{cases} \bar{u}(t,a) & \text{for } (t,a)\in B(\tau,b;\alpha), \\ \hat{u}(t,a) & \text{for } (t,a)\notin B(\tau,b;\alpha), \end{cases} \qquad v_\alpha(t) = \begin{cases} \bar{v}(t) & \text{for } t\in[\tau-\alpha,\tau], \\ \hat{v}(t) & \text{for } t\notin[\tau-\alpha,\tau]. \end{cases}, \quad (3.7)$$

and denote by (y_α,z_α) the corresponding solution.

Proposition 3.3. *For each $\tau>0$, $b\in(0,\omega]$, and compact sets $\bar{U}\subset U$ and $\bar{V}\subset V$ there exists a constant c_0 such that for every $\alpha\in(0,\alpha_0]$ and measurable $\bar{u}:B(\tau,b;\alpha)\to\bar{U}$ and $\bar{v}:[\tau-\alpha,\tau]\to\bar{V}$ it holds that*

$$\begin{aligned} \|z_\alpha-\hat{z}\|_{L_\infty(0,\tau)} &\le c_0(\alpha+\|\bar{v}-\hat{v}\|_{L_\infty([\tau-\alpha,\tau])}), \\ \|y_\alpha-\hat{y}\|_{L_\infty(D_\tau)} &\le c_0(\alpha+\|\bar{v}-\hat{v}\|_{L_\infty([\tau-\alpha,\tau])}), \\ \|y_\alpha(t,\cdot)-\hat{y}(t,\cdot)\|_{L_1([0,\omega])} &\le c_0\alpha\left(\alpha+\|\bar{v}-\hat{v}\|_{L_\infty([\tau-\alpha,\tau])}\right), \quad t\in[\tau-\alpha,\tau]. \end{aligned}$$

Both propositions follow from [25, Proposition 1]. In the cited proposition, the non-distributed control does not enter the right hand side of (3.2) and (3.4). However, it can be treated as additional distributed control variable with $u_2(t,a)\equiv v(t)$, for which the results can be applied. Using then the triangular inequality and, for Proposition 3.3, the specific needle variation form of the disturbance (u_α,v_α), and the local Lipschitz property, proves our claims.

3.1.3 Fundamental Matrix Solution for Age-Structured Systems

Let $F(t,a)$ be a locally bounded matrix-valued function of corresponding dimensions and consider the differential equation

$$\mathscr{D}y(t,a) = F(t,a)\,y(t,a). \quad (3.8)$$

Define the set

$$\Gamma_0 := \{(t_0,a_0)\in D:\ \text{either } t_0=0 \text{ or } a_0=0\},$$

that is, Γ_0 is the lower left boundary of D. The *fundamental matrix solution* of (3.8), $X \in L_\infty^{\mathrm{loc}}(D)$, is defined as the $(n \times n)$-matrix solution of the equation

$$\mathscr{D}X(t,a) = F(t,a)X(t,a), \qquad X(t,a) = I \text{ for } (t,a) \in \Gamma_0, \tag{3.9}$$

where I is the identity matrix. The definition is correct, since the characteristic lines of (3.8) emanating from Γ_0 cover in a disjunctive way the domain D. For every $(t_0,a_0) \in \Gamma_0$ the function X can be defined on the characteristic line passing through (t_0,a_0) as $X(t_0+s,a_0+s) = Z(s)$, $s \in [0,\omega - a_0]$, where Z is determined by the equation

$$\dot{Z}(s) = F(t_0+s,a_0+s)Z(s), \quad Z(0) = I,$$

that is, it is the fundamental matrix solution as defined in Section 2, equation (2.1). Thus, for given side conditions on the lower-left boundary Γ_0, one can represent the solution of (3.2) in terms of X by the Cauchy formula for ODEs.

Moreover,

$$\mathscr{D}\left(X^{-1}(t,a)\right) = -X^{-1}(t,a)F(t,a), \quad (t,a) \in D. \tag{3.10}$$

3.2 A Maximum Principle for an Age-structured Optimal Control Problem on an Infinite Horizon[3]

In this section we deal with an age-structured optimal control problem on an infinite horizon. The dynamics is affine in the states to allow an analytic representation of the solutions, while the dependence on the controls may be non-linear. The objective function may be non-linear in both, the states and the controls. The boundary condition depends on a non-local integral state and a control. An extension of the presented results to non-linear systems is possible and requires some known stability results for non-local age-structured systems, in a similar way as in Chapter 4.

The main result gives necessary optimality conditions of Pontryagin's type for weakly overtaking optimal solutions. The adjoint variables for which the maximization condition holds are defined explicitly in a unique way, eliminating the need of transversality conditions. Furthermore, the maximum principle is in normal form, that is, with the Lagrange multiplier of the objective function equal to one. The approach is based on results in [5, 7, 6] for the case of ODEs. The extension to age-structured system, however, requires substantial work, some of which is rather technical.

This section is organized as follows. Section 3.2.1 presents the problem and some basic assumptions. In Section 3.2.2, the propagation of a variation in the initial condition is studied. The main result is formulated in Section 3.2.3. The proof follows in Section 3.2.4 while Section 3.2.6 presents some selected applications. Some technical proofs are moved to Section 3.2.5 for a better readability.

3.2.1 Formulation of the Problem

Consider the following optimization problem

$$\max_{u,v} \int_0^\infty \int_0^\omega L(t,a,y(t,a),z(t),u(t,a),v(t))\,\mathrm{d}a\,\mathrm{d}t, \tag{3.11}$$

[3]This section is based on the joint publication with V. Veliov, published in [45].

subject to

$$\mathscr{D}y(t,a) = F(t,a,u(t,a),v(t))\,y(t,a) + f(t,a,u(t,a),v(t)), \tag{3.12}$$

$$y(t,0) = \Phi(t,v(t))\,z(t) + \varphi(t,v(t)), \qquad y(0,a) = y_0(a), \tag{3.13}$$

$$z(t) = \int_0^\omega [H(t,a,u(t,a),v(t))\,y(t,a) + h(t,a,u(t,a),v(t))]\,\mathrm{d}a, \tag{3.14}$$

$$u(t,a) \in U, \tag{3.15}$$

$$v(t) \in V. \tag{3.16}$$

Here $(t,a) \in D := [0,\infty) \times [0,\omega]$, $\omega > 0$. The functions $y : D \to \mathbb{R}^n$ and $z : [0,\infty) \to \mathbb{R}^m$ represent the states of the system; $u : D \to U$ and $v : [0,\infty) \to V$ are control functions with values in the subsets U and V of finite-dimensional Euclidean spaces. The matrix- or vector-valued functions F, f, Φ, φ, H, h have corresponding dimensions. The considered system is affine in the states, while the integrand L in the objective functional (3.11) and the dependence on the controls can be non-linear. Note that $\mathscr{D}$ denotes the directional derivative as in (3.6).

The following assumptions are standing in this section.

Assumption 3.1. *The set V is convex. The functions F, f, Φ, φ, H, h, and L, together with the partial derivatives L_y, L_z and the partial derivatives with respect to v of all the above functions, are locally bounded, measurable in (t,a) for every (y,z,u,v), and locally Lipschitz continuous in (y,z,u,v).*[4]

The sets of *admissible controls*, $\mathscr{U}$ and $\mathscr{V}$, consist of all functions $u : D \to U$ and $v : [0,\infty) \to V$ belonging to the spaces $L_\infty^{\mathrm{loc}}(D)$ and $L_\infty^{\mathrm{loc}}(0,\infty)$ of measurable and locally bounded functions, respectively. By $\mathscr{U}_0$ we denote the set of functions $u \in L_\infty([0,\omega];U)$.

Let us define what we mean by a *solution* of system (3.1)–(3.5). Denote by $\mathscr{A}(D)$ the set of all n-dimensional functions $y \in L_\infty^{\mathrm{loc}}(D)$ which are absolutely continuous on almost every characteristic line $t - a = \text{const}$. For $y \in \mathscr{A}(D)$ the traces $y(t,0)$ and $y(0,a)$ in (3.3) are well-defined almost everywhere. Given $u \in \mathscr{U}$ and $v \in \mathscr{V}$, a couple of functions $y \in \mathscr{A}(D)$, $z \in L_\infty^{\mathrm{loc}}(0,\infty)$ is a solution of (3.2)–(3.4) if y satisfies (3.2) almost everywhere on almost every characteristic line intersecting

[4] The last part of the assumption means that, for every compact sets Y, Z, $\bar{U} \subset U$, $\bar{V} \subset V$ and $T > 0$, there exists a constant C such that for each of the functions listed above (take $g(t,a,y,z,u,v)$ as a representative)

$$|g(t,a,y_1,z_1,u_1,v_1) - g(t,a,y_2,z_2,u_2,v_2)| \le C(|y_1 - y_2| + |z_1 - z_2| + |u_1 - u_2| + |v_1 - v_2|), \tag{3.17}$$

for every $(t,a,y_i,z_i,u_i,v_i) \in [0,T] \times [0,\omega] \times Y \times Z \times \bar{U} \times \bar{V}$, $i = 1,2$.

D, and (3.3), (3.4) are also satisfied almost everywhere. For more detailed explanations of the notion of solution of (3.2)–(3.4) see e.g. [3, 25, 33, 55].

Then for any admissible pair of controls (u,v) with corresponding trajectories (y,z), and every $T > 0$, the integral

$$J_T(u,v) := \int_0^T \int_0^\omega L(t,a,y,z,u,v)\,\mathrm{d}a\,\mathrm{d}t,$$

is finite. Therefore, we can use the notion of weakly overtaking optimality (see Definition 2.1).

We do not investigate the issue of existence of a WOO solution but assume that such exists. In what follows, we fix a WOO solution $(\hat{u},\hat{v},\hat{y},\hat{z})$, for which we obtain necessary optimality conditions of Pontryagin's type. We also remind of the notational convention to skip functions with a "hat" when they appear as arguments of other functions.

In addition, we introduce the following simplifying assumption.

Assumption 3.2. *There exists a measurable function* $\rho : [0,\infty) \to [0,\infty)$ *such that*

$$|L_y(t,a,y,z)| + |L_z(t,a,y,z)| \le \rho(t) \quad \textit{for every } (t,a) \in D \textit{ and } (y,z).$$

The above condition is made only for simplification, and it is fulfilled for most applications in economics and population dynamics. It can be removed (following Assumption (A2) in [7]) at the price of a more implicit Assumption 3.3 than the one introduced in Section 3.2.3.

3.2.2 Variation of the Initial Data in System (3.12)–(3.14)

Let us fix a number $\tau > 0$, and consider a variation $\delta(a)$ of the state $\hat{y}(\tau,\cdot)$. We shall study the propagation of this variation on the domain $[\tau,\infty)\times[0,\omega]$. That is, we consider on $[\tau,\infty)\times[0,\omega]$ the system

$$\mathscr{D}y(t,a) = F(t,a)\,y(t,a) + f(t,a), \tag{3.18}$$

$$y(t,0) = \Phi(t)\,z(t) + \varphi(t), \qquad y(\tau,a) = \hat{y}(\tau,a) + \delta(a), \tag{3.19}$$

$$z(t) = \int_0^\omega [H(t,a)\,y(t,a) + h(t,a)]\,\mathrm{d}a, \tag{3.20}$$

where $\delta \in L_\infty(0,\omega)$. Denote the corresponding solution by $(y,z) \in \mathscr{A}([\tau,\infty)\times[0,\omega]) \times L_\infty^{\mathrm{loc}}(0,\omega)$. It exists and is unique [14, Lemma 5.3]. For the variations $\Delta y := y - \hat{y}$ and $\Delta z := z - \hat{z}$ we have

$$\mathscr{D}\Delta y(t,a) = F(t,a)\,\Delta y(t,a), \tag{3.21}$$

$$\Delta y(t,0) = \Phi(t)\,\Delta z(t), \qquad \Delta y(\tau,a) = \delta(a), \tag{3.22}$$

$$\Delta z(t) = \int_0^\omega H(t,a)\,\Delta y(t,a)\,\mathrm{d}a. \tag{3.23}$$

Due to (3.9), we have

$$\Delta y(t,a) = \begin{cases} X(t,a)X^{-1}(\tau,a-t+\tau)\,\delta(a-t+\tau), & \text{if } a-t+\tau \ge 0, \\ X(t,a)\,\Phi(t-a)\,\Delta z(t-a), & \text{if } a-t+\tau < 0. \end{cases}$$

For convenience, we extend the definition of δ and Δz setting $\delta(a) = 0$ for $a \notin [0,\omega]$ and $\Delta z(t) = 0$ for $t \in [0,\tau)$. Then

$$\Delta y(t,a) = X(t,a)X^{-1}(\tau,a-t+\tau)\,\delta(a-t+\tau) + X(t,a)\,\Phi(t-a)\,\Delta z(t-a). \tag{3.24}$$

Moreover, we set $H(t,a) = 0$ for $a \notin [0,\omega]$, and abbreviate $HX(t,a) := H(t,a)X(t,a)$. Using the equation for Δz, the above extensions and notation, and changing the variables, we have

$$\begin{aligned} \Delta z(t) = \int_0^\omega H(t,a)\,\Delta y(t,a)\,\mathrm{d}a &= \int_0^\omega HX(t,a)X^{-1}(\tau,\tau-t+a)\,\delta(\tau-t+a)\,\mathrm{d}a \\ &+ \int_0^\omega HX(t,a)\,\Phi(t-a)\,\Delta z(t-a)\,\mathrm{d}a \\ &= \int_0^\omega HX(t,s+t-\tau)X^{-1}(\tau,s)\,\delta(s)\,\mathrm{d}s + \int_\tau^t HX(t,t-s)\,\Phi(s)\,\Delta z(s)\,\mathrm{d}s. \end{aligned}$$

With the notations

$$K(t,s) := HX(t,t-s)\,\Phi(s), \quad Q(\tau,t,s) := HX(t,s+t-\tau)X^{-1}(\tau,s), \tag{3.25}$$

and $q(\tau,t) := \int_0^\omega Q(\tau,t,s)\,\delta(s)\,\mathrm{d}s$, we obtain that Δz satisfies on $[\tau,\infty)$ equation (2.10) with the measurable and locally bounded kernel $K(t,s)$ (3.25). Notice that $K(t,s) = 0$ for $t > s+\omega$. Denote by $R(t,s)$ its resolvent (see Section 2.5 for more details). Then, changing the order of integration below, we obtain that

$$\begin{aligned}\Delta z(t) &= q(\tau,t) + \int_\tau^t R(t,s)\, q(\tau,s)\, \mathrm{d}s \\ &= \int_0^\omega \left[Q(\tau,t,s) + \int_\tau^t R(t,x)\, Q(\tau,x,s)\, \mathrm{d}x \right] \delta(s)\, \mathrm{d}s. \end{aligned} \tag{3.26}$$

Thus we have the explicit representations (3.26) and (3.24) of the variations Δy and Δz as linear functions of δ.

The resolvent R and the fundamental matrix solution X defined above will be involved in all the subsequent analysis.

3.2.3 Main Result: The Maximum Principle

Papers [25, 14, 54] contain necessary optimality conditions in the form of the Pontryagin maximum principle for general age-structured systems on a finite time-horizon $[0,T]$. These conditions involve adjoint functions $\xi : D_T \to \mathbb{R}^n$ and $\zeta : [0,T] \to \mathbb{R}^m$ corresponding to the state variables y and z. These functions satisfy the following *adjoint system*:

$$-\mathscr{D}\xi(t,a) = \xi(t,a)\, F(t,a) + \zeta(t)\, H(t,a) + L_y(t,a), \tag{3.27}$$

$$\zeta(t) = \xi(t,0)\, \Phi(t) + \int_0^\omega L_z(t,a)\, \mathrm{d}a, \tag{3.28}$$

where we use the notational convention made in Section 2.1 that we skip variables fixed with a "hat". This system is complemented by the boundary condition $\xi(t,\omega) = 0$ and an appropriate transversality condition at $t = T$. In the infinite-horizon case, the adjoint equations are the same, but the transversality condition at $t = \infty$ is problematic due to several reasons (some of them are present also for ODE control problems). The notion of solution of the adjoint system is the same as above for the primal system, so we are looking for solutions $(\xi,\zeta) \in L_\infty^{\mathrm{loc}}(D)^n \times L_\infty^{\mathrm{loc}}(0,T)^m$. In the result below, we avoid the necessity of transversality conditions, since we explicitly define a unique solution of the adjoint system for which the maximum principle holds.

It will be convenient to define the pre-Hamiltonian

$$\begin{aligned}\mathscr{H}(t,a,y,z,u,v,\xi,\zeta) := {} & L(t,a,y,z,u,v) + \xi\,[F(t,a,u,v)\,y + f(t,a,u,v)] \\ & + \zeta\,[H(t,a,u,v)\,y + h(t,a,u,v)].\end{aligned}$$

With this notation, the adjoint equation (3.27) can be written in the shorter form

$$\mathscr{D}\xi(t,a) = -\mathscr{H}_y(t,a,\xi(t,a),\zeta(t)). \tag{3.29}$$

We now use the notations introduced in Section 3.1 and in the last subsection, in particular we remind that X denotes the fundamental matrix solution of the differential equation (3.8), and R is the resolvent corresponding to an integral equation of type (2.10) with kernel (3.25). We define for $(t,a) \in D$ and $t \in [0,\infty)$ the following functions

$$\hat{\xi}(t,a) := \left[\int_a^\omega L_y X(t-a+x,x)\,\mathrm{d}x + \int_t^{t+\omega-a} \hat{\zeta}(\theta) H X(\theta,\theta-t+a)\,\mathrm{d}\theta\right] X^{-1}(t,a), \tag{3.30}$$

$$\hat{\zeta}(t) := \psi(t) + \int_t^\infty \psi(\theta) R(\theta,t)\,\mathrm{d}\theta, \tag{3.31}$$

where

$$\psi(t) := \int_0^\omega \left[L_y X(t+s,s)\,\Phi(t) + L_z(t,s)\right] \mathrm{d}s, \tag{3.32}$$

and as before we shorten $L_y X(t,a) := L_y(t,a)X(t,a)$ and $HX(t,a) := H(t,a)X(t,a)$.

In order to justify the utilization of the infinite-horizon integral in the definition of $\hat{\zeta}$, we introduce the following additional assumption.

Assumption 3.3. *There exists a measurable function $\lambda(t,\theta)$, $\lambda : [0,\infty)\times[0,\infty) \to [0,\infty)$, such that*

$$\int_0^\omega \big(\rho(t+s)\,|X(t+s,s)|\,|\Phi(t)| + \rho(t)\big)\,\mathrm{d}s\,|R(t,\theta)| \le \lambda(t,\theta), \quad \forall t \ge \theta \ge 0,$$

and the integral $\int_\theta^\infty \lambda(t,\theta)\,\mathrm{d}t$ is finite and locally bounded as a function of θ.

Essentially, the above assumption poses some restriction on the combined growth of the resolvent R, the fundamental matrix solution X (which both depend on the optimal controls), and the data of the problem. It can be formulated in different ways, out of which we chose the one that is most convenient in the proof. Sufficient conditions for Assumption 3.3 that are easier to check are given in the end of this section.

Lemma 3.4. *On Assumptions 3.1–3.3, the integral in* (3.31) *is absolutely convergent and the functions $\hat{\xi}$ and $\hat{\zeta}$, defined by* (3.30) *and* (3.31) *(regarding* (3.32)*), belong to the spaces $\mathscr{A}(D)$ and $L_\infty^{\mathrm{loc}}(0,\infty)$, respectively, and satisfy the adjoint system* (3.27)–(3.28)*.*

This lemma will be proved in Section 3.2.5. Next, we present the main result of this Section.

Theorem 3.5. *Let Assumptions 3.1 and 3.2 be satisfied. Let $(\hat{u},\hat{v},\hat{y},\hat{z})$ be a WOO solution of problem* (3.11)–(3.16)*, for which Assumption 3.3 is fulfilled.*

Then the functions $\hat{\xi}$ and $\hat{\zeta}$, defined in (3.30) *and* (3.31) *(regarding* (3.32)*), satisfy the adjoint system* (3.27), (3.28)*, and the following maximization conditions are fulfilled:*

$$\mathscr{H}(t,a,\hat{u}(t,a)) = \sup_{u\in U} \mathscr{H}(t,a,u),$$
$$\left[\int_0^\omega \mathscr{H}_v(t,a,\hat{v}(t))\,\mathrm{d}a + \xi(t,0)(\Phi_v(t,\hat{v}(t))\hat{z}(t) + \varphi_v(t,\hat{v}(t)))\right](v - \hat{v}(t)) \le 0, \quad \forall v \in V.$$

A few remarks follow. First, we mention that the above maximum principle is of *normal form*, that is, the objective integrand appears in the definition of the pre-Hamiltonian with a multiplier equal to 1. This is typical for finite-horizon problems without state constraints, but not for infinite-horizon problems (notice that our definition of optimality goes even beyond the classical one). Second, the maximization condition with respect to v is local (in contrast to that for u). This is the case also for finite-horizon problems, and it is an open question if a global maximum principle with respect to the boundary control really holds. The formal reason for this localness is in Proposition 3.3, where the L_∞-norm of the disturbance of v appears in the estimation, rather the L_1-norm, as it is for u.

It is important to mention that the adjoint variable $\hat{\xi}(t,\cdot)$ defined in (3.30)–(3.31) does not necessarily converge to zero when $t \to \infty$. That is, the "classical" transversality conditions

$$\lim_{t\to\infty} \hat{\xi}(t,\cdot) = 0, \qquad \lim_{t\to\infty} \int_0^\omega \hat{\xi}(t,a)\,\hat{y}(t,a)\,\mathrm{d}a = 0,$$

the ODE-counterpart of which are used in the literature, do not necessarliy hold. In the first example of Section 3.2.6, a WOO solution exists but none of the classical transversality conditions hold. It is also possible to embed Halkin's example (see the Discussion in [7, Section 5.1], [31] or Remark 2.4) in an age-structured system, which shows that although the objective functional is finite, the classical transversality conditions are violated.

The above theorem is formulated and proved for affine systems. The extension to nonlinear systems, where the aggregate state z does not appear in the state equation (3.12) is a matter of technicality. However, if z appears in (3.12), then the problem becomes substantially more difficult.

Now, we elaborate on Assumption 3.3. If the growth estimations

$$|L_y(t,a,y,z)| \le ce^{\lambda_1 t}, \quad |X(t,a)| \le ce^{\lambda_2 t}, \quad |\Phi(t)| \le ce^{\lambda_3 t}, \quad |L_z(t,a,y,z)| \le ce^{\lambda_4 t}, \qquad (3.33)$$

hold for some constants c and λ_i, then the integral in Assumption 3.3 is estimated from above by

$$\tilde{c}\, e^{\lambda_0 t} |R(t,\theta)|, \quad \text{where } \lambda_0 := \max\{\lambda_1 + \lambda_2 + \lambda_3,\, \lambda_4\}, \tag{3.34}$$

and $\tilde{c}$ is another constant. Then Assumption 3.3 will be satisfied if for $\lambda(t,\theta) := \tilde{c}\, e^{\lambda_0 t}\, |R(t,\theta)|$ the integral $\int_\theta^\infty \lambda(t,\theta)\, \mathrm{d}t$ is finite and locally bounded as a function of θ. The following is a sufficient condition for that, which does not involve the resolvent R.

Assumption 3.4. *The inequalities* (3.33) *hold for any* $(t,a,y,z) \in D \times \mathbb{R}^n \times \mathbb{R}^m$ *and*

$$\operatorname{ess\,sup}_{t\in[0,\infty)} \int_t^{t+\omega} e^{\lambda_0 (s-t)} |K(s,t)|\, \mathrm{d}s < 1, \tag{3.35}$$

where λ_0 *is as in* (3.34).

Let us denote by $L_\infty^{\lambda_0}(0,\infty)$ the weighted L_∞-space with the weight $e^{-\lambda_0 t}$. In order to show that Assumption 3.4 implies Assumption 3.3 with $\lambda(t,\theta) := \tilde{c}\, e^{\lambda_0 t} |R(t,\theta)|$, we use Proposition 3.10 in [28, Chapter 9], according to which the operator $\mathscr{R}$ defined as $(\mathscr{R}\mu)(\theta) := \int_\theta^\infty \mu(t) R(t,\theta)\, \mathrm{d}t$ maps $L_\infty^{\lambda_0}(0,\infty)$ into itself and is bounded (see also Section 2.5). Applying this fact for $\mu(t) = \tilde{c} e^{\lambda_0 t}$, we obtain that the function $\theta \mapsto \int_\theta^\infty \lambda(t,\theta)\, \mathrm{d}t$ is finite and bounded in $L_\infty^{\lambda_0}(0,\infty)$, thus, locally bounded.

Therefore, Assumption 3.4 implies Assumption 3.3. Inequality (3.35) has a clear interpretation for population models, as it will be indicated in the second example in Section 3.2.6.

We also mention that λ_1 and λ_4 are usually negative due to discounting (which is implicitly included in L), and $\lambda_3 = 0$. Then λ_0 can happen to be negative, which helps for the validity of (3.35). The last models mentioned in Section 3.2.6 crucially employ this fact.

3.2.4 Proof of the Maximum Principle

The proof of Theorem 3.5 is somewhat long and technical, therefore, we first briefly present the idea, which builds on [7]. We follow the general understanding that the adjoint function $\hat{\xi}(t,\cdot)$ evaluated at time t gives the principle term of the effect of a disturbance $\delta = \delta(a)$ of the state $\hat{y}(t,\cdot)$ on the objective value. Therefore, for an arbitrary $\tau \in (0,\infty)$, we consider a disturbance $\delta(\cdot)$ of $\hat{y}(\tau,\cdot)$. Then, by linearization, one can represent the difference in the objective value on an interval $[\tau, T]$, $T > \tau$, corresponding to the perturbed $\hat{y}(\tau,\cdot) + \delta(\cdot)$ (with the same controls $\hat{u}$ and $\hat{v}$) as

$$\int_0^\omega \xi^T(\tau,a)\, \delta(a)\, \mathrm{d}a \;+\; \text{“rest terms”},$$

with some $\xi^T(\tau,\cdot)$ for which we will obtain a representation in terms of the fundamental matrix solution X and the resolvent R. Then, utilizing Assumption 3.3, we prove that $\xi^T(\tau,\cdot)$ converges to the adjoint function $\hat{\xi}$ defined in (3.30), and that Lemma 3.4 holds. This is the first part of the proof. In the second part, we apply a needle-type variation of the controls on $[\tau-\alpha,\tau]$, which results in a specific disturbance δ of $\hat{y}(\tau,\cdot)$. Then we represent the direct effect of this variation on the objective value (that is, on $[\tau-\alpha,\tau]$) and the indirect effect (resulting from δ) in terms of the pre-Hamiltonian $\mathscr{H}$. Finally, we use the definition of WOO to obtain the maximization conditions in Theorem 3.5.

Now we begin with the detailed proof.

Part 1

Let us fix an arbitrary $\tau>0$ and consider any two numbers $T>\tau+\omega$ and $T'>T+\omega$. For any $\delta\in L_\infty(0,\omega)$, we consider the disturbed system (3.18)–(3.20). Using the same notation (y,z) and $(\Delta y,\Delta z)$ as in Section 3.2.2, we obtain in a standard way the representation

$$\begin{aligned}\Delta_{T'}(y,z) &:= \int_\tau^{T'}\int_0^\omega [L(t,a,y(t,a),z(t))-L(t,a)]\,\mathrm{d}a\,\mathrm{d}t\\ &= \int_\tau^{T'}\int_0^\omega [\bar{L}_y(t,a)\,\Delta y(t,a)+\bar{L}_z(t,a)\,\Delta z(t)]\,\mathrm{d}a\,\mathrm{d}t,\end{aligned}$$

where $\bar{L}_y(t,a):=L_y(t,a,\bar{y}(t,a),\bar{z}(t))$, $\bar{L}_z(t,a):=L_z(t,a,\bar{y}(t,a),\bar{z}(t))$, and $\bar{y}$, $\bar{z}$ are measurable functions satisfying

$$(\bar{y}(t,a),\bar{z}(t))\in\operatorname{co}\{(y(t,a),z(t)),(\hat{y}(t,a),\hat{z}(t))\}. \tag{3.36}$$

Now, we use representation (3.26) of Δz, and representation (3.24) of Δy, with (3.26) inserted in (3.24). After some elementary calculus (changing variables and order of integration), we obtain that

$$\Delta_{T'}(y,z)=\int_0^\omega \xi^{T'}(\tau,s)\,\delta(s)\,\mathrm{d}s,$$

where

$$\xi^{T'}(\tau,s)=\int_s^\omega \bar{L}_yX(\tau+a-s,a)\,\mathrm{d}a X^{-1}(\tau,s) \tag{3.37}$$

$$+\int_\tau^{T'}\left[\int_0^\omega \big(\chi(T'-a-\theta)\bar{L}_yX(\theta+a,a)\Phi(\theta)+\bar{L}_z(\theta,a)\big)\,\mathrm{d}a\right] \tag{3.38}$$

$$\times\left[Q(\tau,\theta,s)+\int_\tau^\theta R(\theta,x)\,Q(\tau,x,s)\,\mathrm{d}x\right]\mathrm{d}\theta, \tag{3.39}$$

and χ is the Heaviside-function: $\chi(s)$ equals 0 for $s<0$ and equals 1 for $s\geq 0$.

In the above expression for $\xi^{T'}(\tau,s)$, we shall split $\int_\tau^{T'}=\int_\tau^T+\int_T^{T'}$ and will investigate the two appearing terms separately. We shall use the symbols c_1, c_2, ... for numbers that are independent of δ, and also of T and T', unless otherwise indicated by an argument of c_i. However, these numbers may depend on τ and s.

According to Proposition 3.2,

$$\|\Delta y\|_{L_\infty(D_T\setminus D_\tau)}+\|\Delta z\|_{L_\infty(\tau,T)}\leq c_0(T)\,\|\delta\|_{L_\infty(0,\omega)}.$$

Then both (y,z) and $(\hat{y},\hat{z})$ remain in a bounded domain when $t\leq T$, and in this domain, L_y and L_z are Lipschitz continuous with a constant depending on T. Moreover, X, Φ, and the term in the brackets in (3.39) are bounded when $\theta\leq T$ (again by a constant depending on T). Therefore, having in mind (3.36), we can replace the functions $\bar{L}_y(t,a)$ and $\bar{L}_z(t,a)$ with $L_y(t,a)$ and $L_z(t,a)$ in the term (3.37), and also in the term (3.38), where the integration is taken only to T. For the resulting residual, $e_1(\tau,s;T)$, we have

$$|e_1(\tau,s;T)|\leq c_1(T)\,\|\delta\|_{L_\infty(0,\omega)}.$$

The integral on $[T,T']$ in (3.38), (3.39) will be estimated differently, using Assumption 3.3. We obtain the following estimation of this integral, denoted by $e_2(\tau,s;T,T')$. First of all, we notice that $Q(\tau,\theta,s)=0$ if $\theta+s-\tau>\omega$ (see Section 3.2.2) and this is the case if $\theta>T$. Thus, the remaining integral on $[T,T']$ in (3.38), (3.39) can be estimated by

$$\begin{aligned}
&|e_2(\tau,s;T,T')|\leq\dots\\
&\quad\leq\int_T^{T'}\int_0^\omega(\rho(\theta+a)|X(\theta+a,a)\,\Phi(\theta)|+\rho(\theta))\,\mathrm{d}a\int_\tau^\theta|R(\theta,x)|\,|Q(\tau,x,s)|\,\mathrm{d}x\,\mathrm{d}\theta\\
&\quad=\int_\tau^{\tau+\omega}\int_T^{T'}\int_0^\omega(\rho(\theta+a)|X(\theta+a,a)\,\Phi(\theta)|+\rho(\theta))\,\mathrm{d}a\,|R(\theta,x)|\,\mathrm{d}\theta\,|Q(\tau,x,s)|\,\mathrm{d}x\\
&\quad\leq\int_\tau^{\tau+\omega}\int_T^{T'}\lambda(\theta,x)\,\mathrm{d}\theta\,\|Q(\tau,\cdot,s)\|_{L_\infty(\tau,\tau+\omega)}\leq c_2\int_\tau^{\tau+\omega}\int_T^\infty\lambda(\theta,x)\,\mathrm{d}\theta\,\mathrm{d}x.
\end{aligned}$$

The last term converges to zero when $T\to\infty$, due to the Assumption 3.3 about λ, and the Lebesgue dominated convergence theorem.

As a result of the above considerations we obtain that

$$\xi^{T'}(\tau,s) = \int_s^\omega L_y X(\tau+a-s,a)\,\mathrm{d}a X^{-1}(\tau,s) + \int_\tau^T \left[\int_0^\omega (L_y X(\theta+a,a)\Phi(\theta) + L_z(\theta,a))\,\mathrm{d}a\right] \times \left[Q(\tau,\theta,s) + \int_\tau^\theta R(\theta,x)\,Q(\tau,x,s)\,\mathrm{d}x\right] \mathrm{d}\theta + e_1(\tau,s;T) + e_2(\tau,s;T,T'),$$

(we used that $\chi(T'-a-\theta) = 1$ for $\theta \le T$, since $T' > T+\omega$). Rearranging the terms, substituting Q from (3.25), using that $Q(\tau,\theta,s) = 0$ for $\theta > \tau+\omega-s$, and that $T > \tau+\omega$, the above expression for $\xi^{T'}$ becomes

$$\xi^{T'}(\tau,s) = \left[\int_s^\omega L_y X(\tau-s+a,a)\,\mathrm{d}a + \int_\tau^{\tau+\omega-s} \zeta^T(\theta) X(\theta,\theta-\tau+s)\,\mathrm{d}\theta\right] X^{-1}(\tau,s) + e_1(\tau,s;T) + e_2(\tau,s;T,T'),$$

where

$$\zeta^T(\theta) := \psi(\theta) + \int_\tau^T \psi(x)\,R(x,\theta)\,\mathrm{d}x,$$

and ψ is given by (3.32). Due to Assumption 3.3,

$$\int_T^\infty |\psi(x)\,R(x,\theta)|\,\mathrm{d}x \le \int_T^\infty \lambda(x,\theta)\,\mathrm{d}x,$$

and the right-hand side is locally bounded in θ. Then we obtain that

$$\xi^{T'}(\tau,s) = \hat{\xi}(\tau,s) + e_1(\tau,s;T) + e_2(\tau,s;T,T') + e_3(\tau;T),$$
$$|e_3(\tau;T)| \le c_3 \int_\tau^{\tau+\omega} \int_T^\infty \lambda(x,\theta)\,\mathrm{d}x\,\mathrm{d}\theta,$$

and the last term converges to zero when $T \to \infty$ due to the dominated convergence theorem. Hence, for the variation of the objective value we have

$$\Delta_{T'}(y,z) = \int_0^\omega \hat{\xi}(\tau,s)\,\delta(s)\,\mathrm{d}s + e_4(\tau;T,T',\delta), \tag{3.40}$$

with

$$|e_4(\tau;T,T',\delta)| \le \left(c_4\bar{\varepsilon}(T) + c_5(T)\|\delta\|_{L_\infty(0,\omega)}\right)\|\delta\|_{L_1(0,\omega)}, \tag{3.41}$$

where $\bar{\varepsilon}(T) \to 0$ when $T \to \infty$. Clearly, the constants c_4 and c_5, and the function $\bar{\varepsilon}$ may depend also on τ.

In order to shorten the notations, further on we abbreviate $F^\sharp(t,a,y,u,v) := F(t,a,u,v)\,y + f(t,a,u,v)$, and similarly $H^\sharp = H\,y + h$, $\Phi^\sharp = \Phi y + \varphi$. Moreover, we apply the notational convention to skip arguments with "hat"-s.

Part 2 for u.

Now we investigate the effect of a needle variation of the control (u,v) on the objective value, starting with u (keeping $v = \hat{v}$). Let us fix an arbitrary $u \in U$, and denote by $\Omega(u)$ the set of all points (τ,b) in the interior of D which are Lebesgue points of each of the following functions

$$(t,a) \mapsto L(t,a,u) - L(t,a), \qquad (t,a) \mapsto \int_0^\omega L_z(t,s)\,\mathrm{d}s\,(H^\sharp(t,a,u) - H^\sharp(t,a)),$$
$$(t,a) \mapsto \hat{\xi}(\tau,\tau - t + a)\,\Big[F^\sharp(t,a,u) - F^\sharp(t,a)\Big],\ (t,a) \mapsto\ \hat{\xi}(t,0)\,\Phi(t)\,(H^\sharp(t,a,u) - H^\sharp(t,a)).$$

This means, that, taking $p = p(t,a)$ as a representative of the above functions,

$$\lim_{\alpha \searrow 0} \frac{1}{\alpha^2} \int_{B(\tau,b;\alpha)} p(t,a)\,\mathrm{d}a\,\mathrm{d}t = p(\tau,b),$$

where $B(\tau,b;\alpha) := [\tau - \alpha,\tau] \times [b - \alpha,b]$.

Let us arbitrarily fix $(\tau,b) \in \Omega(u)$ and let $\alpha > 0$ be such, that $2\alpha < \tau$, $2\alpha < b$, and $2\alpha < \omega - b$. Define the control u_α as in (3.7) with $\bar{u}(t,a) = u$. Let (y_α, z_α) be the solution of (3.12)–(3.14) corresponding to $(u_\alpha,\hat{v})$. According to Proposition 3.3, we have for the resulting differences, $\Delta y = y_\alpha - \hat{y}$, $\Delta z = z_\alpha - \hat{z}$,

$$\|\Delta y\|_{L_\infty(D_\tau)} + \|\Delta z\|_{L_\infty(0,\tau)} \le c_0(\tau)\,\alpha, \qquad \|\Delta y(\tau,\cdot)\|_{L_1(0,\omega)} \le c_0(\tau)\,\alpha^2. \tag{3.42}$$

From the first estimation and the absolute continuity of y along the characteristic lines $t - a =$ const, it follows that $\delta(\cdot) := \Delta y(\tau,\cdot)$ satisfies

$$\|\delta\|_{L_\infty(0,\omega)} \le c_0(\tau)\,\alpha. \tag{3.43}$$

Hence, from (3.41) we have

$$|e_4(\tau;T,T',\delta)| \le (c_4\bar{\varepsilon}(T) + c_5(T)\,c_0(\tau)\,\alpha)\,c_0(\tau)\,\alpha^2 = c_6\bar{\varepsilon}(T)\,\alpha^2 + c_7(T)\,\alpha^3. \tag{3.44}$$

According to Proposition 3.2, the second inequality in (3.42), and (3.43), we obtain that for every

$T > \tau + \omega$,

$$\|\Delta y\|_{L_\infty(D_T \setminus D_\tau)} + \|\Delta z\|_{L_\infty(\tau,T)} \le c_0(T)^2\,\alpha, \quad \|\Delta y\|_{L_1(D_T \setminus D_\tau)} + \|\Delta z\|_{L_1(\tau,T)} \le c_0(T)^2\,\alpha^2. \tag{3.45}$$

Now, for arbitrarily fixed $T > \tau + \omega$ and $T' > T + \omega$, we consider the variation

$$\begin{aligned} J_{T'}(u_\alpha, \hat{v}) - J_{T'}(\hat{u}, \hat{v}) = & \int_{\tau-\alpha}^{\tau} \int_0^\omega [L(t,a,y_\alpha(t,a), z_\alpha(t), u_\alpha(t,a)) - L(t,a)]\,\mathrm{d}a\,\mathrm{d}t + \Delta_{T'}(y_\alpha, z_\alpha) \\ = & \Delta_\tau(u_\alpha) + \int_0^\omega \hat{\xi}(\tau,s)\,\delta(s)\,\mathrm{d}s + e_4(\tau; T, T', \delta), \end{aligned} \tag{3.46}$$

where $\Delta_\tau(u_\alpha)$ is a notation for the above double integral, and the last term results from (3.40) with $\delta(a) = \Delta y(\tau, a)$.

In the sequel $o(\varepsilon)$ denotes any function (independent of T and T' but possibly depending on τ) such that $|o(\varepsilon)|/\varepsilon \to 0$ with $\varepsilon \to 0$.

Lemma 3.6. *The term $\Delta_\tau(u_\alpha)$ in* (3.46) *has the representation*

$$\Delta_\tau(u_\alpha) = \alpha^2\,(g(\tau,b,u) - g(\tau,b)) + \alpha^2 \int_0^\omega g_z(\tau,a)\,\mathrm{d}a\,(H^\sharp(\tau,b,u) - H^\sharp(\tau,b)) + o(\alpha^2).$$

Lemma 3.7. *The term $\int_0^\omega \hat{\xi}(\tau,a)\,\delta(a)\,\mathrm{d}a$ in* (3.46)*, with $\delta(a) = \Delta y(\tau,a)$ has the representation*

$$\begin{aligned} \int_0^\omega \hat{\xi}(\tau,a)\,\delta(a)\,\mathrm{d}a = & \alpha^2\,\hat{\xi}(\tau,b)\,(F^\sharp(\tau,b,u) - F^\sharp(\tau,b)) \\ & + \alpha^2\,\hat{\xi}(\tau,0)\,\Phi(\tau)\,(H^\sharp(\tau,b,u) - H^\sharp(\tau,b)) + o(\alpha^2). \end{aligned}$$

The proof of the lemmas is moved to Section 3.2.5. Combining the representations in the last two lemmas and (3.46), and taking into account the definition of the pre-Hamiltonian $\mathscr{H}$, we obtain that

$$J_{T'}(u_\alpha, \hat{v}) - J_{T'}(\hat{u}, \hat{v}) = \alpha^2\,[\mathscr{H}(\tau,b,u) - \mathscr{H}(\tau,b)] + e_4(\tau; T, T', \delta) + o(\alpha^2). \tag{3.47}$$

Part 3 for u.

In the above parts of the proof we have fixed an arbitrary $u \in U$, and arbitrary $(\tau, b) \in \Omega(u)$. For all sufficiently small $\alpha > 0$ we have defined the variation u_α of $\hat{u}$. Now, let us take an arbitrary $\varepsilon_0 > 0$ and an arbitrary $T > \tau + \omega$, such that $\bar{\varepsilon}(T) \le \varepsilon_0$ (see the line below (3.41)). We shall apply the definition of WOO (see Definition 2.1) for $(u_\alpha, \hat{v})$, $\varepsilon = \alpha^3$, and T. It says that there exists $T' > T$ (without any restriction we may assume $T' > T + \omega$), such that $J_{T'}(\hat{u}, \hat{v}) \ge J_{T'}(u_\alpha, \hat{v}) - \varepsilon$.

Then, according to (3.47), (3.41), and (3.44),

$$\begin{aligned}\varepsilon \geq J_{T'}(u_\alpha,\hat v) - J_{T'}(\hat u,\hat v) =& \alpha^2\,[\mathscr{H}(\tau,b,u)-\mathscr{H}(\tau,b)] + e_4(\tau;T,T',\delta) + o(\alpha^2)\\ \geq& \alpha^2\,[\mathscr{H}(\tau,b,u)-\mathscr{H}(\tau,b)] - c_6\,\varepsilon_0\,\alpha^2 - c_7(T)\alpha^3 - |o(\alpha^2)|.\end{aligned}$$

Replacing $\varepsilon = \alpha^3$, and dividing by α^2 we obtain that

$$\alpha \geq \mathscr{H}(\tau,b,u) - \mathscr{H}(\tau,b) - c_6\,\varepsilon_0 - c_7(T)\alpha - \theta(\alpha),$$

where $\theta(\alpha)\to 0$ with $\alpha \to 0$. Passing to a limit with $\alpha\to 0$, we have $0 \geq \mathscr{H}(\tau,b,u) - \mathscr{H}(\tau,b) - c_6\,\varepsilon_0$, and since $\varepsilon_0 > 0$ was arbitrarily fixed, we obtain the inequality

$$\mathscr{H}(\tau,b,u) \leq \mathscr{H}(\tau,b,\hat u(\tau,b))$$

for the considered $u \in U$ and $(\tau,b)\in\Omega(u)$.

Now, consider a countable and dense subset $U^d \subset U$. Since $\Omega(u)$ has full measure in D for every $u\in U^d$, then $D' := \cap_{u\in U^d}\Omega(u)$ also has full measure in D. Thus, the inequality $\mathscr{H}(t,a,u)\leq \mathscr{H}(t,a,\hat u(t,a))$ is fulfilled for every $(t,a)\in D'$ and every $u\in U^d$. Since U^d is dense in U and $\mathscr{H}$ is continuous in u, this inequality holds for every $u\in U$ and $(t,a)\in D'$. This implies the first relation in Theorem 3.5.

Part 2 for v.

Now, we fix $u=\hat u$ and consider a variation v_α as in (3.7), with $\bar v(t) = v_\alpha(t) := \hat v(t) + \alpha(v-\hat v(t))$. Here, $v\in V$ is arbitrarily chosen, $\tau\in\Omega(v)$, $\alpha\in(0,\tau)$, where now $\Omega(v)$ is the set of all τ that are Lebesgue points of the following functions:

$$\begin{aligned}&t\mapsto \int_0^\omega L_z(t,s)\,\mathrm{d}s\int_0^\omega H_v^\sharp(t,a)(v-\hat v(t))\,\mathrm{d}a,\; t\mapsto \hat\xi(t,0)\Phi(t)\int_0^\omega H_v^\sharp(t,a)(v-\hat v(t))\,\mathrm{d}a,\\ &t\mapsto \hat\xi(t,0)\Phi_v^\sharp(t)(v-\hat v(t)),\; t\mapsto \int_0^\omega \hat\xi(t,a)F_v^\sharp(t,a)(v-\hat v(t))\,\mathrm{d}a,\; t\mapsto \int_0^\omega L_v(t,a)(v-\hat v(t))\,\mathrm{d}a.\end{aligned}$$

Let (y_α, z_α) be the solution of (3.12)–(3.14) corresponding to $(\hat u, v_\alpha)$. Similarly as in "Part 2 for u", one can obtain estimations (3.42)–(3.45), thanks to the inequality $\|v_\alpha - \hat v\|_{L_\infty(0,\infty)} \leq c\,\alpha$.

Now, for any $T > \tau + \omega$ and $T' > T + \omega$, we consider the objective value

$$\begin{aligned} J_{T'}(\hat{u}, v_\alpha) - J_{T'}(\hat{u}, \hat{v}) = &\int_{\tau-\alpha}^{\tau} \int_0^{\omega} [L(t,a,y_\alpha(t,a), z_\alpha(t), v_\alpha(t)) - L(t,a)] \, \mathrm{d}a \, \mathrm{d}t + \Delta_{T'}(y_\alpha, z_\alpha) \\ = &\Delta_\tau(v_\alpha) + \int_0^{\omega} \hat{\xi}(\tau,a) \, \delta(a) \, \mathrm{d}a + e_4(\tau; T, T', \delta), \end{aligned} \tag{3.48}$$

where $\Delta_\tau(v_\alpha)$ is a notation for the above double integral.

Lemma 3.8. *The term $\Delta_\tau(v_\alpha)$ in* (3.48) *has the representation*

$$\Delta_\tau(v_\alpha) = \alpha^2 \int_0^{\omega} L_z(\tau,a) \, \mathrm{d}a \int_0^{\omega} H_v^\sharp(\tau,a)(v - \hat{v}(\tau)) \, \mathrm{d}a + \alpha^2 \int_0^{\omega} L_v(\tau,a)(v - \hat{v}(\tau)) \, \mathrm{d}a + o(\alpha^2).$$

Lemma 3.9. *The term $\int_0^{\omega} \hat{\xi}(\tau,s) \, \delta(s) \, \mathrm{d}s$ in* (3.48)*, with $\delta(a) = \Delta y(\tau,a)$ has the representation*

$$\begin{aligned} \int_0^{\omega} \hat{\xi}(\tau,a) \, \delta(a) \, \mathrm{d}a = &\alpha^2 \int_0^{\omega} \hat{\xi}(\tau,a) F_v^\sharp(\tau,a)(v - \hat{v}(\tau)) \, \mathrm{d}a + \alpha^2 \hat{\xi}(\tau,0) \, \Phi_v^\sharp(\tau)(v - \hat{v}(\tau)) \\ &+ \alpha^2 \hat{\xi}(\tau,0) \, \Phi(\tau) \int_0^{\omega} H_v^\sharp(\tau,a)(v - \hat{v}(\tau)) \, \mathrm{d}a + o(\alpha^2). \end{aligned}$$

Part 3 for v.

Thanks to the above lemmas, and using that $\hat{\zeta}$ satisfies (3.28) we obtain the following representation for $T > \tau + \omega$ and $T' > T + \omega$:

$$\begin{aligned} &J_{T'}(\hat{u}, v_\alpha) - J_{T'}(\hat{u}, \hat{v}) = \ldots \\ = &\alpha^2 \left[\int_0^{\omega} \mathscr{H}_v(t,a,\hat{v}(t)) \, \mathrm{d}a + \xi(t,0) \, \Phi_v^\sharp(t, \hat{v}(t)) \right] (v - \hat{v}(t)) + e_4(\tau; T, T', \delta) + o(\alpha^2). \end{aligned}$$

Then the proof of the second inequality in Theorem 3.5 goes in essentially the same way as in "Part 3 for u".

The proof is complete.

3.2.5 Proof of the Lemmas

Below we give the proofs of Lemmas 3.4, 3.6–3.9.

Proof of Lemma 3.4. For a fixed $t > 0$, the integrand in (3.31) can be estimated using (3.32), Assumption 3.2 and 3.3 as follows

$$
\begin{aligned}
|\psi(\theta)R(\theta,t)| &= \left| \int_0^{\omega} [L_y X(\theta+s,s)\Phi(\theta) + L_z(\theta,s)]\,\mathrm{d}s\, R(\theta,t) \right| \\
&\le \int_0^{\omega} [\rho(\theta+s)\,|X(\theta+s,s)|\,|\Phi(\theta)| + \rho(\theta)]\,\mathrm{d}s\,|R(\theta,t)| \le \lambda(\theta,t). \qquad (3.49)
\end{aligned}
$$

The first claim of Lemma 3.4 follows from the integrability of $\lambda(\cdot,t)$. The local boundedness of $\hat{\zeta}$ follows from the local boundedness of ψ and $\int_t^{\infty}\lambda(\theta,t)\,\mathrm{d}\theta$.

Now consider $\hat{\xi}$. Denote $\xi_1(t,a) := \int_a^{\omega} L_y X(t-a+s,s)\,\mathrm{d}s X^{-1}(t,a)$, which is the first term in (3.30), including the multiplication with X^{-1}. First, we show that ξ_1 is Lipschitz continuous along the characteristic lines $t-a = const$:

$$
\begin{aligned}
&\hat{\xi}_1(t+\varepsilon,a+\varepsilon) - \hat{\xi}_1(t,a) = \ldots \\
&\int_{a+\varepsilon}^{\omega} L_y X(t-a+s,s)\,\mathrm{d}s X^{-1}(t+\varepsilon,a+\varepsilon) - \int_a^{\omega} L_y X(t-a+s,s)\,\mathrm{d}s X^{-1}(t,a) = \ldots \\
&\int_{a+\varepsilon}^{\omega} L_y X(t-a+s,s)\,\mathrm{d}s \left[X^{-1}(t+\varepsilon,a+\varepsilon) - X^{-1}(t,a)\right] + \int_a^{a+\varepsilon} L_y X(t-a+s,s)\,\mathrm{d}s X^{-1}(t,a).
\end{aligned}
$$

The Lipschitz continuity follows from the Lipschitz continuity of X^{-1} along the characteristic lines and the local boundedness of $g_y X$ and X^{-1}. Applying the differentiation $\mathscr{D}$ to ξ_1 and using the expression (3.10) for $\mathscr{D}X^{-1}$, we obtain

$$
-\mathscr{D}\xi_1(t,a) = \int_a^{\omega} L_y X(t-a+s,s)\,\mathrm{d}s X^{-1}(t,a)F(t,a) + L_y(t,a) = \xi_1(t,a)F(t,a) + L_y(t,a).
$$

The proof that the second term, ξ_2, in the definition of $\hat{\xi}$ in (3.27) is Lipschitz continuous along the characteristic lines $t-a = const$ and satisfies the equation $-\mathscr{D}\xi_2(t,a) = \xi_2(t,a)F(t,a) + \hat{\zeta}(t)H(t,a)$ is similar and therefore omitted. Then $\hat{\xi} = \xi_1 + \xi_2$ belongs to the space $\mathscr{A}(D)$ and satisfies (3.27).

The definition (3.31) of $\hat{\zeta}$ has the form (2.12), with $T = \infty$. We know that $K(t,s) = 0$ for $t \notin [s, s+\omega]$. Moreover, from (3.49) and Assumption 3.3 we obtain that the integral in (2.12) with $T = \infty$ is locally bounded in t. Then we may apply the implication in the end of Section 3.1, which claims that

$$
\hat{\zeta}(t) = \psi(t) + \int_t^{\infty} \hat{\zeta}(\theta)K(\theta,t)\,\mathrm{d}\theta = \psi(t) + \int_t^{t+\omega} \hat{\zeta}(\theta)K(\theta,t)\,\mathrm{d}\theta.
$$

Inserting the expressions (3.25) and (3.32) for ψ and K, respectively, we obtain that

$$\begin{aligned}\hat{\zeta}(t) = &\left[\int_0^\omega L_y X(t+x,x)\,\mathrm{d}x + \int_t^{t+\omega} \hat{\zeta}(\theta)\, HX(\theta,\theta-t)\,\mathrm{d}\theta\right]\Phi(t) + \int_0^\omega L_z(t,a)\,\mathrm{d}a \\ =&\hat{\xi}(t,0)\,\Phi(t) + \int_0^\omega L_z(t,a),\end{aligned}$$

that is, (3.28) is fulfilled by $(\hat{\xi},\hat{\zeta})$. The proof is complete. □

Proof of Lemma 3.6. We represent

$$\begin{aligned}\Delta_\tau(u_\alpha) = \int_{\tau-\alpha}^{\tau}\int_0^\omega &[(L(t,a,y_\alpha,z_\alpha,u_\alpha) - L(t,a,z_\alpha,u_\alpha)) + (L(t,a,z_\alpha,u_\alpha) - L(t,a,u_\alpha)) \\ &+ (L(t,a,u_\alpha) - L(t,a))]\,\mathrm{d}a\,\mathrm{d}t =: I_1 + I_2 + I_3.\end{aligned}$$

We remind that $\|\Delta y\|_{L_\infty(D_\tau)} + \|\Delta z\|_{L_\infty(0,\tau)} \le c_0\,\alpha$, see (3.42). Moreover, $u_\alpha(t,a) \ne \hat{u}(t,a)$ only on the set $B(\tau,b;\alpha)$ (where $u_\alpha(t,a) = u$), and $y_\alpha(t,a) = \hat{y}(t,a)$ except on a set of measure $2\alpha^2$. Using, in addition, that L is Lipschitz continuous with respect to (y,z) in the domain where (y_α,z_α) and $(\hat{y},\hat{z})$ take values for $(t,a) \in D_\tau$, we apparently have $|I_1| = o(\alpha^2)$. For I_3 we have

$$I_3 = \iint_{B(\tau,b;\alpha)} (L(t,a,u) - L(t,a))\,\mathrm{d}a\,\mathrm{d}t = \alpha^2(L(\tau,b,u) - L(\tau,b)) + o(\alpha^2) \tag{3.50}$$

due to the Lebesgue property of (τ,b), see the beginning of “Part 2 for u” in Section 3.2.4. Finally,

$$I_2 = \int_{\tau-\alpha}^{\tau}\left(\int_0^\omega L_z(t,a,u_\alpha(t,a))\,\Delta z(t)\,\mathrm{d}a + o(\alpha;t)\right)\mathrm{d}t = \int_{\tau-\alpha}^{\tau}\int_0^\omega L_z(t,a)\,\Delta z(t)\,\mathrm{d}a\,\mathrm{d}t + o(\alpha^2),$$

where here and below $o(\alpha;t)/\alpha$ converges to zero uniformly in t in the interval of interest (in this case, it is $[0,\tau]$). For Δz we have

$$\begin{aligned}\Delta z(t) &= \int_0^\omega \left(H^\sharp(t,a,y_\alpha(t,a),u_\alpha(t,a)) - H^\sharp(t,a)\right)\,\mathrm{d}a \\ &= \int_0^\omega \left(H^\sharp(t,a,u_\alpha(t,a)) - H^\sharp(t,a)\right)\,\mathrm{d}a + o(\alpha) \\ &= \int_{b-\alpha}^{b} \left(H^\sharp(t,a,u) - H^\sharp(t,a)\right)\,\mathrm{d}a + o(\alpha).\end{aligned} \tag{3.51}$$

Inserting this in the expression for I_2 and changing the order of integration, we obtain

$$I_2 = \iint_{B(\tau,b;\alpha)} \int_0^{\omega} L_z(t,s)\,\mathrm{d}s\,(H^\sharp(t,a,u) - H^\sharp(t,a))\,\mathrm{d}a\,\mathrm{d}t + o(\alpha^2)$$
$$= \alpha^2 \int_0^{\omega} L_z(\tau,s)\,\mathrm{d}s\,(H^\sharp(\tau,b,u) - H^\sharp(\tau,b)) + o(\alpha^2),$$

where for the last equality we use the Lebesgue property of (τ,b), see the beginning of "Part 2 for u" in Section 3.2.4.

Summing the obtained expressions for I_1, I_2, and I_3, we obtain the claim of the lemma. □

Proof of Lemma 3.7. We remind of the inequalities $2\alpha < \tau$, $2\alpha < b$, and $2\alpha < \omega - b$ posed for α in "Part 2 for u" in Section 3.2.4. Observe that $\Delta y(\tau,a) = 0$ for all a except for $a \in [0,\alpha] \cup [b-\alpha, b+\alpha]$. Therefore, we consider the integral in the formulation of the lemma separately on these two intervals.

Beginning with $[0,\alpha]$, we represent

$$\int_0^{\alpha} \hat{\xi}(\tau,a)\,\Delta y(\tau,a)\,\mathrm{d}a$$
$$= \int_0^{\alpha} \left[\hat{\xi}(\tau-a,0) + \int_0^{a} \mathscr{D}\hat{\xi}(\tau-a+s,s)\,\mathrm{d}s\right] \left[\Delta y(\tau-a,0) + \int_0^{a} \mathscr{D}\Delta y(\tau-a+s,s)\,\mathrm{d}s\right] \mathrm{d}a.$$

We remind that $\|\Delta y\|_{L_\infty(D_\tau)} \le c_0\,\alpha$. For any $t \ge 0$ and $s \in [0,\alpha]$, we have $u_\alpha(t,s) = \hat{u}(t,s)$, hence $|\mathscr{D}\Delta y(t,s)| = |F(t,s)\,\Delta y(t,s)| \le c\,\alpha$. Moreover, due to Lemma 3.4 and the local boundedness of F, $\hat{\xi}$, $\hat{\zeta}$, H, and L_y, we have $\|\mathscr{D}\hat{\xi}\|_{L_\infty(D_\tau)} \le c$. Hence,

$$\int_0^{\alpha} \hat{\xi}(\tau,a)\,\Delta y(\tau,a)\,\mathrm{d}a = \int_0^{\alpha} \hat{\xi}(\tau-a,0)\,\Delta y(\tau-a,0)\,\mathrm{d}a + o(\alpha^2).$$

Since $\Delta y(\tau-a,0) = \Phi(\tau-a)\,\Delta z(\tau-a)$, using representation (3.51) and changing the integration variable, we obtain that

$$\int_0^{\alpha} \hat{\xi}(\tau,a)\,\Delta y(\tau,a)\,\mathrm{d}a = \iint_{B(\tau,b;\alpha)} \hat{\xi}(t,0)\,\Phi(t)\,(H^\sharp(t,a,u) - H^\sharp(t,a))\,\mathrm{d}a\,\mathrm{d}t + o(\alpha^2).$$

Due to the Lebesgue point property of (τ,b), the expression in the right-hand side equals the second term in the right hand side in the assertion of the lemma.

Now, we consider $E := \int_{b-\alpha}^{b+\alpha} \hat{\xi}(\tau,a)\,\Delta y(\tau,a)\,\mathrm{d}a$. For a in the interval of integration,

$$\Delta y(\tau,a) = \Delta y(\tau-\alpha, a-\alpha) + \int_0^\alpha \mathscr{D}\Delta y(\tau-x, a-x)\,\mathrm{d}x,$$

and the first term in the right-hand side is zero (due to $a-\alpha \geq 0$). Then E is equal to

$$\begin{aligned}
&\int_{b-\alpha}^{b+\alpha}\int_0^\alpha \hat{\xi}(\tau,a)\left[F^\sharp(\tau-x,a-x,y_\alpha(\tau-x,a-x),u_\alpha(\tau-x,a-x)) - F^\sharp(\tau-x,a-x)\right]\mathrm{d}x\,\mathrm{d}a\\
=&\int_{\tau-\alpha}^{\tau}\int_{b-\tau+t-\alpha}^{b-\tau+t+\alpha} \hat{\xi}(\tau,\tau-t+s)\left[F^\sharp(t,s,u_\alpha(t,s)) - F^\sharp(t,s)\right]\mathrm{d}s\,\mathrm{d}t + o(\alpha^2),
\end{aligned}$$

where we passed to the new variables $t = \tau - x$ and $s = a - x$, and used that $\|\Delta y\|_{L_\infty(D_\tau)} \leq c_0\,\alpha$. Notice that if $s < b-\alpha$ or $s > b$ the last integrand is zero, since $u_\alpha(t,s) = \hat{u}(t,s)$. Otherwise $u_\alpha(t,s) = u$. Then

$$E = \iint_{B(\tau,b;\alpha)} \hat{\xi}(\tau,\tau-t+s)\left[F^\sharp(t,s,u) - F^\sharp(t,s)\right]\mathrm{d}s\,\mathrm{d}t + o(\alpha^2),$$

Using the Lebesgue property of (τ,b), (see the beginning of “Part 2 for u” in Section 3.2.4) we obtain the first term in the right hand side in the assertion of the lemma. □

Proof of Lemma 3.8. By definition of v_α we have $|v_\alpha(t) - \hat{v}(t)| \leq c\,\alpha$. According to Proposition 3.3, $\|\Delta y(t,\cdot)\|_{L_1(0,\omega)} \leq c\alpha^2$. Moreover,

$$\Delta z(t) = \int_0^\omega \alpha H_v^\sharp(t,a)(v - \hat{v}(t))\,\mathrm{d}a + o(\alpha;t).$$

Then we obtain

$$\begin{aligned}
\Delta_\tau(v_\alpha) &= \int_{\tau-\alpha}^{\tau}\int_0^\omega [L(t,a,z_\alpha(t),v_\alpha(t)) - L(t,a,v_\alpha(t)) + L(t,a,v_\alpha(t)) - L(t,a)]\,\mathrm{d}a\,\mathrm{d}t + o(\alpha^2)\\
&= \int_{\tau-\alpha}^{\tau}\int_0^\omega [L_z(t,a)\Delta z(t) + L(t,a,v_\alpha(t)) - L(t,a)]\,\mathrm{d}a\,\mathrm{d}t + o(\alpha^2)\\
&= \int_{\tau-\alpha}^{\tau}\left\{\int_0^\omega L_z(t,s)\,\mathrm{d}s \int_0^\omega H_v^\sharp(t,a)\,\mathrm{d}a + \int_0^\omega L_v(t,a)\,\mathrm{d}a\right\}\alpha(v-\hat{v}(t))\,\mathrm{d}t + o(\alpha^2),
\end{aligned}$$

which proves the claim of the lemma due to the Lebesgue property of τ (see the beginning of “Part 2 for v” in Section 3.2.4). □

Proof of Lemma 3.9. The proof uses similar arguments as that of Lemma 3.7 and therefore is somewhat shortened. From (3.13) we have

$$\Delta y(t,0) = \Phi(t,\hat{v})\Delta z(t) + \alpha \Phi_v^\sharp(t)(v - \hat{v}(t)) + o(\alpha;t).$$

For $t = \tau$ and $a < \alpha$,

$$\Delta y(\tau,a) = \Delta y(\tau - a, 0) + \int_{-a}^{0} \mathscr{D}\Delta y(\tau + s, a + s)\,\mathrm{d}s = \Delta y(\tau - a, 0) + o(\alpha;a).$$

while for $a > \alpha$,

$$\begin{aligned}\Delta y(\tau,a) &= \int_{-\alpha}^{0} [F^\sharp(\tau+s,a+s,y_\alpha,v_\alpha) - F^\sharp(\tau+s,a+s)]\,\mathrm{d}s \\ &= \int_{-\alpha}^{0} \alpha F_v^\sharp(\tau+s,a+s)(v - \hat{v}(\tau+s))\,\mathrm{d}s + o(\alpha;a).\end{aligned}$$

This is obtained by splitting the difference in two parts: one for the difference between y_α and $\hat{y}$, and one for v_α and $\hat{v}$. Due to Proposition 3.3, the difference with respect to y can be estimated by $o(\alpha;a)$.

Consider the integral $\int_0^\omega \hat{\xi}(\tau,a)\Delta y(\tau,a)\,\mathrm{d}a$ on $[0,\alpha)$. Because $|\Delta y(\tau,a)| \le c\alpha$, and the above representation,

$$\int_0^\alpha \hat{\xi}(\tau,a)\Delta y(\tau,a)\,\mathrm{d}a = \int_0^\alpha \hat{\xi}(\tau - a, 0)\Delta y(\tau - a, 0)\,\mathrm{d}a + o(\alpha^2).$$

Since τ is a Lebesgue point, using the representation for $\Delta y(\tau,0)$ and for $\Delta z(\tau)$ from the proof of the Lemma 3.8 we obtain the expression

$$\alpha^2 \hat{\xi}(\tau,0)\Phi(\tau)\int_0^\omega H_v^\sharp(\tau,a)(v - \hat{v}(\tau))\,\mathrm{d}a + \alpha^2 \hat{\xi}(\tau,0)\Phi_v^\sharp(\tau)(v - \hat{v}(\tau)) + o(\alpha^2).$$

With the representation for $\Delta y(\tau,a)$ for $a > \alpha$, and the absolute continuity of $\hat{\xi}$ along the characteristic lines, we obtain

$$\begin{aligned}\int_\alpha^\omega \hat{\xi}(\tau,a)\Delta y(\tau,a)\,\mathrm{d}a &= \int_\alpha^\omega \int_{\tau-\alpha}^{\tau} \hat{\xi}(t, a - \tau + t)\alpha F_v^\sharp(t, a - \tau + t)(v - \hat{v}(t))\,\mathrm{d}t\,\mathrm{d}a + o(\alpha^2) \\ &= \alpha \int_\alpha^{\omega-\alpha} \int_{\tau-\alpha}^{\tau} \hat{\xi}(t,a)\,F_v^\sharp(t,a)(v - \hat{v}(t))\,\mathrm{d}t\,\mathrm{d}a + o(\alpha^2) \\ &= \alpha \int_{\tau-\alpha}^{\tau} \int_0^\omega \hat{\xi}(t,a)\,F_v^\sharp(t,a)(v - \hat{v}(t))\,\mathrm{d}a\,\mathrm{d}t + o(\alpha^2).\end{aligned}$$

Since τ is a Lebesgue point, this implies the claim of the lemma. □

3.2.6 Selected Applications

In this section, we apply the obtained result to a few models from the economic literature, and, in particular, we shed some light on Assumption 3.3. The first two examples have been analyzed on a finite horizon although the natural formulation is on an infinite horizon. We show that our Assumption 3.3 is satisfied in these examples.

A Problem of Optimal Investment

In many cases, e.g. [23, 38, 39], the boundary condition (3.13) does not involve the integral state z. In these cases, Assumption 3.3 is trivially fulfilled because the resolvent is zero (see (2.9) and also (3.25), where $\Phi = 0$, thus, $R = 0$).

Consider for example the optimal investment problem in [23] (where we change notations to fit to our general model). The objective is to maximize the discounted net profit,

$$\max_{u,v} \int_0^\infty e^{-rt} \Big(p(z(t)) - (b_0\, v(t) + c_0\, v(t)^2) - \int_0^\omega (b(a)\, u(t,a) + c(a)\, u(t,a)^2)\, \mathrm{d}a \Big)\, \mathrm{d}t,$$

subject to

$$\begin{aligned} \mathscr{D}y(t,a) &= u(t,a) - \mu(a)\, y(t,a), \quad y(t,0) = v(t), \quad y(0,a) = y_0(a), \\ z(t) &= \int_0^\omega H(t-a)\, y(t,a)\, \mathrm{d}a, \end{aligned}$$

where $y(t,a)$ is the capital stock of machines of age a at time t, u and v are investments in old and current vintages, respectively, and $H(s)$ is the productivity of technologies (machines) of vintage s.

Here the fundamental solution X has the form

$$X(t,a) = e^{-\int_0^{\min\{t,a\}} \mu(a-s)\, \mathrm{d}s}.$$

The adjoint functions $\hat{\xi}$ and $\hat{\zeta}$ defined in (3.30) and (3.31) (taking into account (3.32)), take the explicit forms

$$\psi(\theta) = e^{-rt} p'(\hat{z}(\theta)), \quad \hat{\zeta}(\theta) = \psi(\theta),$$
$$\hat{\xi}(t,a) = \int_t^{t+\omega-a} e^{-rt} p'(\hat{z}(\theta)) H(t-a) X(\theta, \theta - t + a) \, \mathrm{d}\theta \, X^{-1}(t,a).$$

A few remarks follow. Clearly, X is bounded, as well as X^{-1}, since the depreciation rate μ can be assumed bounded in the life-span of the machines. The revenue function $p(z)$ is defined on $(0, \infty)$ and is non-negative, increasing and concave. Then $p'(\hat{z}(t))$ is bounded, because $\hat{z}(t)$ does not approach zero in the considered model. Consequently,

$$|\hat{\xi}(t,a)| \leq c \int_t^{t+\omega-a} e^{-rt} H(t-a) \, \mathrm{d}\theta.$$

If the productivity function satisfies $H(t) < c_1 e^{\rho t}$ with $\rho < r$, then $\hat{\xi}(t, \cdot) \to 0$ when $t \to \infty$. Thus, $\hat{\xi}$ satisfies the usual transversality condition. This result is consistent with that in [10], where $r > 0$ and $\rho = 0$. However, if $\rho \geq r$, the objective functional may be infinite and the considered problem still has a WOO solution. Theorem 3.5 holds with the above defined adjoint functions $\hat{\xi}$ and $\hat{\zeta}$, although neither of the standard transversality conditions, $\hat{\xi}(t, \cdot) \to 0$ and $\hat{\xi}(t, \cdot) \hat{y}(t, \cdot) \to 0$, is fulfilled.

Optimal Harvesting

Consider the model of optimal harvesting in [3, p. 75]. The problem reads as

$$\max_{v(t)} \int_0^\infty e^{-rt} \int_0^\omega v(t)\, p(a)\, y(t,a)\, \mathrm{d}a\, \mathrm{d}t,$$

subject to

$$\begin{aligned} \mathscr{D} y(t,a) &= -(\mu(a)+v(t))\, y(t,a), \\ y(0,a) &= y_0(a) > 0, \qquad y(t,0) = z(t), \\ z(t) &= \int_0^\omega \beta(a)\, y(t,a)\, \mathrm{d}a, \\ v(t) &\in [0,\bar{v}]. \end{aligned}$$

Here, $y(t,a)$ is interpreted as a stock of biological resource of age a and $v(t)$ is the harvesting effort. The mortality rate $\mu(a)$, fertility rate $\beta(a)$, and profit function $p(a)$ are all non-negative, measurable and bounded. The discount rate r is non-negative.

Assumption 3.3 is not trivially fulfilled for this model because $\Phi(t,v) = 1$. However, below we show that it is generically non-restrictive.

The fundamental solution X reads as

$$X(t,a) = \exp\left(- \int_0^{\min\{t,a\}} (\mu(a-s) + \hat{v}(t-s))\, \mathrm{d}s \right).$$

Regarding (3.28), the adjoint functions defined in 3.30–(3.32) take the form

$$\begin{aligned} \hat{\xi}(t,a) &= \int_a^\omega \left[X(t+s-a,s) e^{-r(t+s-a)} \hat{v}(t+s-a)\, p(s) + \hat{\zeta}(t+s-a)\, \beta(s) \right] \mathrm{d}s\, X(t,a)^{-1}, \\ \hat{\zeta}(t) &= \hat{\xi}(t,0). \end{aligned}$$

Consider the function

$$\Theta(\nu) := \int_0^\omega e^{-\nu a} e^{-\int_0^a \mu(s)\mathrm{d}s} \beta(a)\, \mathrm{d}a, \quad \nu \in \mathbb{R}. \tag{3.52}$$

Lemma 3.10. *If $\Theta(r) < 1$, Assumption 3.3 is fulfilled. If $\Theta(r) > 1$, no WOO solution exists.*

Proof. **Case $\Theta(r) < 1$.** The functions $X(t,a)$, $\hat{v}(t)$ and $p(a)$ are essentially bounded, thus, $\psi \in$

$L^r_\infty(0,\infty)$. Obviously, for any admissible control v it holds that

$$\operatorname{ess\,sup}_{t\in[0,\infty)} \int_0^\omega |e^{-ra} e^{-\int_0^a (\mu(s)+v(t+s))\mathrm{d}s} \beta(a)| \,\mathrm{d}a < 1, \tag{3.53}$$

which is condition (3.35). Thus, Assumption 3.4 is fulfilled, which implies Assumption 3.3. Therefore, Theorem 3.5 is applicable.

Case $\Theta(r) > 1$. According to Theorem 1.3 and equation (1.13) in [33, Chapter 2], the population grows in the long run with rate ν, for which $\Theta(\nu) = 1$. Since $\Theta(\cdot)$ is decreasing, $\nu > r$. Therefore, there exists a constant control $v > 0$, such that the population is growing with a rate $\nu > r$. This implies (see again [33, Chapter 2]), that for sufficiently large t, we have $y(t,a) \geq \frac{1}{2} y(0,a) e^{\nu t}$. Hence, $\lim_{T\to\infty} J_T(v) \to \infty$. Thus, the objective value is infinite for any WOO control (if such exists).

However, we show now that perpetual "postponement" of harvesting is beneficial in terms of the WOO criterion of optimality, thus, no WOO control exists. Indeed, assume that $\hat{v}(t)$ is optimal, and denote by $\hat{y}(t,a)$ the corresponding trajectory. Clearly, $\hat{v}$ is not a.e. identical to zero, since otherwise the objective value will also be zero. Let $\hat{v}$ be not identically zero on the interval $[0,\tau]$. We modify $\hat{v}$ as $v(t) = 0$ for $t \in [0,\tau]$ and $v(t) = \hat{v}(t)$ for $t > \tau$. Then there is a constant c such that $y(\tau,a) \geq (1+c)\hat{y}(\tau,a)$ and due to the linear homogeneous structure of the system, this inequality is preserved for all $t \geq \tau$.

For any $T > \tau$,

$$J_T(v) - J_T(\hat{v}) = c \int_\tau^T e^{-rt} p(a)\, \hat{v}(t)\, \hat{y}(t,a) \,\mathrm{d}a\,\mathrm{d}t - \int_0^\tau e^{-rt} p(a)\, \hat{v}(t)\, \hat{y}(t,a) \,\mathrm{d}a\,\mathrm{d}t.$$

Since the first integral above tends to infinity with T, the above difference also tends to infinity, which contradicts the WOO of $\hat{v}$. This completes the proof. □

In the borderline case, $\Theta(r) = 1$, it is not clear whether Assumption 3.3 holds, but this case involves a relation between the intrinsic population vital rates and the economic discount, which is non-generic.

Populations of Fixed Size and Optimal Age-patterns of Immigration

The application potential of the approach presented in this paper goes beyond the particular class of problems considered here.

The papers [27, 42] investigate the issue of optimal age-structured recruitment/immigration policies of organizations/countries, where the goal is to keep the size of the population constant, while optimizing combinations of certain demographic characteristics (such as average age, size of the inflow, dependency ratio). The involved optimal control problems are not covered by the consideration in the present paper due to fact that the aggregated state variable z appears in the distributed state equation (3.12). As mentioned in the discussions after Theorem 3.5, this case is technically more difficult. However, the approach utilized for the models in [27, 42] is essentially the same as the one in the present paper. In particular, the verification of an appropriate analog of Assumption 3.3 was a key issue accomplished for the specific models in these papers. In [42], the existence of an optimal solution was proved (Proposition 1), as well as that there exists a unique solution of the adjoint equation in feedback form (Theorem 2). Thus, although this paper only necessary conditions, they can be very helpful to identify the optimal control.

3.3 Optimal Immigration Age-patterns in Populations of Fixed Size[5]

Many countries face low fertility levels combined with an increase in life expectancy especially in older ages. These demographic developments influence the populations' well-being in many ways and lead, for example, to severe challenges for their social security systems. One possible way to counteract these developments is to steer immigration in an appropriate way.

In this section, we consider a human population where immigration is allowed, although subjected to restrictions. It is assumed that the intensity of the migration inflow and, to a certain extent, the age-structure of the migrants can be used as control (policy) instruments. The problem we consider is to keep the size of the population constant by choosing appropriate immigration policy which, in addition, optimizes a certain objective function. Of course, the problem is meaningful only if the population would steadily decrease without migration, that is, for a population with *below-replacement fertility*.

The relevance of this issue is underlined by the United Nations in [51], where the authors investigate whether immigration can be used to hinder a decline or aging of the populations of eight industrialized countries. It is concluded that immigration alone cannot stop the aging of these populations. In [41], the authors determine for a stationary population with below-replacement fertility the optimal age-specific immigration profile that minimizes the dependency ratio while fixing either the population size or the immigration quota. In contrast to [41], in this paper non-stationary populations are investigated leading to the formulation and study of a distributed control problem on an infinite horizon. The population dynamics in this case can be modeled by the McKendrick-von Foerster equation.

From mathematical point of view, the considered problem is challenging for three reasons: (i) it has the form of a distributed optimal control problem with state constraints (although rather specific); (ii) the time horizon is infinite and a theory for infinite-horizon optimal control problems for age-structured systems is missing ([17] and [27] are exceptions, as well as a few non-sound papers that we do not mention); also the result from the previous section is not applicable, because the integral state enters the dynamics and not only in the boundary condition; (iii) we deal with a maximization problem for a non-concave functional, where the existence of a solution and the well-posedness are problematic.

The main results of the section are as follows. We obtain a Pontryagin type maximum principle with a transversality condition in the form of boundedness of the adjoint variable. This is

[5]This section is based on the joint work with C. Simon and V. Veliov, published in [42].

done under suitable stability assumptions, which are fulfilled for populations with sufficiently low fertility. Existence of an optimal solution is also proved. The most striking result is that, under an additional generic well-posedness condition, for a population with time-invariant mortality and fertility the optimal age-density of the migration turns out to be time-invariant and independent of the initial data. This makes it possible to find it by solving the associated steady-state problem, which is an optimal control problem for an ordinary differential equation and was studied in details in [41]. Thanks to this property, also qualitative results for the optimal policy are obtained.

As an application, we consider the Austrian female population in 2009 and determine from the available data the age-specific mortality and fertility rates and the initial age structure of the population and of the migration. Then we consider the age-profile of the migration as a control (policy) variable, allowing for modifications of the age-profile from 2009. Using the results in this section, we determine numerically the immigration policy that maximizes the aggregate number of workers over time. It turns out that the optimal migration intensity is at its upper bound on a single age-interval, and on its lower bound at all other ages.

The problem at hand has a similar structure as those studied in [14, 25, 54], with the substantial difference that here the time-horizon is infinite. A distributed control problem on an infinite horizon is considered in [27], where the authors investigate the recruitment problem of organizations of fixed size. However, the problem considered here is substantially more complicated due to the involvement of births in the boundary condition, but some ideas from this paper are used in this Section. The result from Section 3.2 is not applicable for the following reasons (i) the maximal age is infinite, (ii) admissible controls must satisfy an integral equation, (iii) the integral state enters the dynamics of the integral state. The first point can be dealt with rather easily requiring some additional assumptions (imposed in 3.5), but the other two points make the analysis substantially more difficult.

The presentation is structured as follows. In Section 3.3.1 we state the population dynamics and some auxiliary results. In Section 3.3.2 we formulate the optimization problem and in Section 3.3.3 we prove necessary optimality conditions. Stationarity, structure, and uniqueness of the optimal solution are investigated in Section 3.3.4. In Section 3.3.5 we provide numerical illustrations with Austrian data. Most of the proofs are moved to Section 3.3.6 for a better readability.

3.3.1 The Dynamic System and some Preliminary Results

Below, $t \geq 0$ denotes time, $a \geq 0$ denotes age, and $D := [0,\infty) \times [0,\infty)$, while $D_T := [0,T] \times [0,\infty)$. The function $a \mapsto y(t,a) \geq 0$ is the (non-probabilistic) age-density of a population[6]. The mortality and the fertility rate at age a of this population are denoted by $\mu(a)$ and $\varphi(a)$, respectively. The immigration flux (number of immigrants) at time t will be denoted by $z_R(t) \geq 0$, and the immigration age-density by $u(t,a)$, $(t,a) \in D$. That is, u satisfies $u(t,a) \geq 0$, $\int_0^\infty u(t,a)\,\mathrm{d}a = 1$ and $z_R(t)u(t,a)$ is the flow of immigrants of age a at time t. Then the evolution of the population is described by the McKendrick-von Foerster equations (see e.g. [55])

$$\mathscr{D}y(t,a) = -\mu(a)\,y(t,a) + z_R(t)\,u(t,a), \qquad (t,a) \in D, \tag{3.54}$$

$$y(0,a) = y_0(a), \qquad a \geq 0, \tag{3.55}$$

$$y(t,0) = \int_0^\infty \varphi(a)\,y(t,a)\,\mathrm{d}a, \qquad t \geq 0, \tag{3.56}$$

where $y_0(\cdot)$ is the (given) initial population density and $\mathscr{D}$ is the directional derivative defined in (3.6). The formal meaning of these equations is essentially the same as in Section 3.2.1, but will be given below in this section.

Given an immigration profile, $u(t,a)$, one can always keep the size of the population constant (equal to $M := \int_0^\infty y_0(a)\,\mathrm{d}a$) by an appropriate choice of the immigration intensity, namely by choosing $z_R(t)$ in the feedback form

$$z_R(t) := \int_0^\infty (\mu(a) - \varphi(a))\,y(t,a)\,\mathrm{d}a. \tag{3.57}$$

This fact has an obvious demographic meaning and can easily be established by integration of equation (3.54) with respect to a, provided that y is a differentiable function and the equation is satisfied in the classical sense. This is not necessarily the case due to a possible inconsistency of the initial and the boundary conditions (3.55), (3.56), or due to discontinuities of u (the optimal control for the problem described in the next section is discontinuous indeed). Therefore we give a strict proof below, see Lemma 3.14.

[6] In demography, the population is denoted by N (as in the published paper). Here, y is chosen in order to use a consistent notation throughout all chapters. The variables z_R and z_B are (in the published versions), denoted by R and z_B, respectively.

Let us introduce also the number of births $z_B(t)$ and deaths $z_D(t)$,

$$z_B(t) := \int_0^\infty \varphi(a)\, y(t,a)\, \mathrm{d}a, \tag{3.58}$$

$$z_D(t) := \int_0^\infty \mu(a)\, y(t,a)\, \mathrm{d}a. \tag{3.59}$$

Now we pass to a strict formulation of the previous consideration, starting with some basic assumptions:

Assumption 3.5. *(i) The functions μ, φ, $y_0 : [0,\infty) \to \mathbb{R}$ are non-negative, bounded and Lipschitz continuous,*
(ii) μ satisfies $\mu(a) \geq \mu_0 > 0$ for all sufficiently large a,
(iii) $\varphi(a)$ and $y_0(a)$ are equal to zero for all sufficiently large a,
(iv) there is $a_0 \geq 0$ such that $\varphi(a_0) > 0$ and $y_0(a) > 0$ for $a \in [0, a_0]$,
(v) y_0 satisfies $\int_0^\infty y_0(a)\, \mathrm{d}a = M$ with some positive $M < \infty$.

Remark 3.11. The Lipschitz continuity assumption (i) is made just for technical convenience and can be relaxed. The remaining assumptions (ii)–(iv) about the fertility $\varphi(a)$, the present age-density of the population, $y_0(a)$, and that the population is non-void until some fertile age a_0 are factual. The boundedness assumption (ii) for the mortality rate needs some explanation. There is no empirical evidence about boundedness or unboundedness of $\mu(a)$. We can assume equally well that the mortality rate is unbounded close to some maximal age $a = \omega$ in such a way that all the population dies till age ω. An alternative (not less plausible, in our opinion) is that $\mu(a)$ is bounded and large enough after a certain age, say 110 years, so that individuals of age above 130 exist only mathematically. We chose the second option about the mortality rate just for a minor technical convenience.

In contrast to the last section, the maximal age is here unbounded. Therefore, the notion of solution has to be adapted.

Define the space $\mathscr{A}_\infty(D)$ which consists of all functions $y : D \to \mathbb{R}$ which are
(i) measurable, and the function $t \mapsto \int_0^\infty |y(t,a)|\, \mathrm{d}a < \infty$ is finite and locally bounded;
(ii) locally absolutely continuous on almost every line $t - a =$ const. intersected with D (these are the characteristic lines of the differential operator in (3.54)). Thus, the directional derivative exists for all $y \in \mathscr{N}(D)$.

Let $u : D \to \mathbb{R}$ be an immigration age-profile, that is u is measurable and locally bounded, $u(t,a) \geq 0$ and $\int_0^\infty u(t,a)\, \mathrm{d}a = 1$. Moreover, let $z_R : [0,\infty) \to \mathbb{R}$ be also measurable and locally bounded. Then, by definition $y \in \mathscr{N}(D)$ is a solution of (3.54)–(3.56) if the equations are satisfied almost everywhere with $\mathscr{D}y$ being the directional derivative, see (3.6). Notice that property (i) of the functions from $\mathscr{N}(D)$ implies that the right-hand side of (3.56) makes sense, and property (ii) implies that the traces $y(0,\cdot)$ and $y(\cdot,0)$ are (a.e.) well-defined and measurable (see [25] for more details; the above definition of a solution is equivalent to the ones commonly used in the literature, e.g. [3, 55]).

Note, that since we consider a human population, negative values for z_R and y make no sense. However, so far it is neither clear whether a solution of the system exists, nor that it is non-negative. In the following lemmas, existence, boundedness and non-negativity as well as the fixed size of a solution are discussed, both for the original system, and z_R chosen in the feedback form (3.57).

Lemma 3.12. *Let u and z_R be fixed as above. Then system (3.54)–(3.56) has a unique solution $y \in \mathscr{N}(D)$ and $y \in L_\infty(D_T)$ for every $T > 0$. The function z_B is locally bounded.*

The proofs of this and of the next lemmas in this section will be given in Section 3.3.6, in order to make the section more readable.

In parallel we consider the system

$$\mathscr{D}y(t,a) = -\mu(a)\,y(t,a) + \int_0^\infty (\mu(s) - \varphi(s))\,y(t,s)\,\mathrm{d}s\,u(t,a) \tag{3.60}$$

with side conditions (3.55) and (3.56). The meaning of a solution $y \in \mathscr{N}(D)$ is the same as for (3.54)–(3.56), regarding the fact that the integral on the right-hand side of (3.60) is well-defined and finite due to property (i) of the space $\mathscr{N}(D)$.

Lemma 3.13. *Let u be fixed as above. Then equation (3.60), with side conditions (3.55)–(3.56), has a unique solution $y \in \mathscr{N}(D)$ and y is (essentially) bounded on every subset $D_T \subset D$, $0 < T < \infty$. Moreover, the functions z_R and z_B, defined by (3.57) and (3.58), are locally Lipschitz continuous.*

Lemma 3.14. *Let u and z_R be as above and let y be the unique solution of (3.54)–(3.56). Then the population y has a fixed size (that is, $\int_0^\infty y(t,a)\,\mathrm{d}a = M$) if and only if the function z_R satisfies (3.57). In this case y coincides with the unique solution of (3.60) with side conditions (3.55)–(3.56).*

Lemma 3.15. *Let u be fixed as above. Assume that for the unique solution $y \in \mathscr{N}(D)$ of (3.60) with side conditions (3.55)–(3.56), it holds that $z_R(t) \geq 0$ for every $t \geq 0$, where z_R is defined by (3.57). Then the functions y, z_B and z_R are non-negative and bounded, uniformly with respect to u as above, for which the assumption $z_R(t) \geq 0$ is fulfilled.*

3.3.2 The Optimization Problem and an Existence Result

The main aim of this work is to determine optimal age patterns of immigrants in a population of fixed size. The specific optimization problem that we will introduce below arises only for populations that need a positive immigration in order to sustain their size, as it is the case for most European countries. Many of these countries had to face below-replacement fertility,

$$\int_0^\infty \varphi(a)\, e^{-\int_0^a \mu(\theta)\,\mathrm{d}\theta}\,\mathrm{d}a < 1, \tag{3.61}$$

for a long period. Below replacement fertility implies that the population decreases at exponential rate in the long run [33, Chapter 2]. Thus, the population would extinct without immigration. Therefore, immigration is needed to sustain the size. However, the asymptotically exponential decrease does not imply that the native population will decrease in the short run, therefore negative immigration, i.e. emigration ($z_R(t) < 0$) may be needed for some t (see [34]). In Figure 3.1–3.3 we provide an illustrative example which shows that the immigration rate $z_R(t)$ determined by (3.57) takes negative values for some t several generations after the initial time (see Figure 3.3), although fertility and mortality satisfy condition (3.61). Figure 3.1 presents the fertility, mortality and the immigration profile $u(a)$. The initial population $y_0(a)$ and the population $y(T,a)$ at time $T = 90$ are depicted in Figure 3.2.

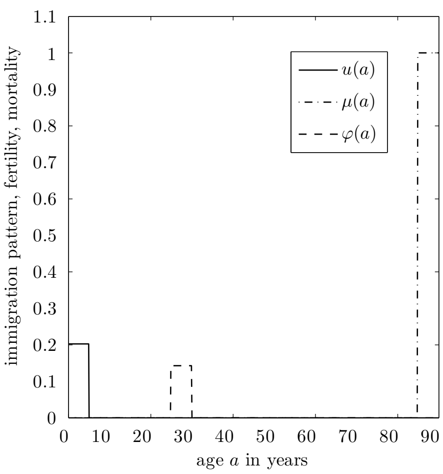

Figure 3.1: Fertility $\varphi(a)$ (dashed line), mortality $\mu(a)$ (dashed-dotted line) and time-invariante age-profile $u(a)$ (solid line). They satisfy the below replacement fertility condition.

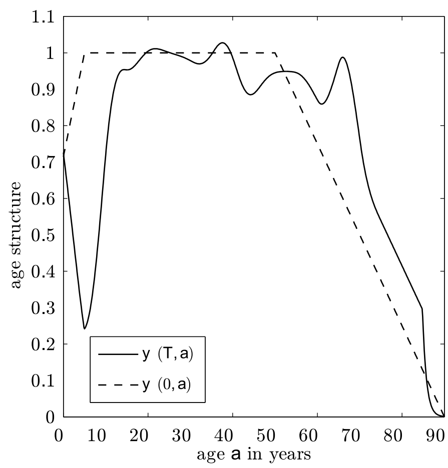

Figure 3.2: The initial $y_0(a)$ (dashed line) and the age structure $y(T,a)$ (solid line) at $T = 90$.

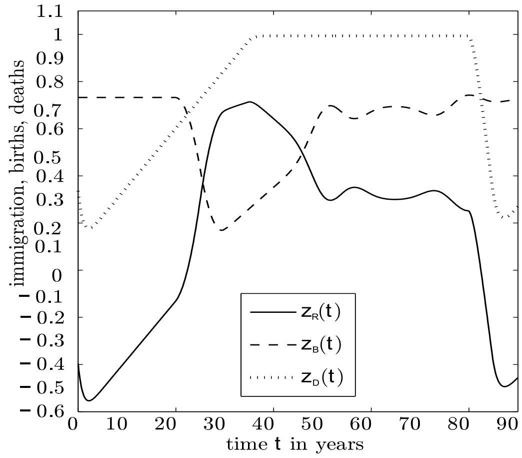

Figure 3.3: Below-replacement fertility is not enough to ensure $z_R(t) > 0$ (solid line) over time. The dashed line is the number of births $z_B(t)$, the dotted line the number of deaths $z_D(t)$.

Since in the present work we use immigration as a policy instrument, and negative immigration is not admissible, we have to eliminate this possibility by introducing Assumption 3.7 below, which is stronger than the below-replacement fertility condition.

In practice, discrimination of immigrants will not happen based on age only. Nevertheless, the age of applicants for a visa is taken into account, for example, by the Australian authorities[7]. There, in the skilled point test, 60 points are needed for a working permit; 30 of those can be gained by being a member of the age group ranging from 25 to 29, while for age 45+ zero points are awarded. Still, countries can not choose immigration arbitrarily but will have to face certain limits due to family bondings and the willingness of migrants to come to one country.

Therefore, let $m(a)$ be the present flow of immigrants, that is, at time $t = 0$, which is historically determined by habits, policies or other factors. Then the present normalized age-density of immigration is given by

$$u_0(a) := \frac{m(a)}{\int_0^\infty m(a)\,\mathrm{d}a}. \tag{3.62}$$

Therefore, when using the age-density of the immigration as a control (policy) variable we can implement only slight changes in $u_0(a)$. For this reason we consider control constraints of the form $\underline{u}(a) \le u(t,a) \le \bar{u}(a)$, where the lower and the upper bounds are not much different from the present values $u_0(a)$, say $\underline{u}(a) = (1-\varepsilon)u_0(a)$ and $\bar{u}(a) = (1+\varepsilon)u_0(a)$ with some small $\varepsilon > 0$.

[7] `www.visabureau.com/australia`, Standing of July 25, 2012

The optimization problem we consider in what follows is:

$$\max_{z_R,u} \int_0^\infty e^{-rt} \left[\int_0^\infty p(a)\, y(t,a)\, \mathrm{d}a - q z_R(t) \right] \mathrm{d}t, \tag{3.63}$$

subject to

$$\mathscr{D}y(t,a) = -\mu(a)y(t,a) + z_R(t)\, u(t,a), \qquad (t,a) \in D, \tag{3.64}$$

$$y(0,a) = y_0(a), \qquad a \geq 0, \tag{3.65}$$

$$y(t,0) = \int_0^\infty \varphi(a)\, y(t,a)\, \mathrm{d}a, \qquad t \geq 0, \tag{3.66}$$

$$\int_0^\infty y(t,a)\, \mathrm{d}a = M, \tag{3.67}$$

$$\underline{u}(a) \leq u(t,a) \leq \bar{u}(a), \qquad \int_0^\infty u(t,a)\, \mathrm{d}a = 1, \tag{3.68}$$

$$z_R(t) \geq 0. \tag{3.69}$$

Here, the function $p(a)$ is a weight function that is higher, if people of a certain age are more valuable from point of view of the policy maker solving this optimization problem. For example, $p(a)$ could be the function taking the value 1 for ages $a \in [20,65]$, representing the working ages, and 0 otherwise. The second term captures the cost or benefit of immigration.

The intertemporal discount rate is r. The constant q represents the benefits or costs of immigration arising, for example, from possible integration or education expenditures. If $q = 0$ then maximizing the performance value is related to minimizing the dependency ratio of the population (considered in a steady state in [41]), that is, the fraction of non-workers to workers in a population, which is a measure of how solvent a social security system is.

Additionally to Assumption 3.5 we make the following assumptions.

Assumption 3.6. *(vi) The discount rate r is strictly positive, the function $p : [0,\infty) \to \mathbb{R}$ is measurable and bounded, and q is a real number;*
(vii) the functions $\underline{u}$, $\bar{u} : [0,\infty) \to \mathbb{R}$ are measurable and bounded, satisfy the relations $0 \leq \underline{u}(a) \leq \bar{u}(a)$ for all $a \geq 0$, and $\bar{u}(a) = 0$ for all sufficiently large a, and $\int_0^\infty \underline{u}(a)\, \mathrm{d}a < 1 < \int_0^\infty \bar{u}(a)\, \mathrm{d}a$.

According to Lemma 3.14, we can reformulate problem (3.63)–(3.69) in the following way:

$$\max_{u \in \mathscr{U}} J(u) := \int_0^\infty e^{-rt} \left[\int_0^\infty [p(a) - q(\mu(a) - \varphi(a))]\, y(t,a)\, \mathrm{d}a \right] \mathrm{d}t, \tag{3.70}$$

$$\mathscr{D}y(t,a) = -\,\mu(a)\,y(t,a) + u(t,a)\int_0^\infty (\mu(s)-\varphi(s))\,y(t,s)\,\mathrm{d}s, \qquad (t,a)\in D, \tag{3.71}$$

$$y(0,a) = y_0(a), \qquad a\ge 0, \tag{3.72}$$

$$y(t,0) = z_B(t) = \int_0^\infty \varphi(a)\,y(t,a)\,\mathrm{d}a, \qquad t\ge 0. \tag{3.73}$$

Here, the set of admissible controls $\mathscr{U}$ is defined as

$$\mathscr{U} := \left\{ u : D \to \mathbb{R} \;\middle|\; \underline{u}(a) \le u(t,a) \le \bar{u}(a), \quad \int_0^\infty u(t,a)\,\mathrm{d}a = 1 \right\}. \tag{3.74}$$

Due to Assumption 3.5.(v), the set of admissible controls $\mathscr{U}$ is nonempty. Further, we denote by $\mathscr{U}_0$ the set of admissible age distributions for a fixed t:

$$\mathscr{U}_0 := \left\{ v : [0,\infty) \to \mathbb{R} \;\middle|\; \underline{u}(a) \le v(a) \le \bar{u}(a), \quad \int_0^\infty v(a)\,\mathrm{d}a = 1 \right\}.$$

The condition $z_R(t)\ge 0$ for a non-negative immigration rate is disregarded in the above reformulation. It will be stipulated by the following additional assumption.

Assumption 3.7. *For any $u\in\mathscr{U}$, the immigration intensity z_R, defined by* (3.57) *for the corresponding solution of (3.71)–(3.73), is strictly positive for all t.*

Even with below-replacement fertility, see (3.61), it could happen that $z_R(t) < 0$ for some t (as depicted in Figures 3.1–3.3). We assume that for the present immigration pattern $u_0(a)$, the resulting immigration size satisfies $z_R(t)\ge R_0 > 0$. Implicitly, this property requires that the initial density $y_0(a)$ results from a population which has experienced below-replacement fertility for quite a while before the present time $t = 0$. This is the situation in most of the European countries in the 21st century, for example, as in our case study in Section 3.3.5. We assume a bit more, namely that $z_R(t) > 0$ for any admissible control u, having in mind that all admissible controls are close to u_0.

Since $z_R(t) > 0$, Lemma 3.15 together with $r > 0$ and the boundedness of p, μ and φ, imply that $J(u)$ is finite for every $u\in\mathscr{U}$, and that $\sup_{u\in\mathscr{U}} J(u)$ is finite. Thanks to this, we can use the standard definition of optimality: $u\in\mathscr{U}$ is optimal if $J(u)\ge J(v)$ for every $v\in\mathscr{U}$.

Proposition 3.16. *Let assumptions* (3.5)–(3.7) *be fulfilled. Then the optimal control problem* (3.70)–(3.73) *has a solution.*

This proof is not routine since we deal with a problem of maximization of a non-concave functional. Indeed, the mapping $\mathscr{U}\ni u \longrightarrow$ "objective value $J(u)$" is not concave, as argued in

[27] even in the substantially simpler case $\varphi = 0$. The proof is a modification of that in [27], while the underlying idea stems from [4].

Proof. Denote by $J(u)$ the objective value for an admissible control u. Due to Lemma 3.15 and $r > 0$, the value $J(u)$ is finite and uniformly bounded with respect to $u \in \mathscr{U}$. Then $\hat{J} = \sup_{u\in\mathscr{U}} J(u)$ is also finite. Pick a maximizing sequence $\{u_k\}$ of admissible controls for which $J(u_k) \geq \hat{J} - \frac{1}{k}$. Denote by y_k the corresponding solution of (3.71)–(3.74), and let z_{R_k} be defined as in (3.57). According to Assumption 3.6 and Lemma 3.15, there is a constant C such that for all k it holds that $0 \leq y_k(t,a) \leq C$ and $0 < z_{R_k}(t) \leq C$ almost everywhere.

The sequence $\{e^{-rt}y_k\}$ of elements of $L_1(D)$ is weakly relatively compact due to the Dunford-Pettis criterion. Therefore, there exists a subsequence, which will also be denoted by y_k, such that $e^{-rt}y_k$ converges $L_1(D)$-weakly to some $e^{-rt}\hat{y}$, and $\hat{y}$ is obviously bounded by the same constant C. According to Mazur's lemma, there exist a sequence

$$e^{-rt}\tilde{y}_k := \sum_{i=k}^{n_k} p_i^k e^{-rt} y_i, \;\; p_i^k \geq 0, \;\; \sum_{i=k}^{n_k} p_i^k = 1,$$

that (strongly) converges to $e^{-rt}\hat{y}$ in $L_1(D)$. Obviously, for every $T > 0$ the sequence $\tilde{y}_k$ converges to $\hat{y}$ in $L_1(D_T)$. With the same weights p_i^k we define

$$\tilde{z}_{R_k}(t) := \sum_{i=k}^{n_k} p_i^k z_{R_i}(t) = \int_0^\infty (\mu(a) - \varphi(a))\tilde{y}_k(t,a)\,\mathrm{d}a. \tag{3.75}$$

Since $z_{R_k}(t) > 0$ holds for all $k > 0$ and $t > 0$, this also holds for $\tilde{z}_{R_k}$ and we can define

$$\tilde{u}_k(t,a) := \frac{1}{\tilde{z}_{R_k}(t)} \sum_{i=k}^{n_k} p_i^k z_{R_i}(t) u_i(t,a).$$

Obviously $\tilde{u}_k$ is also an admissible control. Moreover, we have that

$$\mathscr{D}\tilde{y}_k = \sum_{i=k}^{n_k} p_i^k(-\mu y_i + z_{Ri}u_i) = -\mu\tilde{y}_k + \tilde{z}_{R_k}\tilde{u}_k, \tag{3.76}$$

$$\tilde{y}_k(t,0) = \sum_{i=k}^{n_k} p_i^k y_i(t,0) = \sum_{i=k}^{n_k} p_i^k \int_0^\infty \varphi(a) y_i(t,a)\,\mathrm{d}a = \int_0^\infty \varphi(a)\tilde{y}_k(t,a)\,\mathrm{d}a, \tag{3.77}$$

which means that $(\tilde{u}_k, \tilde{y}_k)$ is an admissible control-trajectory pair in problem (3.70)–(3.74).

Since $\tilde{y}_k$ converges to $\hat{y}$ in $L_1(D_T)$, we may pass to an almost everywhere converging subsequence, which we denote again by $\tilde{y}_k$. Moreover, we may assume (passing again to a subsequence) that $e^{-rt}\tilde{u}_k$ converges to some $e^{-rt}\hat{u}$ weakly in $L_1(D)$. Now we will show that $\hat{u}$ is an admissible control. For every measurable and bounded set $\Gamma \subset [0,\infty)$ it holds that

$$\int_\Gamma \int_0^{\bar{a}} \tilde{u}_k(t,a)\,\mathrm{d}a\,\mathrm{d}t \to \int_\Gamma \int_0^{\bar{a}} \hat{u}(t,a)\,\mathrm{d}a\,\mathrm{d}t$$

where $\bar{a}$ is such that $\bar{u}(a) = 0$ for $a \geq \bar{a}$, hence also $\hat{u}(t,a) = 0$ (see (A1)). Since $\tilde{u}_k$ are admissible controls the left hand side is equal to meas(Γ), and thus also the right hand side. Since this holds for any measurable and bounded set Γ, this implies that $\bar{u}$ satisfies the integral constraint in (3.74). The inequality constraints are obviously also satisfied. Therefore, $\hat{u}$ is an admissible control. In the next paragraph we shall prove that $\hat{y}$ solves (3.71)–(3.73) with $u = u^*$.

Let us define

$$\hat{z}_R(t) = \int_0^\infty (\mu(a) - \varphi(a))\hat{y}(t,a)\,\mathrm{d}a.$$

Due to the pointwise convergence of $\tilde{y}_k$ in D_T, we obtain by passing to a limit in (3.75) that $\tilde{z}_{R_k}(t) \to \hat{z}_R(t)$ for a.e. $t \in [0,T)$, and since T is arbitrary this holds for a.e. $t \geq 0$. Moreover, for a.e. t the mapping $[0,T-t] \ni s \to \tilde{y}_k(t+s,s)$ is uniformly Lipschitz continuous. From here it easily follows (see [25] for more details) that $\hat{y}(t,0)$ is well defined for a.e. t and $\tilde{y}_k(t,0) \to \hat{y}(t,0)$. Then, by passing to a limit in (3.77) we obtain that $\hat{y}$ satisfies the boundary condition (3.73) for a.e. $t \in [0,T]$, hence for a.e. $t \geq 0$. In a similar way one can prove that $\hat{y}$ satisfies the initial condition (3.72). In order to show that (3.71) is also satisfied, we take an arbitrary measurable set $\Gamma \subset [0,T]$, and integrate with respect to $a \in \Gamma$ the representation (3.103) of the solution $\tilde{y}_k$ for $u = \tilde{u}_k$. Due to the established properties, we can pass on to the limit. Since Γ is arbitrary, we obtain that $(\hat{u},\hat{y},\hat{z}_R)$ satisfy (3.103) on D_T, hence $\hat{y}$ is a solution of (3.71) on D for $u = \hat{u}$. Thus, $(\hat{u},\hat{y})$ is an admissible control-trajectory pair.

Now let us show that $J(\hat{u}) \geq \hat{J}$. We have

$$\begin{aligned} J(\tilde{u}_k) &= \int_0^\infty e^{-rt}\left[\int_0^\infty p(a)\tilde{y}_k(t,a)\,\mathrm{d}a - q\tilde{z}_{R_k}(t)\right]\mathrm{d}t \\ &= \int_0^\infty e^{-rt}\left[\int_0^\infty p(a)\sum_{i=k}^{n_k} p_i^k y_i(t,a)\,\mathrm{d}a - q\sum_{i=k}^{n_k} p_i^k z_{Ri}(t)\right]\mathrm{d}t \\ &= \sum_{i=k}^{n_k} p_i^k J(u_i) = \sum_{i=k}^{n_k} p_i^k\left(\hat{J} - \frac{1}{i}\right) \geq \hat{J} - \frac{1}{k}. \end{aligned}$$

Using this, we obtain

$$
\begin{aligned}
\hat{J} \leq & \limsup_k \left(J(\tilde{u}_k) + \frac{1}{k} \right) = \limsup_k J(\tilde{u}_k) \\
= & \limsup_k \int_0^\infty e^{-rt} \left[\int_0^\infty p(a) \tilde{N}_k(t,a) \, \mathrm{d}a - q\tilde{R}_k(t) \right] \mathrm{d}t \\
= & \int_0^\infty e^{-rt} \left[\int_0^\infty p(a) \hat{y}(t,a) \, \mathrm{d}a - q\hat{z}_R(t) \right] \mathrm{d}t \\
= & J(\hat{u}).
\end{aligned}
$$

□

3.3.3 Necessary Optimality Conditions

In this section, necessary optimality conditions of Pontryagin's type for problem (3.70)-(3.73) are formulated and proved. They include (i) an appropriate *adjoint equation*; (ii) an appropriate *transversality condition* that uniquely determines a solution of the adjoint equation; (iii) a *maximization condition* for each t separately. The word "appropriate" in (i) and (ii) means that the maximization condition in (iii) holds true with the "appropriate" adjoint function.

The appropriate adjoint equation associated with our problem will be shown to have the form

$$
\begin{aligned}
\mathscr{D}\xi(t,a) = & (r + \mu(a))\,\xi(t,a) - \varphi(a)\,\xi(t,0) - (\mu(a) - \varphi(a)) \int_0^\infty \xi(t,\alpha)\,u(t,\alpha)\,\mathrm{d}\alpha \\
& - p(a) + q(\mu(a) - \varphi(a)).
\end{aligned} \tag{3.78}
$$

A main challenge is to define an appropriate transversality condition. Following ideas originating in [8], and developed in [6] for ordinary differential systems, and also in [27] for a problem which is similar but substantially simpler than (3.70)-(3.73), we introduce the "transversality" condition $\|\xi\|_{L_\infty(D)} < \infty$. This is justified by Proposition 3.20 and Theorem 3.21 below.

Note that the results from the previous section can not be applied here. A minor reason for this is that here we do not consider a maximal age. The major reason is the fact that the integral state enters the dynamics of the population, which is not considered in the previous model due to the complexity of this problem.

We start with some preliminary results and an additional assumption.

We introduce the notations

$$\rho(a) := r + \mu(a), \qquad \nu(a) := \mu(a) - \varphi(a), \qquad f(a) := p(a) - q\,(\mu(a) - \varphi(a))$$

and the auxiliary (adjoint) variables

$$\lambda(t) := \xi(t,0), \qquad \eta(t) := \int_0^\infty \xi(t,a)\,u(t,a)\,\mathrm{d}a. \tag{3.79}$$

Then the adjoint equation becomes

$$\mathscr{D}\,\xi(t,a) = \rho(a)\,\xi(t,a) - \varphi(a)\,\lambda(t) - \nu(a)\,\eta(t) - f(a). \tag{3.80}$$

Lemma 3.17. *Let Assumptions 3.5–3.7 be fulfilled. Then, for any given functions λ, $\eta \in L_\infty(0,\infty)$, equation (3.80) has a unique bounded solution on D, and it is given by the formula*

$$\xi(t,a) = \int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}\,[\varphi(s)\,\lambda(s+t-a) + \nu(s)\,\eta(s+t-a) + f(s)]\,\mathrm{d}s, \tag{3.81}$$

where the integral is absolutely convergent.

Proof. The integral in (3.81) is absolutely convergent and the function ξ is bounded due to the boundedness of the term in brackets and the inequality

$$\int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \le \frac{1}{r}.$$

One can verify by substitution that ξ defined by (3.81) satisfies (3.80).

To prove the uniqueness assertion we consider the difference $\Delta\xi(t,a)$ between two bounded solutions, which is also bounded. It satisfies the equation

$$\mathscr{D}\,\Delta\xi(t,a) = \rho(a)\,\Delta\xi(t,a).$$

If $\Delta\xi(t,a) \neq 0$ for some (t,a), then the function $x(s) := \Delta\xi(t+s,a+s)$, $s \geq 0$, satisfies $\dot{x}(s) = \rho(a+s)x(s)$ with $x(0) = \Delta\xi(t,a)$. Due to $\rho(a+s) \geq r > 0$, $x(s)$ is unbounded since $x(0) = \Delta\xi(t,a) \neq 0$. This contradiction completes the proof. □

Similarly as in the proof of Lemma 3.13 in Section 3.3.6, one can obtain by substituting (3.81) in (3.79) that ξ is a bounded solution of (3.78) if and only if it is the unique bounded solution of

(3.81) with the functions λ and η determined as bounded solutions of the following equations:

$$\lambda(t) = \int_t^\infty e^{-\int_0^{s-t}\rho(\tau)\,\mathrm{d}\tau}\left[\varphi(s-t)\,\lambda(s) + \nu(s-t)\,\eta(s) + f(s-t)\right]\mathrm{d}s, \tag{3.82}$$

$$\eta(t) = \int_t^\infty \int_0^\infty u(t,a)\,e^{-\int_a^{a+s-t}\rho(\tau)\,\mathrm{d}\tau} \times \ldots \tag{3.83}$$

$$\times\left[\varphi(a+s-t)\,\lambda(s) + \nu(a+s-t)\,\eta(s) + f(a+s-t)\right]\mathrm{d}a\,\mathrm{d}s.$$

This system can be written as integral equation (cf. Section 2.5)

$$x(t) = \int_t^\infty K(t,s)x(s)\,\mathrm{d}s + F(t), \tag{3.84}$$

where $x = (\lambda, \eta)$, $K(t,s) = (k_{i,j}(t,s))$ is the matrix

$$k(t,s) = \begin{pmatrix} e^{-\int_0^{s-t}\rho(\tau)\,\mathrm{d}\tau}\varphi(s-t) & e^{-\int_0^{s-t}\rho(\tau)\,\mathrm{d}\tau}\nu(s-t) \\ \int_0^\infty u(t,a)\,e^{-\int_a^{a+s-t}\rho(\tau)\,\mathrm{d}\tau}\varphi(a+s-t)\,\mathrm{d}a & \int_0^\infty u(t,a)\,e^{-\int_a^{a+s-t}\rho(\tau)\,\mathrm{d}\tau}\nu(a+s-t)\,\mathrm{d}a \end{pmatrix}, \tag{3.85}$$

$$F(t) = \begin{pmatrix} \int_t^\infty e^{-\int_0^{s-t}\rho(\tau)\,\mathrm{d}\tau} f(s-t)\,\mathrm{d}s \\ \int_t^\infty \int_0^\infty u(t,a)\,e^{-\int_a^{a+s-t}\rho(\tau)\,\mathrm{d}\tau} f(a+s-t)\,\mathrm{d}a\,\mathrm{d}s \end{pmatrix}.$$

A key point in the subsequent analysis is that integral equation (3.84) has a unique bounded solution. This, however requires an additional assumption about the kernel, which is formulated in terms of the numbers κ introduced below:

$$\kappa_{11} := \int_0^\infty e^{-\int_0^a \rho(\theta)\,\mathrm{d}\theta}\varphi(a)\,\mathrm{d}a, \qquad \kappa_{12} := \int_0^\infty e^{-\int_0^a \rho(\theta)\,\mathrm{d}\theta}|\nu(a)|\,\mathrm{d}a, \tag{3.86}$$

$$\kappa_{21} := \max_{a\geq 0}\int_0^\infty e^{-\int_a^{a+\tau}\rho(\theta)\,\mathrm{d}\theta}\varphi(a+\tau)\,\mathrm{d}\tau, \qquad \kappa_{22} := \max_{a\geq 0}\int_0^\infty e^{-\int_a^{a+\tau}\rho(\theta)\,\mathrm{d}\theta}|\nu(a+\tau)|\,\mathrm{d}\tau. \tag{3.87}$$

The following condition ensures that the integral operator in (3.84) is contractive in an appropriate norm (cf. Section 2.5):

Assumption 3.8. *Let either one of the following be true:*
(i) The following inequality is fulfilled

$$\frac{1}{2}\left[\kappa_{11} + \kappa_{22} + \sqrt{(\kappa_{11}-\kappa_{22})^2 + 4\,\kappa_{12}\,\kappa_{21}}\right] < 1. \tag{3.88}$$

(ii) Let u_0 be a reference time-invariant control (see (3.62) and the explanations there). Define

$$\bar{\kappa}_{21} := \int_0^\infty u_0(a) \int_0^\infty e^{-\int_a^{a+\tau} \rho(\theta)\,\mathrm{d}\theta} \varphi(a+\tau)\,\mathrm{d}\tau\,\mathrm{d}a,$$
$$\bar{\kappa}_{22} := \int_0^\infty u_0(a) \int_0^\infty e^{-\int_a^{a+\tau} \rho(\theta)\,\mathrm{d}\theta} |\nu(a+\tau)|\,\mathrm{d}\tau\,\mathrm{d}a,$$

For some $\varepsilon > 0$ it holds that

$$\bar{u}(a) \leq (1+\varepsilon)u_0(a), \;\; a \geq 0,$$

and the following inequality is fulfilled:

$$\frac{1}{2}\left[\kappa_{11} + (1+\varepsilon)\bar{\kappa}_{22} + \sqrt{(\kappa_{11} - (1+\varepsilon)\bar{\kappa}_{22})^2 + 4(1+\varepsilon)\kappa_{12}\,\bar{\kappa}_{21}}\right] < 1. \tag{3.89}$$

Remark 3.18. Assumption 3.8 implicitly requires that $\kappa_{11} < 1$, which is equivalent to the below-replacement fertility condition if $r = 0$. For $r > 0$ the below-replacement fertility condition is stronger than $\kappa_{11} < 1$.

While condition (i) is an assumption solely on the data of the problem, it may be too strict in some cases. If ε is sufficiently small, condition (ii) may be weaker because $\bar{\kappa}_{21} \leq \kappa_{21}$ and $\bar{\kappa}_{22} \leq \kappa_{22}$.

Lemma 3.19. *Under Assumptions 3.5–3.8, system (3.82)–(3.83) has a unique solution in $L_\infty(0,\infty)$.*

The proof will be given in the Section 3.3.6.

As a consequence of the above two lemmas in combination, we obtain the following proposition.

Proposition 3.20. *Under Assumptions 3.5–3.8, the adjoint equation (3.78) has a unique solution in $L_\infty(D)$.*

Proof. According to Lemma 3.19 system (3.82), (3.83) has a bounded solution. Then according to Lemma 3.17 equation (3.78) also has a bounded solution, obtained by substitution of the solution (λ, η) of (3.82), (3.83) in (3.80).

For any bounded solution ξ of (3.78) the functions λ and η defined by (3.79) are bounded and satisfy (3.82), (3.83). Therefore λ and η are uniquely determined (Lemma 3.19), hence ξ is unique (Lemma 3.17). □

The next theorem gives a Pontryagin-type necessary optimality condition in form of a maximum principle for problem (3.70)-(3.73). In the following $(\hat{u}, \hat{y})$ denotes an optimal solution of problem (3.70)–(3.73) and, corresponding to the pair $(\hat{u}, \hat{y})$.

Theorem 3.21. *Let Assumptions 3.5–3.8 be fulfilled, let* $(\hat{u}, \hat{y})$ *be an optimal solution of problem* (3.70)-(3.73)*, and denote by* $\hat{z}_R$ *and* $\hat{z}_B$ *the corresponding number of immigrants and births given by (3.57) and (3.58). Let* $\hat{\xi}$ *be the unique solution in* $L_\infty(D)$ *of the adjoint equation (3.78) with* $u = \hat{u}$*. Then for a.e.* $t \geq 0$ *the optimal control* $u(t,\cdot)$ *maximizes the integral*

$$\int_0^\infty \xi(t,a)v(a)\,\mathrm{d}a,$$

on the set of measurable functions $v \in \mathscr{U}_0$*.*

Proof. Let $\hat{J}$ be the optimal objective value and let $\hat{\xi}$ be the unique bounded solution of the adjoint equation (3.78) on D (see Proposition 3.20). Let us fix an arbitrary $\theta > 0$, let $h > 0$ be arbitrary (and presumably small), and $T > 0$ be such that $\theta - h \geq 0$ and $\theta + h \leq T$. Denote $\Theta_h := [\theta - h, \theta + h] \times [0,\infty) \subset D$ and define a "disturbed" control

$$\tilde{u}(t,a) := \begin{cases} \hat{u}(t,a) & \text{for } (t,a) \notin \Theta_h, \\ v(a) & \text{for } (t,a) \in \Theta_h, \end{cases} \tag{3.90}$$

where v is any measurable function in $\mathscr{U}_0$.

Then $\tilde{u}$ satisfies the control constraints. Let $\tilde{y}$ be the corresponding solution of (3.71)–(3.73) and $\tilde{z}_R$, $\tilde{z}_B$ be corresponding functions (immigration and birth flows) defined by (3.57) and (3.58), while $\hat{z}_R$ and $\hat{z}_B$ correspond to $\hat{y}$. Denote $\Delta J = J(\tilde{u}) - J(\hat{u})$, $\Delta u = \tilde{u} - \hat{u}$, $\Delta y = \tilde{y} - \hat{y}$, $\Delta z_R = \tilde{z}_R - \hat{z}_R$, all depending on the chosen h and v. According to Assumption 3.7, $\tilde{R}$ is non-negative, and all the functions introduced above are bounded (see Lemma 3.15).

Clearly,

$$\Delta J = \int_0^\infty e^{-rt} \int_0^\infty f(a)\,\Delta y(t,a)\,\mathrm{d}a\,\mathrm{d}t. \tag{3.91}$$

In order to obtain an expression for Δy we multiply the equation

$$\begin{aligned} \mathscr{D}\Delta y(t,a) = &- \mu(a)\,\Delta y(t,a) \\ &+ \int_0^\infty \nu(\alpha)\,[\Delta y(t,s)\,\hat{u}(t,a) + \hat{y}(t,s)\,\Delta u(t,a) + \Delta y(t,s)\Delta u(t,a)]\,\mathrm{d}s, \end{aligned} \tag{3.92}$$

which results from (3.71), by $e^{-rt}\hat{\xi}(t,a)$ and integrate on D. Since $D = \{(s, x+s) \,|\, s,\, x \geq 0\} \cup \{(x+s,s) : s,x \geq 0\}$ and the two sets on the right intersect only on a set of measure zero, we may represent

$$\int_0^\infty\int_0^\infty \mathscr{D}\Delta y(t,a)e^{-rt}\hat{\xi}(t,a)\,\mathrm{d}t\,\mathrm{d}a = \int_0^\infty\int_0^\infty e^{-rs}\hat{\xi}(s,x+s)\frac{\mathrm{d}}{\mathrm{d}s}\Delta y(s,x+s)\,\mathrm{d}s\,\mathrm{d}x + \int_0^\infty\int_0^\infty e^{-r(x+s)}\hat{\xi}(x+s,s)\frac{\mathrm{d}}{\mathrm{d}s}\Delta y(x+s,s)\,\mathrm{d}s\,\mathrm{d}x. \tag{3.93}$$

By integration by parts, the first term on the right-hand side gives

$$\int_0^\infty \Big[e^{-rs}\hat{\xi}(s,x+s)\Delta y(s,x+s)|_{s=0}^\infty - \int_0^\infty \Delta y(s,x+s)e^{-rs}(-r\hat{\xi}(s,x+s)+\frac{\mathrm{d}}{\mathrm{d}s}\hat{\xi}(s,x+s))\,\mathrm{d}s\Big]\,\mathrm{d}x.$$

The term with $s\to\infty$ is zero because both $\hat{\xi}$ and Δy are bounded, and the term with $s=0$ is zero because $\Delta y(0,a)=0$. The second term on the right-hand side of (3.93) is treated in the same way and combining the two terms, we obtain that the right-hand side of (3.93) is equal to

$$-\int_0^\infty e^{-rt}\hat{\xi}(t,0)\Delta y(t,0)\,\mathrm{d}t - \int_0^\infty\int_0^\infty e^{-rt}(\mathscr{D}\hat{\xi}(t,a)-r\hat{\xi}(t,a))\Delta y(t,a)\,\mathrm{d}a\,\mathrm{d}t. \tag{3.94}$$

Then, taking into account that $\hat{y}(t,0)=\int_0^\infty \varphi(a)\hat{y}(t,a)\,\mathrm{d}a$, we obtain from (3.93), by using (3.60) and rewriting it again in the (t,a)-plane, the equality

$$0=\int_0^\infty\int_0^\infty e^{-rt}\Big[(\hat{\xi}(t,0)\varphi(a)+\mathscr{D}\hat{\xi}(t,a)-r\hat{\xi}(t,a)-\mu(a)\hat{\xi}(t,a))\Delta y(t,a) + \int_0^\infty \nu(s)\hat{\xi}(t,a)\,(\Delta y(t,s)\hat{u}(t,a)+\hat{y}(t,s)\Delta u(t,a)+\Delta y(t,s)\Delta u(t,a))\,\mathrm{d}s\Big]\,\mathrm{d}a\,\mathrm{d}t.$$

Using the adjoint equation (3.78), we obtain that

$$0=\int_0^\infty\int_0^\infty e^{-rt}\Big[-f(a)\Delta y(t,a) + \int_0^\infty \Big(\nu(s)\hat{\xi}(t,a)\hat{y}(t,s)\Delta u(t,a)+\nu(s)\hat{\xi}(t,a)\Delta y(t,s)\Delta u(t,a)\Big)\,\mathrm{d}s\Big]\,\mathrm{d}a\,\mathrm{d}t.$$

Adding this to 3.91, we get

$$\begin{aligned}\Delta J &= \int_0^\infty\int_0^\infty\int_0^\infty e^{-rt}\nu(s)\hat{\xi}(t,a)\Big[\hat{y}(t,s)\Delta u(t,a)+\Delta y(t,s)\Delta u(t,a)\Big]\,\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}t \\ &= \int_0^\infty\int_0^\infty e^{-rt}z_R(t)\hat{\xi}(t,a)\Delta u(t,a)\,\mathrm{d}a\,\mathrm{d}t + \int_0^\infty\int_0^\infty\int_0^\infty e^{-rt}\nu(s)\hat{\xi}(t,a)\Delta y(t,s)\Delta u(t,a)\,\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}t.\end{aligned} \tag{3.95}$$

Next, we shall show that the second term on the right-hand side above is of second order with respect to h (cf. (3.90)). Note, that $\Delta u(t,a)=0$ for $t\notin[\theta-h,\theta+h]$, thus we need an estimation of Δy only on the time-horizon $[0,T]$, with some $T>\theta$, say $T=\theta+1$.

By solving equation (3.71) along the characteristic lines, we obtain the representation

$$\hat{y}(t,a) = \hat{z}_B(t-a)e^{-\int_{0\wedge(a-t)}^{a}\mu(\tau)\,\mathrm{d}\tau} + \int_{0\wedge(t-a)}^{t} e^{-\int_{a-t+s}^{a}\mu(\tau)\,\mathrm{d}\tau}\hat{z}_R(s)u(s,a-t+s)\,\mathrm{d}s,$$

where $\hat{z}_B$ is extended as $\hat{z}_B(t) = y_0(-t)$ for $t < 0$, and $0 \wedge \alpha := \max\{0,\alpha\}$. Due to Assumption 3.5, we can estimate $e^{-\int_0^a \mu(\tau)\,\mathrm{d}\tau} \leq 1$. A similar equality holds for $\tilde{y}(t,a)$ corresponding to the control $\tilde{u}$. Subtracting the two expressions we obtain

$$\|\Delta y(t,a)\| \leq \|\Delta z_B(t-a)\| + \int_{0\wedge(t-a)}^{t} \left\|\Delta z_R(s)\tilde{u}(s,a-t+s) + z_R(s)\Delta u(s,a-t+s)\right\| \mathrm{d}s. \quad (3.96)$$

From (3.57) and (3.58) we can estimate

$$|\Delta z_R(t)| + |\Delta z_B(t)| \leq c_1 \int_0^\infty |\Delta y(t,a)|\,\mathrm{d}a = c_1\|\Delta y(t,\cdot)\|_{L_1(0,\infty)},$$

where c_1 is a constant depending only on φ and μ. Then it is a matter of routine estimations (taking into account that Δu is non-zero on a set of measure proportional to h) to obtain the inequality

$$\|\Delta y(t,\cdot)\|_1 \leq \int_0^t c_2\|\Delta y(s,\cdot)\|_1 e^{-(t-s)\mu_0}\,\mathrm{d}s + c_3 h, \quad t \in [0,\theta+1],$$

where c_2 and c_3 are independent of h (although they may depend on θ and the data of the problem).

Since $T = \theta + 1$ is finite, Gronwall's lemma gives

$$\|\Delta y(t,\cdot)\|_1 \leq Ch, \qquad t \leq T.$$

Now it is straightforward to estimate the last term in (3.95) by

$$\int_{\theta-h}^{\theta+h} e^{-rt} C h \|v\|_{L_\infty} \|\hat{\xi}\|_{L_\infty} \int_0^{\bar{a}} |\Delta u(t,\alpha)|\,\mathrm{d}\alpha\,\mathrm{d}t \leq Ch^2.$$

Using this in the estimation (3.95), and having in mind that $\Delta J \leq 0$ due to the optimality of u, we obtain that

$$\frac{1}{2h}\int_{\theta-h}^{\theta+h} e^{-rt}\int_0^\infty z_R(t)\,\xi(t,a)\,u(t,a)\,\mathrm{d}a\,\mathrm{d}t \geq \frac{1}{2h}\int_{\theta-h}^{\theta+h} e^{-rt}\int_0^\infty z_R(t)\,\xi(t,a)\,v(a)\,\mathrm{d}a\,\mathrm{d}t - \frac{C}{2}h.$$

Almost every s is a Lebesgue point of the function $t \to \int_0^\infty \hat{z}_R(t)\hat{\xi}(t,a)\hat{u}(t,a)\,\mathrm{d}a$, and $\hat{z}_R(t) > 0$. Therefore, we can conclude the proof of the theorem by taking the limit $h \to 0$. □

3.3.4 Uniqueness, stationarity, and Structure of the Optimal Control

In this section, we use Theorem 3.21 to obtain some qualitative properties of the optimal solution of problem (3.70)-(3.73). The most interesting one is that the optimal control $\hat{u}$ (that is, the optimal immigration profile) is unique and time-invariant: $\hat{u}(t,a) \equiv \hat{u}(a)$. This fact is not evident. Its proof is based on stability condition 3.8, and on an additional well-posedness condition, which implies also a bang-bang structure of the optimal control $u(a)$.

To prove uniqueness and stationarity of the optimal solution, we rewrite the adjoint equation in a feedback form. To do this, we introduce the functional $\sigma(\cdot)$:

$$\sigma(g) := \max_{v \in \mathscr{U}_0} \int_0^\infty g(a)v(a)\,\mathrm{d}a, \quad g \in L_\infty(0,\infty). \tag{3.97}$$

Then, using the optimization condition in Theorem 3.21, we can rewrite the adjoint equation (3.78) in feedback form,

$$\mathscr{D}\xi(t,a) = (r+\mu(a))\xi(t,a) - \varphi(a)\xi(t,0) - (\mu(a)-\varphi(a))\sigma(\xi(t,\cdot)) - f(a). \tag{3.98}$$

The existence of a solution in $L_\infty(D)$ to this equation follows from the necessity of the maximum principle.

Lemma 3.22. *If Assumptions 3.5–3.6 are fulfilled, then equation* (3.98) *has a unique bounded solution.*

The proof is given in Section 3.3.6.

Now we introduce a *regularity assumption* that ensures that the maximization condition in Theorem 3.21 determines a unique control. As shown in [27] in a simpler version of the problem considered here, without a certain regularity assumption the uniqueness fails and this is due to non-concavity of the problem. On the other hand, the regularity assumption is in a reasonable sense generic and easy to check.

Assumption 3.9. *For all real numbers d_0, d_1 and d_2 it holds that*

$$\mathrm{meas}\{a \in [0,\infty] : d_0 + d_1\mu(a) + d_2\varphi(a) - p(a) = 0\} = 0.$$

This assumption requires that μ, φ, and p must not be linearly related on a set of positive measure.

Theorem 3.23. *Let Assumptions 3.5–3.9 be fulfilled. Then the optimal control problem* (3.70)–(3.74) *has a unique optimal control $\hat{u}$ and it is time-invariant: $\hat{u}(t,a) \equiv \hat{u}(a)$.*

Proof. First we shall prove that (3.98) has a stationary bounded solution $\hat{\xi}(t,a) = \hat{\xi}(a)$. To do this, we show that the equation

$$\xi'(a) = (r+\mu(a))\xi(a) - \varphi(a)\xi(0) - (\mu(a)-\varphi(a))\sigma(\xi(\cdot)) - f(a) \tag{3.99}$$

has a bounded solution. Denote $\lambda = \xi(0)$ and $\eta = \sigma(\xi(\cdot))$, then we can write the solution of the differential equation as

$$\xi(a) = \int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}[\varphi(s)\lambda + \nu(s)\eta + f(s)]\,\mathrm{d}s.$$

Using the definition of $\sigma(\xi(\cdot))$, (3.97), the equations for η and λ are

$$\begin{aligned}\lambda &= \int_0^\infty e^{-\int_0^s \rho(\tau)\,\mathrm{d}\tau}[\varphi(s)\lambda + \nu(s)\eta + f(s)]\,\mathrm{d}s \\ \eta &= \max_{v\in\mathscr{U}_t}\int_0^\infty v(a)\int_a^\infty e^{-\int_a^s \rho(\tau)\,\mathrm{d}\tau}[\varphi(s)\lambda + \nu(s)\eta + f(s)]\,\mathrm{d}s\,\mathrm{d}a\end{aligned}$$

Denoting the terms independent from λ and η by (b_1,b_2), and by K the matrix defined by the right hand side, we can write the equations above as

$$(I-K)\begin{pmatrix}\lambda\\ \eta\end{pmatrix} = \begin{pmatrix}b_1\\ b_2\end{pmatrix}, \tag{3.100}$$

where I is the 2×2 identity matrix. As in the proof of Lemma 3.19 and 3.22, define a norm on $\mathbb{R}^2$ as $\|(x,y)\| = \max\{\|x\|, \alpha\|y\|\}$, $\alpha > 0$. Estimating the operator norm of K in the same way as in the proof of Lemma 3.19 gives that the norm is smaller or equal to the left hand side in (3.89). Then Assumption 3.8 states that the norm is smaller than one, thus $(I-K)$ is invertible and therefore a unique solution of (3.100) exists. Thus, a bounded solution $\hat{\xi}(a)$ of (3.99) exists and it is obviously a stationary bounded solution of (3.98).

According to Lemma 3.22 the stationary function $\hat{\xi}(a)$ is the unique bounded solution of (3.98). On the other hand, Theorem 3.21 claims that for every optimal control u, the adjoint equation (3.78) has a unique bounded solution $\hat{\xi}(t,a)$, and for a.e. $t \geq 0$

$$\int_0^\infty \hat{\xi}(t,a)\hat{u}(t,a)da = \max_{v\in\mathscr{U}_0}\int_0^\infty \hat{\xi}(t,a)v(a)\,\mathrm{d}a = \sigma(\hat{\xi}(t,\cdot)).$$

Then ξ is a bounded solution also of (3.98), which implies that $\xi = \hat{\xi}$. The above maximization condition reads now as

$$\int_0^\infty \hat{\xi}(a)u(t,a)da = \max_{v\in\mathscr{U}_0} \int_0^\infty \hat{\xi}(a)v(a)\,\mathrm{d}a, \tag{3.101}$$

where $\hat{\xi}$ is the unique bounded solution of (3.99). This implies the time-invariance of the solution. □

Assumption 3.9 obviously implies that the solution $\hat{\xi}$ of (3.99) cannot be constant on a set of positive measure. Then, similarly as in Corollary 5.1. in [27], one can prove that (3.101) uniquely determines (modulo a set of measure zero) a control $u \in \mathscr{U}$, it is time-invariant and has the following structure: there is a real number l such that

$$\hat{u}(t,a) = \begin{cases} \underline{u}(a) & \text{if } \hat{\xi}(a) \le l, \\ \bar{u}(a) & \text{if } \hat{\xi}(a) > l. \end{cases} \tag{3.102}$$

We formulate the last finding in the proof of the above theorem as a corollary.

Corollary 3.24. *Let $\hat{\xi}$ be the unique bounded solution of (3.99). Then, there is $l \in \mathbb{R}$ such that the unique optimal control $\hat{u}(t,a) \equiv \hat{u}(a)$ is determined by (3.102). This number l is the only one for which the resulting u_0 satisfies $\int_0^\infty \hat{u}(a)\,\mathrm{d}a = 1$.*

Thus the optimal solution is of bang-bang type. A related result is obtained in [41] for a static counterpart of the problem considered in this section. Since $\hat{\xi}(a)$ can be interpreted marginally as "shadow price" of an a-year-old individual, the above corollary asserts that there is a critical value l such that it is optimal to encourage as much as possible migration in ages for which the shadow price is higher than l ($\hat{u}(a) = \bar{u}(a)$), and restrict as much as possible migration in ages for which the shadow price is smaller than l. The remarkable fact here is that the shadow price is time-invariant.

3.3.5 A Case Study: the Austrian Population

In this section, we numerically determine the optimal immigration policy of (3.70)–(3.73) for the case study of the Austrian population. The numerical results for the optimal time-invariant immigration profile and the population's age structure obtained in this section are based on the analytical results above. In all the numerical calculations below, we specify $p(a)$ in (3.70) as the characteristic function of the age interval $[20,65]$. If we additionally set $q = 0$, the objective function (3.70) is the discounted and aggregated number of workers over time. It is related to the so-called dependency ratio, which is the ratio of nonworking age population to the working age population. The dependency ratio is an important demographic indicator for the solvency of the social security system of a population. The case of $q > 0$, which is also discussed below, accounts for possible costs for the integration of immigrants.

For the computations, we initialize the age structure of demographic variables referring to Austrian data as of 2009, and interpolate these data piecewise linearly to obtain continuous representations of the vital rates, $\varphi(a)$, $\mu(a)$. As already mentioned in Remark 3.11, we assume that $\mu(a) \equiv \mu(95)$ for $a \geq 95$. These demographic data, together with an intertemporal discount rate of $r = 0.04$, satisfy Assumption 3.8 with $\kappa_{11} = 0.0737$, $\kappa_{12} = 0.0774$, $\kappa_{21} = 0.1480$, $\kappa_{22} = 0.1506$. For these values, the quantity in the left hand side of (3.88) equals 0.2259 and is therefore well below 1. For the initial age structure $y_0(a)$ we take the annual average numbers of the Austrian female population in 2009, see Figure 3.5 (solid line). The normalized immigration age-density of 2009 is denoted by $u_0(a)$, see Figure 3.4. We set the lower and upper age-specific limits for immigration to

$$\underline{u}(a) = 0 \text{ and } \overline{u}(a) = 2\hat{u}(a).$$

In the following, we analyze three scenarios: in the uncontrolled case, the immigration age density remains the same in the future $u(t,a) \equiv u_0(a)$; then we assume that q in (3.70) takes the value zero and the immigration age density is chosen optimally $u(t,a) = \hat{u}(a)$; additionally, we set $q = 200$ where again $u(t,a) = \hat{u}(a)$ is chosen optimally. With the last scenario we analyze the effect of immigration costs on the optimal immigration age-pattern, see Figure 3.4. As it can be seen in this figure, the optimal age profile of immigrants is at its upper bound from slightly before the lowest working age of $a = 20$ until the mid thirties. It is on its lower bound at any other ages. Note that increasing the costs of immigration shifts the optimal age pattern to the left.

In Figure 3.5, we compare the age structure of the initial population with the stationary population at $t = 400$, which results in the uncontrolled case, and when applying the optimal $\hat{u}(a)$ for $q = 200$. The sharp increase of the optimal population $\hat{y}(400,a)$ at the low working ages is due to the annual inflow of immigrants at these ages.

In Figure 3.6, we plot the evolution of the number of newborns $z_B(t)$, the number of deaths $z_D(t)$ and the recruitment rate $z_R(t)$ on the time horizon $[0,400]$, where $z_D(t) = z_R(t) + z_B(t)$. Note, that for the controlled as well as for the controlled immigration, there is a huge increase in the number of immigrants $z_R(t)$ at the beginning, caused by the high number of deaths as a result

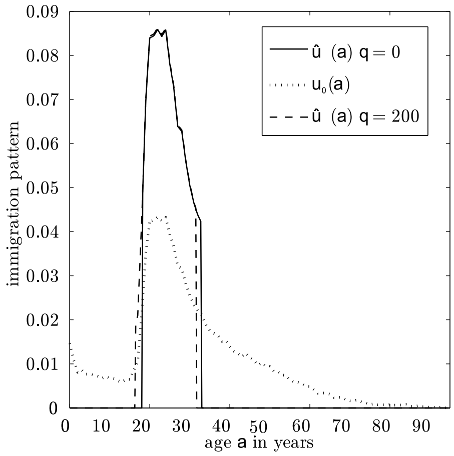

Figure 3.4: The actual age density $u_0(a)$ (dottet line) and the optimal immigration density $u^*(a)$ for $q = 0$ (solid) and $q = 200$ (dashed line).

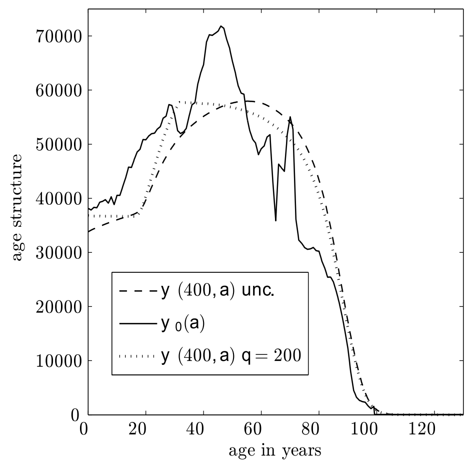

Figure 3.5: The initial age structure $y_0(a)$ (solid line) and $y(400, a)$ for the uncontrolled case (dashed) and for the optimal control with $q = 200$ (dotted).

of the baby boom that occurred in Austria in the 50s and 60s.

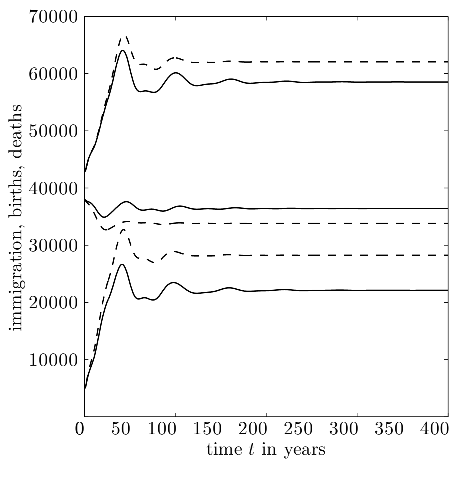

Figure 3.6: The evolution of the number of deaths $z_D(t)$ (upper solid line), the number of births $z_B(t)$ (middle solid line) and the number of immigrants $z_R(t)$ (lower solid line) over time for the optimal control and $q = 0$ compared with the uncontrolled case (corresponding dashed lines).

In Figure 3.7, the change of the number of workers, and in Figure 3.8, the dependency ratio over time are shown. We compare the scenario with $q = 0$ to the case, where current age-specific immigration rates would remain the same in the future. Clearly, we can sustain a higher number of workers and simultaneously a lower dependency ratio when applying the optimal immigration pattern $\hat{u}(a)$.

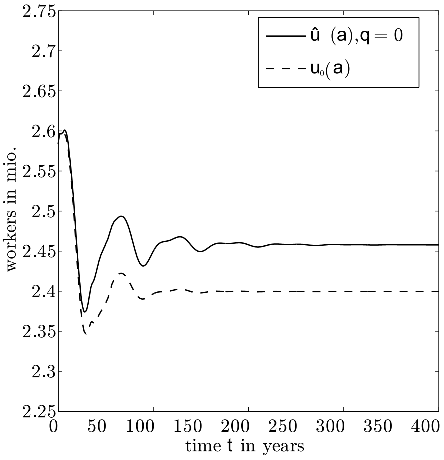

Figure 3.7: The evolution of the number of workers over time for $q = 0$ (solid line) and the uncontrolled case (dashed line).

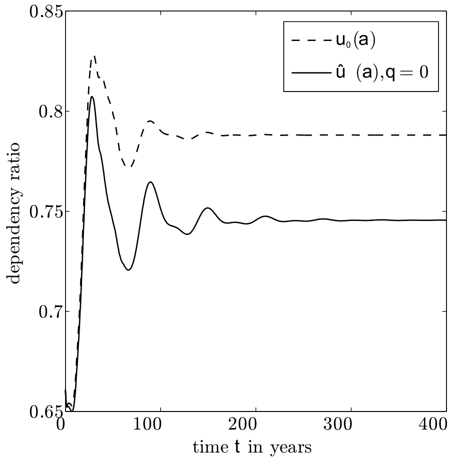

Figure 3.8: The evolution of the so-called dependency ratio over time for $q = 0$ (solid line) and the uncontrolled case (dashed line).

3.3.6 Proofs of the Lemmas

We start with the proof of Lemma 3.13, then we prove Lemma 3.12, and the rest of the proofs are in the order as they appear in the paper.

Proof of Lemma 3.13. Let us start with the uniqueness. Let $y \in \mathcal{N}$ be a solution of (3.60), (3.55), (3.56). Let $z_R(t)$ and $z_B(t)$ be defined by (3.57) and (3.58), respectively. Both are measurable and locally bounded, according to property (i) of $\mathcal{N}$.

The function y has for $(t,a) \in D$ the following representation, resulting from solving (3.54) along the characteristic lines:

$$y(t,a) := \begin{cases} e^{-\int_0^a \mu(\tau)\,\mathrm{d}\tau} z_B(t-a) + \int_{t-a}^t e^{-\int_{a-t+s}^a \mu(\tau)\,\mathrm{d}\tau} z_R(s)\, u(s,a-t+s)\,\mathrm{d}s, & \text{if } a < t, \\ e^{-\int_{a-t}^a \mu(\tau)\,\mathrm{d}\tau} y_0(a-t) + \int_0^t e^{-\int_{a-t+s}^a \mu(\tau)\,\mathrm{d}\tau} z_R(s)\, u(s,a-t+s)\,\mathrm{d}s, & \text{if } a \geq t. \end{cases} \tag{3.103}$$

on $[0,\infty)$. Inserting this expression for y in (3.58) and (3.57), and changing the order of integration in the double integrals, we obtain the following system of Volterra equations of the second kind for z_B and z_R:

$$z_B(t) = \int_0^t z_R(s) \int_0^\infty e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} \varphi(a+t-s)\, u(s,a)\, \mathrm{d}a\, \mathrm{d}s \tag{3.104}$$
$$+ \int_0^t z_B(s)\, e^{-\int_0^{t-s} \mu(\tau)\,\mathrm{d}\tau} \varphi(t-s)\, \mathrm{d}s + \int_0^\infty e^{-\int_s^{t+s} \mu(\tau)\,\mathrm{d}\tau}\, \varphi(s+t)\, y_0(s)\, \mathrm{d}s,$$
$$z_R(t) = \int_0^t z_R(s) \int_0^\infty e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} \nu(a+t-s)\, u(s,a)\, \mathrm{d}a\, \mathrm{d}s$$
$$+ \int_0^t z_B(s)\, e^{-\int_0^{t-s} \mu(\tau)\,\mathrm{d}\tau} \nu(t-s)\, \mathrm{d}s + \int_0^\infty e^{-\int_s^{t+s} \mu(\tau)\,\mathrm{d}\tau}\, \nu(s+t)\, y_0(s)\, \mathrm{d}s,$$

where $\nu(a) := \mu(a) - \varphi(a)$. Notice that all the four components of the kernel of this system are bounded due to the properties of u and the data. Indeed, take for example the most complicated component

$$\left| \int_0^\infty e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} \nu(a+t-s)\, u(s,a)\, \mathrm{d}a \right| \le \sup_{a \ge 0} \left\{ e^{-\int_a^{a+t-s} \mu(\tau)\,\mathrm{d}\tau} |\nu(a+t-s)| \right\} \int_0^\infty u(s,a)\, \mathrm{d}a$$
$$\le \bar{\sup}_{a \ge 0} |\nu(a)| < \infty, \qquad 0 \le s \le t < \infty.$$

According to Theorems 5.4 and 5.5 in Chapter 9 of [28], this system has a unique locally bounded solution (z_B, z_R), so that z_B and z_R are uniquely determined, hence y is also uniquely determined by (3.103).

On the other hand, from the existence of the locally bounded solution (z_B, z_R), we obtain a function y from (3.103). Due to the local boundedness of z_B and z_R, and due to $\int_0^\infty y_0(a)\, \mathrm{d}a = M$, we have that $y \in \mathscr{N}$. It is straightforward to check that y satisfies (3.60), (3.55), (3.56), which proves the existence.

It remains to prove that the functions z_R and z_B are locally Lipschitz continuous. We have

$$z_B(t+h) - z_B(t) = \int_0^\infty \varphi(a)(y(t+h,a) - y(t+h,a+h))\, \mathrm{d}a$$
$$+ \int_0^\infty \varphi(a)(y(t+h,a+h) - y(t,a))\, \mathrm{d}a.$$

Notice that both y and $\mathscr{D}y$ are bounded on every set D_T due to property (i) of $\mathscr{N}$ and (3.6). Then the second integral is proportional to h because of the absolute continuity of y along the characteristic lines. For the first integral it holds that

$$\begin{aligned}
&\int_0^\infty \varphi(a)(y(t+h,a)-y(t+h,a+h))\,\mathrm{d}a \\
&= \int_0^h \varphi(a)\,y(t+h,a)\,\mathrm{d}a + \int_h^\infty \varphi(a)\,y(t+h,a)\,\mathrm{d}a - \int_0^\infty \varphi(a)\,y(t+h,a+h)\,\mathrm{d}a \\
&= \int_0^h \varphi(a)\,y(t+h,a)\,\mathrm{d}a + \int_0^\infty (\varphi(a+h)-\varphi(a))\,y(t+h,a+h)\,\mathrm{d}a.
\end{aligned}$$

The last integral is proportional to h due to the Lipschitz continuity of φ and the boundedness of y. Therefore, the Lipschitz continuity follows. The proof for z_R is analogous. □

Proof of Lemma 3.12. The proof is essentially the same as the proof of Lemma 3.13, but easier because $z_R(t)$ is given and we deal with only on Volterra equation.

Let $z_B(t) := y(t,0)$. Then equations (3.103) for $y(t,a)$ and (3.104) for $z_B(t)$ hold true, where in the last one z_R is given. Due to the local boundedness of the kernel of the integral equation (3.104), a locally bounded solution $z_B(t)$ exists. Then the uniqueness of a solution $y \in \mathscr{N}$ and the local Lipschitz continuity of $z_B(t)$ are derived in the same way as in the proof of Lemma 3.13. □

Proof of Lemma 3.14. The function $t \mapsto M(t) := \int_0^\infty y(t,a)\,\mathrm{d}a$ is locally Lipschitz (which can be concluded analogously as for z_R and z_B in Lemma 3.13) and thus almost everywhere differentiable. Then the fixed size property is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}M(t) = 0$ for almost every t. The latter is equivalent to having the weak derivative of $M(\cdot)$ equal to zero. That is, having

$$\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}M(t)\,\mathrm{d}t = 0$$

for every $\Psi(t) \in C_0^\infty(0,\infty)$ (the space of all infinitely differentiable function with compact support and $\Psi(0) = 0$).

We have

$$\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}M(t)\,\mathrm{d}t = [\Psi(t)M(t)]_{t=0}^\infty - \int_0^\infty M(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)\,\mathrm{d}t$$

The first term is zero because of the properties of $\Psi(t)$. The second we can rewrite along the characteristic lines of (3.54) as follows:

$$\begin{aligned}
&\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}M(t)\,\mathrm{d}t = -\int_0^\infty\int_0^\infty y(t,a)\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}a\,\Psi(t)\,\mathrm{d}t\\
&= -\int_0^\infty\int_0^\infty y(s,\tau+s)\frac{\mathrm{d}}{\mathrm{d}s}\Psi(s)\,\mathrm{d}s\,\mathrm{d}\tau - \int_0^\infty\int_0^\infty y(\tau+s,s)\frac{\mathrm{d}}{\mathrm{d}s}\Psi(\tau+s)\,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned}$$

Integrating again by parts and using (3.56) we obtain

$$\begin{aligned}
\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}M(t)\,\mathrm{d}t &= -\left[\int_0^\infty y(s,\tau+s)\Psi(s)\right]_0^\infty \mathrm{d}\tau + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty y(s,\tau+s)\,\Psi(s)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\quad -\left[\int_0^\infty y(\tau+s,s)\Psi(\tau+s)\right]_0^\infty \mathrm{d}\tau + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty y(\tau+s,s)\,\Psi(\tau+s)\,\mathrm{d}s\,\mathrm{d}\tau\\
&= \int_0^\infty\int_0^\infty \varphi(a)y(\tau,a)\,\mathrm{d}a\,\Psi(\tau)\,\mathrm{d}\tau\\
&\quad +\int_0^\infty\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}y(s,\tau+s)\,\Psi(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^\infty\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}y(\tau+s,s)\,\Psi(\tau+s)\,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned}$$

Using (3.54) and rewriting the integral again in the (t,a)-plane, we obtain

$$\int_0^\infty \Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}M(t)\,\mathrm{d}t = \int_0^\infty\int_0^\infty \Psi(t)[-\mu(a)y(t,a)+\varphi(a)y(t,a)+z_R(t)u(t,a)]\,\mathrm{d}a\,\mathrm{d}t.$$

The fact $\int_0^\infty u(t,a)\,\mathrm{d}a = 1$ and the arbitrary choice of $\Psi \in C_0^\infty(0,\infty)$ imply that the left hand side is zero if and only if $z_R(t) = \int_0^\infty [\mu(a)-\varphi(a)]y(t,a)\,\mathrm{d}a$ for almost every t.

The last claim of the lemma is evident. □

Proof of Lemma 3.15. First we shall prove that under the conditions in Lemma 3.15, we have $z_B(t) \geq 0$ for all $t \geq 0$. We recall that z_B is a continuous function due to Lemma 3.13. Moreover, from (3.13) and Assumption 3.5, we have $z_B(0) = \int_0^\infty \varphi(s)y_0(s)\,\mathrm{d}s > 0$. Denote

$$\theta = \sup\{t \geq 0 : z_B(s) > 0 \text{ on } [0,t)\}.$$

Assume that θ is finite (otherwise we are done). Then $z_B(\theta) = 0$. On the other hand, we have from (3.13) that

$$0 = z_B(\theta) \geq \int_0^\theta z_B(s)\,e^{-\int_0^{\theta-s}\mu(\tau)\,\mathrm{d}\tau}\varphi(\theta-s)\,\mathrm{d}s + \int_0^\infty e^{-\int_s^{\theta+s}\mu(\tau)\,\mathrm{d}\tau}\varphi(s+\theta)\,y_0(s)\,\mathrm{d}s, \quad (3.105)$$

where we use the assumption $z_R(t) \geq 0$. Obviously both terms are non-negative. We shall show that at least one of them is strictly positive, which contradicts (3.105). If $\varphi(a) > 0$ for some $a \in [0, \theta)$, then the first integral in (3.105) is strictly positive since $z_B(s) > 0$ on $[0, \theta)$. Alternatively, let $\varphi(a) = 0$ for all $a \in [0, \theta)$. Then $a_0 \geq \theta$ (see Assumption 3.5). Take $s = a_0 - \theta$. Then the integrand $e^{-\int_s^{\theta+s} \mu(\tau)\,\mathrm{d}\tau} \varphi(s+\theta) y_0(s)$ is strictly positive, hence the second integral in (3.105) is strictly positive, too. The obtained contradiction proves that $z_B(t) \geq 0$ for all $t \geq 0$. From (3.103) it follows also that $y(t,a) \geq 0$.

Then the boundedness follows easily:

$$\begin{aligned} z_R(t) \leq & \int_0^\infty |\mu(a) - \varphi(a)|\, |y(t,a)|\,\mathrm{d}a \leq (\bar{\mu} + \bar{\varphi}) \int_0^\infty |y(t,a)|\,\mathrm{d}a \\ =& (\bar{\mu} + \bar{\varphi}) \int_0^\infty y(t,a)\,\mathrm{d}a = (\bar{\mu} + \bar{\varphi}) M, \end{aligned}$$

where $\bar{\varphi}$ is the upper bound for φ. The same argument proves also boundedness of z_B. Then the boundedness of $y(t,a)$ follows from (3.103) and the fact that $u(t,a) = 0$ for all sufficiently large a.
□

Proof of Lemma 3.19. Consider the kernel $K(t,s)$ of the integral equation (3.84) defined in (3.85). It defines an operator $\mathscr{K} : (L_\infty(0,\infty))^2 \to (L_\infty(0,\infty))^2$. The operator depends on u, so we need the existence of a solution of the integral equation for every admissible u. If the operator norm is smaller than one, a resolvent $\mathscr{R} : (L_\infty(0,\infty))^2 \to (L_\infty(0,\infty))^2$ with kernel $R(t,s)$ exists according to Corollary 3.10 and Theorem 3.6 in [28], and can be written as

$$x(t) = F(t) - \int_0^\infty R(t,s) F(s)\,\mathrm{d}s.$$

To show that the norm is smaller than unity, we define for $\alpha > 0$ a new norm on $(L_\infty(0,\infty))^2$.

$$\|(x_1, x_2)\| = \max\{\|x_1\|_{L_\infty}, \alpha \|x_2\|_{L_\infty}\}.$$

Take $x \in (L_\infty(0,\infty))^2$ with $\|x\| = 1$, and estimate the norm of $y = \mathscr{K} x$.

$$\begin{aligned}
\|y\| &= \max\{y_1, \alpha y_2\} \\
&= \max\Big\{ \sup_{t\geq 0} \int_t^\infty [k_{11}(t,s)x_1(s) + k_{12}(t,s)x_2(s)]\,\mathrm{d}s, \\
&\qquad \alpha \sup_{t\geq 0} \int_t^\infty [k_{21}(t,s)x_1(s) + k_{22}(t,s)x_2(s)]\,\mathrm{d}s \Big\} \\
&\leq \max\Big\{ \sup_{t\geq 0} \int_t^\infty |k_{11}(t,s)|\,\mathrm{d}s\,\|x_1\|_\infty + \sup_{t\geq 0} \int_0^\infty \frac{1}{\alpha}|k_{12}(t,s)|\,\mathrm{d}s\,\alpha\|x_2\|_\infty, \qquad (3.106)\\
&\qquad \sup_{t\geq 0} \int_t^\infty \alpha|k_{21}(t,s)|\,\mathrm{d}s\,\|x_1\|_\infty + \sup_{t\geq 0} \int_0^\infty |k_{22}(t,s)|\,\mathrm{d}s\,\alpha\|x_2\|_\infty \Big\}.
\end{aligned}$$

Since $\int_0^\infty u(t,a)\,\mathrm{d}a = 1$, with κ_{ij} defined in (3.87), it holds that:

$$\sup_{t\geq 0} \int_0^\infty |k_{ij}(t,s)|\,\mathrm{d}s \leq \kappa_{ij}, \qquad i,j \in \{1,2\}.$$

With this, (3.106), and $\|x\| = 1$ it can be concluded that

$$\|y\| \leq \max\left\{ \kappa_{11} + \frac{1}{\alpha}\kappa_{12}, \alpha\kappa_{21} + \kappa_{22} \right\}.$$

Thus, the right hand side is an estimation for the operator norm of $\mathscr{K}$. The operator norm being smaller than unity is implied by the existence of $\theta_0 < 1$ and $\alpha > 0$ such that

$$\kappa_{11} + \frac{1}{\alpha}\kappa_{12} \leq \theta_0, \qquad (3.107)$$

$$\alpha\kappa_{21} + \kappa_{22} \leq \theta_0. \qquad (3.108)$$

Since the first line is monotonously decreasing and the second increasing in α, for the optimal α (which allows for the smallest possible θ_0), both equations are fulfilled as equality. Therefore, we solve the equation

$$\kappa_{11} + \frac{1}{\alpha}\kappa_{12} = \alpha\kappa_{21} + \kappa_{22}$$

for α, and obtain

$$\alpha_{1,2} = \frac{1}{2\kappa_{21}} \left[\kappa_{11} - \kappa_{22} \pm \sqrt{(\kappa_{11} - \kappa_{22})^2 + 4\kappa_{12}\kappa_{21}} \right].$$

We insert the positive solution for α into the second line of (3.108) to obtain θ_0:

$$\theta_0 = \frac{1}{2} \left[\kappa_{11} + \kappa_{22} + \sqrt{(\kappa_{11} - \kappa_{22})^2 + 4\kappa_{12}\kappa_{21}} \right].$$

The requirement $\theta_0 < 1$ is exactly inequality (3.89) in condition (i) of Assumption 3.8.

The sufficiency of condition (ii) follows because the assumption implies, that for all admissible u, it holds that $\sup_{t\geq 0}\int_0^\infty |k_{ij}(t,s)|\,\mathrm{d}s \leq (1+\varepsilon)\bar{\kappa}_{2j}$ for $j=1,2$. System (3.107)–(3.108) then reads as

$$\begin{aligned}\kappa_{11}+\frac{1}{\alpha}\kappa_{12} &\leq \theta_0,\\ (1+\varepsilon)(\alpha\bar{\kappa}_{21}+\bar{\kappa}_{22}) &\leq \theta_0.\end{aligned} \tag{3.109}$$

By following the same steps as above, we obtain that (3.88) is sufficient for the operator norm of $\mathscr{K}$ to be smaller than one. □

Proof of Lemma 3.22. The proof is similar to the one of Lemma 3.19. Let us take two bounded solutions, ξ_1 and ξ_2, and denote by $\Delta\xi(t,a)$ the difference between the two. The solutions ξ_i can be written as (cf. (3.81))

$$\xi_i(t,a) = \int_a^\infty e^{-\int_a^s \rho(\theta)\,\mathrm{d}\theta}[\varphi(s)\xi_i(s+t-a,0)+\nu(s)\sigma(\xi_i(s+t-a,\cdot))+f(s)]\,\mathrm{d}s,\;\; i=1,2.$$

Let $\Delta\lambda(t) := \xi_1(t,0)-\xi_2(t,0)$ and $\Delta\sigma(t) := \sigma(\xi_1(t,\cdot))-\sigma(\xi_2(t,\cdot))$, then we obtain the homogeneous system of integral equations

$$\begin{aligned}\Delta\xi(t,a) &= \int_0^\infty e^{-\int_a^{s+a}\rho(\theta)\,\mathrm{d}\theta}(\varphi(s+a)\Delta\lambda(s+t)+\nu(s+a)\Delta\sigma(s+t))\,\mathrm{d}s,\\ \Delta\lambda(t) &= \int_0^\infty e^{-\int_0^s\rho(\theta)\,\mathrm{d}\theta}[\varphi(s)\Delta\lambda(s+t)+\nu(s)\Delta\sigma(s+t)]\,\mathrm{d}s.\end{aligned} \tag{3.110}$$

Denote by $\mathscr{K} : (L_\infty(0,\infty))^2 \to (L_\infty(0,\infty))^2$ the integral operator representing the system of integral equations above. The existence of a unique solution of the system is guaranteed if $\|K\| < 1$. To show this, take the norm in the system of equations and use that σ is Lipschitz with Lipschitz constant 1 (cf. Lemma 4.1 in [27]),

$$\begin{aligned}\|\Delta\xi\|_\infty &\leq \kappa_{21}\|\Delta\lambda\|_\infty+\kappa_{22}\|\Delta\xi\|_\infty\\ \|\Delta\lambda\|_\infty &\leq \kappa_{11}\|\Delta\lambda\|_\infty+\kappa_{12}\|\Delta\xi\|_\infty.\end{aligned}$$

As in the proof of Lemma 3.19 we define a norm $\|(\Delta\xi,\Delta\lambda)\| := \max\{\|\Delta\xi\|_\infty, a\|\Delta\lambda\|_\infty\}$ for $a>0$.

We choose $a > 0$ in such a way, that the norm of the operator $\mathscr{K}$ is minimized. The minimum is exactly the left hand side of (3.88), and Assumption 3.8 guarantees that it is smaller than one. Therefore, a unique solution exists to the homogeneous system, which is obviously $\Delta\xi = 0$. $\square$

Chapter 4

Fixed Domain of Heterogeneity

In this chapter, we study a trait-structured model on the infinite horizon. Compared to the previous chapter, the characteristic lines of the partial differential equation are horizontal. We study necessary optimality conditions of first order and the existence of a Hamiltonian formulation.

Distributed parameter systems are used, for example, in epidemic models taking into account the heterogeneity of the population [53, 19, 18]. The parameter σ, which may take values in some fixed interval $[0, \Sigma]$, describes a certain trait existing in the population, which could account for a natural resistance towards the disease, or social behaviour favouring the spreading of the disease.

Trait-structured models also find application in economics, for example in economic growth models (see, e.g. [46, 50], where decentralised equilibria are analyzed). The parameter of heterogeneity describes the diversity of agents with respect to physical capital, human capital, time preference rate, or abilities.

The mathematical model considered in this section consists of distributed state variables, governed by infinitely many differential equations (one for each trait), coupled by non-local integral states. The natural formulation is (for many if not the most applications) on an infinite horizon.

As mentioned in the introduction and the previous chapter, infinite-horizon optimal control is still challenging, even in the case of ordinary differential equations. We follow in this chapter the same approach as in the previous one, which is based on [5, 6, 7], where we define explicitly an appropriate solution of the adjoint system, similar as in [45]. In contrast to the mentioned paper, this work deals with a different problem setting: the dynamic system may be non-linear in the states, and the non-local integral states may enter not only the objective function, but also the dynamics of the distributed state.

For optimal control problems of ordinary differential equations, a Hamiltonian formulation is well known, which has many properties of the Hamiltonian in classical mechanics: the primal and the adjoint system can be reproduced, with the right adjoint functions the maximum principle holds, and the functional is constant along the optimal trajectory. Finding a Hamiltonian which is constant along optimal trajectories for autonomous systems is not a trivial task since the adjoint integral state may not be absolutely continuous and the proof of constancy is therefore difficult.

The contribution is twofold: (i) a maximum principle of Pontryagin's type in normal form is proved. Since the objective value may be infinite for some or even all controls, the notion of weak overtaking optimality is used. (ii) The necessary optimality conditions obtained have a Hamiltonian representation. In case of stationary data, the constancy of the Hamiltonian is proved.

The remainder of this chapter is as follows. In Section 4.1, the problem is introduced. The linearization of the primal system is studied in Section 4.2. Necessary optimality conditions of Pontryagin's type are formulated and discussed in Section 4.3 and proved in Section 4.4. The stationarity of the Hamiltonian is proved in Section 4.5. In the final Section 4.6, the applicability of our results to a problem arising in epidemiology is shown.

4.1 Formulation of the Problem

Let U be a measurable and bounded subset of $\mathbb{R}^{m_u}$. Denote by $L_2([0,\Sigma];U)$ the subset of functions in $L_2([0,\Sigma];\mathbb{R}^{m_u})$ with values in U and let $\mathscr{U}_0$ be a non-empty subset of $L_2([0,\Sigma];U)$. Then $\mathscr{U}_0$ has a countable and dense subset, as we will show below. Note that due to the boundedness of U, all functions $u(\cdot) \in \mathscr{U}_0$ are also elements of $L_\infty([0,\Sigma];U)$, and due to $\Sigma < \infty$ and Jensen's inequality, they are also elements of $L_1([0,\Sigma];U)$.

Consider the following optimization problem

$$\max_u J(u) := \int_0^\infty \int_0^\Sigma L(t,\sigma,y(t,\sigma),z(t),u(t,\sigma))\,\mathrm{d}\sigma\,\mathrm{d}t, \tag{4.1}$$

subject to

$$\dot{y}(t,\sigma) = f(t,\sigma,y(t,\sigma),z(t),u(t,\sigma)), \qquad y(0,\sigma) = y_0(\sigma), \tag{4.2}$$

$$z(t) = \int_0^\Sigma h(t,\sigma,y(t,\sigma))\,\mathrm{d}\sigma, \tag{4.3}$$

$$u(t,\cdot) \in \mathscr{U}_0. \tag{4.4}$$

By a dot, we denote the differentiation with respect to time. The domain, in which the problem is considered, is $D := [0,\infty) \times [0,\Sigma]$, with a finite $\Sigma > 0$. The vector-valued functions $y : D \to \mathbb{R}^{n_y}$

and $z : [0,\infty) \to \mathbb{R}^{n_z}$ represent the states of the system.

We now show the separability of $\mathscr{U}_0$. The space $L_2([0,\Sigma];\mathbb{R}^{m_u})$ is separable, therefore there exists a countable and dense subset $G = \{g_1, g_2, \dots\}$. Denote for $a \in L_2([0,\Sigma];\mathbb{R}^{m_u})$ and $B \subset L_2([0,\Sigma];\mathbb{R}^{m_u})$ the distance by $d(a,B) := \inf_{b\in B} \|a-b\|_{L_2(0,\Sigma)}$. Then for every $n \in \mathbb{N}$ choose a $\{u_n\} \in \mathscr{U}_0$ such that $\|g_n - u_n\|_{L_2(0,\Sigma)} < 2d(g_n, \mathscr{U}_0)$. The set $\{u_n\}$, $n \in \mathbb{N}$, is a countable and dense subset of $\mathscr{U}_0$. Indeed, take $u \in \mathscr{U}_0$ and let $\{g_{n_j}\}$ be the series of elements such that $g_{n_j} \to u$ in L_2 as $j \to \infty$. Then

$$\|u - u_{n_j}\|_{L_2(0,\Sigma)} \le \|u - g_{n_j}\|_{L_2(0,\Sigma)} + \|g_{n_j} - u_{n_j}\|_{L_2(0,\Sigma)} \le 3\|u - g_{n_j}\|_{L_2(0,\Sigma)},$$

thus $u_{n_j} \to u$ in L_2 as $j \to \infty$.

In particular the definition allows for the following three sets $\mathscr{U}_0$:

(i) *Non-distributed controls:*

$$\mathscr{U}_0 := \{u(\cdot) \in L_2([0,\Sigma];U) \,|\, \forall \sigma \in [0,\Sigma] : u(\sigma) = u_0 \in U\}.$$

(ii) *Controls with σ-dependend control constraints:* Let $\underline{u}(\sigma) \le \bar{u}(\sigma)$ be two measurable and essentially bounded functions with values in U. Then,

$$\mathscr{U}_0 := \{u(\cdot) \in L_2([0,\Sigma];U) \,\Big|\, \underline{u}(\sigma) \le u(\sigma) \le \bar{u}(\sigma)\}, \tag{4.5}$$

where the inequality has to hold for every component of u, is a possible set for constraint 4.4.

(iii) *Controls that satisfy integral constraints:* Let $0 \le \underline{u}(\sigma) \le \bar{u}(\sigma)$ be two U-valued measurable and bounded functions with $\int_0^\Sigma \underline{u}(\sigma)\,\mathrm{d}\sigma < 1 < \int_0^\Sigma \bar{u}(\sigma)\,\mathrm{d}\sigma$, where the inequality has to hold for every component of $\underline{u}$ and $\bar{u}$. Then the set of controls that satisfy for every t additionally an integral constraint is defined by

$$\mathscr{U}_0 := \left\{u(\cdot) \in L_2([0,\Sigma];U) \,\Big|\, \underline{u}(\sigma) \le u(\sigma) \le \bar{u}(\sigma) \text{ and } \int_0^\Sigma u(\sigma)\,\mathrm{d}\sigma = 1\right\}.$$

The set of *admissible controls* $\mathscr{U}$ consists of all measurable functions $u : [0,\infty) \to \mathscr{U}_0$, for which the solution of (4.2), (4.3) exists on $[0,\infty)$.

Assumption 4.1. *The functions f, h and L, together with the partial derivatives L_y, L_z, f_y, f_z, and h_y, are locally bounded, measurable in (t,σ) for every (y,z,u), and locally Lipschitz continuous* *in* (y,z,u).[1]

Denote by $\mathscr{A}(D)$ the set of all functions in $L_\infty^{loc}(D)$ that are absolutely continuous in t for almost every σ. For a control $u \in \mathscr{U}$, a couple of functions $(y,z) \in \mathscr{A}(D) \times L_\infty^{loc}(0,\infty)$ is a solution of (4.2)–(4.4) if y satisfies (4.2) almost everywhere for almost every $\sigma \in [0,\Sigma]$, and (4.3) is also satisfied almost everywhere. By D_T we denote the truncated set $[0,T] \times [0,\Sigma]$, and by $J_T(u)$ the finite-horizon objective function where the time-integral goes only to T.

The notion of optimality used is the weakly overtaking optimality, see Definition 2.1.

We remind of the notational convention made in Section 2.1, we skip (t,σ), as well as functions with a "hat", when they appear as argument of other functions. For example, $f(t,\sigma,u) = f(t,\sigma,\hat{y},\hat{z},u)$.

4.2 Linearization of System (4.2)–(4.3)

In this section, we consider the linearization of system (4.2)–(4.3), and prove some Lipschitz property of both the original system and the linearized system.

Let $\hat{u}$ denote a weakly overtaking optimal control and let $(\hat{y},\hat{z})$ be the corresponding state trajectory. Then for every fixed $\sigma \in [0,\Sigma]$, we consider the ordinary differential equation

$$\dot{y}(t,\sigma) = f_y(t,\sigma)y(t,\sigma), \qquad y(0,\sigma) = y_0(\sigma).$$

A fundamental solution exists (see Section 2.2) which we denote by $X_\sigma(t)$. The state transition matrix is denoted by $X_\sigma(t,\tau)$.

Consider the linearization of system (4.2)–(4.3) around the optimal trajectory,

$$\delta\dot{y}(t,\sigma) = f_y(t,\sigma)\delta y(t,\sigma) + f_z(t,\sigma)\delta z(t), \tag{4.6}$$

$$\delta z(t) = \int_0^\Sigma [h_y(t,\sigma)\delta y(t,\sigma)]\,\mathrm{d}\sigma, \tag{4.7}$$

with initial condition

$$\delta y(\tau,\sigma) = \delta_0(\sigma).$$

[1] Let g be a representative function. The local Lipschitz continuity requires that for every compact sets Y, Z, $\bar{U} \subset U$, and $T > 0$ there exists a constant C such that

$$|g(t,\sigma,y_1,z_1,u_1) - g(t,\sigma,y_2,z_2,u_2)| \leq C(|y_1-y_2| + |z_1-z_2| + |u_1-u_2|)$$

for every $(t,\sigma,y_i,z_i,u_i) \in [0,T] \times [0,\Sigma] \times Y \times Z \times \bar{U}$, $i = 1,2$.

With the state transition matrix from above, we represent the solution of the differential equation (4.6) as

$$\delta y(t,\sigma) = X_\sigma(t,\tau)\delta_0(\sigma) + \int_\tau^t X_\sigma(t,s) f_z(s,\sigma)\delta z(s)\,\mathrm{d}s.$$

Inserting this in the equation for δz, and changing the order of integration, we obtain a Volterra integral equation of the second kind:

$$\delta z(t) = \int_0^\Sigma h_y(t,\sigma)X_\sigma(t,\tau)\delta_0(\sigma)\,\mathrm{d}\sigma + \int_\tau^t \int_0^\Sigma h_y(t,\sigma)X_\sigma(t,s) f_z(s,\sigma)\,\mathrm{d}\sigma\,\delta z(s)\,\mathrm{d}s.$$

Denote the first term by $F(t,\tau)$ and the inner integral in the second term by $K(t,s)$. Denote by $R(t,s)$ the resolvent corresponding to the kernel $K(t,s)$ (cf. Section 2.5). Then

$$\begin{aligned} \delta z(t) =& F(t,\tau) + \int_\tau^t R(t,s)F(s,\tau)\,\mathrm{d}s \\ =& \int_0^\Sigma \left[h_y(t,\sigma)X_\sigma(t) + \int_\tau^t R(t,s)h_y(s,\sigma)X_\sigma(s)\,\mathrm{d}s \right] X_\sigma(\tau)^{-1}\delta_0(\sigma)\,\mathrm{d}\sigma. \end{aligned} \tag{4.8}$$

Denote the expression in the brackets [...] by $R^z(t,\sigma)$. Then the solution for δy is

$$\delta y(t,\sigma) = X_\sigma(t,\tau)\delta_0(\sigma) + \int_0^\Sigma \int_\tau^t X_\sigma(t,s) f_z(s,\sigma) R^z(s,\tilde\sigma)\,\mathrm{d}s\, X_{\tilde\sigma}(\tau)^{-1}\delta_0(\tilde\sigma)\,\mathrm{d}\tilde\sigma.$$

Denoting by $R^y(t,\sigma,\tilde\sigma)$ the integral with respect to s, we have

$$\delta y(t,\sigma) = X_\sigma(t)X_\sigma(\tau)^{-1}\delta_0(\sigma) + \int_0^\Sigma R^y(t,\sigma,\tilde\sigma)X_{\tilde\sigma}(\tau)^{-1}\delta_0(\tilde\sigma)\,\mathrm{d}\tilde\sigma. \tag{4.9}$$

Next, we prove some Lipschitz properties of the linearized and the original system.

Lemma 4.1. *Let $\hat u$ denote an optimal control, then the trajectories $(y(t,\cdot),z(t))$ exist on $[0,\infty)$. Denote by u any other admissible control which deviates from $\hat u$ only on the set $t \in [\tau-\alpha,\tau]$, and for which the corresponding trajectories $(y(t,\cdot),z(t))$ exist also on $[0,\infty)$. Denote $\Delta y(t,\sigma) := y(t,\sigma) - \hat y(t,\sigma)$ and $\Delta z(t) := z(t) - \hat z(t)$. Furthermore, let $(\delta y(t,\cdot),\delta z(t))$ be the solution of system* (4.6)–(4.7) *with initial condition $\delta y(\tau,\cdot) = \Delta y(\tau,\cdot)$. Then for every $T > 0$, the following estimations hold uniformly in $[0,T]$:*

$$\|\Delta y\|_{L_\infty(D_T)} + \|\Delta z\|_{L_\infty(0,T)} \le C(T)\alpha. \tag{4.10}$$

$$\|\delta y\|_{L_\infty(D_T\setminus D_\tau)} + \|\delta z\|_{L_\infty(\tau,T)} \le C(T)\alpha. \tag{4.11}$$

$$\|\Delta y - \delta y\|_{L_\infty(D_T\setminus D_\tau)} + \|\Delta z - \delta z\|_{L_\infty(\tau,T)} \le o(\alpha). \tag{4.12}$$

Proof. Let us show the first assertion of the Lemma. Fix $T > 0$ and denote by $C_i(T)$ constants that depend on T, but are independent of all other functions. Furthermore denote $\Delta f(t,\sigma) := f(t,\sigma,u) - f(t,\sigma)$. For $t \le T$ and $\sigma \in [0,\Sigma]$ we have

$$\begin{aligned}\Delta y(t,\sigma) &= \int_{\tau-\alpha}^{t} [f(s,\sigma,y,z,u) - f(s,\sigma)]\,\mathrm{d}s \\ &= \int_{\tau-\alpha}^{t} [f_y(s,\sigma,\bar{y},\bar{z},u)\Delta y(s,\sigma) + f_z(s,\sigma,\bar{y},\bar{z},u)\Delta z(s) + \Delta f(s,\sigma)]\,\mathrm{d}s,\end{aligned}$$

where $(\bar{y}(t,\sigma),\bar{z}(t)) \in \mathrm{co}\{(y(t,\sigma),z(t)),(\hat{y}(t,\sigma),\hat{z}(t))\}$. Due to the local boundedness of f_y and f_z, it follows that

$$|\Delta y(t,\sigma)| \le \int_{\tau-\alpha}^{t} [C_1(T)|\Delta y(s,\sigma)| + C_2(T)|\Delta z(s)| + |\Delta f(s,\sigma)|]\,\mathrm{d}s, \tag{4.13}$$

$$|\Delta z(t)| = \left| \int_0^{\Sigma} [h(t,\sigma,y) - h(t,\sigma)]\,\mathrm{d}\sigma \right| \le \int_0^{\Sigma} C_3(T)|\Delta y(t,\sigma)|\,\mathrm{d}\sigma. \tag{4.14}$$

Inserting the equation for Δz in (4.13), integrating over $\sigma \in [0,\Sigma]$, and changing the order of integration,

$$\begin{aligned}\|\Delta y(t,\cdot)\|_{L_1(0,\Sigma)} \le& \int_{\tau-\alpha}^{t} \int_0^{\Sigma} |\Delta f(s,\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}s + \int_{\tau-\alpha}^{t} C_4(T)\|\Delta y(s,\cdot)\|_{L_1(0,\Sigma)}\,\mathrm{d}s \\ \le& C_5(T) \int_{\tau-\alpha}^{t} \int_0^{\Sigma} |\Delta f(s,\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}s,\end{aligned}$$

where in the last inequality, the lemma of Gronwall was used. Note that $\Delta f(t,\sigma)$ is zero for $t > \tau$, and uniformly bounded for $t \in [\tau-\alpha,\tau]$. This proves $\|\Delta y(t,\cdot)\|_{L_1(0,\Sigma)} \le C(T)\alpha$, uniformly for $t \in [\tau-\alpha,T]$.

Inserting this into equation (4.14), it follows $\|\Delta z\|_{L_\infty(\tau-\alpha,T)} \le C(T)\alpha$, and equation (4.13) implies that also $\|\Delta y\|_{L_\infty(D_T)} \le C(T)\alpha$.

The assertions for δy and δz follow in the same way and are therefore omitted.

For the third assertion, note that for $t \in [\tau,T]$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta y(t,\sigma) - \delta y(t,\sigma)) =& f_y(t,\sigma)(\Delta y(t,\sigma) - \delta y(t,\sigma)) + f_z(t,\sigma)(\Delta z(t) - \delta z(t)) \\ &+ o(t,\sigma,\Delta y) + o(t,\Delta z), \\ \Delta z(t) - \delta z(t) =& \int_0^{\Sigma} [h_y(t,\sigma)(\Delta y(t,\sigma) - \delta y(t,\sigma)) + o(t,\sigma,\Delta y)]\,\mathrm{d}s.\end{aligned}$$

By $o(t,\sigma,x)$ or $o(t,x)$ we denote a term that satisfies $\lim_{|x|\to 0} \frac{|o(x)|}{|x|} = 0$ uniformly in $(t,\sigma) \in D_T$

or $t \in [\tau, T]$. Solving the differential equation, the following system is obtained:

$$\begin{aligned}\Delta y(t,\sigma) - \delta y(t,\sigma) &= \int_\tau^t [f_y(s,\sigma)(\Delta y(s,\sigma) - \delta y(s,\sigma)) + f_z(s,\sigma)(\Delta z(s) - \delta z(s))]\,\mathrm{d}s \\ &\quad + o(\|\Delta y\|_{L_\infty(D_T)}) + o(\|\Delta z\|_{L_\infty(0,T)}), \\ \Delta z(t) - \delta z(t) &= \int_0^\Sigma [h_y(t,\sigma)(\Delta y(t,\sigma) - \delta y(t,\sigma))]\,\mathrm{d}\sigma + o(\|\Delta y\|_{L_\infty(D_T)}).\end{aligned}$$

Note that $\Delta y(\tau,\cdot) - \delta y(\tau,\cdot) = 0$. Repeating the same steps as above (inserting one equation in the other, integrating on $[0,\Sigma]$, using Gronwall's inequality regarding equation (4.10)), this proves the final assertion of the lemma. □

4.3 A Pontryagin Maximum Principle

In this section, we will prove necessary optimality conditions of Pontryagin's type. At first, some auxiliary results are proved and the adjoint system introduced.

In [54, 35], necessary optimality conditions in the form of a Pontryagin maximum principle are stated for the system (4.1) subject to (4.2)–(4.4) on a finite time-horizon $[0,T]$. The optimality conditions involve adjoint functions $\xi(t,\sigma) : D_T \to \mathbb{R}^{n_y}$ and $\zeta(t) : [0,T] \to \mathbb{R}^{n_z}$ which satisfy the following adjoint system

$$\dot{\xi}(t,\sigma) = \xi(t,\sigma) f_y(t,\sigma) + \zeta(t) h_y(t,\sigma) + L_y(t,\sigma), \tag{4.15}$$

$$\zeta(t) = \int_0^\Sigma [\xi(t,\sigma) f_z(t,\sigma) + L_z(t,\sigma)]\,\mathrm{d}\sigma, \tag{4.16}$$

together with an appropriate transversality condition for $\xi(t,\sigma)$ at $t = T$. When considering the system on the infinite horizon $[0,\infty)$, the adjoint system remains the same but stating an appropriate transversality condition is more problematic. Therefore, we use an approach (which was developed for ODEs in [5, 6, 7]) in which the adjoint variables for which the maximum principle holds, are explicitly defined.

Define for $(t,\sigma) \in D$

$$\hat{\xi}(t,\sigma) := \int_t^\infty [L_y(s,\sigma) + \zeta(s) h_y(s,\sigma)] X_\sigma(s)\,\mathrm{d}s X_\sigma(t)^{-1}, \tag{4.17}$$

$$\hat{\zeta}(t) := \hat{\psi}(t) + \int_t^\infty \hat{\psi}(s) R(s,t)\,\mathrm{d}s, \tag{4.18}$$

where

$$\hat{\psi}(t) := \int_0^\Sigma \left[L_z(t,\sigma) + \int_t^\infty L_y(s,\sigma) X_\sigma(s)\, \mathrm{d}s X_\sigma(t)^{-1} f_z(t,\sigma) \right] \mathrm{d}\sigma, \tag{4.19}$$

and X and R are the fundamental matrix solution and resolvent from Section 4.2. The use of the infinite-horizon integrals in the above definitions is justified by the following assumptions and proved in the subsequent lemma.

Assumption 4.2. *There exist non-negative functions $\lambda_i(t)$, such that for every $\tau \in [0,\infty)$, there exists a number $\gamma(\tau) > 0$, such that for every $\delta_0(\sigma) \in L_\infty(0,\Sigma)$ with $\|\delta_0\|_{L_\infty(0,\Sigma)} < \gamma(\tau)$, system* (4.2)–(4.3) *with initial condition $y(\tau,\sigma) = \hat{y}(\tau,\sigma) + \delta_0(\sigma)$ has a solution on $[\tau,\infty)$ and the following relations hold uniformly in σ:*

$$\int_0^\Sigma |L_y(t,\sigma,\bar{y},\bar{z},\hat{u})\Delta y(t,\sigma) + L_z(t,\sigma,\bar{y},\bar{z},\hat{u})\Delta z(t)|\, \mathrm{d}\sigma \le \lambda_1(t) \|\delta_0(\cdot)\|_{L_\infty(0,\Sigma)}, \tag{4.20}$$

$$|L_y(t,\sigma) X_\sigma(t)| \le \lambda_2(t), \tag{4.21}$$

$$\int_0^\Sigma \left(|L_z(t,\sigma)| + \left| \int_t^\infty L_y(s,\sigma) X_\sigma(s)\, \mathrm{d}s X_\sigma(t)^{-1} f_z(t,\sigma) \right| \right) \mathrm{d}\sigma \le \lambda_3(t), \tag{4.22}$$

$$\left(\lambda_3(t) + \int_t^\infty \lambda_3(s) |R(s,t)|\, \mathrm{d}s \right) |h_y(t,\sigma) X_\sigma(t)| \le \lambda_4(t), \tag{4.23}$$

where $R(t,s)$ is the resolvent from Section 4.2, $(\bar{y}(t,\sigma),\bar{z}(t)) \in \mathrm{co}\{(y(t,\sigma),z(t)),(\hat{y}(t,\sigma),\hat{z}(t))\}$ and the functions $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)|R(t,s)|$, and $\lambda_4(t)$ are integrable in t on $[0,\infty)$, and the function $s \mapsto \int_\tau^\infty \lambda_3(t)|R(t,s)|\, \mathrm{d}t$ is locally bounded in s.

Lemma 4.2. *Let Assumption 4.2 hold. Then the functions $\hat{\xi}$ and $\hat{\zeta}$ defined in* (4.17) *and* (4.18) *(regarding* (4.19)*) belong to the sets $\mathscr{A}(D)$ and $L_\infty^{loc}(0,\infty)$, respectively, and satisfy the adjoint system* (4.15)–(4.16).

Proof. According to Assumption 4.2, the integrals in the definition of $\hat{\xi}$ are absolutely convergent. We show that $\hat{\xi} \in \mathscr{A}(D)$ and start with the Lipschitz continuity of the first term, $\hat{\xi}_1(t,\sigma) := \int_t^\infty L_y(s,\sigma) X_\sigma(s)\, \mathrm{d}s X_\sigma(t)^{-1}$:

$$\begin{aligned}
\hat{\xi}_1(t+\varepsilon,\sigma) - \hat{\xi}_1(t) &= \int_{t+\varepsilon}^\infty L_y(s,\sigma) X_\sigma(s)\, \mathrm{d}s X_\sigma(t+\varepsilon)^{-1} - \int_t^\infty L_y(s,\sigma) X_\sigma(s)\, \mathrm{d}s X_\sigma(t)^{-1} \\
&= \int_{t+\varepsilon}^\infty L_y(s,\sigma) X_\sigma(s)\, \mathrm{d}s \left[X_\sigma(t+\varepsilon)^{-1} - X_\sigma(t)^{-1} \right] - \int_t^{t+\varepsilon} L_y(s,\sigma) X_\sigma(s)\, \mathrm{d}s X_\sigma(t)^{-1}
\end{aligned}$$

The Lipschitz continuity follows from the Lipschitz continuity of $X_\sigma(t)^{-1}$ and the local boundedness of L_y and X together with the fact that the length of the integral is ε. Thus we can differentiate with respect to t, and obtain

$$-\dot{\hat{\xi}}_1(t,\sigma) = L_y(t,\sigma) + \int_t^\infty L_y(s,\sigma)X_\sigma(s)\,\mathrm{d}s X_\sigma(t)^{-1} f_y(t,\sigma) = L_y(t,\sigma) + \hat{\xi}_1(t,\sigma) f_y(t,\sigma).$$

The proof that the second term in $\hat{\xi}$ is Lipschitz continuous and satisfies $-\dot{\hat{\xi}}_2(t,\sigma) = \hat{\zeta}(t)h_y(t,\sigma) + \hat{\xi}_2(t,\sigma)f_y(t,\sigma)$ is essentially the same and therefore omitted. Then also $\hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2$ is Lipschitz in t, thus belongs to the space $\mathscr{A}(D)$ and satisfies the differential equation (4.15).

Due to the definition of $\hat{\zeta}$, and Assumption 4.2, it is clear that $\zeta(t) \in L_\infty^{\mathrm{loc}}(0,\infty)$. Also, it satisfies the adjoint equation (4.16), which follows from the considerations in Section 2.5, equations (2.12) and (2.11). □

Assumption 4.2 imposes restrictions on the combined growth of the fundamental solution X, the resolvent R, the differences Δy and Δz, and on the data. A more convenient sufficient condition is presented at the end of the Section.

Define the pre-Hamiltonian

$$H(t,\sigma,y,z,u,\xi,\zeta) := L(t,\sigma,y,z,u) + \xi f(t,\sigma,y,z,u) + \zeta h(t,\sigma,y),$$

and the Hamiltonian $\mathscr{H} : [0,\infty) \times (L_\infty(0,\Sigma))^{n_y} \times \mathbb{R}^{n_z} \times \mathscr{U}_0 \times (L_\infty(0,\Sigma))^{n_y} \times \mathbb{R}^{n_z} \to \mathbb{R}$ as

$$\mathscr{H}(t,y(\cdot),z,u(\cdot),\xi(\cdot),\zeta) := \int_0^\Sigma H(t,\sigma,y(\sigma),z,u(\sigma),\xi(\sigma),\zeta)\,\mathrm{d}\sigma - \zeta z. \tag{4.24}$$

Now we are ready to formulate a Pontryagin maximum principle.

Theorem 4.3. *Let Assumption 4.1 be fulfilled and let* $(\hat{u},\hat{y},\hat{z})$ *be a WOO solution of problem* (4.1) *subject to* (4.2)–(4.4)*, for which Assumption 4.2 is fulfilled. With the functions* $\hat{\xi}$ *and* $\hat{\zeta}$ *defined in* (4.17)–(4.18) *(regarding* (4.19)*), the following maximization condition is fulfilled for almost every* $t \in [0,\infty)$*:*

$$\mathscr{H}(t,\hat{u}(t,\cdot)) = \sup_{u(\cdot)\in\mathscr{U}_0} \mathscr{H}(t,u). \tag{4.25}$$

The complete proof is given in the next section. Here, we only sketch the main ideas: We consider certain types of needle variations of the controls, that is admissible controls which deviate from the optimal control only on some small set $[\tau-\alpha,\tau]$, where τ is arbitrarily chosen in $(0,\infty)$. Such needle controls induce a certain disturbance in the trajectory at time τ, which we denote by $\delta y(\cdot)$. Then we show that the effect of such a disturbance on the objective function can be represented by terms involving the Hamiltonian functional $\mathscr{H}$ (which involves the adjoint variable $\xi(\tau,\cdot)$), plus some error terms. Finally, we use the definition of weak overtaking optimality to obtain the maximization condition in the theorem.

Now a few remarks and an alternative to Assumption 4.2 are presented.

Remark 4.4. (i) The maximum principle is in normal form, that is, the multiplier of the objective function L is equal to one. While this is typical for finite-horizon problems without state constraints, it is not for infinite-horizon problems.

(ii) The adjoint variable $\hat{\xi}(t,\cdot)$ does not necessarily fulfil the "classical" transversality conditions $\lim_{t\to\infty}\hat{\xi}(t,\sigma)=0$ or $\lim_{t\to\infty}\int_0^\Sigma \hat{\xi}(t,\sigma)y(t,\sigma)\,\mathrm{d}\sigma=0$. The example by Halkin (see Remark 2.4 or [31]) can be embedded in this distributed setting as a counterexample.

(iii) Note that the definition of the adjoint variables is explicit. However, the solution is implicit from a practical point of view, as it involves the (a priori unknown) optimal solution.

We now elaborate on the Assumption 4.2. The following is a sufficient condition, which might be easier to check in many applications, see, for example, in Section 4.6:

Assumption 4.3. *For every $\tau\in[0,\infty)$ there exists a number $\gamma(\tau)>0$ such that for every $\delta_0(\sigma)\in L_\infty(0,\Sigma)$ with $\|\delta_0\|_{L_\infty(0,\Sigma)}<\gamma(\tau)$, system* (4.2)–(4.3) *with initial condition $y(\tau,\sigma)=\hat{y}(\tau,\sigma)+\delta_0(\sigma)$ has a solution on $[\tau,\infty)$. Furthermore, let the following growth estimations hold for all (y,z), where the norm is always the L_∞ norm on $[0,\Sigma]$,*

$$
\begin{aligned}
&\|L_y(t,\sigma,y,z)\|\le Ce^{\rho_1 t}, \quad \|L_z(t,\sigma,y,z)\|\le Ce^{\rho_2 t},\\
&\|\Delta y(t,\sigma)\|\le Ce^{\rho_3 t}\|\Delta y(\tau,\cdot)\|, \quad \|\Delta z(t)\|\le Ce^{\rho_4 t}\|\Delta y(\tau,\cdot)\|,\\
&\|X_\sigma(t,s)\|\le Ce^{\rho_5(t-s)}, \quad \|f_z(t,\sigma,y,z)\|\le Ce^{\rho_6 t}, \quad \|h_y(t,\sigma,y)\|\le Ce^{\rho_7 t}.
\end{aligned}
$$

Let

$$\max\{\rho_1+\rho_3,\rho_2+\rho_4,\rho_1+\rho_5\}<0, \tag{4.26}$$

$$\rho_0:=\max\{\rho_2,\rho_1+\rho_6\}<0, \tag{4.27}$$

$$\rho_0+\rho_5+\rho_7<0, \tag{4.28}$$

and

$$\operatorname{ess\,sup}_{t\in[0,\infty)}\int_t^\infty e^{\rho_0(s-t)}|K(s,t)|\,\mathrm{d}s<1. \tag{4.29}$$

Note that the objective function may include discounting, then ρ_1 and ρ_2 are usually negative, and likely ρ_0 is negative, too. This helps for the validity of Assumption 4.3.

We now show that Assumption 4.3 implies Assumption 4.2: By estimation, we obtain that equation (4.26) implies the existence and integrability of λ_1 and λ_2. Then

$$\left| \int_t^\infty L_y(s,\sigma) X_\sigma(s)\,\mathrm{d}s X_\sigma(t)^{-1} \right| \leq \int_t^\infty C_1 e^{\rho_1 s} e^{\rho_5 (s-t)}\,\mathrm{d}s \leq C_2 e^{\rho_1 t},$$

where here and in the following C_i denotes some positive constant. Therefore, $\lambda_3(t) \leq C_3 e^{\rho_0 t}$, with ρ_0 defined as in (4.27). The integrability follows by the assumption $\rho_0 < 0$.

Equation (4.29) ensures that the resolvent $R(s,t)$ defines an operator $(\mathscr{R}\mu)(t) := \int_t^\infty \mu(s) R(s,t)\,\mathrm{d}s$, which maps the weighted space $L_\infty^{\rho_0}(0,\infty)$ into itself and is bounded (see equation (2.13) or Proposition 3.10 in Chapter 9 of [28]). Therefore, $\lambda_3(s)R(s,t)$ is integrable, and $(\mathscr{R}\lambda_3)(t) \leq C_4 e^{\rho_0 t}$. The function λ_4 is then integrable because of equation (4.28). Thus, Assumption 4.3 implies Assumption 4.2.

Remark 4.5. If the set $\mathscr{U}_0$ is defined as in (4.5), that is, if the control is distributed and the constraints are only pointwise, then the maximum principle can also be formulated as pointwise (with respect to σ) maximization of $H(t,\sigma)$: for almost every $t \in [0,T]$ and almost every $\sigma \in [0,\Sigma]$:

$$H(t,\sigma,\hat{u}(t,\sigma)) = \sup_{\underline{u}(\sigma) \leq u \leq \bar{u}(\sigma)} H(t,\sigma,u).$$

This is a standard consequence of (4.25).

4.4 Proof of Theorem 4.3

Denote by U^B a countable and dense subset of $\mathscr{U}_0$. Take $u(\cdot) \in U^B$ and denote by $\Omega(u) \subset [0,\infty)$ the set of Lebesgue points of the following functions:

$$t \mapsto \int_0^\Sigma [L(t,\sigma,u) - L(t,\sigma)]\,\mathrm{d}\sigma, \qquad t \mapsto \int_0^\Sigma \xi(t,\sigma)[f(t,\sigma,u) - f(t,\sigma)]\,\mathrm{d}\sigma$$

Take an arbitrary $\tau \in \Omega(u)$ and consider the control u_α, which deviates from $\hat{u}$ only on the set $[\tau-\alpha,\tau]$:

$$u_\alpha(t,\cdot) = \begin{cases} u(\cdot) & \text{for } t \in [\tau-\alpha,\tau], \\ \hat{u}(t,\cdot) & \text{for } t \notin [\tau-\alpha,\tau], \end{cases}$$

Denote the solution of (4.2)–(4.3) corresponding to u_α by (y_α, z_α) and fix $T' > \tau$. The difference between the corresponding objective values is

$$\Delta J_{T'} := \int_0^{T'} \int_0^{\Sigma} [L(t,\sigma,y_\alpha,z_\alpha,u_\alpha) - L(t,\sigma)] \, \mathrm{d}\sigma \, \mathrm{d}t.$$

Denote $\Delta y(t,\sigma) := y_\alpha(t,\sigma) - \hat{y}(t,\sigma)$, $\Delta z(t) := z_\alpha(t) - \hat{z}(t)$, and let $(\delta y_\alpha, \delta z_\alpha)$ be the solution of system (4.6)–(4.7) on $[\tau, T']$ with initial condition $\delta y(\tau,\sigma) = \Delta y(\tau,\sigma)$. Taylor's theorem implies

$$L(t,\sigma,y_\alpha,z_\alpha) - L(t,\sigma) = \bar{L}_y(t,\sigma)\Delta y(t,\sigma) + \bar{L}_z(t,\sigma)\Delta z(t),$$

where $\bar{L}_x(t,\sigma) := L_x(t,\sigma,\bar{y}(t,\sigma),\bar{z}(t))$ for $x \in \{y,z\}$ and $\bar{y},\bar{z}$ are measurable functions satisfying

$$(\bar{y}(t,\sigma),\bar{z}(t)) \in \mathrm{co}\{(y_\alpha(t,\sigma),z_\alpha(t)),(\hat{y}(t,\sigma),\hat{z}(t))\}.$$

Taking into account the specific form of the needle variation u_α, $\Delta J_{T'}$ can be written for every $T \in [\tau, T']$ as follows:

$$\begin{aligned}
\Delta J_\tau^{T'} = & \int_{\tau-\alpha}^{\tau} \int_0^{\Sigma} [L(t,\sigma,y_\alpha,z_\alpha,u) - L(t,\sigma)] \, \mathrm{d}\sigma \, \mathrm{d}t \\
& + \int_{\tau}^{T} \int_0^{\Sigma} [\bar{L}_y(t,\sigma)\Delta y(t,\sigma) - L_y(t,\sigma)\delta y(t,\sigma) + \bar{L}_z(t,\sigma)\Delta z(t) - L_z(t,\sigma)\delta z(t)] \, \mathrm{d}\sigma \, \mathrm{d}t \\
& + \int_{T}^{T'} \int_0^{\Sigma} [\bar{L}_y(t,\sigma)\Delta y(t,\sigma) + \bar{L}_z(t,\sigma)\Delta z(t)] \, \mathrm{d}\sigma \, \mathrm{d}t \\
& + \int_{\tau}^{\infty} \int_0^{\Sigma} [L_y(t,\sigma)\delta y(t,\sigma) + L_z(t,\sigma)\delta z(t)] \, \mathrm{d}\sigma \, \mathrm{d}t \\
& - \int_{T}^{\infty} \int_0^{\Sigma} [L_y(t,\sigma)\delta y(t,\sigma) + L_z(t,\sigma)\delta z(t)] \, \mathrm{d}\sigma \, \mathrm{d}t.
\end{aligned}$$

Denote the double integral in the first line by I_1, the second by I_2, the third by I_3, the fourth by I_4 and the last by I_5. We summarize representations for these terms in the following Lemma. The proof is rather long and therefore moved to the end of this section.

Lemma 4.6. *The following representations and estimations hold for $\|\Delta y(\tau,\cdot)\| < \gamma(\tau)$, where γ is* *as in Assumption 4.2:*

$$
\begin{aligned}
&I_1 = \alpha \int_0^{\Sigma} [L(\tau,\sigma,u) - L(\tau,\sigma)]\, \mathrm{d}\sigma + o(\alpha), \\
&I_2 \le o(T;\alpha), \\
&I_3 \le \|\Delta y(\tau,\cdot)\|_{L_\infty(0,\omega)} \int_T^\infty \lambda_1(t)\, \mathrm{d}t = \alpha \int_T^\infty \lambda_1(t)\, \mathrm{d}t + o(\alpha), \\
&I_4 = \int_0^{\Sigma} \xi(\tau,\sigma)\Delta y(\tau,\sigma)\, \mathrm{d}\sigma = \alpha \int_0^{\Sigma} \xi(\tau,\sigma)[f(\tau,\sigma,u) - f(\tau,\sigma)]\, \mathrm{d}\sigma + o(\alpha), \\
&I_5 \le \varepsilon(T)\|\Delta y(\tau,\cdot)\|_{L_\infty(0,\omega)} = \alpha\varepsilon(T) + o(\alpha),
\end{aligned}
$$

where $\varepsilon(T) \to 0$ *as* $T \to \infty$ *and* $\lim_{\alpha\to 0} \frac{o(T;\alpha)}{\alpha} \to 0$ *for every fixed* $T > 0$.

We continue with the proof of the theorem. Take an arbitrary $\varepsilon_0 > 0$ and choose T large enough, such that $(\varepsilon(T) + \int_T^\infty \lambda_1(t)\,\mathrm{d}t) < \varepsilon_0$. According to the weakly overtaking optimality of $\hat{u}$, for $\varepsilon = \alpha^2$ and T, there exists $T' > T$ such that $J_{T'}(\hat{u}) \ge J_{T'}(u_\alpha) - \varepsilon$, which implies $\Delta_{T'} J \le \alpha^2$. Thus, we have

$$
\alpha^2 \ge \alpha[\mathscr{H}(\tau,u(\cdot)) - \mathscr{H}(\tau)] - c\alpha\left(\varepsilon(T) + \int_T^\infty \lambda_1(t)\,\mathrm{d}t\right) - o(T;\alpha).
$$

Take an arbitrary Divide by α and take the limit $\alpha \to 0$, it follows

$$
\mathscr{H}(\tau) \ge \mathscr{H}(\tau,u(\cdot)) - c\varepsilon_0.
$$

Since $\varepsilon_0 > 0$ and $u \in U^B$ were arbitrary, the inequality $\mathscr{H}(\tau) \ge \mathscr{H}(\tau,u(\cdot))$ holds for every $u \in U^B$ and almost every $\tau \in \cap_{u\in U}\Omega(u)$, that is, almost everywhere. Due to the continuity of $\mathscr{H}$ in u, and since U^B is dense in $\mathscr{U}_0$, the inequality holds almost everywhere for every $u \in \mathscr{U}_0$, which concludes the proof.

Proof of Lemma 4.6. We start with deriving a representation for $\Delta y(t,\sigma)$, $t \in [\tau-\alpha,\tau]$. We have

$$
\begin{aligned}
\Delta y(t,\sigma) &= \int_{\tau-\alpha}^{t} [f(s,\sigma,y_\alpha,z_\alpha,u_\alpha) - f(t,\sigma)]\, \mathrm{d}s \\
&= \int_{\tau-\alpha}^{t} [f_y(s,\sigma)\Delta y(s,\sigma) + f_z(s,\sigma)\Delta z(s) + f(t,\sigma,u) - f(t,\sigma)]\, \mathrm{d}s \\
&= \alpha[f(\tau,\sigma,u) - f(\tau,\sigma)] + o(\alpha)
\end{aligned}
$$

Let us start with I_1. It is equal to

$$\int_{\tau-\alpha}^{\tau}\int_0^{\Sigma}[L(t,\sigma,y_\alpha,z_\alpha,u_\alpha)-L(t,\sigma,\hat{y},\hat{z},u_\alpha)+L(t,\sigma,\hat{y},\hat{z},u_\alpha)-L(t,\sigma)]\,\mathrm{d}\sigma\,\mathrm{d}t.$$

For the second term of the double integral above, τ is a Lebesgue point, therefore it is equal to

$$\alpha\int_0^{\Sigma}[L(t,\sigma,u)-L(t,\sigma)]\,\mathrm{d}\sigma+o(\alpha).$$

The first difference is equal to

$$L_y(t,\sigma,\bar{y},\bar{z},u_\alpha)\Delta y(t,\sigma)+L_z(t,\sigma,\bar{y},\bar{z},u_\alpha)\Delta z(t).$$

Since $|\Delta y(t,\sigma)|+|\Delta z(t)|\le C\alpha$, this (integrated on $[\tau-\alpha,\tau]$) is $o(\alpha)$, which concludes the assertion for I_1.

The estimation for integral I_2 follows from Lemma 4.1 and the Lipschitz continuity of L_y using

$$\bar{L}_y\Delta y-L_y\delta y=(\bar{L}_y-L_y)\Delta y+L_y(\Delta y-\delta y),$$

and the same for L_z, Δz and δz.

The assertion for I_3 is a direct consequence of Assumption 4.2.

Consider now I_4. Using the representation of a solution for $\delta y(t,\sigma)$, (4.9), and for $\delta z(t)$, (4.8), we obtain the following:

$$\begin{aligned}\lim_{T\to\infty}\int_\tau^T\int_0^\Sigma\Big\{&L_y(t,\sigma)[X_\sigma(t,\tau)^{-1}\delta_0(\sigma)+\int_0^\Sigma\int_\tau^t X_\sigma(t,s)f_z(s,\sigma)R^z(s,\tilde{\sigma})\,\mathrm{d}sX_{\tilde{\sigma}}(\tau)^{-1}\delta_0(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}]\\&+L_z(t,\sigma)\int_0^\Sigma R^z(t,\tilde{\sigma})X_{\tilde{\sigma}}(\tau)^{-1}\delta_0(\tilde{\sigma})\,\mathrm{d}\sigma\Big\}\,\mathrm{d}\sigma\,\mathrm{d}t.\end{aligned}\tag{4.30}$$

The first integrand is bounded from above by $\lambda_2(t)$, thus integrable. For $T\to\infty$ it equals the first term in the definition of (4.17). Thus, consider the second term and insert the definition of R^z (see (4.7)),

$$\begin{aligned}\int_\tau^T\int_0^\Sigma L_y(t,\sigma)\int_0^\Sigma\int_\tau^t X_\sigma(t,s)f_z(s,\sigma)\Big[h_y(s,\tilde{\sigma})X_{\tilde{\sigma}}(s)+\int_\tau^s R(s,x)h_y(x,\tilde{\sigma})X_{\tilde{\sigma}}(x)\,\mathrm{d}x\Big]\,\mathrm{d}s\times\dots\\\times X_{\tilde{\sigma}}(\tau)^{-1}\delta_0(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\sigma\,\mathrm{d}t.\end{aligned}$$

Split the term again, and in the first one, change the order of integration,

$$\int_0^\Sigma \int_\tau^T \int_0^\Sigma \int_s^T L_y(t,\sigma)X_\sigma(t,s)\,\mathrm{d}t f_z(s,\sigma)\,\mathrm{d}\sigma h_y(s,\tilde{\sigma})X_{\tilde{\sigma}}(s)\,\mathrm{d}s X_{\tilde{\sigma}}(\tau)^{-1}\delta_0(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\sigma.$$

Also in the second part above, we change the order of integration and obtain the expression

$$\int_0^\Sigma \int_\tau^T \int_x^T \int_0^\Sigma \int_s^T L_y(t,\sigma)X_\sigma(t,s)\,\mathrm{d}t f_z(s,\sigma)R(s,x)\,\mathrm{d}s h_y(x,\tilde{\sigma})X_{\tilde{\sigma}}(x,\tau)\,\mathrm{d}x\delta_0(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}.$$

Now combining the two terms and changing the name of the variables, we obtain

$$\int_0^\Sigma \int_\tau^T \left[\tilde{\psi}_T(t) + \int_t^T \tilde{\psi}_T(s)R(s,t)\,\mathrm{d}s\right] h_y(t,\tilde{\sigma})X_{\tilde{\sigma}}(t)\,\mathrm{d}t X_{\tilde{\sigma}}(\tau)^{-1}\delta_0(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}, \tag{4.31}$$

where $\tilde{\psi}_T(t) := \int_0^\Sigma \int_t^T L_y(s,\sigma)X_\sigma(s)\,\mathrm{d}s X_\sigma(t)^{-1} f_z(t,\sigma)\,\mathrm{d}\sigma$. The integrals above are well defined due to Assumption 4.2: equation (4.21) implies the existence of $\lim_{T\to\infty}\tilde{\psi}_T(t)$, (4.22) proves the uniform integrability of $t \to \psi_T(t)$, then (4.23) ensures that we can pass to the limit $T \to \infty$ in (4.31). Passing to the limit $T \to \infty$, $\tilde{\psi}(t)$ is equal to the integral in the definition of $\hat{\psi}(t)$, (4.19).

The proof that the terms multiplied by L_z are equal to the first term in the definition of $\hat{\psi}$ is essentially the same and therefore omitted. Combining these two assertions proves the first equality for I_4. Inserting the representation for $\Delta y(\tau,\sigma)$ derived at the beginning of the integral, and using the Lebesgue point property proves the assertion for I_4.

To see that the estimation for I_5 holds true, consider again equation 4.30, where now the outer integral is from T to ∞. This integral is well defined due to the considerations for I_4. Changing the order of integration implies that we can estimate it by $\int_0^\Sigma \int_T^\infty \lambda_6(t,\sigma)\,\mathrm{d}t\delta_0(\sigma)\,\mathrm{d}\sigma$. Now the considerations for I_4 imply that this λ_6 is integrable, therefore the integral goes to zero when $T \to \infty$, which proves the claim. □

4.5 A Hamiltonian Formulation

With the definition (4.24) of $\mathscr{H}$, the primal system (4.2)–(4.3) and the adjoint system (4.15)–(4.16) can be written in their Hamiltonian representation:

$$\dot{y}(t,\cdot) = \mathscr{H}_\xi(t,y(t,\cdot),z(t),u(t,\cdot),\xi(t,\cdot),\zeta(t)), \qquad y(0,\sigma) = y_0(\sigma) \tag{4.32}$$

$$0 = \mathscr{H}_\zeta(t,y(t,\cdot),z(t),u(t,\cdot),\xi(t,\cdot),\zeta(t)), \tag{4.33}$$

$$-\dot{\hat{\xi}}(t,\cdot) = \mathscr{H}_y(t,\hat{y}(t,\cdot),\hat{z}(t),\hat{u}(t,\cdot),\hat{\xi}(t,\cdot),\hat{\zeta}(t)), \tag{4.34}$$

$$0 = \mathscr{H}_z(t,\hat{y}(t,\cdot),\hat{z}(t),\hat{u}(t,\cdot),\hat{\xi}(t,\cdot),\hat{\zeta}(t)). \tag{4.35}$$

The derivatives $\mathscr{H}_\xi$ and $\mathscr{H}_y$ are a priori only elements of the dual space of $L_\infty(0,\Sigma)$, but they turn out to be representable by L_∞-functions which are equal (in the L_∞-sense) to the right hand side of the corresponding differential equations. The proof is the same as in Section 5.5 and therefore omitted.

The following theorem justifies the use of the notion "Hamiltonian", as it completes the proof that $\mathscr{H}$ satisfies the reproducibility of the primal and the adjoint system, the maximum principle, and that $\mathscr{H}$ is constant along the optimal trajectory for autonomous problems.

Theorem 4.7. *Let the assumptions of Theorem 4.3 be fulfilled, and let the functions L, f and h do not depend explicitly on t. Then for every $T > 0$ the maximized Hamiltonian*

$$\hat{\mathscr{H}}(t) := \max_{u(\cdot)\in\mathscr{U}_0} \mathscr{H}(t,\hat{y}(t,\cdot),\hat{z}(t),u(\cdot),\hat{\xi}(t,\cdot),\hat{\zeta}(t))$$

is constant on the interval $[0,T]$.

Proof. Obviously $\hat{y}(t,\sigma)$ and $\hat{\xi}(t,\sigma)$ are, for fixed σ, absolutely continuous functions of time as solutions to differential equations. This implies that also $z(t)$ is absolutely continuous, because $h(\sigma,y(t,\sigma))$ is differentiable with respect to y. However, $\hat{\zeta}(t)$ as well as the control $\hat{u}$ may not be continuous but only bounded and measurable.

The optimal control $\hat{u}(t,\cdot)$ is bounded on the compact set $[0,T]$. Thus, there exists a set $\mathscr{U}_c \subset \mathscr{U}_0$, such that $\hat{u}(t,\cdot) \in \mathscr{U}_c$ for a.e. $t \in [0,T]$ and $\mathscr{U}_c$ is compact in the topology induced by the L_2-norm on $\mathscr{U}_0$. Obviously

$$\max_{u(\cdot)\in\mathscr{U}_0} \mathscr{H}(\hat{y}(t,\cdot),\hat{z}(t),\hat{\xi}(t,\cdot),\hat{\zeta}(t),u(\cdot)) \geq \max_{u(\cdot)\in U_0} \mathscr{H}(\hat{y}(t,\cdot),\hat{z}(t),\hat{\xi}(t,\cdot),\hat{\zeta}(t),u(\cdot))$$

and the equality holds almost everywhere in $[0,T]$ because $\hat{u}(t,\cdot) \in \mathscr{U}_c$ for almost every t. It is clear that it is enough to show that $\max_{u\in U_0}\mathscr{H}(t,u)$ is constant in t, where the maximization is taken over the smaller set U_0.

At first, we establish that $\mathscr{H}(t,u)$ is absolutely continuous for a fixed u. Then for $t,\tau\in[0,T]$:

$$\mathscr{H}(t,u)-\mathscr{H}(\tau,u)=\mathscr{H}(\hat{y}(t,\cdot),\hat{z}(t),\hat{\xi}(t,\cdot),\hat{\zeta}(\tau),u)-\mathscr{H}(\hat{y}(\tau,\cdot),\hat{z}(\tau),\hat{\xi}(\tau,\cdot),\hat{\zeta}(\tau),u) \quad (4.36)$$
$$+(\zeta(t)-\zeta(\tau))\left(\int_0^\Sigma h(\sigma,\hat{y}(t,\sigma))\,\mathrm{d}\sigma-\hat{z}(t)\right). \quad (4.37)$$

The last bracket is equal to zero and therefore the non-differentiable term $\zeta(t)$ disappears. Then the differentiability (with Lipschitz partial derivatives) of L and f with respect to y and z, as well as the absolute continuity of $\hat{y}$, $\hat{\xi}$, and $\hat{z}$ conclude the proof of absolute continuity of $\mathscr{H}$. The absolute continuity is even uniform in $t\in[0,T]$ and $u(\cdot)\in U_0$ because U_0 is compact and $(\hat{y}(t,\cdot),\hat{z}(t),\hat{\xi}(t,\cdot))$ are continuous on the compact set $[0,T]$, therefore take values only in a compact subset of $L_\infty(0,\Sigma)^{n_y}\times\mathbb{R}^{m_y}\times L_\infty(0,\Sigma)^{n_y}$. Thus, there exists a constant $C>0$ such that for every $t,\tau\in[0,T]$ and every $u\in U_0$

$$|\mathscr{H}(\hat{y}(t,\cdot),\hat{z}(t),\hat{\xi}(t,\cdot),\hat{\zeta}(\tau),u)-\mathscr{H}(\hat{y}(\tau,\cdot),\hat{z}(\tau),\hat{\xi}(\tau,\cdot),\hat{\zeta}(\tau),u)|\le C|t-\tau|.$$

Using equation (4.37) and that $\hat{u}(t)$ maximizes the Hamiltonian, we obtain

$$\begin{aligned}-C|t-\tau|\le&\mathscr{H}(t,\hat{u}(\tau))-\mathscr{H}(\tau,\hat{u}(\tau))\le\\ \le&\mathscr{H}(t,\hat{u}(t))-\mathscr{H}(\tau,\hat{u}(\tau))\le\\ \le&\mathscr{H}(t,\hat{u}(t))-\mathscr{H}(\tau,\hat{u}(t))\le C|t-\tau|,\end{aligned}$$

which implies $|\hat{\mathscr{H}}(t)-\hat{\mathscr{H}}(\tau)|\le C|t-\tau|$, and thus the Lipschitz continuity and differentiability of $t\to\hat{\mathscr{H}}(t)$.

Fix $\tau\in[0,T]$ in the subset of $[0,T]$ with full measure, in which the state equations and the adjoint equations in Hamiltonian form (4.32)–(4.35) are satisfied. Then, by definition,

$$\mathscr{H}(t)-\mathscr{H}(\tau)\ge\mathscr{H}(t,\hat{u}(\tau))-\mathscr{H}(\tau,\hat{u}(\tau)),$$

for almost every $t \in [0,T]$. Divide by $(t-\tau)$ and take the limes $t \to \tau$, where $t > \tau$, then

$$\begin{aligned}
\mathscr{H}(t) - \mathscr{H}(\tau) \geq \lim_{t \searrow \tau} \frac{1}{(t-\tau)} \big(\mathscr{H}(t,\hat{u}(\tau)) - \mathscr{H}(\tau,\hat{u}(\tau))\big) \\
= \lim_{t \searrow \tau} \frac{1}{(t-\tau)} \Big[\mathscr{H}(\hat{y}(t,\cdot),\hat{z}(t),\hat{\xi}(t,\cdot),\hat{\zeta}(\tau),\hat{u}(\tau)) - \mathscr{H}(\hat{y}(\tau,\cdot),\hat{z}(\tau),\hat{\xi}(\tau,\cdot),\hat{\zeta}(\tau),\hat{u}(\tau)) + \dots \\
+ (\zeta(t) - \zeta(\tau)) \left(\int_0^{\Sigma} h(\sigma,\hat{y}(t,\sigma))\,\mathrm{d}\sigma - \hat{z}(t)\right)\Big].
\end{aligned}$$

The last term is equal to zero for all t, see (4.37). In the first difference, all functions are absolutely continuous. Differentiate to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t,\hat{u}(\tau))\Big|_{t=\tau} = \Big(\langle \mathscr{H}_y, \dot{\hat{y}}\rangle + \langle \mathscr{H}_z, \dot{\hat{z}}\rangle + \langle \mathscr{H}_\xi, \dot{\hat{\xi}}\rangle\Big)_{t=\tau},$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product on the appropriate spaces (the point of evaluation is omitted for clarity). Using the Hamiltonian representations (4.32)–(4.35) for $\mathscr{H}_y$, $\mathscr{H}_z$, and $\mathscr{H}_\xi$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathscr{H}}(t)\Big|_{t=\tau} \geq \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t,\hat{u}(\tau))\Big|_{t=\tau} = \Big(\langle -\dot{\hat{\xi}}, \dot{\hat{y}}\rangle + \langle 0, \dot{\hat{z}}\rangle + \langle \dot{\hat{y}}, \dot{\hat{\xi}}\rangle\Big)_{t=\tau} = 0$$

Taking $\lim_{t\nearrow\tau}$ but with t converging from below, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathscr{H}}(t)\Big|_{t=\tau} \leq 0.$$

Recall that the above consideration holds for a.e. $\tau \in [0,T]$, thus $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathscr{H}}(t) = 0$ almost everywhere, which implies that $\hat{\mathscr{H}}(t)$ is constant. Since T was arbitrary, the proof is concluded. □

4.6 An Example: A Control Problem Arising in Epidemiology

In [53], a model for the spreading of an infectious disease with heterogeneous agents is considered. The population (indexed by $\sigma \in [0,\Sigma]$) is heterogeneous with respect to their contact rate $p(\sigma)$. For every σ, the population can be divided in susceptible and infected individuals. Susceptible (healthy) individuals reproduce with rate λ but become infected based upon the strength of the infection μ, the individual contact rate $p(\sigma)$, and the probability of meeting an infected person, $\frac{z_I(t)}{z_S(t)+z_I(t)}$. The number of infected individuals increases with the susceptible individuals becoming infected and decreases by deaths, $\delta I(t,\sigma)$.

We extend this formulation to a control model by making the strength of the infection μ and the death rate θ controllable, $\mu = \mu(u_1(t,\sigma))$, $\theta = \theta(u_2(t,\sigma))$, and adding some externality $\varphi(t)$.

The strength of infection can be reduced, for example, by prevention through education (in case of HIV), or through preventive measures (grip masks or vaccination to prevent spreading). The death rate can be decreased by medication. All these measures are associated with costs, while a healthy population may be valued positive by the policy maker, who maximizes the total value:

$$\max_{u_1,u_2} \int_0^\infty \int_0^\Sigma e^{-rt} \left(q(t,\sigma) y_S(t,\sigma) - c_1(u_1(t,\sigma)) - c_2(u_2(t,\sigma)) \right) \mathrm{d}\sigma \, \mathrm{d}t,$$

subject to

$$\begin{aligned}
\dot{y}_S(t,\sigma) &= \left(-\mu(u_1(t,\sigma)) \frac{z_I(t)}{z_S(t)+z_I(t)} p(\sigma) + \lambda \right) y_S(t,\sigma) + \varphi(t,\sigma), \qquad S(0,\sigma) = S_0(\sigma),\\
\dot{y}_I(t,\sigma) &= \left(\mu(u_1(t,\sigma)) \frac{z_I(t)}{z_S(t)+z_I(t)} p(\sigma) \right) y_S(t,\sigma) - \theta(u_2(t,\sigma)) y_I(t,\sigma), \qquad I(0,\sigma) = I_0(\sigma),\\
z_S(t) &= \int_0^\Sigma p(\sigma) y_S(t,\sigma) \, \mathrm{d}\sigma,\\
z_I(t) &= \int_0^\Sigma p(\sigma) y_I(t,\sigma) \, \mathrm{d}\sigma,\\
u_i(t,\sigma) &\in [\underline{u}, \bar{u}], \qquad i \in \{1,2\}.
\end{aligned}$$

Let q be an essentially bounded function, c_1 and c_2 be increasing in u_1 and u_2 with strictly positive first derivatives, and let $\lambda > 0$. Furthermore, let μ and θ be decreasing functions in the controls and let there exist positive numbers $(\underline{\mu}, \bar{\mu})$ such that $\underline{\mu} \leq \mu(u) \leq \bar{\mu}$ for $u \in [\underline{u}, \bar{u}]$, and let the same hold for $\delta(u)$. Assume that for a σ in a subset of $[0,\Sigma]$ with positive measure, $\lambda + \underline{\delta} > \bar{\mu} p(\sigma) > 0$.

Denote the optimal control by $(\hat{u}_1(t,\sigma), \hat{u}_2(t,\sigma))$, and the trajectories by $(\hat{y}_S, \hat{y}_I, \hat{z}_S, \hat{z}_I)$. Then $L_y(t,\sigma) = e^{-rt} p(\sigma)$, $L_z(t,\sigma) = 0$, $h_y(t,\sigma) = p(\sigma)$ uniformly for all (y,z). Define $\beta(t,\sigma) := \left(\mu(\hat{u}_1(t,\sigma)) \frac{z_I(t)}{z_S(t)+z_I(t)} p(\sigma) \right)$. Then the linearized system (as in Section 4.2) is equivalent to setting $\varphi = 0$, and the solution is

$$y_S(t,\sigma) = e^{-\int_\tau^t \beta(s,\sigma) \, \mathrm{d}s} e^{\lambda(t-\tau)} y_S(\tau,\sigma).$$

Inserting this solution in the equation for $\dot{y}_I(t,\sigma)$, we obtain

$$y_I(t,\sigma) = e^{-\theta(t-\tau)} y_I(\tau,\sigma) + \int_\tau^t e^{-\theta(t-s)} e^{-\int_\tau^s \beta(x,\sigma) \, \mathrm{d}x} e^{\lambda(s-\tau)} y_S(\tau,\sigma) \, \mathrm{d}s.$$

Thus, $\|X_\sigma(t,s)\| \leq C e^{\lambda(t-s)}$ uniformly in σ. Note that estimating $X(t)$ and $X(t)^{-1}$ separately is worse, because the best estimate for the inverse is $\|X(t)^{-1}\| \leq Ce^{\theta t}$. We have

$$f_z(t,\sigma) = \frac{\mu(u_1(t,\sigma))p(\sigma)}{(z_I(t)+z_S(t))^2} \begin{pmatrix} z_I(t) & -z_S(t) \\ -z_I(t) & z_S(t) \end{pmatrix} y_S(t,\sigma),$$

which implies $\|f_z(t,\sigma)\| \leq C$ because $z_S(t)$ grows in the long run with the same rate as $y_S(t,\sigma)$. This follows for $\varphi = 0$ from equations (19) and (20) in [53] (in which case the growth rate is λ), but holds true also for $\varphi(t) \neq 0$.

Therefore, the growth estimations in Assumption 4.3 are fulfilled with $\rho_1 = -r$, $\rho_2 = -\infty$, $\rho_6 = \rho_7 = 0$, $\rho_3 = \rho_4 = \rho_5 = \lambda$. Assume $\lambda - r < 0$ small enough such that

$$\operatorname{ess\,sup}_{t\in[0,\infty)} \int_t^\infty e^{-r(s-t)} \left| \int_0^\Sigma h_y(t,\sigma)X_\sigma(t,s)f_z(t,\sigma)\,\mathrm{d}\sigma \right| \mathrm{d}s < 1,$$

that is, such that (4.29) holds. Then Assumption 4.3 is satisfied and the results from the maximum principle, Theorem 4.3, can be applied.

Define $\hat{\xi}(t,\sigma) = (\hat{\xi}_S(t,\sigma), \hat{\xi}_I(t,\sigma))$ and $\hat{\zeta}(t) = (\hat{\zeta}_S(t), \hat{\zeta}_I(t))$ as in (4.17) and (4.18) (regarding (4.19)). Then the optimal controls maximize

$$\begin{aligned} &-c_1(u_1(t,\sigma)) + (\hat{\xi}_I(t,\sigma) - \hat{\xi}_S(t,\sigma)) \left(\mu(u_1(t,\sigma)) \frac{\hat{z}_I(t)}{\hat{z}_S(t)+\hat{z}_I(t)} p(\sigma) \right) \hat{y}_S(t,\sigma), \\ &-c_2(u_2(t,\sigma)) - \hat{\xi}_I(t,\sigma)\theta(u_2(t,\sigma))I(t,\sigma). \end{aligned}$$

In case $\varphi = 0$, the objective functional is finite because y_S grows at rate $\lambda < r$. However, Assumption 4.3 is also satisfied for $\varphi(t) > 0$. The assertions about λ_2, λ_3, and λ_4 deal with the linearized system, which are independent of φ. The assertion concerning λ_1 also remains valid, as $\Delta y(t,\sigma) = \delta y(t,\sigma)$, independently of $\varphi(t)$. Therefore, the maximum principle holds although the objective value may be infinite, and the "classical" transversality condition $\lim_{t\to\infty} \hat{\xi}(t,\sigma)\hat{y}(t,\sigma) = 0$ may be violated for $\varphi(t)$ growing fast enough.

Chapter 5

Endogeneous Domain of Heterogeneity[1]

The problem considered in this chapter can be viewed as an extension of the model in the previous chapter. The domain of heterogeneity (e.g. the number of products a company can produce) is not fixed, but the decision maker can invest to enlarge the variety, for example, invest in research to develop new products. Thus, the domain of heterogeneity changes endogenously. As in the previous chapter, we analyze first order necessary conditions and a Hamiltonian formulation, but the time horizon is finite.

We begin with a motivating example for the system considered in this section. Consider a company that produces a variety of products, indexed by a parameter σ. At time t, $[0,\Sigma(t)]$ is the interval of those σ, for which a product with index σ exists. The firm can invest in production capital $y(t,\sigma)$ to increase output, but can also invest in R&D to increase the variety of products $\Sigma(t)$, that is, the technology frontier. For a given σ, the capital stock follows a linear ODE involving depreciation and investments (a distributed control $u(t,\sigma)$). There exist aggregated states which may influence the payout, but also the dynamics of the technology frontier.

That is, the model consists of a family of ODEs parameterized by σ. This distributed control system is only meaningful in the domain $\{(t,\sigma) : \sigma \in [0,\Sigma(t)]\}$, because production facilities can only be built up for existing technologies. Due to the dynamics of $\Sigma(t)$, which is control-dependent, the control problem is on a controlled domain. The described problem is not only interesting from the perspective of a company, but also occurs in modeling of endogenous economic growth.

There are several facts that complicate the analysis of an optimization problem on a controlled domain: the objective value of the maximization problem is possibly non-differentiable as

[1]This Chapter is based on the joint work with Tsvetomir Tsachev and Vladimir Veliov, [44].

a function of the control variable (shown in [12]) as well as non-concave (shown in Section 5.3). Furthermore, technical complications arise due to the dependence of the spatial domain $[0,\Sigma(t)]$ on the control.

Optimality conditions for a more general problem than the one considered here are obtained in [12] and summarized in Section 5.2. However, in the general form (cf. Theorem 5.1), these optimality conditions are not enough for efficient numerical approaches. The more useful "regular" formulation (cf. Theorem 5.2) is only true under a priori "regularity assumptions" on the optimal control (cf. Assumption 5.3). The verification of this turns out to be complicated even for simple cases, as it can not make use of the optimality conditions, which rely on the regularity itself.

In this chapter, we prove that for the considered problem, the optimal control satisfies the regularity conditions discussed above. Some additional work for the non-distributed control is required to obtain optimality conditions as a consequence of the results in [12]. In this sense, the results presented in this chapter are substantial complements and enhancements of [12].

The optimality conditions are obtained in an appropriate Hamiltonian form, which turned out to be non-trivial because the Hamiltonian involves the dynamics outside of the intrinsically relevant spatial domain. For an autonomous control problem, we also prove that, under some additional condition, this Hamiltonian is constant along the optimal trajectory.

Additionally, we prove the existence of an optimal control for the considered problem, even though the maximization problem is non-concave.

We make a short link and comparison with the existing economic literature. We consider true heterogeneous products, in contrast to [20, 30, 48], where a variety of products is present but they are viewed as identical. On the other hand, [11, 47] investigate truly heterogeneous goods, but in a different context, where dynamics of the capital stock is ignored.

The structure of this chapter is the following. In Section 5.1 the problem is stated. Section 5.2 summarizes the findings and important results from [12]. The existence of an optimal solution as well as the non-concavity of the problem are shown in Section 5.3. The regularity of the optimal control is proved in Section 5.4. In Section 5.5 the Hamiltonian formulation of this control problem and its adjoint system is presented, the Maximum Principle and the constancy of the Hamiltonian along the optimal solutions are shown.

5.1 Statement of the Problem

In this section, we formulate a problem which is a particular case of the general one considered in [12]. Let $[0,T]$ be a fixed time-interval and let $[0,\bar{\Sigma}]$ be an interval where the parameter of heterogeneity σ will take values ($T > 0$ and $\bar{\Sigma} > 0$ are given). Denote $D_T = [0,T] \times [0,\bar{\Sigma}]$. State variables in the model below are the functions

$$\Sigma : [0,T] \to [0,\bar{\Sigma}], \quad y : D_T \to \mathbb{R}^1, \quad z : [0,T] \to \mathbb{R}^1,$$

while $u : D_T \to U := [\underline{u},\overline{u}] \subset \mathbb{R}^1$, and $v : [0,T] \to V := [\underline{v},\overline{v}] \subset \mathbb{R}^1$ are control functions, with $\overline{u} > \underline{u} \geq 0$, $\overline{v} > \underline{v} \geq 0$ being given. For a given $\Sigma : [0,T] \to [0,\bar{\Sigma}]$, we denote $D_\Sigma := \{(t,\sigma) \in D_T : \sigma \in [0,\Sigma(t)]\}$.

The optimal control problem we consider reads as follows:

$$\max_{(u(\cdot,\cdot),v(\cdot))} J[u,v] := \int_0^T \left[\int_0^{\Sigma(t)} \Big(L(t,\sigma,\Sigma(t),y(t,\sigma),z(t)) - c_1(t,u(t,\sigma)) \Big) \,\mathrm{d}\sigma - c_2(t,v(t)) \right] \mathrm{d}t, \tag{5.1}$$

subject to

$$\dot{\Sigma}(t) = g_1(t,\Sigma(t))z(t) + g_2(t,\Sigma(t))v(t) \qquad \text{for a.e. } t \in [0,T],\ \Sigma(0) = \Sigma^0, \tag{5.2}$$

$$\dot{y}(t,\sigma) = -\delta y(t,\sigma) + u(t,\sigma) \qquad \text{for a.e. } (t,\sigma) \in D_\Sigma, \tag{5.3}$$

$$y(0,\sigma) = y^0(\sigma) \qquad \text{for a.e. } q \in [0,Q^0], \tag{5.4}$$

$$y(t,\Sigma(t)) = y^b(t) \qquad \text{for } t \in (0,T], \tag{5.5}$$

$$z(t) = \int_0^{\Sigma(t)} d(t,\sigma)u(t,\sigma)\,\mathrm{d}\sigma \qquad \text{for a.e. } t \in [0,T], \tag{5.6}$$

$$u(t,\sigma) \in U \qquad \text{for a.e. } (t,\sigma) \in D, \tag{5.7}$$

$$v(t) \in V \qquad \text{for a.e. } t \in [0,T].,\tag{5.8}$$

where L, c_1, c_2, g_1, g_2, and d are given functions, and $\delta > 0$ is a given number. The exact meaning of a solution of system (5.2)–(5.8) is given below.

The economic interpretation of the variables is the following: $\Sigma(t)$ is the technological frontier at time t, which changes in accordance with (5.2); $y(t,\sigma)$ is the amount of physical capital (or, alternatively, a quality measure) of technology σ; $v(t)$ is a direct investment in development of new technologies; $u(t,\sigma)$ is the investment in the technology $\sigma \in [0,\Sigma(t)]$, and δ the depreciation of capital; while $z(t)$ is the indirect effect of the capital investments on the development of new technologies.

The objective functional represents the aggregated net revenue, where the dependence on t allows for discounting. The numbers $\Sigma^0 \in [0,\bar{\Sigma})$, and $y^0(\sigma)$, $\sigma \in [0,\Sigma^0]$, are given initial data,

$y^b(t)$ is a boundary condition, which represents the amount of capital stock (or quality level) of technology σ, at the time this technology is developed, that is, when $\Sigma(t) = \sigma$.

The set of admissible controls is $\mathscr{U} \times \mathscr{V}$, where $\mathscr{U} = \{u \in L_\infty(D_T) : u(t,\sigma) \in U \text{ for a.e. } (t,\sigma) \in D_T\}$ and $\mathscr{V} = \{v \in L_\infty(0,T) : v(t) \in V \text{ for a.e. } t \in [0,T]\}$. We denote $\mathscr{U}_0 := L_\infty([0,\Sigma(t)];U)$, which is the subset of functions in $L_\infty([0,\Sigma(t)];\mathbb{R})$ with values in U, cf. the explanations in Chapter 4 above equation (4.5).

We study problem (5.2)–(5.8) under the following assumptions, which are similar to the ones imposed in [12] (see Assumption 5.2 in Section 5.2):

Assumption 5.1. *(i) The functions L, g_1, g_2, c_1 and c_2 are measurable in (t,σ), locally essentially bounded, differentiable in (Σ,y,z), with locally Lipschitz partial derivatives, uniformly with respect to $(t,\sigma) \in D_T$; the function L is concave in (y,z); the function c_1 is strongly convex and twice continuously differentiable in u on an open set containing U, uniformly in $t \in [0,T]$; the function c_2 is convex and differentiable in v on an open set containing V, with locally Lipschitz derivative, uniformly in $t \in [0,T]$; the function d is measurable on D_T.*

(ii) There exist $\overline{g}_i > \underline{g}_i \geq 0$, $i = 1,2$, such that for almost every $t \in [0,T]$ we have $\underline{g}_i \leq g_i(t,\Sigma) \leq \overline{g}_i$ for $i = 1,2$, $\Sigma \in [\Sigma^0, \bar{\Sigma}]$.

(iii) There exist $\overline{d} > \underline{d} > 0$, such that $\underline{d} \leq d(t,q) \leq \overline{d}$ for a.e. $(t,\sigma) \in D_T$.

(iv) For the above mentioned parameters, $\underline{g}_1\, \underline{u} + \underline{g}_2\, \underline{v} > 0$ holds.

(v) $y^b : [0,T] \to \mathbb{R}^1$ is continuously differentiable, $y^0 : [0,\Sigma^0] \to \mathbb{R}^1$ is continuous, and they satisfy $y^0(\Sigma^0) = y^b(0)$.

(vi) For every $(u,v) \in \mathscr{U} \times \mathscr{V}$, the solution $\Sigma[u,v]$ exists on the whole interval $[0,T]$ and takes values in $[\Sigma^0, \bar{\Sigma}]$.

In this chapter, we refer e.g. to Assumption 5.1.(iii) also in short as Assumption (iii). Some comments on this assumptions follow.

For any given $u \in \mathscr{U}$ and $v \in \mathscr{V}$, one can represent

$$\dot{\Sigma}(t) = g_1(t,\Sigma(t)) \int_0^{\Sigma(t)} d(t,\sigma)u(t,\sigma)\,\mathrm{d}\sigma + g_2(t,\Sigma(t))v(t), \tag{5.9}$$

and the function in the right-hand side is locally Lipschitz in Σ. Therefore, equation (5.2) has locally an absolute continuous solution $\Sigma = \Sigma[u,v]$, and it is unique on its maximal interval of existence in $[0,T]$. Assumption (vi) is only needed to ensure that this interval of existence is $[0,T]$.

Given $u \in \mathscr{U}$ and $v \in \mathscr{V}$, we define for $\sigma \in [0, \bar{\Sigma}]$

$$\theta[u,v](\sigma) := \begin{cases} 0 & \text{if } \sigma \in [0, \Sigma^0], \\ \Sigma[u,v]^{-1}(\sigma) & \text{if } \sigma \in (\Sigma^0, \Sigma[u,v](T)), \\ T & \text{if } \sigma \in [\Sigma[u,v](T), \bar{\Sigma}]. \end{cases}$$

By Assumptions (ii)–(iv), $\Sigma[u,v]$ is invertible, and its image is $[\Sigma^0, \Sigma[u,v](T)]$. Thus, the definition is correct.

We extend the definition of $y^0(\cdot)$ to $[0, \bar{\Sigma}]$ as $y^0(\sigma) = y^b(0)$ for $\sigma \in (\Sigma^0, \bar{\Sigma}]$, and extend the dynamics of the distributed space variable $y(\cdot,\cdot)$ to the whole region D_T by replacing (5.3)–(5.5) with

$$\dot{y}(t,\sigma) = -\delta y(t,\sigma) + u(t,\sigma) \qquad \text{for a.e. } \sigma \in [0,\bar{\Sigma}], \text{ and a.e. } t \in [\theta[u,v](\sigma), T], \tag{5.10}$$

$$\dot{y}(t,\sigma) = \dot{y}^b(t) \qquad \text{for a.e. } \sigma \in [0,\bar{\Sigma}], \text{ and a.e. } t \in [0, \theta[u,v](\sigma)), \tag{5.11}$$

$$y(0,\sigma) = y^0(\sigma) \qquad \text{for a.e. } \sigma \in [0,\bar{\Sigma}]. \tag{5.12}$$

By solution of (5.10)–(5.12) we mean a function $y(\cdot,\cdot)$, measurable on D_T, such that for almost every $\sigma \in [0,\bar{\Sigma}]$ it holds that (5.12) is satisfied, $y(\cdot,\sigma)$ is absolutely continuous on $[0,T]$ and satisfies (5.10) and (5.11) almost everywhere in $t \in [0,T]$; $y(t,\cdot) \in L_\infty(0,\bar{\Sigma})$ for every $t \in [0,T]$.

We thus view (5.10)–(5.12) as a family of ODEs (one for each $\sigma \in [0,\bar{\Sigma}]$), where the functions $z = z[u,v]$ and $\Sigma = \Sigma[u,v]$ are already defined in (5.2) and (5.6) as described above.

For each $\sigma \in [0,\bar{\Sigma}]$ we define

$$y^*[u,v](\sigma) := \begin{cases} y^0(\sigma) & \text{if } \sigma \in [0, \Sigma^0], \\ y^b(\theta[u,v](\sigma)) & \text{if } \sigma \in (\Sigma^0, \Sigma[u,v](T)), \\ y^b(T) & \text{if } \sigma \in [\Sigma[u,v](T), \bar{\Sigma}]. \end{cases} \tag{5.13}$$

Then, for given controls (u,v), the solution of (5.10)–(5.12) on the domain $D_{\Sigma[u,v]}$ is

$$y[u,v](t,\sigma) = e^{-\delta(t-\theta[u,v](\sigma))} y^*[u,v](\sigma) + \int_{\theta[u,v](\sigma)}^{t} e^{\delta(\tau-t)} u(\tau,\sigma)\,\mathrm{d}\tau. \tag{5.14}$$

In accordance with (5.10)–(5.12), we extend $y[u,v]$ as

$$y[u,v](t,\sigma) := y^b(t) \quad \text{for } (t,\sigma) \in [0,T] \times [\Sigma[u,v](t), \bar{\Sigma}]. \tag{5.15}$$

An example satisfying Assumption 5.1 (as well as Assumption 5.4 which we will introduce before Proposition 5.6 in Section 5.4 below) is the investment problem in Section 6 in [12], after adding the control constraint $0 < \underline{u} \leq u(t,\sigma) \leq \bar{u} < \infty$. The strictly positive lower bound on the investment $u(\cdot,\cdot)$ can be justified by the fact that governments often require minimal investments when granting licenses to private companies.

5.2 Preliminary Results

As already mentioned in the introduction, [12] presents two different forms of optimality conditions for a system very similar to the one considered here. In Section 5.5, we will make use of [12, Theorem 2], for completeness we present it here in this section. Before that, we describe the considered system and the general form of the maximum principle, to emphasize the importance of the regularity assumption.

Fix $T > 0$, $\bar{\Sigma} > 0$, and denote $D_T = [0,T] \times [0,\bar{\Sigma}]$. The state variables are the functions

$$\Sigma : [0,T] \to [0,\bar{\Sigma}], \qquad y : D_T \to \mathbb{R}^n, \qquad z : [0,T] \to \mathbb{R}^m,$$

while the control is $u : D_T \to \mathbb{R}^r$. Denote $\mathscr{U} = \{u \in L_\infty(D_T) : u(t,\sigma) \in U\}$.

The optimal control problem (OCP) reads as follows:

$$\max_u \int_0^T \int_0^{Q(t)} L(t,\sigma,\Sigma(t),y(t,\sigma),z(t),u(t,q))\,\mathrm{d}q\,\mathrm{d}t,$$

subject to the equations

$$\begin{aligned}
\dot{\Sigma}(t) &= g(t,\Sigma(t),z(t)), \qquad \Sigma(0) = \Sigma_0 \geq 0,\\
\dot{y}(t,\sigma) &= f(t,\sigma,\Sigma(t),y(t,\sigma),z(t),u(t,\sigma)),\\
& y(0,\sigma) = y_0(\sigma), \qquad 0 \leq \sigma \leq \Sigma_0,\\
& y(t,\Sigma(t)) = x_b(t), \quad t \in [0,T],\\
z(t) &= \int_0^{\Sigma(t)} h(t,\sigma,u(t,\sigma))\,\mathrm{d}\sigma,\\
& u(t,q) \in U.
\end{aligned}$$

Assumption 5.2. *[12, Standing Assumptions (i)–(vi)] The set $U \subset \mathbb{R}^r$ is compact. The functions L, f, g, and h are measurable in (t,σ) and continuous in the rest of the variables; locally essen-*

tially bounded; differentiable in (Σ, y, z), with locally Lipschitz partial derivatives, uniformly with respect to $u \in U$ and $(t,\sigma) \in D$. The function h is locally Lipschitz continuous in u uniformly with respect to $(t,\sigma) \in D$.

$g(t,\Sigma,z) \geq \alpha_0 > 0$ *for every* $\Sigma \in [0,\bar{\Sigma}]$ *and* $z \in \int_0^Q h(t,\sigma,U)\,\mathrm{d}\sigma$.

$y^b : [0,T] \to \mathbb{R}^n$ *is continuously differentiable;* $y_0 : [0,\Sigma_0] \to \mathbb{R}^n$ *is measurable and bounded.*

For every $u \in \mathscr{U}$, the solution of the differential equation for Σ exists in $[0,\bar{\Sigma}]$ on the whole interval $[0,T]$, and for almost every $\sigma \in [0,\bar{\Sigma}]$, the solution $x[u](\cdot,\sigma)$ exists on $[0,T]$.

For comments on the assumptions, or the exact notion of solution, see the special case in the previous section.

Define the adjoint equations

$$-\dot{\xi}(t,\sigma) = L_y(t,\sigma) + \xi(t,\sigma) f_x(t,\sigma), \qquad \xi(T,\sigma) = 0, \tag{5.16}$$

$$\zeta[\psi](t) = \psi g_y(t) + \int_0^{\Sigma(t)} [L_z(t,\sigma) + \xi(t,\sigma) f_z(t,\sigma)]\,\mathrm{d}\sigma, \tag{5.17}$$

and the differential inclusion (which is also part of the adjoint system),

$$-\dot{\psi}(t) \in \left\{ \psi(t) g_\Sigma(t) + \int_0^{\Sigma(t)} [L_\Sigma(t,\sigma) + \xi(t,\sigma) f_\Sigma(t,\sigma)]\,\mathrm{d}\sigma + \Gamma(t,\psi) \right\}, \quad \psi(T) = 0, \tag{5.18}$$

where

$$\Gamma(t,\psi) := \limsup_{\alpha \to 0, \alpha \neq 0} \frac{1}{\alpha} \int_{\Sigma(t)}^{\Sigma(t)+\alpha} \left[L(t,\sigma) + \xi(t,\sigma)(f(t,\sigma) - \dot{y}^b(t)) + \zeta[\psi](t) h(t,\sigma) \right] \mathrm{d}\sigma.$$

Define $\bar{H} : D_T \times \mathbb{R}^{n+1+m+r+n+m} \to \mathbb{R}$ as

$$\bar{H}(t,\sigma,\Sigma,y,z,u,\xi,\zeta) := L(t,\sigma,\Sigma,y,z,u) + \xi f(t,\sigma,\Sigma,y,z,u) + \zeta h(t,\sigma,u)$$

Denote by $R(t)$ the reachable set of the differential inclusion, and, if it is non-empty, define the set $\nu(t) := \zeta[R(t)]$. The theorem now states a maximum principle in a general form:

Theorem 5.1. *[12, Theorem 1] Let $\hat{u} \in L_\infty(D_T)$ be an optimal control of problem (OCP), and let $(\hat{\Sigma}, \hat{y}, \hat{z})$ be the corresponding trajectory. Let $\hat{\xi}$ be the solution of the adjoint equation (5.16) (which exists and is unique). Then the reachable set of the differential inclusion 5.18 is non-empty, and for almost every $(t,\sigma) \in D_\Sigma$,*

$$\max_{u \in U} \min_{\zeta \in \nu(t)} \left(\bar{H}(t,\sigma,\hat{\Sigma},\hat{y},\hat{z},u,\hat{\xi},\zeta) - \bar{H}(t,\sigma,\hat{\Sigma},\hat{y},\hat{z},\hat{u},\hat{\xi},\zeta) \right) \leq 0.$$

However, if the optimal control satisfies some additional regularity assumption, then a simplified (more familiar) version of the maximum principle holds.

Assumption 5.3 (Regularity Assumption)**.** *[12, Standing Assumption (vii)] (vii) L, f, and h are continuous with respect to σ, uniformly in the rest of the variables.*

(viii) The optimal control $\hat{u}$ is continuous from the left with respect to σ at $\sigma = \Sigma(t)$ for a.e. $t \in [0,T]$.

Under this assumption, $\Gamma(t,\psi)$ is a singleton, and the differential inclusion collapses to the differential equation

$$\begin{aligned} -\dot{\psi}(t) =& \psi(t)g_\Sigma(t) + L(t,\Sigma(t)) + \lambda(t,\Sigma(t))(f(t,\Sigma(t)) - \dot{y}^b(t)) + \zeta(t)h(t,\Sigma(t)) \\ &+ \int_0^{\Sigma(t)} [L_\Sigma(t,\sigma) + \xi(t,\sigma)f_\Sigma(t,\sigma)]\,\mathrm{d}\sigma, \qquad \psi(T) = 0, \end{aligned} \tag{5.19}$$

and, therefore, also the set $\nu(t)$ is a singleton. Then the "regular" Maximum Principle takes the following, more usual form.

Theorem 5.2. *[12, Theorem 2] Let $\hat{u} \in L_\infty(D_T)$ be an optimal control of the problem (OCP) and $(\hat{\Sigma},\hat{y},\hat{z})$ be the corresponding trajectory. Under Assumption 5.2 and 5.3, the adjoint system has a unique solution $(\hat{\psi},\hat{\xi},\hat{\zeta})$ and for almost every $t \in [0,T]$ and almost every $\sigma \in [0,\hat{\Sigma}(t)]$,*

$$\bar{H}(t,\sigma,\hat{\Sigma},\hat{y},\hat{z},\hat{u},\hat{\xi},\hat{\zeta}) = \max_{u\in U} \bar{H}(t,\sigma,\hat{S},\hat{y},\hat{z},u,\hat{\xi},\hat{\zeta}).$$

5.3 Existence of a Solution

In this section, we prove existence of a solution of problem (5.1)–(5.8). We present the proof in detail, since the standard Lebesgue-Tonelly approach has to be appropriately adapted to the considered maximization problem. The reason is that the problem turns out to be non-concave, as shown in the end of the section.

Proposition 5.3. *The optimal control problem (5.1)–(5.8) has at least one solution.*

Proof. Due to our assumptions (in particular the boundedness of the controls), the supremum of the cost functional is finite, $J^* := \sup\{J[u,v] : (u,v) \in \mathscr{U} \times \mathscr{V}\} < \infty$. Let $\{(u_k,v_k)\}$ be a maximizing sequence for (5.1)–(5.8). The corresponding variables are also indexed by k, e.g. $\Sigma_k := \Sigma[u_k,v_k]$ or $\theta_k := \theta[u_k,v_k]$.

Since all control functions are equibounded, there exist (after extracting subsequences, which we index again by k) functions $\hat{u} \in L_2(D_T)$ and $\hat{v} \in L_2(0,T)$, such that $\{u_k\}$ converges weakly in $L_2(D_T)$ to $\hat{u}$, and $\{v_k\}$ converges weakly in $L_2(0,T)$ to $\hat{v}$. Since $\mathscr{U}$ and $\mathscr{V}$ are weakly closed in $L_2(D_T)$ and $L_2(0,T)$, respectively, we have that $(\hat{u},\hat{v}) \in \mathscr{U} \times \mathscr{V}$.

By Assumption (vi), $\{\Sigma_k\}$ is equibounded on $[0,T]$. From here and from Assumptions (ii) and (iii), it follows that $\{\dot{\Sigma}_k\}$ is equibounded a.e. on $[0,T]$, meaning that $\{\Sigma_k\}$ is equicontinuous. According to the Arzelà–Ascoli theorem, there is $\hat{\Sigma} \in C([0,T])$ such that (after extracting a subsequence) $\{\Sigma_k\}$ converges to $\hat{\Sigma}$ uniformly on $[0,T]$.

Further, because of (5.8) and of Assumptions (iii) and (vi), $\{z_k\}$ is equibounded a.e. on $[0,T]$, so there is $\hat{z} \in L_2(0,T)$ such that (after extracting a subsequence) $\{z_k\}$ converges to $\hat{z}$ weakly in $L_2(0,T)$. Hence, for each $t \in [0,T]$

$$\int_0^t y_k(\tau)\,\mathrm{d}\tau \to \int_0^t \hat{y}(\tau)\,\mathrm{d}\tau \quad \text{as } k \to \infty.$$

Also, from the equiboundedness of all integrands on D_T, uniformly on $[0,T]$, the convergence of $\{\Sigma_k\}$ to $\hat{\Sigma}$, and the weak in $L_2(D_T)$ convergence of $\{u_k\}$ to $\hat{u}$, it follows for $k \to \infty$ that

$$\int_0^t z_k(\tau)\,\mathrm{d}\tau = \int_0^t \int_0^{\Sigma_k(\tau)} d(\tau,\sigma) u_k(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \to \int_0^t \int_0^{\hat{\Sigma}(\tau)} d(\tau,\sigma)\hat{u}(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Combining the two equations, we obtain

$$\int_0^t \hat{z}(\tau)\,\mathrm{d}\tau = \int_0^t \int_0^{\hat{\Sigma}(\tau)} d(\tau,\sigma)\hat{u}(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Differentiation with respect to t results in

$$\hat{z}(t) = \int_0^{\hat{\Sigma}(t)} d(t,\sigma)\hat{u}(t,\sigma)\,\mathrm{d}\sigma \quad \text{for a.e. } t \in [0,T],$$

i.e. $\hat{z} = z[\hat{u},\hat{v}]$.

Further on, from (5.2) we obtain for every $k \in \mathbb{N}$ and for every $t \in [0,T]$

$$\Sigma_k(t) - \Sigma^0 = \int_0^t \dot{\Sigma}_k(\tau)\,\mathrm{d}\tau = \int_0^t \left(g_1(\tau,\Sigma_k(\tau)) \int_0^{\Sigma_k(\tau)} d(\tau,\sigma)u_k(\tau,\sigma)\,\mathrm{d}\sigma + g_2(\tau,\Sigma_k(\tau))v_k(\tau) \right) \mathrm{d}\tau.$$

From the uniform continuity of $g_i(\tau,\cdot)$ on $[\Sigma^0,\bar{\Sigma}]$ (uniformly in $\tau \in [0,T]$), $i = 1,2$, the uniform on $[0,T]$ convergence of $\{\Sigma_k\}$ to $\hat{\Sigma}$, and from the weak convergence of $\{u_k\}$ and $\{v_k\}$, we obtain that for $k \to \infty$

$$\Sigma_k(t) \to \Sigma^0 + \int_0^t \left(g_1(\tau, \hat{\Sigma}(\tau)) \int_0^{\hat{\Sigma}(\tau)} d(\tau,\sigma)\hat{u}(\tau,\sigma)\,\mathrm{d}\sigma + g_2(\tau,\hat{\Sigma}(\tau))\hat{v}(\tau) \right) \mathrm{d}\tau,$$

for every $t \in [0,T]$. Taking into account that $\Sigma_k(t)$ converges to $\hat{\Sigma}(t)$ for all $t \in [0,T]$, and differentiating in t, we see that $\hat{\Sigma}(t)$ satisfies (5.2), i.e. $\hat{\Sigma} = \Sigma[\hat{u},\hat{v}]$.

Note that θ_k converges uniformly on $[0,\bar{\Sigma}]$ to $\hat{\theta} := \theta[\hat{u},\hat{v}]$ and this, together with the continuity of y^b on $[0,T]$, yields that y_k^* (cf. (5.13)) converges uniformly on $[0,\bar{\Sigma}]$ to $\hat{y}^*$. It is clear from (5.14) and (5.15), that the sequence $\{y_k\}$ is equibounded on D_T, hence, there is $\hat{y} \in L_2(D_T)$, which is the weak in $L_2(D_T)$ limit of (a subsequence of) $\{y_k\}$. So, for each $t \in [0,T]$ and each $\sigma \in [0,\bar{\Sigma}]$, we have

$$\int_0^t \int_0^\sigma y_k(\tau,\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\tau \to \int_0^t \int_0^\sigma \hat{x}(\tau,\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\tau \quad \text{as} \quad k \to \infty. \tag{5.20}$$

Taking $(t,\sigma) \in [0,T] \times [0,\hat{\Sigma}(t))$, we first obtain from (5.15) that

$$\int_0^\sigma y_k(t,\tilde{\sigma})\,\mathrm{d}\tilde{\sigma} \to \int_0^\sigma e^{-\delta(t-\hat{\theta}(\tilde{\sigma}))}\hat{y}^*(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma} + \int_0^\sigma \int_{\hat{\theta}(\tilde{\sigma})}^t e^{\delta(s-t)}\hat{u}(s,\tilde{\sigma})\,\mathrm{d}s\,\mathrm{d}\tilde{\sigma}$$

as $k \to \infty$, because of the uniform on $[0,\bar{\Sigma}]$ convergence of $\{\theta_k^*\}$ and $\{y_k^*\}$, and the weak in $L_2(D_T)$ convergence of $\{u_k\}$. Next, the Lebesgue bounded convergence theorem yields

$$\int_0^t \int_0^\sigma y_k(\tau,\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\tau \to \int_0^t \int_0^\sigma e^{-\delta(\tau-\hat{\theta}(\tilde{\sigma}))}\hat{x}^*(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\tau + \int_0^t \int_0^\sigma \int_{\hat{\theta}(\tilde{\sigma})}^\tau e^{\delta(s-\tau)}\hat{u}(s,\tilde{\sigma})\,\mathrm{d}s\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\tau \tag{5.21}$$

as $k \to \infty$. Taking into account (5.20) and (5.21), and differentiating first in t, then in σ, we obtain

$$\hat{y}(t,\sigma) = e^{-\delta(t-\hat{\theta}(\sigma))}\hat{y}^*(\sigma) + \int_{\hat{\theta}(\sigma)}^t e^{\delta(s-t)}\hat{u}(s,\sigma)\,\mathrm{d}s$$

for $(t,\sigma) \in [0,T] \times [0,\hat{\Sigma}(t))$, i.e. $\hat{y} = y[\hat{u},\hat{v}]$.

Next we define the linear mappings $u \to (\bar{y}[u],\bar{z}[u])$ in the following way:

$$\bar{y}[u](t,\sigma) := e^{-\delta(t-\hat{\theta}(\sigma))}\hat{y}^*(\sigma) + \int_{\hat{\theta}(\sigma)}^t e^{\delta(s-t)}u(s,\sigma)\,\mathrm{d}s,$$

$$\bar{z}[u](t) := \int_0^{\hat{Q}(t)} d(t,\sigma)u(t,\sigma)\,\mathrm{d}\sigma.$$

One easily obtains that $y_k(t,\sigma)=\bar{y}[u_k](t,\sigma)+\gamma_{y,k}(t,\sigma)$ and $z_k(t)=\bar{z}[u_k](t)+\gamma_{z,k}(t)$, with $\gamma_{y,k}(t,\sigma)\to 0$ uniformly on D_T and $\gamma_{z,k}(t)\to 0$ uniformly on $[0,T]$ as $k\to\infty$. The functional

$$\bar{J}[u,v]:=\int_0^T\left[\int_0^{\hat{\Sigma}(t)}\Big(L(t,\sigma,\hat{\Sigma}(t),\bar{y}(t,\sigma),\bar{z}(t))-c_1(t,u(t,\sigma))\Big)\,\mathrm{d}\sigma-c_2(t,v(t))\right]\mathrm{d}t$$

is strongly in $L_2(D_T)\times L_2(0,T)$ continuous and concave, hence, it is weakly upper semicontinuous. Also, one easily obtains that $J[u_k,v_k]=\bar{J}[u_k,v_k]+\gamma_{J,k}$ with $\gamma_{J,k}\to 0$ as $k\to\infty$. So, we have

$$J^*=\lim_{k\to\infty}J[u_k,v_k]=\lim_{k\to\infty}\bar{J}[u_k,v_k]\le\bar{J}[\hat{u},\hat{v}]=J[\hat{u},\hat{v}],$$

i.e. $(\hat{u},\hat{v})$ is a solution of the optimal control problem (5.1)–(5.8). □

Remark 5.4. We note that problem (5.1)–(5.8) is *not* concave, so the obtained existence result is not an entirely trivial issue. The non-concavity is shown in the following example in which a particular case of (5.1)–(5.8) is considered. We show that the cost functional $J(\cdot)$ considered over the one-dimensional line segment in $\mathscr{U}$ consisting of all constant functions in $\mathscr{U}$ (the control v is absent) is not concave as a function of one variable.

Example 5.5. Consider the problem

$$\max_u J(u):=\int_0^T\int_0^{\Sigma(t)}\big(y(t,\sigma)-\alpha u^2(t,\sigma)\big)\,\mathrm{d}\sigma\,\mathrm{d}t,$$

subject to the equations

$$\begin{aligned}
\dot{\Sigma}(t)&=\beta z(t) && \text{for a.e. } t\in[0,T], \qquad \Sigma(0)=1,\\
z(t)&=\int_0^{\Sigma(t)}u(t,\sigma)\,\mathrm{d}\sigma && \text{for a.e. } t\in[0,T],\\
\dot{y}(t,\sigma)&=u(t,\sigma) && \text{for a.e. } (t,\sigma)\in D_\Sigma,\\
y(0,\sigma)&=0 && \text{for a.e. } \sigma\in[0,\Sigma^0],\\
y(t,\Sigma(t))&=0 && \text{for } t\in(0,T],\\
u(t,\sigma)&\in[\underline{u},\bar{u}] \text{ for a.e. } (t,\sigma)\in D.
\end{aligned}$$

Here, $\alpha>0$, $\beta>0$, and $\bar{u}>0$ are given. The other parameter are also given and have to satisfy $0<\underline{u}<\bar{u}$, $\underline{u}^{-2}>\alpha\beta$, $T>2/(\beta\underline{u})$, and $\bar{\Sigma}>e^{\beta\bar{u}T}$.

Consider $u(t,\sigma)\equiv u\in[\underline{u},\bar{u}]$. Then $\Sigma[u](t)=e^{\beta ut}$ for $t\in[0,T]$, and $\theta[u](\sigma)=\frac{1}{\beta u}\ln(\sigma)$, for

$\sigma \in [1, e^{\beta u T}]$. Hence,

$$y[u](t,\sigma) = \begin{cases} tu & \text{if} \quad (t,\sigma) \in [0,T] \times [0,1], \\ \left(t - \frac{1}{\beta u}\ln(\sigma)\right) u & \text{if} \quad (t,\sigma) \in [\frac{1}{\beta u}\ln(\sigma), T] \times [1, e^{\beta u T}], \end{cases}$$

and

$$J(u) = -\frac{\alpha u}{\beta} e^{\beta u T} + \frac{1}{\beta^2 u}\left(e^{\beta u T} - 1\right) + \frac{1}{\beta}(\alpha u - T).$$

From here we obtain that

$$J''(u) = \Phi(T,u) e^{\beta u T} + \frac{2}{\beta^2 u^3}\left(e^{\beta u T} - 1\right),$$

where

$$\Phi(T,u) = \left[\left(-2\alpha - \frac{2}{\beta u^2}\right) + \left(-\alpha\beta u + \frac{1}{u}\right) T\right] T.$$

The second term of $J''(u)$ is positive. Take now $u_0 \in (\underline{u}, \overline{u})$ such that $-\alpha\beta u_0 + u_0^{-1} > 0$. For $u \in [\underline{u}, u_0]$, we have $-\alpha\beta u + \frac{1}{u} > -\alpha\beta u_0 + \frac{1}{u_0}$ and $2\alpha + \frac{2}{\beta \underline{u}^2} > 2\alpha + \frac{2}{\beta u^2}$. Since $T > 2/(\beta \underline{u})$, $\Phi(T,\cdot) > 0$ holds true on $[\underline{u}, u_0]$, i.e $J''(\cdot) > 0$ on $[\underline{u}, u_0]$, i.e. $J(\cdot)$ is strongly convex, hence non-concave, on the set of constant controls $u \in [\underline{u}, u_0]$.

5.4 Regularity of the Problem

As it was shown in [12] (cf. Theorem 5.2), Pontryagin's type optimality conditions for an optimal control $\hat{u}$ are valid if this control satisfies the "regularity condition" (Assumption 5.3), that for almost every t, $\hat{u}(t,\cdot)$ is continuous from the left at $\sigma = \Sigma(t)$. In this section, it shall be proved that any optimal control in problem (5.1)–(5.8) satisfies the "regularity assumption". We need the following additional regularity assumption on the data.

Assumption 5.4. *The functions L, L_x and d are continuous with respect to σ, uniformly in the rest of the variables.*

Proposition 5.6. *Let Assumptions 5.1 and 5.4 be fulfilled. Then any optimal control $\hat{u}$ of problem (5.1)–(5.8) is continuous in σ for almost every $t \in [0,T]$.*

Before proving the proposition, we will consider an auxiliary problem and prove some auxiliary results.

Let $(\hat{\Sigma}, \hat{y}, \hat{z}, \hat{u}, \hat{v})$ be a solution of problem (5.1)–(5.8), the existence of which was proved in the previous section. Consider the new optimal control problem with additional control constraints

$$\max_u J(u) = \int_0^T \int_0^{\hat{\Sigma}(t)} [L(t,\sigma,\hat{\Sigma}(t),y(t,\sigma),\hat{z}(t)) - c_1(t,u(t,\sigma))]\,\mathrm{d}\sigma\,\mathrm{d}t \tag{5.22}$$

subject to

$$\dot{y}(t,\sigma) = -\delta y(t,\sigma) + u(t,\sigma) \qquad \text{for a.e. } (t,\sigma) \in D_{\hat{\Sigma}}, \tag{5.23}$$

$$y(0,\sigma) = y^0(\sigma) \qquad \text{for a.e. } \sigma \in [0,\Sigma^0], \tag{5.24}$$

$$y(t,\hat{\Sigma}(t)) = y^b(t) \qquad \text{for } t \in [0,T], \tag{5.25}$$

$$u \in U \qquad \text{for a.e. } \sigma \in [0,\hat{\Sigma}(t)], \tag{5.26}$$

$$\int_0^{\hat{\Sigma}(t)} d(t,\sigma)u(t,\sigma)\,\mathrm{d}\sigma = \hat{z}(t) \qquad \text{for a.e. } t \in [0,T]. \tag{5.27}$$

This is a reduction of the original problem, where Σ, z, and v are fixed at their optimal values. Obviously $\hat{u}$ is an optimal control for the reduced problem.

Consider the following (decoupled) family of ODEs parameterized by $\sigma \in [0,\hat{\Sigma}(T)]$:

$$\dot{\xi}(t,\sigma) = \delta\xi(t,\sigma) - L_x(t,\sigma), \quad \xi(T,\sigma) = 0, \quad t \in [\theta(\sigma),T]. \tag{5.28}$$

Clearly, on $D_{\hat{\Sigma}}$ there exists a unique solution $\hat{\xi}(t,\sigma)$ which is absolutely continuous in t, and measurable and uniformly bounded in σ.

Lemma 5.7. *Let Assumptions 5.1 and 5.4 be fulfilled. Let $\hat{u}$ be an optimal control of problem (5.22)–(5.27), and let $\hat{\xi}$ be the solution of (5.28). Then for almost every $t \in [0,T]$ the function $\hat{u}(t,\cdot)$ maximizes*

$$\int_0^{\hat{Q}(t)} [\hat{\xi}(t,\sigma)u(\sigma) - c_1(t,u(\sigma))]\,\mathrm{d}\sigma \tag{5.29}$$

over the set of $u \in L_\infty(0,\hat{\Sigma}(t))$ satisfying

$$\int_0^{\hat{\Sigma}(t)} d(t,\sigma)u(\sigma)\,\mathrm{d}\sigma = \hat{z}(t), \quad \underline{u} \le u(\sigma) \le \bar{u}, \quad \textit{for a.e.} \quad \sigma \in [0,\hat{\Sigma}(t)]. \tag{5.30}$$

Proof. Let $u(t,\sigma)$ be any measurable function on D_T satisfying the constraints (5.26)–(5.27). Denote by $\hat{y}(t,\sigma)$ and $y(t,\sigma)$ the trajectories corresponding to $\hat{u}$ and u, respectively. Further, define $\Delta u(t,\sigma) := u(t,\sigma) - \hat{u}(t,\sigma)$, $\Delta y(t,\sigma) := y(t,\sigma) - \hat{y}(t,\sigma)$ and $\Delta J := J(u) - J(\hat{u})$. In a first step, we derive a representation for ΔJ; in a second step, inequality (5.29) will be proved.

Step 1: With some $\tilde{y}(t,\sigma) \in \mathrm{co}\{y(t,\sigma), \hat{y}(t,\sigma)\}$,

$$\Delta J = \int_0^T \int_0^{\hat{\Sigma}(t)} [L(t,\sigma,y) - L(t,\sigma,\hat{y}) - c_1(t,u(t,\sigma)) + c_1(t,\hat{u}(t,\sigma))] \,\mathrm{d}\sigma\,\mathrm{d}t,$$
$$= \int_0^T \int_0^{\hat{\Sigma}(t)} [L_y(t,\sigma,\tilde{y}(t,\sigma))\Delta y(t,\sigma) - c_1(t,u(t,\sigma)) + c_1(t,\hat{u}(t,\sigma))] \,\mathrm{d}\sigma\,\mathrm{d}t,$$
$$= \int_0^T \int_0^{\hat{\Sigma}(t)} [L_y(t,q,\hat{y}), \Delta y(t,\sigma) - c_1(t,u(t,q)) + c_1(t,\hat{u}(t,q))] \,\mathrm{d}\sigma\,\mathrm{d}t + e(\Delta u), \tag{5.31}$$

where

$$e(\Delta u) := \int_0^T \int_0^{\hat{\Sigma}(t)} [L_y(t,\sigma,\tilde{y}) - L_y(t,\sigma)]\Delta y(t,\sigma) \,\mathrm{d}\sigma\,\mathrm{d}t.$$

The difference $\Delta y(t,\sigma)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta y(t,\sigma) = -\delta \Delta y(t,\sigma) + \Delta u(t,\sigma),$$

with initial conditions $\Delta y(0,\sigma) = \Delta y(t,\hat{\Sigma}(t)) = 0$. Furthermore, L_y is Lipschitz in y, which implies the existence of some constant $C_e > 0$ such that

$$|e(\Delta u)| \leq C_e \left(\mathrm{meas}\{t \in [0,T] : u(t,\cdot) \neq \hat{u}(t,\cdot)\}\right)^2. \tag{5.32}$$

Using (5.28) for $\hat{\xi}$, the following holds true:

$$\int_0^T \int_0^{\hat{\Sigma}(t)} L_y(t,\sigma,\hat{y})\Delta y(t,\sigma) \,\mathrm{d}\sigma\,\mathrm{d}t = \int_0^T \int_0^{\hat{\Sigma}(t)} [\delta\hat{\xi}(t,\sigma) - \dot{\hat{\xi}}(t,\sigma)]\Delta y(t,\sigma) \,\mathrm{d}\sigma\,\mathrm{d}t$$
$$= \int_0^T \int_0^{\hat{\Sigma}(t)} \delta\hat{\xi}(t,\sigma)\Delta y(t,\sigma) \,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\hat{\Sigma}(t)} \hat{\xi}(t,\sigma)\Delta y(t,\sigma) \,\mathrm{d}\sigma\,\mathrm{d}t$$
$$+ \int_0^T \hat{\xi}(t,\hat{\Sigma}(t))\Delta x(t,\hat{\Sigma}(t))\dot{\hat{\Sigma}}(t) \,\mathrm{d}t + \int_0^T \int_0^{\hat{\Sigma}(t)} \hat{\xi}(t,\sigma)[-\delta\Delta y(t,\sigma) + \Delta u(t,\sigma)] \,\mathrm{d}\sigma\,\mathrm{d}t$$
$$= \int_0^T \int_0^{\hat{\Sigma}(t)} \hat{\xi}(t,\sigma)\Delta u(t,\sigma) \,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^{\hat{\Sigma}(T)} \hat{\xi}(T,\sigma)\Delta y(T,\sigma) \,\mathrm{d}\sigma + \int_0^{\Sigma^0} \hat{\xi}(0,\sigma), \Delta y(0,\sigma) \,\mathrm{d}\sigma$$
$$= \int_0^T \int_0^{\hat{\Sigma}(t)} \hat{\xi}(t,\sigma), \Delta u(t,\sigma) \,\mathrm{d}\sigma\,\mathrm{d}t.$$

Inserting the last expression in (5.31), we obtain

$$\Delta J = \int_0^T \int_0^{\hat{Q}(t)} [\hat{\xi}(t,q)\Delta u(t,q) - c_1(t,u(t,\sigma)) + c_2(t,\hat{u}(t,\sigma))] \,\mathrm{d}\sigma\,\mathrm{d}t + e(\Delta u).$$

Step 2: We now prove the assertion of the Lemma. The optimality of $\hat{u}$ implies $\Delta J \leq 0$, that is, according to the calculations in the first step,

$$\int_0^T \int_0^{\hat{\Sigma}(t)} [\hat{\xi}(t,\sigma)\hat{u}(t,\sigma) - c_1(t,\hat{u}(t,\sigma))] \, \mathrm{d}\sigma \, \mathrm{d}t \geq \int_0^T \int_0^{\hat{\Sigma}(t)} [\hat{\xi}(t,\sigma)u(t,\sigma) - c_2(t,u(t,\sigma))] \, \mathrm{d}\sigma \, \mathrm{d}t + e(\Delta u). \tag{5.33}$$

Assume that the assertion of the Lemma is not true, i.e. there exist an $\varepsilon > 0$ and a subset $A \subset [0,T]$ with $\mathrm{meas}(A) > 0$ such that for every $t \in A$ there exists a $u_t(\cdot) \in \mathscr{U}_t$ such that

$$\int_0^{\hat{\Sigma}(t)} [\hat{\xi}(t,\sigma)u_t(\sigma) - c_1(t,u_t(\sigma))] \, \mathrm{d}\sigma \geq \int_0^{\hat{\Sigma}(t)} [\hat{\xi}(t,\sigma)\hat{u}(t,\sigma) - c_1(t,\hat{u}(t,\sigma))] \, \mathrm{d}\sigma + \varepsilon. \tag{5.34}$$

Next we want to show that we can choose u_t to be measurable in (t,σ). Consider the set valued mapping $\mathscr{G} : A \rightrightarrows L_1([0,\bar{\Sigma}];U)$, defined as

$$\mathscr{G}(t) := \left\{u(\cdot) \in L_1([0,\bar{\Sigma}];U) : (G_1(t,u(\cdot)), G_2(t,u(\cdot))) \in [0,\infty) \times \{0\}\right\}, \tag{5.35}$$

where

$$G_1(t,u(\cdot)) = \int_0^{\hat{\Sigma}(t)} [\hat{\xi}(t,\sigma)u(\sigma) - c_1(t,u(\sigma)) - \hat{\xi}(t,\sigma)\hat{u}(t,\sigma) + c_1(t,\hat{u}(t,\sigma))] \, \mathrm{d}\sigma - \varepsilon,$$

$$G_2(t,u(\cdot)) = \int_0^{\hat{\Sigma}(t)} d(t,\sigma)u(\sigma) \, \mathrm{d}\sigma - \hat{z}(t).$$

The so defined function fulfills the assumptions of Theorem 8.2.9 (p. 315 of [9]), because the space $L_1([0,\bar{\Sigma}];U)$ is a complete separable metric space, $(G_1(t,u(\cdot)), G_2(t,u(\cdot)))$ is Caratheodory, $\mathscr{G}(t) \neq \emptyset$ for all $t \in A$. Therefore, a measurable selection $w(t) \in \Gamma(t)$ exists. Since w is a measurable function from A to $L_1([0,\bar{Q}];U)$, there exists an equivalent function $u(t,\sigma)$ measurable in (t,σ) (Lusin's theorem and Lemma 2.1 on page 25 in [55]).

Now choose m big enough such that $\varepsilon m > C_e$ and $m\,\mathrm{meas}(A) > 1$, then choose a subset A_m of A with $\mathrm{meas}(A_m) \leq 1/m$ and define

$$u_m(t,\sigma) = \begin{cases} u(t,\sigma) & \text{if} \quad t \in A_m \\ \hat{u}(t,\sigma) & \text{if} \quad t \in [0,T] \backslash A_m \end{cases}$$

The so defined control u_m is admissible because $\hat{u}$ and u are measurable and fulfill the conditions (5.26)–(5.27). It differs only on a set of measure $1/m$ from the optimal control and therefore, using (5.32), $|e(\Delta u_m)| \leq C_e m^{-2}$.

Using (5.34) and the definition of A_m and u_m, it follows that

$$\begin{aligned}
&\int_{A_m}\int_0^{\hat{\Sigma}(t)}[\hat{\xi}(t,\sigma)\hat{u}(t,\sigma)-c_1(t,\hat{u}(t,\sigma))]\,\mathrm{d}\sigma\,\mathrm{d}t+\frac{\varepsilon}{m}\\
&\le \int_{A_m}\int_0^{\hat{\Sigma}(t)}[\hat{\xi}(t,\sigma)u_m(t,\sigma)-c_1(t,u_m(t,\sigma))]\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\le \int_{A_m}\int_0^{\hat{\Sigma}(t)}[\hat{\xi}(t,\sigma)\hat{u}(t,\sigma)-c_1(t,\hat{u}(t,\sigma))]\,\mathrm{d}\sigma\,\mathrm{d}t-e(\Delta u_m)\\
&< \int_{A_m}\int_0^{\hat{\Sigma}(t)}[\hat{\xi}(t,\sigma)\hat{u}(t,\sigma)-c_1(t,\hat{u}(t,\sigma))]\,\mathrm{d}\sigma\,\mathrm{d}t+\frac{\varepsilon}{m},
\end{aligned}$$

where the second inequality comes from (5.33) and the last one holds true because of the choice of m. The obtained contradiction completes the proof. □

In what follows, c_1' denotes differentiation with respect to u; while by $(c_1'(t))^{-1}(\eta)$ we denote $\left(\frac{\partial c_1}{\partial u}(t,\cdot)\right)^{-1}(\eta)$.

Lemma 5.8. *Let Assumptions 5.1 and 5.4 be fulfilled, then the function*

$$U(t,\sigma,\xi,\zeta):=\begin{cases}\underline{u} & \text{if} \quad \xi+\zeta d(t,\sigma)<c_1'(t,\underline{u}),\\ (c_1'(t))^{-1}(\xi+\zeta\,d(t,\sigma)) & \text{if} \quad c_1'(t,\underline{u})\le\xi+\zeta d(t,\sigma)\le c_1'(t,\overline{u}),\\ \overline{u} & \text{if} \quad \xi+\zeta d(t,\sigma)>c_1'(t,\overline{u}),\end{cases} \tag{5.36}$$

is continuous in σ, and Lipschitz in ξ. Denote by $\hat{\xi}$ the solution of (5.28), then there exists a measurable on $[0,T]$ function $\hat{\zeta}(t)$, such that the optimal control $\hat{u}(t,\sigma)$ of problem (5.1)–(5.8) fulfills $\hat{u}(t,\sigma)=U(t,\sigma,\hat{\xi}(t,q),\hat{\zeta}(t))$.

Proof. According to Lemma 5.7, for almost every $t\in[0,T]$, the function $\hat{u}(t,\cdot)$ maximizes (5.29), subject to (5.26). The Theorem on page 218 in Section 4.2 in [1] ensures the existence of $\zeta_0(t)\ge 0$ and $\zeta^*(t)\in\mathbb{R}$ with $\zeta_0^2(t)+\zeta^*(t)^2>0$, such that for a.e. $\sigma\in[0,\hat{\Sigma}(t)]$, $\hat{u}(t,\sigma)$ solves the problem

$$\max_{u\in[\underline{u},\overline{u}]}\left\{\zeta_0(t)[\hat{\xi}(t,\sigma)u-c_1(t,u)]+\zeta^*(t)\left[d(t,\sigma)u-\frac{\hat{z}(t)}{\hat{\Sigma}(t)}\right]\right\}. \tag{5.37}$$

It can be shown that $\zeta_0(t)$ can be taken equal to 1.

Fix $(t,\sigma)\in D_{\hat{\Sigma}}$. For $\zeta_0(t)=1$, problem (5.37) is equivalent to the maximization of $[\zeta^*(t)d(t,\sigma)+\hat{\xi}(t,\sigma)]u-c_1(t,u)$ over $u\in[\underline{u},\overline{u}]$.

Since $c_1(t,\cdot)$ is strongly convex, that is, $c_1''(t,u) \geq \bar{\varepsilon} > 0$ for all u, the function $c_1'(t,\cdot)$ is invertible. From $(c_1'(t))^{-1}(c_1'(t,u)) = u$, it follows that $[(c_1'(t))^{-1}(c_1'(t,u))]' = 1$, and therefore,

$$[(c_1'(t))^{-1}]'(c_1'(t,u)) = \frac{1}{c_1''(t,u)} \leq \frac{1}{\bar{\varepsilon}}. \tag{5.38}$$

The maximizer is thus obtained by the sign of

$$\hat{\xi}(t,\sigma) + \zeta^*(t)d(t,\sigma) - c_1'(t,u)$$

for $u \in [\underline{u}, \overline{u}]$. The optimal control is therefore $(c_1'(t))^{-1}(\hat{\xi}(t,\sigma) + \zeta^*(t)d(t,\sigma))$ projected on $[\underline{u}, \overline{u}]$. That is, the optimal control can be written in feedback form (5.36), $\hat{u}(t,\sigma) = U(t,\sigma,\hat{\xi}(t,\sigma),\zeta^*(t))$. It is clear from (5.36) and (5.38), that $U(t,\sigma,\xi,\zeta)$ is Lipschitz with respect to ξ with constant $(\bar{\varepsilon})^{-1}$. The continuity in σ follows from that of $d(t,\sigma)$.

The Lagrange multiplier ζ^* from above may not be measurable. We will now prove that there exists a measurable multiplier $\hat{\zeta}(\cdot)$, such that $U(t,\sigma,\hat{\xi}(t,\sigma),\hat{\zeta}(t)) = U(t,\sigma,\hat{\xi}(t,\sigma),\zeta^*(t))$ a.e. on $D_{\hat{\Sigma}}$, hence $\hat{u}(t,\sigma) = U(t,\sigma,\hat{\xi}(t,\sigma),\hat{\zeta}(t))$.

Consider the set valued mapping

$$G(t) := \left\{ \zeta \in \mathbb{R} : \int_0^{\hat{\Sigma}(t)} d(t,\sigma)U(t,\sigma,\hat{\xi}(t,\sigma),\zeta)\,\mathrm{d}\sigma - \hat{z}(t) = 0 \right\},\ t \in [0,T]. \tag{5.39}$$

Since $\hat{z}(\cdot)$ is measurable, from Theorem 8.2.9 in [9], it follows that $G(\cdot)$ is measurable, therefore, a measurable selections exists which we denote by $\hat{\zeta}(\cdot)$.

Note that the mapping $\zeta \to U(t,\sigma,\hat{\xi}(t,\sigma),\zeta)$ is monotone increasing. Let $\hat{\zeta}(t) \geq (\leq)\, \zeta^*(t)$ hold for some $t \in [0,T]$. Due to the monotonicity of U, this implies the corresponding inequality $U(t,\sigma,\hat{\xi}(t,\sigma),\hat{\zeta}(t)) \geq (\leq) U(t,\sigma,\hat{\xi}(t,q),\zeta^*(t))$ for $\sigma \in [0,\hat{\Sigma}(t)]$. Since the equality in (5.39) is satisfied with both $\zeta = \hat{\zeta}(t)$ and $\zeta = \zeta^*(t)$, and since $d(t,\sigma) > 0$, we conclude that $U(t,\sigma,\hat{\xi}(t,\sigma),\zeta^*(t)) = U(t,\sigma,\hat{\xi}(t,\sigma),\hat{\zeta}(t))$ for a.e. $\sigma \in [0,\hat{\Sigma}(t)]$. This completes the proof. □

Proof of Proposition 5.6. Let $\hat{\xi}(t,\sigma)$ be the solution of (5.28), and let $U(\cdot,\cdot,\cdot,\cdot)$ and $\hat{\zeta}(\cdot)$ be the functions defined in Lemma 5.8. We know from that Lemma, that $\hat{u}(t,\sigma) = U(t,\sigma,\hat{\xi}(t,\sigma),\hat{\zeta}(t))$.

Let us consider the following boundary value problem on $D_{\hat{\Sigma}}$:

$$\dot{y}(t,\sigma) = -\delta y(t,\sigma) + U(t,\sigma,\xi(t,\sigma),\hat{\zeta}(t)), \qquad y(\hat{\theta}(\sigma),\sigma) = x^0(\sigma) \tag{5.40}$$

$$\dot{\xi}(t,\sigma) = \delta\xi(t,\sigma) - L_y(t,\sigma,y(t,\sigma)), \qquad \xi(T,\sigma) = 0. \tag{5.41}$$

Obviously $(\hat{y},\hat{\xi})$ is a solution of this system. Our goal below will be to prove that it is the only solution and that it depends continuously on σ, hence $\hat{u}$ (via the feedback U) also depends continuously on σ.

Consider the initial value problem (5.40)–(5.41), but instead of the end point condition, take the initial condition $\xi(\hat{\theta}(\sigma),\sigma) = p$, and denote the solution by $\xi(t,\sigma;p)$. A solution of the boundary value problem is a solution of the initial value problem satisfying $\xi(T,\sigma;p) = 0$, if such exists.

Due to Assumptions 5.1 and Lemma 5.8, the right-hand side of the differential system in (5.40) and (5.41) is Lipschitz continuous in (y,ξ). Then for every σ, the initial value problem has a unique solution $(y(t,\sigma;p),\xi(t,\sigma;p))$ on $[\hat{\theta}(\sigma),T]$. Let us fix σ and suppress it, as well as $\hat{\xi}(t)$, in the notations below.

To prove uniqueness of the solution of (5.40)–(5.41), assume that there exist two solutions, (y_1,ξ_1) and (y_2,ξ_2). If $\xi_1(\hat{\theta}) = \xi_2(\hat{\theta})$, then both solutions coincide with $(y(t;p),\xi(t;p))$ for $p = \xi_1(\hat{\theta})$.

Therefore, let us (w.l.o.g.) assume that $\xi_2(\hat{\theta}) - \xi_1(\hat{\theta}) > \varepsilon$ for some $\varepsilon > 0$. Let τ be the maximal number in $[\hat{\theta},T]$, such that $\xi_2(t) - \xi_1(t) \geq \varepsilon$ for all $t \in [\hat{\theta},\tau]$. Using that the function $\xi \to U(t,\xi)$ is non-decreasing (cf. the definition (5.36)), we obtain that for $t \in [\hat{\theta},\tau]$

$$\dot{y}_2(t) - \dot{y}_1(t) = -\delta(y_2(t) - y_1(t)) + U(t,\xi_2(t)) - U(t,\xi_1(t)) \geq -\delta(y_2(t) - y_1(t)).$$

Since $y_1(\hat{\theta}) = y_2(\hat{\theta})$, the above inequality implies $y_2(t) - y_1(t) \geq 0$ for all $t \in [\hat{\theta},\tau]$. Using the last inequality and the fact that the function $y \to L_y(t,y)$ is non-increasing due to the concavity of L, we obtain

$$\begin{aligned}\dot{\xi}_2(t) - \dot{\xi}_1(t) &= \delta(\xi_2(t) - \xi_1(t)) - (L_y(t,y_2(t)) - L_y(t,y_1(t))) \\ &\geq \delta(\xi_2(t) - \xi_1(t)) \geq 0, \qquad t \in [\hat{\theta},\tau],\end{aligned}$$

which implies $\xi_2(\tau) - \xi_1(\tau) > \varepsilon$. This implies that $\tau = T$, and, in particular, $\xi_2(T) - \xi_1(T) \geq \varepsilon$. This contradicts the boundary condition in (5.41), and implies that the solution of (5.40)–(5.41) is unique: $(\hat{y}(\cdot,\sigma),\hat{\xi}(\cdot,\sigma))$.

Next, we shall prove the continuity of $\hat{\xi}$ with respect to σ. Due to the boundedness of $\hat{u}$, there is a compact interval $I \subset \mathbb{R}$ containing all values $\hat{\xi}(\hat{\theta}(\sigma),\sigma)$, $\sigma \in [0,\hat{\Sigma}(T)]$. Let us prove the following property, which claims some continuity of the solutions with respect to σ:

Property (P): for every $\varepsilon > 0$, there exists $\delta > 0$, such that for any σ_1, $\sigma_2 \in [0,\hat{\Sigma}(T)]$ with $|\sigma_1 - \sigma_2| < \delta$, and any p_1, $p_2 \in I$ satisfying $\xi(T,\sigma_1;p_1) = \xi(T,\sigma_2;p_2) = 0$, it holds that $|p_1 - p_2| \le \varepsilon$.

According to the continuous dependence of the solution of an ODE with Lipschitz continuous right-hand side on the initial data and parameters (see e.g. [49, Theorem 2], where the required continuity in t is not necessary), the mapping $[0,\hat{\Sigma}(T)] \times \mathbb{R} \ni (\sigma,p) \to \xi(\cdot,\sigma;p) \in C([\hat{\theta}(\sigma),T])$ is continuous, hence it is uniformly continuous on $[0,\hat{\Sigma}(T)] \times I$.

Assume that property (P) does not hold. Then there exists $\varepsilon > 0$, such that for every $\delta > 0$, there exist σ_1, $\sigma_2 \in [0,\hat{\Sigma}(T)]$ and p_1, $p_2 \in P$ such that $|\sigma_1 - \sigma_2| < \delta$, $\xi(T,\sigma_1;p_1) = \xi(T,\sigma_2;p_2) = 0$, and $p_2 - p_1 > \varepsilon$. Due to the (uniform) continuous dependence we may choose $\delta > 0$ so small that

$$|\xi(T,\sigma_2;p_2) - \xi(T,\sigma_1;p_2)| \le \varepsilon/2.$$

We have

$$\xi(\hat{\theta}(\sigma_1),\sigma_1;p_2) - \xi(\hat{\theta}(\sigma_1),\sigma_1;p_1) = p_2 - p_1 \ge \varepsilon, \quad y(\hat{\theta}(\sigma_1),\sigma_1;p_2) = y(\hat{\theta}(\sigma_1),\sigma_1;p_1).$$

Then we can prove in exactly the same way as a few paragraphs above that $\xi(t,\sigma_1;p_2) - \xi(t,\sigma_1;p_1) \ge \varepsilon$ for all $t \in [\hat{\theta}(\sigma_1),T]$. Hence

$$\xi(T,\sigma_2;p_2) - \xi(T,\sigma_1;p_1) \ge \xi(T,\sigma_1;p_2) - \xi(T,\sigma_1;p_1) - \varepsilon/2 \ge \varepsilon/2.$$

This contradicts the equality $\xi(T,\sigma_1;p_1) = \xi(T,\sigma_2;p_2)$ and proves property (P).

Applying property (P) for $p_i = \hat{\xi}(\hat{\theta}(\sigma_i),\sigma_i)$, $i = 1, 2$, we obtain that for every $\varepsilon > 0$, there exists $\delta > 0$, such that $|\sigma_1 - \sigma_2| < \delta$ implies $|\hat{\xi}(\hat{\theta}(\sigma_1),\sigma_1) - \hat{\xi}(\hat{\theta}(\sigma_2),\sigma_2)| < \varepsilon$, that is, continuity in σ. Then using again the continuous dependence of the solution of ODEs, we conclude that $\sigma \to \hat{\xi}(\cdot,\sigma)$ is continuous. From the equality $\hat{u}(t,\sigma) = U(t,\sigma,\hat{\xi}(t,\sigma),\hat{\zeta}(t))$ and Lemma 5.8, we obtain the desired continuity of $\hat{u}(t,\cdot)$. □

5.5 The Hamiltonian Formulation

In this section we present a Hamiltonian functional for the problem (5.1)–(5.8) and show that it satisfies the conditions stated in Section 2.4:

(i) the reproducibility of the primal and the adjoint system,

(ii) the maximum principle,

(iii) the Hamiltonian for system (for an autonomous problem under additional assumptions) is constant along the optimal trajectory.

In the proofs it will become apparent that the claims (i) and (ii) are also true for the more general problem (OCP) from Section 5.2 under the a priori "regularity assumption" 5.3.

We remind of the representation (5.10)–(5.12), where the definition of $y(t,\sigma)$ is extended to the whole domain $[0,T]\times[0,\bar{\Sigma}]$. Consistent with this redefinition, we also extend the adjoint system to the whole domain by setting $\dot{\xi}(t,\sigma)=0$ on $D\backslash D_{\hat{\Sigma}}$, because $\dot{y}$ is independent of y in that region.

We define the Hamiltonian as a functional depending on the following variables: $t\in[0,T]$, $\Sigma\in[0,\bar{\Sigma}]$, $y\in L_\infty(0,\bar{\Sigma})^n$, $z\in\mathbb{R}^m$, $u\in L_\infty(0,\bar{\Sigma})^{r_1}$, $v\in\mathbb{R}^{r_2}$, $\psi\in\mathbb{R}^1$, $\xi\in L_\infty(0,\bar{\Sigma})^n$, $\zeta\in\mathbb{R}^m$. The definition is as follows:

$$\begin{aligned}\mathscr{H}(t,\Sigma,y,z,u,v,\psi,\xi,\zeta) := &\int_0^{\Sigma} L(t,\sigma,\Sigma,y(\sigma),z,u(\sigma),v)\,\mathrm{d}\sigma + \int_0^{\Sigma}\xi(\sigma)f(t,\sigma,\Sigma,y(\sigma),z)\,\mathrm{d}\sigma\\ &+\int_{\Sigma}^{\bar{\Sigma}}\xi(\sigma)\,\mathrm{d}\sigma\,\dot{y}^b(t)+\psi g(t,\Sigma,z,v)+\zeta\left(\int_0^{\Sigma}d(t,\sigma)u(\sigma)\,\mathrm{d}\sigma-y\right).\end{aligned} \tag{5.42}$$

Using this Hamiltonian functional, the primal and the dual system can be written as

$$\begin{aligned}\dot{\Sigma}(t)&=\mathscr{H}_\psi(t,\Sigma(t),y(t,\cdot),z(t),u(t,\cdot),v(t),\psi(t),\xi(t,\cdot),\zeta(t)), & \Sigma(0)&=\Sigma^0\\ \dot{y}(t,\cdot)&=\mathscr{H}_\xi(t,\Sigma(t),y(t,\cdot),z(t),u(t,\cdot),v(t),\psi(t),\xi(t,\cdot),\zeta(t)), & y(0,\cdot)&=y^0(\cdot),\\ 0&=\mathscr{H}_\zeta(t,\Sigma(t),y(t,\cdot),z(t),u(t,\cdot),v(t),\psi(t),\xi(t,\cdot),\zeta(t)).\end{aligned}$$

$$\begin{aligned}\dot{\psi}(t)&=\mathscr{H}_\Sigma(t,\Sigma(t),y(t,\cdot),z(t),u(t,\cdot),v(t),\psi(t),\xi(t,\cdot),\zeta(t)), & \psi(T)&=0,\\ \dot{\xi}(t,\cdot)&=\mathscr{H}_y(t,\Sigma(t),y(t,\cdot),z(t),u(t,\cdot),v(t),\psi(t),\xi(t,\cdot),\zeta(t)), & \xi(T,\cdot)&=0(\cdot),\\ 0&=\mathscr{H}_z(t,\Sigma(t),y(t,\cdot),z(t),u(t,\cdot),v(t),\psi(t),\xi(t,\cdot),\zeta(t)).\end{aligned}$$

We show this exemplary for $\mathscr{H}_\xi$. First, let us clarify the spaces and notion of differentiation we use. A function $f : A \to C$ (B and C are Banach spaces and A is an open subset of B) is Fréchet-differentiable, if there exists a bounded linear operator $\mathscr{A}_x$, such that for all $\varepsilon \in B$

$$\lim_{\varepsilon \to 0} \frac{\|f(x+\varepsilon) - f(x) - \mathscr{A}_x(\varepsilon)\|}{\|\varepsilon\|} = 0.$$

Then we say that $\mathscr{A}_x = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$. This operator is an element of the dual space, that is in our case, the space of all signed finitely additive measures (the Ba space), which are absolutely continuous on the Lebesgue-Borel measure μ_L (see Riesz representation theorem), $L_\infty[0,\bar{\Sigma}]^* = \{\mu \in Ba : \mu_L(A) = 0 \Rightarrow \mu(A) = 0,\ \forall A \subset [0,\bar{\Sigma}]\}$. The L^1 is included in the dual space in a natural way, any $f \in L^1$ can be interpreted a Radon-Nikodym derivative of a measure with respect to μ_L:

$$\int_0^{\bar{\Sigma}} \varepsilon(\sigma)\,\mathrm{d}\sigma_f(\sigma) = \int_0^{\bar{\Sigma}} \varepsilon(\sigma) f(\sigma)\,\mathrm{d}\sigma_L(\sigma) = \int_0^{\bar{\Sigma}} \varepsilon(\sigma) f(\sigma)\,\mathrm{d}\sigma.$$

The $L_\infty(0,\Sigma)$ itself is a subset of $L_1(0,\Sigma)$, because the set $[0,\Sigma]$ is finite. We now skip all arguments, except ξ. Thus,

$$\mathscr{H}(t,\xi(\cdot)+\varepsilon(\cdot)) - \mathscr{H}(t,\xi(\cdot)) = \int_0^{\Sigma} \varepsilon(\sigma) f(t,\sigma)\,\mathrm{d}\sigma + \int_{\Sigma}^{\bar{\Sigma}} \varepsilon(\sigma)\dot{y}^b(t)\,\mathrm{d}\sigma.$$

Extend the definition of $\hat{f}(t,\sigma) := \dot{y}^b(t)$ for $\sigma > \hat{\Sigma}(t)$, then the above equation can be written as $\int_0^{\bar{\Sigma}} \varepsilon(\sigma)\hat{f}(t,\sigma)\,\mathrm{d}\sigma$, which is a linear operator from $(L_\infty(0,\bar{Q}))^n$ to $\mathbb{R}$. It is bounded because f and $y^b(t)$ are bounded and $\bar{\Sigma} < \infty$, thus $\hat{f} \in L^1$. Therefore, the derivative $\mathscr{H}_\xi$ exists and can be identified with $\hat{f}$, which proves our claim for the general case of (OCP). This also includes our case, as the additional control v does not play a role in the differentiation.

Next, we prove the necessary optimality conditions:

Theorem 5.9. *Let Assumptions 5.1 and 5.4 be fulfilled. Consider problem (5.1)–(5.8), in the reformulation (5.10)–(5.12). Let $(\hat{u},\hat{v})$ be the optimal solution, $(\hat{\Sigma},\hat{y},\hat{z})$ the optimal trajectory and $(\hat{\psi},\hat{\xi},\hat{\zeta})$ the solution of the adjoint system. Then the following maximization condition holds:*

$$\mathscr{H}(t,\hat{\Sigma},\hat{y},\hat{z},\hat{u},\hat{v},\hat{\psi},\hat{\xi},\hat{\zeta}) = \max_{u(\cdot),v} \mathscr{H}(t,\hat{\Sigma},\hat{y},\hat{z},u(\cdot),v,\hat{\psi},\hat{\xi},\hat{\zeta}), \tag{5.43}$$

where the maximization takes place over $u(\cdot) \in L_\infty(0,\Sigma(t))$ and $v \in V$.

Proof. Essentially, this results follows from the more general Theorem 2 in [12], cf. Theorem 5.2 in Section 5.2. The key is to prove the regularity condition for the optimal u, which was done in the previous Section. Another technicality is to include the non-distributed control v.

Therefore, we introduce the artificial control $u_2(t,\sigma) \in [\underline{v}, \overline{v}]$, and an additional integral state $y_2(t) = \int_0^{\Sigma(t)} u_2(t,\sigma)\,\mathrm{d}\sigma$. Obviously, for any $\Sigma(t) > 0$, we can represent every measurable $v(t)$ by $v(t) = y_2(t)/\Sigma(t)$, with $u_2(t,\sigma) \equiv v(t)$, and u_2 satisfies the "regularity assumption", Assumption 5.3. Clearly, $\hat{u}_2(t,\sigma) = \hat{v}(t)$ is optimal for the artificial problem. Thus, Theorem 5.2 can be applied and the maximum principle holds.

The way back is straightforward: For a given $u_2(t,\cdot)$, measurable in (t,σ), $v(t) := y_2(t)/\Sigma(t)$ is measurable. Due to the specific functional forms of L, f, g, and h, it is possible to separate the conditions:

$$\max\nolimits_{u(\cdot)} \int_0^{\hat{\Sigma}(t)} [-c_1(t,u(\sigma)) + (\xi(t,\sigma) + \zeta(t)d(t,\sigma))\,u(\sigma)]\,\mathrm{d}\sigma, \qquad u(\sigma) \in [\underline{u}, \overline{u}]$$

$$\max\nolimits_v \{-c_2(t,v) + \psi(t)g_2(t)\,v\}, \qquad v \in [\underline{v}, \overline{v}].$$

Obviously, the maximization above is equivalent to the pointwise maximization, compare the proof for u_t in Section 5.4, using equation (5.35).

□

For an autonomous problem, under an additional condition stated in the theorem, the Hamiltonian $\mathscr{H}$ (defined as in (5.42)) is constant:

Theorem 5.10. *Consider problem (5.1)–(5.8): Let Assumptions 5.1, 5.4, and the following assumptions hold. L is linear in z; the functions L, c_1, c_2, g_1, g_2, d and $\dot{y}^b$ do not depend on t.*

Let $(\hat{u},\hat{v})$ be a set of optimal controls with the trajectory $(\hat{\Sigma},\hat{y},\hat{z})$ and let $(\hat{\psi},\hat{\xi},\hat{\zeta})$ be the respective solutions to the adjoint system (5.16)–(5.17), (5.19). Then the maximized Hamiltonian

$$\hat{\mathscr{H}}(t) := \max_{u(\cdot),v} \mathscr{H}(\hat{\Sigma}(t), \hat{y}(t,\cdot), \hat{z}(t), u(\cdot), v, \hat{\psi}(t), \hat{\xi}(t), \hat{\zeta}(t)),$$

where the maximization takes place over $u(\cdot) \in \mathscr{U}_t$ and $v \in [\underline{v}, \overline{v}]$, is constant on $[0,T]$.

Remark: The assumptions made are suitable for the specific problem considered in this work. The difficulties arise from the integral state and co-state being possibly not Lipschitz. In our case, the assumed properties on L and the feedback solution of u turn out to be enough to ensure Lipschitz continuity. Other assumptions that ensure the absolute continuity are also sufficient, compare with the assumptions of Theorem 4.7.

Proof. In a first step, we show the uniform boundedness of all variables. Next, we proof Lipschitz continuity of the maximized Hamiltonian with fixed controls. In Step 3, the constancy is proved.

Step 1: First of all we shall prove uniform boundedness of the trajectories and of the adjoint variables.

Denote for brevity $S := (\Sigma, y, z)$ and $\pi := (\psi, \xi, \zeta)$. Due to the boundedness of x^0, $\dot{x}^b$, g_1, g_2, d, $[\underline{u}, \overline{u}]$, and $[\underline{v}, \overline{v}]$, there is a compact set $Z \in \mathbb{R}^5$ such that $(S(t,\sigma), u(t,\sigma), v(t)) \in Z$ for every admissible control (u,v) and corresponding trajectory S. Then the right-hand sides of the adjoint equations (5.16)–(5.17), (5.19), are uniformly bounded, hence there exists a compact set $\Pi \in \mathbb{R}^3$ such that $\pi(t,\sigma) \in \Pi$, for the adjoint variables $\pi = (\psi, \xi, \zeta)$ corresponding to any admissible control and the corresponding trajectories. Then the set $\mathscr{A} := \{(S(t,\cdot), u(t,\cdot), v(t), \pi(t,\cdot)) : t \in [0,T],\ (u(\cdot,\cdot), v(\cdot)) \in \mathscr{U} \times \mathscr{V}\}$ is bounded in $L_\infty(0,\bar{\Sigma}) \times [0,\bar{\Sigma}] \times \mathbb{R}^1 \times L_\infty(0,\bar{\Sigma}) \times \mathbb{R}^1 \times L_\infty(0,\bar{\Sigma}) \times \mathbb{R}^1 \times \mathbb{R}^1$.

Since L is linear in z, we can represent $L(\sigma,\Sigma,y,z) = L_1(\sigma,\Sigma,y)z + L_2(\sigma,\Sigma,y)$, where L_1 and L_2 have the same properties as L in the assumptions. From the corresponding differential equations and the boundedness of the right-hand sides, we have that $\hat{\Sigma}(\cdot)$, $\hat{y}(\cdot,\sigma)$, $\hat{\psi}(\cdot)$, and $\hat{\xi}(\cdot,\sigma)$ are Lipschitz continuous, uniformly in $\sigma \in [0,\bar{\Sigma}]$. Further, we have $L_z = L_1$ and, hence,

$$\hat{\zeta}(t) = \hat{\psi}(t)\, g_1(\hat{\Sigma}(t)) + \int_0^{\hat{\Sigma}(t)} L_1(\sigma, \hat{\Sigma}(t), \hat{y}(t,\sigma))\, \mathrm{d}\sigma. \tag{5.44}$$

The first term in the right-hand side above is Lipschitz continuous in t because $\hat{\psi}$ and $\hat{\Sigma}$ are such, and g_1 is continuously differentiable in Σ, hence Lipschitz on $[0,\bar{\Sigma}]$. The second term is also Lipschitz continuous due to the Lipschitz continuity of $\hat{\Sigma}$, the boundedness of the integrand, the Lipschitz continuity of L_1 in (Σ, y), and the Lipschitz continuity of $\hat{y}(\cdot,\sigma)$ uniformly in σ. Thus $\hat{\zeta}(\cdot)$ is absolutely continuous.

From (5.26), taking into account that now c_1 and d do not depend on t, we obtain that

$$\hat{u}(t,\sigma) = \tilde{U}(\sigma, \hat{\xi}(t,\sigma), \hat{\zeta}(t)),$$

where

$$\tilde{U}(\sigma,\xi,\zeta) := \begin{cases} \underline{u} & \text{if} \quad \xi + \zeta\, d(\sigma) < c_1'(\underline{u}) \\ (c_1')^{-1}(\xi + \zeta\, d(\sigma)) & \text{if} \quad c_1'(\underline{u}) \le \xi + \zeta\, d(\sigma) \le c_1'(\overline{u}) \\ \overline{u} & \text{if} \quad \lambda + \nu\, d(\sigma) > c_1'(\overline{u}). \end{cases}$$

Obviously, $\tilde{U}$ is Lipschitz in both ξ and ζ, uniformly in σ. The uniform Lipschitz continuity of

$\hat{\xi}(\cdot,\sigma)$ implies that $\hat{u}$ is Lipschitz continuous, uniformly in σ. Thus, for $0 \le \tau < t \le T$ we obtain

$$\begin{aligned}
|\hat{z}(t) - \hat{z}(\tau)| &= \left| \int_0^{\hat{\Sigma}(t)} d(\sigma)\hat{u}(t,\sigma)\,\mathrm{d}\sigma - \int_0^{\hat{\Sigma}(\tau)} d(\sigma)\hat{u}(\tau,\sigma)\,\mathrm{d}\sigma \right| \\
&\le \left| \int_0^{\hat{\Sigma}(\tau)} d(\sigma)\left(\hat{u}(t,\sigma) - \hat{u}(\tau,\sigma)\right)\mathrm{d}\sigma + \int_{\hat{\Sigma}(\tau)}^{\hat{\Sigma}(t)} d(\sigma)\hat{u}(t,\sigma)\,\mathrm{d}\sigma \right| \\
&\le \overline{d}\,\bar{Q}\,\|u(t,\cdot) - u(\tau,\cdot)\|_{L_\infty(0,\hat{\Sigma})} + \overline{u}\,\overline{d}|\hat{\Sigma}(t) - \hat{\Sigma}(\tau)|.
\end{aligned}$$

The (uniform) absolute continuity of $\hat{u}(t,\cdot)$ and $\hat{\Sigma}(t)$ yield the absolute continuity of $\hat{z}(t)$.

Step 2: We next show that the function $t \to \hat{\mathscr{H}}(t)$ is absolutely continuous on $[0,T]$, although $\hat{v}$ might not be. First we shall show that for any fixed $\tau \in [0,T]$ the function $t \to \mathscr{H}(\hat{z}(t,\cdot),\hat{u}(\tau,\cdot),\hat{v}(\tau),\hat{\pi}(t,\cdot))$ is Lipschitz continuous, uniformly in τ. Due to the uniform boundedness and the Lipschitz continuity in t of the arguments of this function, it is enough to prove that it is Lipschitz continuous on $\mathscr{A}$. Although this is a routine task, we show this property with respect to $y(\cdot)$:

$$\begin{aligned}
&\left| \mathscr{H}(\Sigma, y_1(\cdot), z, u(\cdot), v, \psi, \xi(\cdot), \zeta) - \mathscr{H}(\Sigma, y_2(\cdot), z, u(\cdot), v, \psi, \xi(\cdot), \zeta) \right| \\
=&\left| \int_0^{\Sigma} \Big[L(\sigma,\Sigma,y_1(\sigma),z) - L(\sigma,\Sigma,y_2(\sigma),z) \Big]\,\mathrm{d}\sigma - \delta \int_0^{\Sigma} \xi(\sigma)\Big(y_1(\sigma) - x_2(\sigma)\Big)\,\mathrm{d}\sigma \right| \\
\le& \int_0^{\Sigma} C_L |y_1(\sigma) - y_2(\sigma)|\,\mathrm{d}\sigma + \delta \int_0^{\Sigma} |\lambda(\sigma)|\,|y_1(\sigma) - y_2(\sigma)|\,\mathrm{d}\sigma \\
\le& (C_L + \delta|\Pi|)\,\bar{\Sigma}\,\|y_1(\cdot) - y_2(\cdot)\|_{L_\infty(0,\bar{\Sigma})},
\end{aligned}$$

where C_L is the Lipschitz constant of L on Z and $|\Pi| := \sup_{\pi\in\Pi} |\pi|$.

In the following, we write $\hat{u}(t,\cdot)$ for $(\hat{u}(t,\cdot),\hat{v}(t))$, as we need not distinguish between the two controls. Due to the Lipschitz continuity of all state and adjoint variables with respect to t, there exists a constant K such that

$$|\mathscr{H}(\hat{S}(t,\cdot),u(\cdot),\hat{\pi}(t,\cdot)) - \mathscr{H}(\hat{S}(\tau,\cdot),u(\cdot),\hat{\pi}(\tau,\cdot))| \le K|t-\tau|,$$

for every $t,\tau \in [0,T]$ and every $u(\cdot) = (u(\cdot),v) \in \mathscr{U} \times \mathscr{V}$. From here, using also (5.43), we obtain that

$$\begin{aligned}
-K|t-\tau| &\le \mathscr{H}(\hat{S}(t,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(t,\cdot)) - \mathscr{H}(\hat{S}(\tau,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(\tau,\cdot)) \le \\
&\le \mathscr{H}(\hat{S}(t,\cdot),\hat{u}(t,\cdot),\hat{\pi}(t,\cdot)) - \mathscr{H}(\hat{S}(\tau,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(\tau,\cdot)) \le \\
&\le \mathscr{H}(\hat{S}(t,\cdot),\hat{u}(t,\cdot),\hat{\pi}(t,\cdot)) - \mathscr{H}(\hat{S}(\tau,\cdot),\hat{u}(t,\cdot),\hat{\pi}(\tau,\cdot)) \le K|t-\tau|,
\end{aligned}$$

hence, $|\mathscr{H}(t)-\mathscr{H}(\tau)| \leq K|t-\tau|$, which yields the Lipschitz continuity of $t \to \mathscr{H}(t)$.

Step 3: Now we shall prove that $\frac{d}{dt}\mathscr{H}(t)=0$ almost everywhere on $[0,T]$. We remind that both $t \to \hat{y}(t,\cdot) \in L_\infty(0,\bar{\Sigma})$ and $t \to \hat{\xi}(t,\cdot) \in L_\infty(0,\bar{\Sigma})$ are Lipschitz continuous, as well as $\hat{\Sigma}$ and $\hat{\psi}$.

Fix τ in the subset (of full measure) of $[0,T]$ on which the maximization condition (5.43), the state equations, and the adjoint equations (in their Hamiltonian form) are satisfied. For almost every $t \in [0,T]$, it holds that

$$\mathscr{H}(t)-\mathscr{H}(\tau) \geq \mathscr{H}(\hat{S}(t,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(t,\cdot)) - \mathscr{H}(\hat{S}(\tau,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(\tau,\cdot)).$$

Assume that t converges to τ from above, divide by $(t-\tau)>0$ and take the limit $t \to \tau$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t)\Big|_{t=\tau} &\geq \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(\hat{S}(t,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(t,\cdot))\Big|_{t=\tau}\\ &= \Big(\langle \mathscr{H}_\Sigma,\dot{\hat{\Sigma}}\rangle + \langle \mathscr{H}_y,\dot{\hat{y}}\rangle + \mathscr{H}_z,\dot{\hat{z}}\rangle + \langle \mathscr{H}_\psi,\dot{\hat{\psi}}\rangle + \langle \mathscr{H}_\xi,\dot{\hat{\xi}}\rangle + \langle \mathscr{H}_\zeta,\dot{\hat{\zeta}}\rangle\Big)_{t=\tau}\\ &= \langle -\dot{\hat{\psi}},\dot{\hat{\Sigma}}\rangle + \langle -\dot{\hat{\xi}},\dot{\hat{y}}\rangle + 0 + \langle \dot{\hat{\Sigma}},\dot{\hat{\psi}}\rangle + \langle \dot{\hat{y}},\dot{\hat{\xi}}\rangle + 0 = 0,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product on the appropriate spaces, and the point of evaluation is omitted for clarity. The two zeros in the left-hand side of the last line result from the facts that the Hamiltonian is linear with respect to y and ν, $\mathscr{H}_y(\hat{S}(t,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(t,\cdot))|_{t=\tau}=0$ (this is the ajoint equation (5.17)) and $\mathscr{H}_\nu(\hat{S}(t,\cdot),\hat{u}(\tau,\cdot),\hat{\pi}(t,\cdot))|_{t=\tau}=0$ (this is the state equation (5.6)).

Taking t to converge to τ from below, results in the opposite inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t)\Big|_{t=\tau} \leq 0.$$

Since both inequalities hold for a.e. $\tau \in [0,T]$, the time derivative of $\mathscr{H}(t)$ is zero almost everywhere, thus, $\mathscr{H}(t)$ is constant.

Note that the differentiation above is for a fixed τ and $u(\tau)$, in general the Hamiltonian is not differentiable, because v may be non-differentiable. □

Chapter 6

Discussion

This thesis summarizes three years of research in the field of optimal control of heterogeneous systems, which was undertaken in close cooperation with Christa Simon, Tsvetomir Tsachev, and Vladimir Veliov. Apart from the common topic of heterogeneity, the methodological foci include infinite-horizon optimal control and Hamiltonian formulations. Therefore, we want to discuss some similarities and differences between the models, and list some potential future work.

Our results show, that the method of defining the adjoint variables explicitly, recently developed for infinite-horizon ODE optimal control problems (see [5, 6, 7]), can be extended to heterogeneous systems, although new challenges arise due to the increasing complexity of such systems arising from non-local state and the boundary condition. Given certain growth assumptions, we prove in Section 3.2 that a maximum principle holds for an age-structured system with dynamics that are affine in the states. Extending this results to systems with non-affine dynamics requires some technicalities which are theoretically not too difficult to overcome.

We show in Section 4 that also for trait-structured systems the above mentioned approach of defining adjoint variables is also feasible for trait-structured systems. The provided proof requires that the aggregate states do not depend explicitly on the control variables. This assumption can be relaxed, however, the maximum principle may then hold not in the global form.

An important future task is to extend this approach to other systems with dynamics described by first order PDEs with non-local dynamics, such as spatial models with diffusion terms and age-structured dynamics at every place.

The second methodological focus deals with Hamiltonian formulations. Such are presented for heterogeneous systems with horizontal characteristics, both on a fixed and on a controlled domain. Under some linearity assumptions in Chapter 5 or assumption of absolute continuity of the integral state in Chapter 4, the constancy of this Hamiltonian functional are proved. In case of age-structured system, a Hamiltonian formulation is still missing. The challenge is to find a formulation which is consistent with both the boundary condition and the potential non-differentiability of $t \mapsto \int_0^{\omega} \xi(t,a) y(t,a) \, \mathrm{d}a$.

Besides the two key aspects mentioned above, the thesis deals also with the existence of optimal solutions. In the two applications considered (Section 3.3 and Chapter 5), we were able to provide a proof of existence. Extending this to a larger class of systems, however, is difficult not only due to the potential non-concavity of the maximization problems.

Another interesting aspect of optimal control deals with sufficient optimality conditions, but literature on it is scarce for heterogeneous systems. Arrow-type sufficient conditions for age-structured systems are proved in [36], but neither necessary nor sufficient Legendre-Clebsch conditions (which would be useful for disturbance and approximation analysis) are available.

We also showed that in many economic applications considered in the literature, the assumptions imposed in this thesis are fulfilled. It would be interesting to apply our results for studies of more complex systems arising, for example, in economics or epidemiology. Proving that the assumptions are fulfilled can be difficult because it involves knowledge about the optimal control and corresponding state trajectories.

List of Figures

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