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Abstract. This paper is about locating ω-limit sets for solutions of differential inclusions with not necessarily continuous right side. Based on the LaSalle principle we assume that as time $t \to \infty$ the set of solutions approach a closed subset $\mathcal{S}$ of $\mathbb{R}^n$ and then consider the dynamics restricted on $\mathcal{S}$ to find a location of the ω-limit set by utilizing nonsmooth Lyapunov type functions over a neighborhood of $\mathcal{S}$; then we prove that this location is also valid for the original dynamics. Our motivation stems from problems of stabilization with discontinuous feedback. We apply our result for nonsmooth differential equations and compare it with some recent results.

Keywords. Differential inclusion, LaSalle principle, Lyapunov function, ω-limit sets, stabilization, discontinuous feedback, nonsmooth dynamics.

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1 Introduction

In this paper we utilize the LaSalle principle together with nonsmooth Lyapunov function techniques to determine the location of the ω-limit set of solutions of a differential inclusion under minimal hypotheses. Our motivation comes from problems of stabilization with discontinuous feedback. We apply the main result obtained to nonsmooth differential equations and give some comparison with previous works.

To put the stage, let us first fix some notation. Throughout $F : \mathbb{R} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ denotes a mapping whose values are nonempty subsets of $\mathbb{R}^n$ which in particular may be single points and then F is a function defined over $\mathbb{R} \times \mathbb{R}^m$. We denote by $\mathbb{B}_r(a)$ the closed ball in $\mathbb{R}^n$ with center a and radius r and set $\mathbb{B} = \mathbb{B}_1(0)$. The distance from a point $x \in \mathbb{R}^n$ to a set $S \subset \mathbb{R}^n$ is denoted by $\mathrm{d}_S(x)$; we also use the notation $\|S\| = \sup_{u \in S} \|u\|$. Given a function $x : \mathbb{R} \to \mathbb{R}^n$, $\dot{x}(t)$ denotes the usual derivative of x at t. Recall that the upper Dini derivative of a Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$ in the direction l is defined as

$$D^+V(x; l) := \limsup_{h \searrow 0} \frac{V(x + hl) - V(x)}{h}.$$

In this paper we consider the following nonautonomous differential inclusion:

$$\dot{x}(t) \in F(t, x(t)). \tag{1}$$

Throughout we consider Carathéodory solutions of (1). Recall that a function $\varphi : I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$, is a Carathéodory solution of (1) on I if φ is absolutely continuous and satisfies (1) for almost every $t \in I$.

Standing assumption. *For every $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exist positive reals r and M such that*

$$\|F(t, x)\| \leq M \quad \text{for every } x \in \mathbb{B}_r(x_0) \text{ and every } t \geq t_0.$$

We define the ω-limit set associated with (1) supplied with the initial condition

$$x(t_0) = x_0, \tag{2}$$

as the collection of points $y \in \mathbb{R}^n$ for each of which there exists a Carathéodory solution $\varphi(\cdot, t_0, x_0)$ of (1),(2) defined on $[t_0, +\infty)$ and bounded on this interval, and a sequence $t_k \to \infty$ such that $\varphi(t_k, t_0, x_0) \to y$ as $k \to \infty$. Throughout the ω-limit set is denoted by $\Omega^+(t_0, x_0)$. Observe that the set $\Omega^+(t_0, x_0)$ may be empty in the case when the initial value problem (1),(2) has no solution on $[t_0, +\infty)$, or every solution of this problem is unbounded. Clearly, each of these cases can be avoided by imposing additional assumptions. For instance, it is known that under appropriate growth conditions there exist solutions on $[t_0, +\infty)$ provided that F is either upper semi-continuous with compact convex values or lower semi-continuous. Global existence of solutions holds for wider classes of mappings, some of which play an important role in control, for more on this topic see [9].

Our main result given next provides a localization of the ω-limit set $\Omega^+(t_0, x_0)$ for the initial value problem (1),(2):

Theorem 1.1. *Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^n$, $\mathcal{U}$ be a relatively open subset of $\mathcal{S}$ (relative with respect to $\mathcal{S}$), V be a locally Lipschitz real-valued function defined on an open set G containing $\mathcal{S}$, and W be a real-valued lower semi-continuous function defined on the set $Z := (G \setminus \mathcal{S}) \cup \mathcal{U}$. Given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, consider the initial value problem* (1),(2) *and suppose that the following conditions hold:*

(B1) *For every $\varepsilon > 0$ and for each bounded solution $\varphi(\cdot, t_0, x_0)$ of* (1),(2) *there exists $T > 0$ such that $\mathrm{d}_{\mathcal{S}}(\varphi(t, t_0, x_0)) < \varepsilon$ for every $t \geq T$;*

(B2) *$W(x) > 0$ for every $x \in \mathcal{U}$;*

(B3) *The following inequality holds:*

$$\sup_{v \in F(t,x)} D^+V(x; v) \leq -W(x) \quad \textit{for every} \;\; x \in Z;$$

(B4) *Every open interval contained in $V(\mathcal{S} \setminus \mathcal{U})$ has empty intersection with $V(\mathcal{U})$.*

Then the set $\Omega^+(t_0, x_0)$ is contained in $\mathcal{S} \setminus \mathcal{U}$.

Theorem 1.1 combines elements of two basic paradigms in system dynamics, the LaSalle principle and Lyapunov functions. Indeed, note that assumption (B1) is actually obtained when we apply the LaSalle invariance principle. For a smooth differential equation statements of this principle are available in textbooks, see, e.g., [7, Theorem 3.4], but there are various more ingenious forms of the LaSalle principle for other models; e.g., a LaSalle principle for differential inclusions was obtained in [4, Theorem 3]. Further, the assumptions (B2) and (B3) clearly resemble conditions that appear in the context of Lyapunov stability; specifically, V can be viewed as a nonsmooth Lyapunov function. Condition (B4) is a somewhat technical assumption which glues everything together: we will see this in the proof of Theorem 1.1. It also allows us to obtain a conclusion concerning the original dynamics outside the set $\mathcal{S}$ based on its properties on $\mathcal{S}$. Note that if the set $V(\mathcal{U}) \setminus V(\mathcal{S} \setminus \mathcal{U})$ is a dense subset of $V(\mathcal{U})$, then the assumption (B4) is satisfied. Indeed, if $(a, b) \subset V(\mathcal{S} \setminus \mathcal{U})$ and $c \in (a, b) \cap V(\mathcal{U})$, then $c \in V(\mathcal{U})$ neither belongs to $V(\mathcal{U}) \setminus V(\mathcal{S} \setminus \mathcal{U})$ nor is a limit point of this set.

Obtaining Theorem 1.1 was inspired by the paper [1] by Arsie and Ebenbauer, where a Lipschitz vector field over a Riemannian manifold is considered. Let us outline the idea of this theorem for the simplest case when the dynamics is described by the ordinary differential equation $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Suppose that there exists a compact subset Σ of $\mathbb{R}^n$ that contains all trajectories of the ODE starting from an arbitrary point of Σ. By the standard LaSalle invariance principle we obtain the existence of a compact subset $\mathcal{S}$ of Σ attracting all solutions (condition (B1)). Then the dynamics is considered only on the set $\mathcal{S}$ by way of finding, e.g., a smooth Lyapunov function V satisfying the condition (B2), (B3) (with $W(x) := V'(x)f(x)$) on $\mathcal{S}$, which allows us to localize the ω-limit set of the system in $\mathcal{S}$. But the continuity of W yields that (B2) and (B3) also hold in an open neighborhood of $\mathcal{S}$; then, by utilizing (B4) we obtain that this location is also valid for the original dynamics.

In the next Section 2 we will present a proof of Theorem 1.1. In Section 3 we obtain some corollaries in special cases. In particular, we specify our main result to the case when the mapping F in (1) is single-valued. An example from the paper [3] which concerns so-called non-pathological Lyapunov functions is used to illustrate our assumptions.

The motivation behind Theorem 1.1 comes from stabilization with discontinuous feedback. Indeed, the inclusion (1) may describe a feedback control system $\dot{x} = f(x, k(x))$ where the feedback function k could be quite arbitrary, e.g. discontinuous, as long as the standing assumption is satisfied. There is a wealth of literature on this topic; here we only mention the basic papers of Artstein [2] and Brockett [5], and the more recent works by Sontag [8] and Clarke [6].

2 Proof of Theorem 1.1

Due to the closedness of $\mathcal{S}$ and the relative openness of $\mathcal{U}$, the set $Z = G \setminus (\mathcal{S} \setminus \mathcal{U})$ is open (as a subset of $\mathbb{R}^n$). By assumption (B2) we have $W(x) > 0$ for each $x \in \mathcal{U}$, then, by the lower semi-continuity of W, there exists an open neighborhood $\tilde{\mathcal{U}}_x$ of x contained in Z such that $W(y) > 0$ for every $y \in \tilde{\mathcal{U}}_x$. Let

$$\tilde{\mathcal{U}} := \bigcup_{x \in \mathcal{U}} \tilde{\mathcal{U}}_x.$$

Then $\tilde{\mathcal{U}} \subset Z$ is an open set containing $\mathcal{U}$ and such that $W(x) > 0$ for each $x \in \tilde{\mathcal{U}}$.

Let us assume that the claim of the theorem is false. Then, since $\Omega^+(t_0, x_0) \subset S$ by (B1), there exists a point $\bar{x} \in \Omega^+(t_0, x_0) \cap \mathcal{U}$. The definition of the set $\Omega^+(t_0, x_0)$ yields that there exists a bounded solution $\varphi(\cdot, t_0, x_0)$ of (1) and a sequence $\{t_k\}_{k=1}^{\infty}$ tending to $+\infty$ with $k \to +\infty$ such that

$$\lim_{k \to +\infty} \varphi(t_k, t_0, x_0) = \bar{x}. \tag{3}$$

Since the pair (t_0, x_0) is fixed, in the sequel we use the simplified notation $\varphi(t) := \varphi(t, t_0, x_0)$.

According to our standing assumption regarding the mapping F in (1), there exist positive r and $M \geq 1$ such that

$$\|F(t, x)\| \leq M \qquad \text{for every } x \in \mathbb{B}_r(\bar{x}) \text{ and every } t \geq t_0. \tag{4}$$

The boundedness of the solution $\varphi(t)$, $t \geq t_0$, implies the existence of a compact set $K \subset \mathbb{R}^n$ such that

$$\varphi(t) \in K \text{ for every } t \geq t_0.$$

Since $\bar{x} \in \mathcal{U}$, we have $W(\bar{x}) > 0$. Then the lower semi-continuity of W yields the existence of ϱ and $\mu > 0$ such that $W(x) \geq \mu$ for each $x \in \mathbb{B}_\varrho(\bar{x}) \cap Z$. Without loss of generality we assume $\mu \leq 1$ and take ϱ smaller if necessary so that $\mathbb{B}_\varrho(\bar{x}) \subset \tilde{\mathcal{U}}$ and $\varrho \in (0, r)$. Set

$$\varepsilon = \frac{\varrho\mu}{5M} \text{ and } \tau = \frac{\varrho}{2M}. \tag{5}$$

Next, we shall prove that there exist $c \in (V(\bar{x}) - \varepsilon, V(\bar{x}) + \varepsilon)$ and $\delta > 0$ such that

$$\{x \in (\mathcal{S} + \delta\mathbb{B}) \cap K \mid V(x) = c\} \subset \tilde{\mathcal{U}}. \tag{6}$$

Assume that the above claim is false. Then for each number $c \in (V(\bar{x}) - \varepsilon, V(\bar{x}) + \varepsilon)$ and for every $\delta = 1/m$, $m = 1, 2, \ldots$, there exists $x_m \in \left(\mathcal{S} + \frac{1}{m}\mathbb{B}\right) \cap K$ such that $V(x_m) =$

c, and $x_m \notin \tilde{U}$. Since K is compact, the sequence $\{x_m\}$ has an accumulation point $\tilde{x}$. Obviously $\tilde{x} \in \mathcal{S} \cap K$, $V(\tilde{x}) = c$, and $\tilde{x} \notin \tilde{U}$. Since the last relation implies that $\tilde{x} \notin \mathcal{U}$, we obtain that $c = V(\tilde{x}) \in V(\mathcal{S} \setminus \mathcal{U})$. This holds for every $c \in (V(\bar{x}) - \varepsilon, V(\bar{x}) + \varepsilon)$, thus $(V(\bar{x}) - \varepsilon, V(\bar{x}) + \varepsilon) \subset V(\mathcal{S} \setminus \mathcal{U})$. Observing that $V(\bar{x}) \in V(\mathcal{U})$, we obtain a contradiction with condition (B4). Thus inclusion (6) is fulfilled for some $c \in (V(\bar{x}) - \varepsilon, V(\bar{x}) + \varepsilon)$ and $\delta > 0$.

Without loss of generality we assume that $\delta \in (0, \min(\varepsilon, \varrho/2))$. According to condition (B1), there exists $T > 0$ such that $\varphi(t) \in \mathcal{S} + \delta I\!\!B$ for every $t \geq T$. Due to (3) and the continuity of V, there exists an index $\bar{k}$ such that $t_{\bar{k}} > T$, $\varphi(t_{\bar{k}}) \in \bar{x} + \delta I\!\!B$ and $|V(\varphi(t_{\bar{k}})) - V(\bar{x})| < \varepsilon$. We will now show that

$$\|\varphi(t) - \bar{x}\| \leq r \quad \text{for} \quad t \in [t_{\bar{k}}, t_{\bar{k}} + \tau]. \tag{7}$$

On the contrary, since $\|\varphi(t_{\bar{k}}) - \bar{x}\| \leq \delta < r$, there exists a minimal $t \in (t_{\bar{k}}, t_{\bar{k}} + \tau]$ such that $\|\varphi(t) - \bar{x}\| = r$. Then

$$\begin{aligned} \|\varphi(t) - \bar{x}\| &= \left\| \varphi(t_{\bar{k}}) + \int_{t_{\bar{k}}}^{t} \frac{d}{ds} \varphi(s)\, \mathrm{d}s - \bar{x} \right\| \\ &\leq \|\varphi(t_{\bar{k}}) - \bar{x}\| + \int_{t_{\bar{k}}}^{t} \|F(s, \varphi(s)\|\, \mathrm{d}s \\ &\leq \delta + (t - t_{\bar{k}}) M \leq \delta + \tau M \leq \delta + \frac{\varrho}{2}, \end{aligned} \tag{8}$$

where we use the inequality $\|\varphi(s) - \bar{x}\| \leq r$ for $s \in [t_{\bar{k}}, t_{\bar{k}} + t]$, the inequality (4), and the choice of τ in (5). Since $\delta < \varrho/2$ and $\varrho < r$, the right-hand side in (8) is strictly smaller than r, while the left-hand side equals r. The obtained contradiction proves (7).

We obtain that (4) holds for every $t \in [t_{\bar{k}}, t_{\bar{k}} + \tau]$, hence (8) holds for every such t. Now we can estimate the right-hand side of (8) by $\varepsilon + \varrho/2 \leq \varrho/5 + \varrho/2 < \varrho$, where we use that $M \geq 1$ and $\mu \leq 1$. Thus $\varphi(t) \in \bar{x} + \varrho I\!\!B$ for every $t \in [t_{\bar{k}}, t_{\bar{k}} + \tau]$. Using this we obtain for every fixed $t \in (t_{\bar{k}}, t_{\bar{k}} + \tau]$ the following chain of inequalities:

$$\begin{aligned} V(\varphi(t)) &= V(\varphi(t_{\bar{k}})) + \int_{t_{\bar{k}}}^{t} \frac{d}{ds} V(\varphi(s))\, \mathrm{d}s \\ &= V(\varphi(t_{\bar{k}})) + \int_{t_{\bar{k}}}^{t} \lim_{h \to 0} \frac{V(\varphi(s+h)) - V(\varphi(s))}{h}\, \mathrm{d}s \\ &= V(\varphi(t_{\bar{k}})) + \int_{t_{\bar{k}}}^{t} \lim_{h \to 0} \frac{V(\varphi(s) + h\dot{\varphi}(s) + o(s;h)) - V(\varphi(s))}{h}\, \mathrm{d}s \\ &\leq V(\varphi(t_{\bar{k}})) + \int_{t_{\bar{k}}}^{t} D^+ V(\varphi(s); \dot{\varphi}(s))\, \mathrm{d}s \\ &\leq V(\varphi(t_{\bar{k}})) - \int_{t_{\bar{k}}}^{t} W(\varphi(s))\, \mathrm{d}s \leq V(\varphi(t_{\bar{k}})) - (t - t_{\bar{k}})\mu \\ &< V(\bar{x}) + \varepsilon - (t - t_{\bar{k}})\mu. \end{aligned}$$

(On the third line of this chain of relations the function $o(s;h)$ satisfies $o(s;h)/h \to 0$ for a.e. s, i.e. on the set of Lebesque points of $\dot{\varphi}$.) Hence, for every $t \in [t_{\bar{k}}, t_{\bar{k}} + \tau]$ we get

$$V(\varphi(t)) < V(\bar{x}) + \varepsilon - (t - t_{\bar{k}})\mu. \tag{9}$$

Applying (9) for $t = t_{\bar{k}} + \tau$ and using the definitions of ε and τ we have

$$\begin{aligned} V(\varphi(t_{\bar{k}} + \tau)) &\leq V(\bar{x}) + \varepsilon - \tau\mu < c + 2\varepsilon - \tau\mu \\ &= c + 2\frac{\varrho\mu}{5M} - \frac{\varrho}{2M}\mu < c. \end{aligned}$$

In further lines we prove that the inequality $V(\varphi(t)) < c$ holds for every $t > t_{\bar{k}} + \tau$. Indeed, let us assume that there exists $\bar{t} > t_{\bar{k}} + \tau$ such that

$$V(\varphi(t)) < c \quad \text{for each} \quad t \in [t_{\bar{k}} + \tau, \bar{t}) \quad \text{and} \quad V(\varphi(\bar{t})) = c. \tag{10}$$

Keeping in mind (6), the equality in (10) combined with $\bar{t} > t_{\bar{k}} + \tau > T$ implies that

$$\varphi(\bar{t}) \in \{x \in (\mathcal{S} + \delta I\!B) \cap K \mid V(x) = c\} \subset \tilde{\mathcal{U}}.$$

Thus $W(\varphi(\bar{t})) > 0$. The continuity of $\varphi(\bar{t})$ yields the existence of $\bar{t}_0 \in (t_{\bar{k}} + \tau, \bar{t})$ such that $\varphi(t) \in \tilde{\mathcal{U}}$ for each $t \in (\bar{t}_0, \bar{t})$, and hence $W(\varphi(t)) > 0$ for each $t \in (\bar{t}_0, \bar{t})$. In parallel to the derivation of (9), one can obtain that

$$\begin{aligned} V(\varphi(\bar{t})) &= V(\varphi(\bar{t}_0)) + \int_{\bar{t}_0}^{\bar{t}} \frac{d}{dt} V(\varphi(t))\,\mathrm{d}t \\ &\leq V(\varphi(\bar{t}_0)) + \int_{\bar{t}_0}^{\bar{t}} D^+V(\varphi(t); \dot{\varphi}(t))\,\mathrm{d}t \\ &\leq V(\varphi(\bar{t}_0)) - \int_{\bar{t}_0}^{\bar{t}} W(\varphi(t))\,\mathrm{d}t < V(\varphi(\bar{t}_0)) < c. \end{aligned}$$

The obtained contradiction with the equality in (10) shows that

$$V(\varphi(t)) < c \ \text{ for every } \ t \geq t_{\bar{k}} + \tau. \tag{11}$$

Since $\varphi(t_k) \to \bar{x}$ as $k \to \infty$, there exists a natural number j such that

$$t_j > t_{\bar{k}} + 2\tau, \quad V(\varphi(t_j)) > V(\bar{x}) - \varepsilon \ \text{ and } \ \|\varphi(t_j) - \bar{x}\| < \delta.$$

Then for each $t \in [t_j - \tau, t_j]$ we have

$$\begin{aligned} \|\varphi(t) - \bar{x}\| &= \left\| \varphi(t_j) + \int_{t_j}^{t} \frac{d}{ds}\varphi(s)\,\mathrm{d}s - \bar{x} \right\| \\ &\leq \|\varphi(t_j) - \bar{x}\| + \int_{t}^{t_j} \|F(s, \varphi(s))\|\,\mathrm{d}s. \end{aligned}$$

By repeating the argument in the lines after (8) we can first prove that $\varphi(t) \in \mathbb{B}_r(\bar{x})$ and then that $\varphi(t) \in \mathbb{B}_\varrho(\bar{x})$ for every $t \in [t_j - \tau, t_j]$. Therefore, in a similar way as in the derivation of (9), we obtain that

$$\begin{aligned} V(\varphi(t_j)) &= V(\varphi(t_j - \tau)) + \int_{t_j-\tau}^{t_j} \frac{d}{dt} V(\varphi(t))\,\mathrm{d}t \\ &\leq V(\varphi(t_j - \tau)) + \int_{t_j-\tau}^{t_j} D^+V(\varphi(t); \dot{\varphi}(t))\,\mathrm{d}t \\ &\leq V(\varphi(t_j - \tau)) - \int_{t_j-\tau}^{t_j} W(\varphi(t))\,\mathrm{d}t \leq V(\varphi(t_j - \tau)) - \tau\mu. \end{aligned}$$

Hence,

$$\begin{aligned} V(\varphi(t_j - \tau)) &\geq V(\varphi(t_j)) + \mu\tau \geq V(\bar{x}) - \varepsilon + \mu\tau \geq c - 2\varepsilon + \mu\tau \\ &\geq c - 2\frac{\varrho\mu}{5M} + \mu\frac{\varrho}{2M} \geq c. \end{aligned}$$

Since $t_j - \tau > t_{\bar{k}} + \tau$, this contradicts (11). The contradiction obtained is a consequence of the assumption we made in the beginning of the proof that the claim of the theorem is false. This completes the proof. ♦

3 Corollaries

As a special case of Theorem 1.1, in our first corollary we obtain conditions under which all bounded trajectories of (1) converge to a single point.

Corollary 3.1. *Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^n$, $\bar{x}$ be a point in $\mathcal{S}$, V be locally Lipschitz continuous real-valued function defined on an open set G containing $\mathcal{S}$, W be a real-valued lower semi-continuous function defined on $G \setminus \{\bar{x}\}$, and also $W(x) > 0$ for every $x \in \mathcal{S} \setminus \{\bar{x}\}$. For $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ consider the initial value problem (1),(2) and suppose that condition (B1) in Theorem 1.1 holds. Furthermore, suppose that for every $x \in G \setminus \{\bar{x}\}$, the following inequality holds:*

$$\max_{v \in F(t,x)} D^+V(x; l) \leq -W(x).$$

Then every bounded Carathéodory solution of (1),(2) converges to $\bar{x}$ as $t \to \infty$.

Proof. We apply Theorem 1.1 with $\mathcal{U} = \mathcal{S} \setminus \{\bar{x}\}$. Assumptions (B2) and (B3) are obviously fulfilled, while (B4) holds since $V(\mathcal{S} \setminus \mathcal{U})$ is just one point, hence, does not contain any open interval. ♦

The next corollary is related to the classical statement of the LaSalle invariance principle:

Corollary 3.2. *On the assumptions of Theorem 1.1, suppose that there exist a compact set K and $\nu_0 > 0$ such that for each $\nu \in (0, \nu_0)$ the set $K \cap (\mathcal{S} + \nu\mathbb{B})$ is forward invariant with* *respect to* (1) *(i.e. each bounded trajectory of* (1) *starting from a point of* $K \cap (\mathcal{S} + \nu I\!B)$ *at some moment of time remains in* $K \cap (\mathcal{S} + \nu I\!B)$*). Then for each* $\varepsilon > 0$ *and each point* $\bar{x} \in \mathcal{U}$ *there exists* $\delta > 0$ *such that for each point* $x_0 \in \mathcal{S} + \delta I\!B$ *with* $V(x_0) < V(\bar{x}) - \varepsilon$ *and for each* $t_0 \in \mathbb{R}$ *the following inclusion holds:*

$$\Omega^+(t_0, x_0) \subset \{x \in \mathcal{S} \setminus \mathcal{U} |\ V(x) \leq V(\bar{x})\}.$$

Proof. Fix $\varepsilon > 0$, $t_0 \in \mathbb{R}$ and a point $\bar{x} \in \mathcal{U}$. Let the set $\tilde{\mathcal{U}}$ be defined in the same way as in the proof of Theorem 1.1. Let $x_0 \in K \cap (\mathcal{S} + \delta I\!B)$ with $V(x_0) < V(\bar{x}) - \varepsilon$. Assume that there exists a point $\bar{y} \in \Omega^+(t_0, x_0)$ such that $V(\bar{y}) > V(\bar{x}) + \varepsilon_1$ for some $\varepsilon_1 > 0$. As in the proof of Theorem 1.1, one can prove there exist $c \in (V(\bar{x}) - \varepsilon, V(\bar{x}) + \varepsilon_1)$ and $\delta > 0$ (without loss of generality we may think that $\delta \in (0, \nu_0)$) such that

$$\{x \in (\mathcal{S} + \delta I\!B) \cap K \mid V(x) = c\} \subset \tilde{\mathcal{U}}. \tag{12}$$

Then the definition of the set $\Omega^+(t_0, x_0)$ and the positive invariance of the set $K \cap (\mathcal{S} + \delta I\!B)$ yield the existence of a bounded solution $\varphi(\cdot, t_0, x_0)$ of (1),(2) and a sequence $\{t_k\}_{k=1}^{\infty}$ tending to $+\infty$ with $k \to +\infty$ such that, denoting as before $\varphi(t) := \varphi(t, t_0, x_0)$, we have

$$\lim_{k \to +\infty} \varphi(t_k) = \bar{y} \text{ and } \varphi(t) \in (\mathcal{S} + \delta I\!B) \cap K \text{ for all } t \in [t_0, +\infty). \tag{13}$$

Because $V(\bar{y}) > V(\bar{x}) + \varepsilon_1$, there exists $t_k > 0$ such that $V(\varphi(t_k)) > V(\bar{x}) + \varepsilon_1$. From here, taking into account (13) and the inequality $V(x_0) < V(\bar{x}) - \varepsilon$, we may conclude that there exists $t \in (t_0, t_k)$ such that $V(\varphi(t)) = c$ and $\varphi(t) \in (\mathcal{S} + \delta I\!B) \cap K$. We denote by $\bar{t}$ the largest t with these properties, i.e. $V(\varphi(\bar{t})) = c$ and $V(\varphi(t)) > c$ for each $t > \bar{t}$. According to (12), the point $\varphi(\bar{t}) \in \tilde{\mathcal{U}}$. Since the set $\tilde{\mathcal{U}}$ is open, there exists $\tau > 0$ such that $\varphi(t) \in \tilde{\mathcal{U}}$ for each $t \in [\bar{t}, \bar{t} + \tau]$. But then

$$\begin{aligned} V(\varphi(\bar{t} + \tau)) &= V(\varphi(\bar{t})) + \int_{\bar{t}}^{t+\tau} \frac{d}{ds} V(\varphi(s))\,\mathrm{d}s \\ &\leq V(\varphi(\bar{t})) + \int_{\bar{t}}^{t+\tau} D^+V(\varphi(s); \dot{\varphi}(s))\,\mathrm{d}s \\ &\leq V(\varphi(\bar{t}) - \int_{\bar{t}}^{t+\tau} W(\varphi(s))\,\mathrm{d}s < V(\varphi(\bar{t})) = c. \end{aligned}$$

which contradicts the definition of $\bar{t}$. This completes the proof. ⧫

In the reminder of this section we present corollaries of Theorem 1.1 for the particular but important case when (1) is represented by the ordinary differential equation

$$\dot{x}(t) = f(x(t)), \tag{14}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the standing assumption. Since f is time-independent, the ω-limit generated by any initial (t_0, x_0) is independent of t_0 and we use the shortened notation $\Omega^+(x_0)$.

The following assumption is a replacement of assumptions (B1)–(B3) in Theorem 1.1.

(B1′) The set $\mathcal{S} \subset \mathbb{R}^n$ is closed and contains $\Omega^+(x_0)$, G is a neighborhood of $\mathcal{S}$, and $V : G \to \mathbb{R}$ is continuously differentiable.

For a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ we use the notation $\dot{V}_f(x) := V'(x)f(x)$.

Corollary 3.3. *Let assumptions (B1′) and (B4) be fulfilled, and let* $\mathcal{U} := \{x \in \mathcal{S} \mid \dot{V}_f(x) < 0\}$. *Then the* ω*-limit set* $\Omega^+(x_0)$ *is contained in the set* $\mathcal{S} \setminus \mathcal{U}$.

Proof. Since $\Omega^+(x_0) \subset S$, condition (B1) is fulfilled. We define $W(x) = -\dot{V}_f(x)$ for $x \in G$, which implies (B2) due to the definition of $\mathcal{U}$. Then $W(\cdot)$ is continuous, and the last part of (B1′) implies (B3). Then the claim of the corollary follows from Theorem 1.1. ♦

Remark 3.4. The statement of Corollary 3.3 can be strengthened if $\Omega^+(x_0)$ is a connected set. Namely, in this case $\Omega^+(x_0)$ is contained in a single connected component of $\mathcal{S} \setminus \mathcal{U}$. This is the case, for example, if f is locally Lipschitz continuous.

Corollary 3.3 implies the following version of a result by Arsie and Ebenbauer [1]:

Corollary 3.5. *Let* f *be locally Lipschitz continuous and let condition (B1′) be fulfilled. Define* $\mathcal{U} := \{x \in \mathcal{S} \mid \dot{V}_f(x) < 0\}$ *and assume that* $V(\mathcal{S} \setminus \mathcal{U})$ *does not contain any open interval. Then the* ω*-limit set* $\Omega^+(t_0, x_0)$ *is contained in a connected subset of the set* $\mathcal{S} \setminus \mathcal{U}$.

Proof. We apply Corollary 3.3 (where (B4) is obviously fulfilled) together with Remark 3.4. ♦

Finally, we relate our result to the work of Bacciotti and Ceragioli [3], who showed that their main result is not applicable to the following example:

Example 3.2. Consider the one-dimensional differential equation

$$\dot{x} = f(x), \quad x(0) = x_0,$$

where $f(0) = 0$, $f(-x) = -f(x)$ and

$$f(x) = \begin{cases} -x + \dfrac{1}{n}, & \text{if } x \in \left(\dfrac{1}{n}, \dfrac{1}{n-1}\right], \ n > 1, \\ -x + 1, & \text{if } x > 1. \end{cases} \tag{15}$$

Set $G = \mathcal{S} = \mathbb{R}$, $V(x) := \dfrac{x^2}{2}$, $W(x) = -\dot{V}_f(x) = -xf(x)$. Clearly, W is not lower semi-continuous at $x = \pm\dfrac{1}{n}$, $n = 1, 2, \ldots$, and hence the result in [3] does not apply. Theorem 1.1, however, allow us to exclude the discontinuity points. Indeed, by taking

$$\mathcal{U} = \mathbb{R} \setminus \left\{0, \ \pm\frac{1}{n}, \ n = 1, 2, \ldots\right\},$$

the function W is continuous on $\mathcal{U}$. Clearly assumptions (B1)–(B3) in Theorem 1.1 are fulfilled. Assumption (B4) also holds, since $V(\mathcal{S} \setminus \mathcal{U})$ consists of a countable number of points, thus it does not contain any open interval. Then according to Theorem 1.1, for any point $x_0 \in \mathbb{R}$, the ω-limit set $\Omega(x_0)$ is contained in the set

$$\mathbb{R} \setminus \mathcal{U} = \left\{ 0,\ \pm\frac{1}{n},\ n = 1, 2, \ldots \right\}.$$

It is to be mentioned that neither of the points $\pm\dfrac{1}{n}$ is invariant (that is, an equilibrium point of the equation). However, each of the intervals $\left(\dfrac{1}{n}, \dfrac{1}{n-1}\right)$ is invariant, hence $\Omega^+(x_0) = \left\{\dfrac{1}{n}\right\}$ for each $x_0 \in \left(\dfrac{1}{n}, \dfrac{1}{n-1}\right)$, $n = 1, 2, \ldots$. Analogously, $\Omega^+(x_0) = \left\{-\dfrac{1}{n}\right\}$ for each $x_0 \in \left(-\dfrac{1}{n-1}, -\dfrac{1}{n}\right)$, $n = 1, 2, \ldots$. ♦

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