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Rational utilisation of a common resource by multiple agents

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Rational utilisation of a common resource by multiple agents

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Abstract

The behaviour of multiple agents who utilise one common resource will be investigated in this thesis, where different growth models for the resource are taken into account. The agents' utility depends on the stock of the resource, their extraction rate and their technological level, which can be increased with investments. The influence of how much one agent is future-oriented will be the key point for the optimal control strategies. The main focus is on the one agent problem, how the optimal solution of this one agent changes if a second agent is introduced, and at last, how the optimal outcome of these two agents can be increased, if instead of a full competition on the market the agents mutually agree on a certain commitment strategy in order to jointly extract the resource optimally. It will turn out that in any case a certain commitment is of benefit for both agents and will increase their output compared to the non-commitment strategy. Solutions will be obtained by applying Pontryagins Maximum Principle and the concept of Nash equilibria in a differential game setup and the obtained numerical solutions for the two agent cases are explicitly given and analysed.
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Contents

Chapter 1

Introduction

The joint extraction of a common resource and its implications for the different agents is discussed ever since the works by Gordon (1954) and Hardin (1968). The theory of differential games and optimal control problems have consistently been used for this problem of a joint extraction by multiple agents, where some examples for continuous time models are Clemhout and Wan (1985), Clark (1990), Dockner and Sorger (1996), Dawid and Kopel (1997), Sorger (1998) and Benchekroun and Long (2002) whereas Levhari and Mirman (1980), Benhabib and Radner (1992) and Dutta and Sundaram (1993a,b) use a discrete time setup.

In this thesis, the rational utilisation of a common resource by one or multiple agents will be explored and a non-cooperative behaviour and a cooperative behaviour of the agents will be investigated. Where the non-cooperative behaviour will result in an over-exploitation of the resource, it will be of interest to see the potential gain for the agents, if they decide to cooperate. The multiple agents have access to only one commonly shared resource, which can be jointly extracted and can then be used for generating a positive production outcome. This common resource will be modelled with three different kinds of resource growth laws, which will turn out to be crucial to the behaviour of the agents' extraction. Extraction will imply costs for the agent and the agents also have the possibility to invest in their technology, which will lead to a more effective extraction and more effective use of the extracted resource in their production. This common resource, acting as the connection between the multiple agents, will lead to a dynamic game theoretic approach, where each agent tries to maximise her own output of production. To maximise each agents' output of production, resource extraction is necessary, therefore it is clear that as soon as the resource is extinct, there cannot be any production anymore. So the mutual motivation of all agents should be to never extinct the resource, if the resource has a positive growth rate. A similar approach was taken by Benhabib and Radner (1992), who assumed that each agents' total utility is only the rate of extraction of the resource, called the consumption. In addition, an upper bound was imposed on each agents' maximum consumption. Here, the output of production does not only depend on the rate of extraction of the resource, but also on the level of technology, which is endogenous and can be increased by investments.

Some references discussing this kind of differential games are Basar and Olsder (1982) and Dockner et al. (2000), and a reference on exhaustible resources is Dasgupta and Heal (1979).

Examples of different types of resource, where this particular problem can be applied, are the oil industry, fishery, forestry, hunting or grazing. Another example, introduced by Lancaster (1973), is to interpret the assets of a capitalist firm as the resource, which are then exploited by the owners of the firm as well as the workforce. In this work, the two parties not being able to cooperate and to agree on a long-term commitment, leads to *the dynamic inefficiency of capitalism*, as it is called in Lancaster (1973). Another example is to interpret the resource as the world climate, where then extraction means to pollute the environment with CO_2 emissions. The problem then is, that the high pollution of CO_2 affect the world climate irreversibly, whereas the common goal of all countries of the world should be a world climate which is as good as possible and which guarantees a healthy environment. A similar example using climate policies in a multi-country setting was already introduced in Bréchet et al. (2012).

After introducing some mathematical tools necessary to solve the upcoming optimal control problems and dynamic games in Chapter 1, Chapter 2 will begin to analyse this problem when there is only one agent present and this one agent has the one resource all to herself. Naturally, having more than one agent extracting from one common resource with no mutual agreement between these agents, each agent will start extraction immediately at a very high rate, in order not to leave the resource stock to the other agents. This behaviour will be analysed in Chapter 3, mainly focusing on a two-agent situation. Chapter 4 will then focus on the possibility that the agents agree on a commitment based on the Nash equilibrium solution introduced in Chapter 3, and analyse how the behaviour of these two agents changes because of the mutually agreed commitment. Some concluding thoughts and possible extensions not discussed in this thesis will then be presented in Chapter 5.

1.1 Preliminaries on Optimal Control

1.1.1 Pontryagins Maximum Principle

Consider the following time-invariant optimal control problem with finite time horizon, $t \in [0,T]$, $T > 0$, $x(t) \in X$, where $X \subseteq \mathbb{R}^n$ is the state space of the system, $u : [0,T] \mapsto U$ is the control and $U \subseteq \mathbb{R}^m$ the control space of the system:

$$\textit{Maximise} \qquad J(u(\cdot)) = \int_0^T e^{-rt} g(x(t), u(t), t)dt + e^{-rT} S(x(T)) \tag{1.1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \tag{1.2}$$

$$x(0) = x^0, \tag{1.3}$$

where x^0 denotes the initial state and $r \geq 0$ denotes the discount rate. The function $g : X \times U \times [0,T] \mapsto \mathbb{R}$ is called the utility function which measures the *instantaneous utility* when in the current state $x(t)$ where the control value $u(t)$ is chosen at time t, and $S : X \mapsto \mathbb{R}$ is called the *scrap value function* or the *terminal value function*. The functions $g(x,u,t)$ and $\partial g(x,u,t)/\partial x$ are assumed to be continuous with respect to their arguments and S is assumed to be continuously differentiable with respect to x. The dynamics of the system is described by $f : X \times U \times [0,T] \mapsto \mathbb{R}^n$. The problem is to obtain an admissible pair $(x(\cdot), u(\cdot))$ such that (1.1) is maximised. Let $x^*(t)$ and $u^*(t)$ denote an optimal state and control respectively, and $J^* = J(u^*(\cdot))$ denote the optimal objective value of the system.

Let

$$H(x, u, \lambda, t) \coloneqq g(x, u, t) + \lambda f(x, u, t) \tag{1.4}$$

denote the Hamiltonian of the problem (1.1)-(1.3).

The following theorem is taken from Pontryagin et al. (1961).

Theorem 1.1 (Pontryagins Maximum Principle). *Let $(x^*(\cdot), u^*(\cdot))$ be an optimal solution of* (1.1)-(1.3) *and let u^* be piecewise continuous. Then there exists a continuous and piecewise continuously differentiable function $\lambda(\cdot)$, with $\lambda(t) \in \mathbb{R}^n$ satisfying for all $t \in$*

$[0, T]$:

$$H(x^*(t), u^*(t), \lambda(t), t) = \max_{u \in U} H(x^*(t), u, \lambda(t), t), \tag{1.5}$$

and at every point t where $u(\cdot)$ is continuous

$$\dot{\lambda}(t) = r\lambda(t) - H_x(x^*(t), u^*(t), \lambda(t), t). \tag{1.6}$$

Furthermore the transversality condition

$$\lambda(T) = S_x(x^*(T)) \tag{1.7}$$

holds.

Let H^* denote the maximised Hamiltonian defined as

$$H^*(x^*, \lambda, t) \coloneqq \max_{u \in U} H(x^*, u, \lambda, t). \tag{1.8}$$

A thorough introduction to this topic is given in Grass et al. (2008) and Ryan (2012) and more details on Pontryagins Maximum Principle can be found in Pontryagin et al. (1961).

1.2 Preliminaries on Game Theory

In this section, a non-cooperative game theoretical approach is used.

The following definition is taken from Dockner et al. (2000) (Chapter 1, Introduction).

Definition 1.2. A *non-cooperative game* is a strategic situation in which decision makers, or agents, cannot or will not make binding agreements to cooperate. In a non-cooperative game, the agents act independently in the pursuit of their own best interest, paying no attention whatsoever to the fortunes of the other agents.

In this thesis, only rational agents are considered. This means, that every agent thinks strategically and decisions are made in a way which is consistent with her objectives, which is to maximise her payoff. Furthermore it includes, that each agent knows the number of opponents and the set of all possible strategies that are available to her opponents. This gives all agents the possibility to form expectations about any uncertainty that may influence the play of the game (Dockner et al. (2000): Chapter 2, Axioms of game theory). Discussions about the strong assumptions of this definition can be found in Dockner et al. (2000).

The following definition is taken from Bressan (2010) (Chapter 3, Differential Games).

Definition 1.3. *Differential game for N agents.* Let $x(t) \in X$, where $X \subseteq \mathbb{R}^n$ is the state space of the game, describe the state of the dynamical system

$$\frac{dx}{dt} = f(t, x, u^1, u^2, \ldots, u^N), \qquad t \in [0, T], \tag{1.9}$$

with initial state

$$x(0) = x_0. \tag{1.10}$$

Here, $[0, T]$ denotes the fixed prescribed duration of the game, x_0 is the initial state known by all players, and $u^i : [0, T] \mapsto \mathbb{R}^{m_i}$ are the controls implemented by the agents, which satisfy $u^i(t) \in U^i, i = 1, 2, \ldots, N$, for some given sets $U^i \subseteq \mathbb{R}^{m_i}$. The goal of the i-th agent is to maximise her own payoff, which is

$$J^i(u^1, u^2, \ldots, u^N) = S^i(x(T)) + \int_0^T F^i(t, x(t), u^1(t), u^2(t), \ldots, u^N(t))dt. \tag{1.11}$$

Here S^i is a terminal payoff, while F^i is the running payoff.

It is essential to specify which information is available for each of the N agents. Here, the open-loop strategy is adopted. Furthermore, the following assumptions are made, that each of the N agents has full knowledge of:

(i) The function f, which determines the evolution of the system.

(ii) The N admissible sets of the controls U^i.

(iii) The N payoff functions J^i.

(iv) The instantaneous time $t \in [0, T]$.

(v) The initial state x^0.

The following definition is taken from Fershtman (1987).

Definition 1.4. An *Open-loop strategy* is a time path $u^i(t)$ that assigns for every t a control in the control space U. The set of all possible open-loop strategies is denoted by ϕ^i and $\phi = \{\phi^1, \phi^2, \ldots, \phi^N\}$.

In an open-loop game, agent i cannot make any observations of the state of the system nor of the strategy which is adopted by the other agents, apart from the initial state of the system.

The following definition is taken from Fershtman (1987).

Definition 1.5. An *open-loop Nash equilibrium* is an N-tuple of open-loop strategies $(u^{1*}, u^{2*}, \ldots, u^{N*})$ satisfying the following condition:

$$J^i(u^{1*}, \ldots, u^{N*}) \geq J^i(u^{1*}, \ldots, u^{i-1*}, u^i, u^{i+1*} \ldots, u^{N*}), \quad i \in \{1, \ldots, N\},$$

for every possible $u^i \in \phi^i$.

If all opponents of agent i use the open-loop Nash strategies $u^{j*}(t), j \neq i$, then agent i faces a control problem of the following form:

$$J^{i*}(u^{i^*}(\cdot)) = \max_{u^i(\cdot)} J^i(u^i(\cdot)) = \max_{u^i(\cdot)} \int_0^T e^{-r^i t} F^i(x(t), u^i(t), t) dt + e^{-r^i T} S^i(x(T))$$

subject to

$$\begin{aligned} \dot{x}(t) &= f^i(x(t), u^i(t), t), \\ x(0) &= x_0, \\ u^i(t) &\in U^i, \end{aligned}$$

where

$$\begin{aligned} F^i(x, u^i, t) &= F^i\left(x, u^{1*}(t), \ldots, u^{i-1*}(t), u^i, u^{i+1*}(t), \ldots, u^{N*}(t), t\right), \\ f^i(x, u^i, t) &= f\left(x, u^{1*}(t), \ldots, u^{i-1*}(t), u^i, u^{i+1*}(t), \ldots, u^{N*}(t), t\right). \end{aligned}$$

To find this Nash equilibrium, it is necessary to simultaneously solve N optimal control problems. The optimal solutions of the other agents

$$\{u^{1*}(t), \ldots, u^{i-1*}(t), u^{i+1*}(t), \ldots, u^{N*}(t)\}$$

enter the problem of the i-th agent as parameters, and vice versa.

The following theorem is from Basar and Olsder (1982).

Theorem 1.6. *For an N-agent differential game of prescribed fixed duration $[0, T]$, let*

(i) $f(t, \cdot, u^1, \dots, u^N)$ *be continuously differentiable on* $\mathbb{R}^n$, $\forall t \in [0, T]$,

(ii) $F^i(t, \cdot, u^1, \dots, u^N)$ *and* $S^i(\cdot)$ *be continuously differentiable on* $\mathbb{R}^n$, $\forall t \in [0, T]$, $i \in \{1, 2, \dots, N\}$.

Then, if $\{u^{i*} : i \in \{1, 2, \dots, N\}\}$ *provides an* open-loop Nash equilibrium solution, *and* $\{x^*(t) : t \in [0, T]\}$ *is the corresponding state trajectory, there exist* N *costate functions* $\lambda^i(\cdot) : [0, T] \mapsto \mathbb{R}^n$, $i \in \{1, 2, \dots, N\}$, *such that the following relations are satisfied:*

$$\begin{aligned}
\dot{x}^*(t) &= f(t, x^*(t), u^{1*}(t), \dots, u^{N*}(t)); \quad x^*(0) = x_0 \\
u^{i*}(t) &= \arg\max_{u^i \in U^i} H^i(t, \lambda^i(t), x^*(t), u^{1*}(t), \dots, u^{i-1*}(t), u^i, u^{i+1*}(t), \dots, u^{N*}(t)) \\
\dot{\lambda}^i(t) &= r^i \lambda^i(t) - \frac{\partial}{\partial x} H^i(t, \lambda^i(t), x^*, u^{1*}, \dots, u^{N*}(t)) \\
\lambda^i(T) &= \frac{\partial}{\partial x} S^i(x^*(T)), \quad i \in \{1, 2, \dots, N\},
\end{aligned}$$

where

$$H^i(t, \lambda, x, u^1, \dots, u^N) := F^i(t, x, u^1, \dots, u^N) + \lambda^i f(t, x, u^1, \dots, u^N),$$
$$t \in [0, T], \quad i \in \{1, 2, \dots, N\}.$$

For a proof this Theorem, see Basar and Olsder (1982). Theorem 1.6 provides a set of necessary conditions that need to be satisfied for the open-loop Nash equilibrium solution.

1.3 Preliminaries on numerics

1.3.1 Solving differential equations: The Heun (Improved Euler) Method

In order to solve an initial value problem (IVP) for an ordinary differential equation numerically, the Euler Method (or the one stage Runge-Kutta method) is the simplest way to obtain the solution. This method produces an approximation of the solution which is as close to the exact solution as desired, where the accuracy is linear with the step size. Improving this method and implementing the Heun method, the accuracy then improves quadratically. This can be seen, when the Euler method is re-interpreted in terms of an integral equation and replacing the rectangular approximation of this integral equation with a trapezoidal approximation, where the Trapezoidal Rule error decreases quadratically with the step size (see Isaacson and Keller (1994)).

So, let ξ be a smooth function on an interval containing the points t_k and t_{k+1}. The Fundamental Theorem (e.g. see Rudin (1986)) implies

$$\xi(t_{k+1}) = \xi(t_k) + \int_{t_k}^{t_{k+1}} \xi'(\theta)\, d\theta.$$

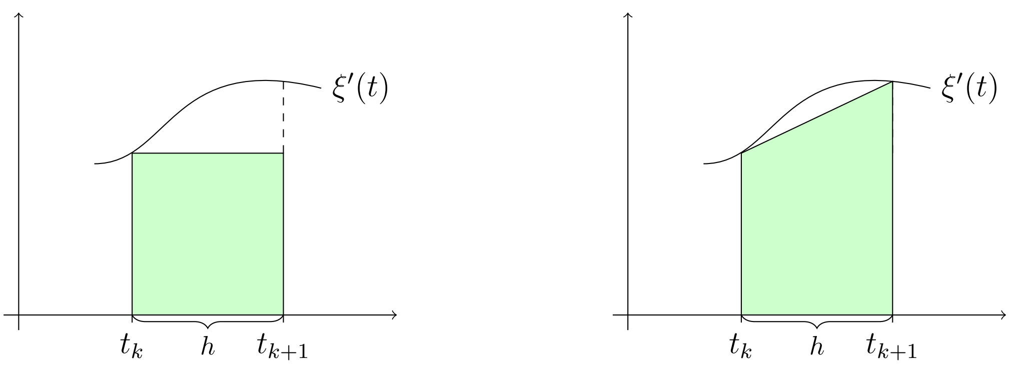
With this result it follows immediately that $\xi(t_{k+1})$ can be approximated only with information about $\xi(t_k)$ and t_k, given that the integral can be approximated. One possibility to approximate this integral is by the value of the left endpoint, which will result in the Euler method, see Figure 1.1a, given by

$$\int_{t_k}^{t_{k+1}} \xi'(\theta)\, d\theta \approx \underbrace{(t_{k+1} - t_k)}_{h}\, \xi'(t_k).$$

If ξ satisfies the differential equation $y'(t) = f(y(t))$, then $\xi'(t_k) = f(\xi(t_k))$ and it follows that

$$\xi(t_{k+1}) \approx \xi(t_k) + h\, f(\xi(t_k)),$$

which is exactly the Euler method.



(a) Area $\approx h\, \xi'(t_k)$

(b) Area $\approx \frac{h}{2}\, (\xi'(t_k) + \xi'(t_{k+1}))$

Figure 1.1: Comparison of rectangular and trapezoidal approximation of an integral.

Another possibility to approximate the integral is to use trapezoids instead of rectangles, see Figure 1.1b, given by

$$\int_{t_k}^{t_{k+1}} \xi'(\theta)\, d\theta \approx \frac{h}{2}\left(\xi'(t_k) + \xi'(t_{k+1})\right),$$

and it follows that

$$\xi(t_{k+1}) \approx \xi(t_k) + \frac{h}{2}\left(f(\xi(t_k)) + f(\xi(t_{k+1}))\right). \tag{1.12}$$

Since $\xi(t_{k+1})$ appears also on the right-hand side of the last equation, $\xi(t_{k+1})$ will first be approximated by the Euler method and then this approximation is used in the right hand side of (1.12) to obtain an improved approximation:

$$\begin{aligned} \tilde{\xi}(t_{k+1}) &= \xi(t_k) + h\, f(\xi(t_k)), \\ \xi(t_{k+1}) &= \xi(t_k) + \frac{h}{2}\Big(f(\xi(t_k)) + f(\tilde{\xi}(t_{k+1}))\Big), \end{aligned}$$

which is the improved Euler method, or the Heun method. Assuming that $\xi(t)$ is a monotonously increasing (decreasing) function, an approximation of the integral with $h\,\xi'(t_k)$ will always result in an underestimation (overestimation) of $\xi(t_{k+1})$, whereas an approximation of the integral with $h\,\xi'(t_{k+1})$ will always result in an overestimation (underestimation). Therefore, the Heun method addresses this problem by taking an approximation halfway between these two prior approximations, and the approximation errors made in either of the two cases cancel themselves out. For a more detailed introduction to this matter, see Kreyszig (2011).

If now the following differential equation $y'(t) = f(y(t))$ with given $y(0) = y_0$ is considered on a given time interval $t \in [0, T]$, $T \in \mathbb{R}$ positive, where $f : X \mapsto \mathbb{R}$, $X \subseteq \mathbb{R}^n$ non-empty, closed and convex, then the solution of this differential equation can be calculated numerically by the iteration

$$\begin{aligned} & y_{i+1} = y_i + \frac{h}{2}\left(f(y_i) + f(\tilde{y}_{i+1})\right), \\ \textit{where} \qquad & \tilde{y}_{i+1} = y_i + h f(y_i), \\ & i \in \{0, 1, \ldots, N-1\}. \end{aligned}$$

The step size h is chosen to be constant and can be calculated by using the number of grid points N of the uniform grid with grid points $\{t_0, t_1, \ldots, t_N\}$. Assuming that the true solution $y(\cdot)$ has a derivative with bounded variation, the sequence $\{y_0, y_1, \ldots, y_N\}$ obtained by the above scheme provides a second order approximation to y, in the sense that $\left|y(t_i) - y_i\right| \leq c\,h^2$.

1.3.2 Calculation of integrals: The Trapezoidal Method

Consider the following integral:

$$\int_a^b f(x)dx,$$

where $[a,b] \subseteq \mathbb{R}$ and $f : X \mapsto \mathbb{R}$, $X \subseteq \mathbb{R}^2$ non-empty, closed and convex. This integral can be approximated by

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{h}{2}\sum_{i=0}^{N-1}\left(f(x_{i+1}) + f(x_i)\right)\\ &= \frac{h}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{N-1}) + f(x_N)\right),\end{aligned}$$

where N is the size of the uniform grid and $h = \frac{b-a}{N}$ the constant stepsize. For more details on this method and further reading, Atkinson (1989) and Isaacson and Keller (1994) are recommended.

1.3.3 Gradient Projection Method

Consider the optimisation problem

$$\begin{aligned}&\textit{Maximise} && f(x)\\ &s.t. && x \in X,\end{aligned} \tag{1.13}$$

where $X \subseteq \mathbb{R}^n$, X non-empty, closed and convex, and $f : X \mapsto \mathbb{R}$ continuously partially differentiable. One method that can be used to solve (1.13) is to apply the Gradient Projection Method.

According to the Projection Theorem (e.g. see Brockwell and Davis (1991)[1]) it follows with X non-empty, closed and convex, that there exists for every $x \in \mathbb{R}^n$ a unique decomposition

$$x = \bar{x} + u$$

[1] $\mathbb{R}^n$ together with the inner product defined by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is a Hilbert space.

such that $\bar{x} \in X$ and $u \in X^{\perp}$. Additionally, $\bar{x}$ is the unique element of X satisfying

$$\|x - \bar{x}\|_2 = \min_{y \in X} \|x - y\|_2.$$

Here, $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$. $\bar{x}$ is called the projection of x into the set X, where the map

$$\pi_X : \mathbb{R}^n \mapsto X, \quad \pi_X(x) = \bar{x},$$

with $x \in \mathbb{R}^n$, is called the projection into X.

Here, only the case of the maximum gain, the gradient method, is described. This method will yield the quickest convergence since the gradient vector of any function always points in the direction of the quickest increase of that function. Therefore, assuming $x^k \in X$ is an admissible solution of (1.13), the new direction will be the gradient $\nabla f(x^k)$ and therefore $x^{k+1} = x^k + s\nabla f(x^k)$, where s is the optimal step size. In order to guarantee that this new solution is also a valid solution and all constraints hold, x^{k+1} will be projected into the admissible space X, which gives the new direction

$$x^{k+1} = \pi_X(x^{k+1}) = \pi_X(x^k + s\nabla f(x^k)).$$

With this given direction, a better solution is possible to be found and the optimal step size needs to be determined next, which will be done by solving the one-dimensional maximisation problem $\max_{s \geq 0} \left\{ f\left(\pi_X(x^k + s\,\nabla f(x^k))\right)\right\}$.

Algorithm for the Gradient Projection Theorem:

(i) Calculate an admissible initial solution $x^0 \in X$ and initialise $k := 0$.

(ii) Calculate

$$x^{k+1} = \pi_X(x^k + s^k \nabla f(x^k)),$$
$$\text{with} \quad s^k := \arg\max_{s \geq 0} f\left(\pi_X(x^k + s\nabla f(x^k))\right),$$

such that

$$f(x^{k+1}) \geq f\left(\pi_X(x^k + s\nabla f(x^k))\right) \qquad \forall s \geq 0.$$

(iii) If $x^{k+1} = x^k$, stop the algorithm. In practise, the algorithm is stopped when $\|x^{k+1} - x^k\| < \varepsilon$. Otherwise increase $k \coloneqq k + 1$ and go to step (ii).

For further reading see Rosen (1960), Rosen (1961) and Gerdts (2011), where this problem is extensively presented and discussed.

On how to implement the above presented Gradient Projection Method for a non-linear optimal control problem, the work by Dontchev et al. (2000) is given as reference, where an error approximation for the discretised problem is also given in detail.

Chapter 2

An optimal growth model with resource constraints: The case of a single agent

In this chapter an optimal growth model with three different types of resource dynamics is discussed: The case of a non-renewable resource, a resource with exponential growth and a resource with logistic growth. The arising optimal control problem with two control variables and two state variables for a single agent is given, with the aim to maximise her production function. This can be done by extracting the resource in an optimal way and deciding on the investment spendings optimally as well. It will be shown that postponing the initial extraction may be optimal. This delay of extraction allows the agent to bring the technology to a high level with excessive investment spendings, which will provide a better efficiency, once extraction starts.

2.1 Problem formulation

Let the economy consist of only one agent, this could be a company, a country or a group of countries. This agent has access to a resource (with the stock of the resource denoted by $R(t)$ at time t), which is either of non-renewable or of renewable nature. The agent can also invest in her technology (with the size of the technological level at time t denoted by $K(t)$)[1]. The aim of the agent is to maximise her economic output by extracting the resource, where the efficiency of extraction depends on the richness of the resource, and the efficiency of utilisation depends on the present technological level. As soon as either the stock of the resource or the level of technology is zero, extraction is not possible anymore. Investment in technology and extraction of the resource will cause costs for the agent, so the aim of the agent - to maximise her economic output, can be achieved by optimally choosing the amount of investment in R&D to be made (denoted by $u_1(t)$ at time t) and the rate of extraction of the resource (denoted by $u_2(t)$ at time t). Hence the agent's optimisation problem is:

$$\textit{Maximise} \qquad J(u) = \int_0^\infty e^{-rt}\Big(K(t)u_2(t)R(t) - c(u_1(t)) - d(u_2(t))\Big)dt \tag{2.1}$$

subject to

$$\dot{K}(t) = -\delta K(t) + u_1(t), \quad K(0) = K_0, \tag{2.2}$$

$$\dot{R}(t) = \varrho(R) - u_2(t)R(t), \quad R(0) = R_0, \tag{2.3}$$

$$u_1(t), u_2(t) \geq 0, \quad t \in [0, T], \tag{2.4}$$

where a dot over variables denote the derivative with respect to time t and $r \geq 0$ is the discount rate. Let $u(t) = \big(u_1(t), u_2(t)\big)^T$ denote the 2-dimensional control vector and $x(t) = \big(K(t), R(t)\big)^T$ the 2-dimensional state vector. In the objective function (2.1), $K(t)u_2(t)R(t)$ is the production function. This production function assumes increasing returns of scale. If both inputs K and R are doubled, the gain of extracting the resource with the same rate is quadrupled. The use of this kind of production function can be defended, for instance, by the example that the resource is a fish-stock in the ocean and the fish is evenly spread over the whole body of water. If the richness of the resource, i.e. the density of the fish in the ocean is kept constant, but the used technology to harvest the fish, i.e. the boats used, fishing rods or nets is doubled, the double amount of fish will be caught with the same effort applied. Vice Versa, if the used technology is kept constant, but the richness of the resource is doubled, again the double amount of fish will be caught with the same effort applied (see Erdlenbruch et al. (2008)). Another example

[1] K can also be interpreted as knowledge, explaining the chosen name of the variable. Another interpretation for K can be capital.

for this kind of production function is the recently very popular shale gas extraction, known as fracking. The reason that this form of extraction won such popularity in the last couple of years is not because fracking was not known before, but because it was far too expensive. The shortages of natural gas in the 1970 led to extensive government spendings and research funding. This new level of technology now allows to apply fracking with a reasonable economic price, and it is predicted that by 2035 already 46% of natural gas supply of the United States will come from shale gas, where in 2000 this rate was only at 1.6% (see Stevens (2012) and Krupnick and Wang (2013)).

The chosen production function also implies the above stated assumption: Extraction does not yield any gain as soon as either the technology or the stock of the resource is zero.

Furthermore in (2.1), the term $c(u_1(t))$ represents the costs for the agent when investing in technology. For reasons of simplicity, this term will from now on be chosen as $c(u_1) = c_1 u_1^2$, where c_1 is a constant. The term $d(u_2(t))$ represents the costs for the agent when the resource is extracted. Again, for the sake of simplicity, this term will from now on be chosen as $d(u_2) = c_2 u_2^2$, where c_2 is another constant.

Equation (2.2) describes the dynamics of the technology, where $\delta > 0$ represents the depreciation rate of the technology. The depreciation rate δ can be interpreted as a loss in value that will occur over time, i.e. machines are getting older and break more easily or technology gets outdated and is not applicable anymore. In order to keep technology at a constant or even to develop it further to a higher rate, investment in technology through the control $u_1(\cdot)$ is possible. Equation (2.3) describes the dynamics of the resource. The resource gets less when the agent starts to extract it through the control $u_2(\cdot)$. But the resource can also grow, which is represented by $\varrho(R)$. Depending on the structure of the function $\varrho(R)$, the behaviour of the resource can be modelled in different ways. In this thesis, three different growth policies for the resources are taken into account:

Case 1: *Non-renewable resource.* In this case, the resource cannot grow at all, leaving the agent to extract only the initially available amount.

$$\varrho(R) = 0. \tag{2.5}$$

Case 2: *Renewable resource with exponential growth.* This case models a growing resource with no upper bound. If the resource is not extracted, it will grow to infinite size.

$$\varrho(R) = \delta_R R, \qquad \delta_R > 0. \tag{2.6}$$

Case 3: *Renewable resource with logistic growth.* This case models a growing resource, where a natural upper bound is imposed.

$$\varrho(R) = \big(b - aR\big)R, \qquad a, b > 0. \tag{2.7}$$

The first case, where the resource has no growth rate at all, can be seen as a special case of either the exponential or the logistic growth rate.

Since the two state equations (2.2) and (2.3) are not depending on each other and there is only one agent in the economy, one can easily see that this one agent has the possibility to postpone the extraction of the resource until the investment has increased the level of technology up to an optimal level, and to start with extraction just then, which will possibly result in a higher objective value. Also, it is completely up to the agent to decide to let the resource grow to a sustainable level or to extinct it completely in any chosen period of time, whichever strategy will maximise her output. The two constraints in equation (2.4) indicate that there can neither be a de-investment of technology nor an artificial acceleration of the growth rate of the resource.

2.2 Mathematical analysis

In order to solve the optimal control problem (2.1)-(2.4) the Maximum Principle of Pontryagin (Theorem 1.1), in current-value notation, will be applied. Therefore, this system needs to be reformulated with a finite time horizon T first, which does not change the objective function (2.1), apart from the upper limit of the integral. Also, the introduced scrap value function is chosen to be $S(x(T)) = 0$. An analysis why a finite time horizon instead of an infinite time horizon and a scrap value function equal to zero can be applied, is given in detail in Section 2.3. For this specific optimal control problem, the

Hamiltonian (1.4) is

$$\begin{aligned} H(x(t),u(t),\eta(t)) =& K(t)u_2(t)R(t) - c_1u_1(t)^2 - c_2u_2(t)^2 + \\ & \eta_1(t)\Big(u_1(t) - \delta K(t)\Big) + \eta_2(t)\Big(\varrho(R) - u_2(t)R(t)\Big), \end{aligned} \tag{2.8}$$

and the necessary conditions (1.5)-(1.7) therefore are

$$\begin{aligned} \textit{(i)} \quad & u_1(t) = \frac{1}{2c_1}\eta_1(t) \\ & u_2(t) = \frac{1}{2c_2}\Big((K(t) - \eta_2(t))R(t)\Big) \end{aligned} \tag{2.9}$$

$$\begin{aligned} \textit{(ii)} \quad & \dot\eta_1(t) = -u_2(t)R(t) + (\delta_K + r)\,\eta_1(t) \\ & \dot\eta_2(t) = -u_2(t)K(t) + \left(u_2(t) - \frac{\partial\varrho(R)}{\partial R} + r\right)\eta_2(t) \end{aligned} \tag{2.10}$$

$$\textit{(iii)} \quad \eta(T) = 0. \tag{2.11}$$

Together with conditions (2.2) and (2.3) this is a system of six equations with six unknowns, and therefore the problem can be solved. For the three different cases of the resource growth rate, this results in the following system of six equations.

Case 1: Non-renewable resource.

$$\begin{aligned} u_1(t) &= \frac{1}{2c_1}\eta_1(t), \\ u_2(t) &= \frac{1}{2c_2}\big((K(t) - \eta_2(t))R(t)\big), \end{aligned} \tag{2.12}$$

$$\begin{aligned} \dot\eta_1(t) &= -u_2(t)R(t) + (\delta_K + r)\,\eta_1(t), \\ \dot\eta_2(t) &= -u_2(t)K(t) + \big(u_2(t) + r\big)\eta_2(t), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \dot K(t) &= -\delta K(t) + u_1(t), \\ \dot R(t) &= -u_2(t)R(t). \end{aligned} \tag{2.14}$$

Case 2: Renewable resource with exponential growth ($\delta_R > 0$).

$$\begin{aligned} u_1(t) &= \frac{1}{2c_1}\eta_1(t), \\ u_2(t) &= \frac{1}{2c_2}\big((K(t)-\eta_2(t))R(t)\big), \end{aligned} \tag{2.15}$$

$$\begin{aligned} \dot{\eta}_1(t) &= -u_2(t)R(t) + (\delta_K + r)\,\eta_1(t), \\ \dot{\eta}_2(t) &= -u_2(t)K(t) + \big(u_2(t) - \delta_R + r\big)\eta_2(t), \end{aligned} \tag{2.16}$$

$$\begin{aligned} \dot{K}(t) &= -\delta K(t) + u_1(t), \\ \dot{R}(t) &= \big(\delta_R - u_2(t)\big)R(t). \end{aligned} \tag{2.17}$$

Case 3: Renewable resource with logistic growth ($a, b > 0$).

$$\begin{aligned} u_1(t) &= \frac{1}{2c_1}\eta_1(t), \\ u_2(t) &= \frac{1}{2c_2}\big((K(t)-\eta_2(t))R(t)\big), \end{aligned} \tag{2.18}$$

$$\begin{aligned} \dot{\eta}_1(t) &= -u_2(t)R(t) + (\delta_K + r)\,\eta_1(t), \\ \dot{\eta}_2(t) &= -u_2(t)K(t) + \big(2aR(t) - b + u_2(t) + r\big)\eta_2(t), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \dot{K}(t) &= -\delta K(t) + u_1(t), \\ \dot{R}(t) &= \big(b - aR(t) - u_2(t)\big)R(t). \end{aligned} \tag{2.20}$$

2.3 A numerical solution

To solve the optimal control problem (2.1)-(2.4) as stated in Section 2.2 numerically, the problem is considered on a finite time horizon $[0, T]$. Therefore, the objective function (2.1) is altered to a finite horizon problem with the scrap value function equal to zero:

$$\textit{Maximise} \qquad J(u) = \int_0^T e^{-rt}\left(K(t)u_2(t)R(t) - c_1u_1(t)^2 - c_2u_2(t)^2\right) dt \tag{2.21}$$

subject to

$$\dot{K}(t) = -\delta K(t) + u_1(t), \quad K(0) = K_0, \tag{2.22}$$

$$\dot{R}(t) = \varrho(R) - u_2(t)R(t), \quad R(0) = R_0, \tag{2.23}$$

$$u_1(t), u_2(t) \in U, \quad t \in [0, T], \tag{2.24}$$

where $U = [0, \infty)$ is the feasible set of controls. Existence of an optimal solution of this problem can be proved in a standard way due to the linear-quadratic structure of the problem with respect to the control (see Cesari (1983), Chapter 9). Let $u^*(\cdot)$ denote an optimal control and let $J^* := J(u^*)$.

Furthermore, a reasonable initial admissible control $u_0(t)$ needs to be chosen:

$$u_0(t) = (u_1(t), u_2(t))^T = (0, 1)^T, \quad t \in [0, T]. \tag{2.25}$$

Choosing a finite time horizon T needs some supporting arguments, which will be discussed on page 22.

The next step necessary to obtain a numerical solution is to discretise the optimal control problem. Therefore, a uniform discretisation with a grid of the size $N+1$ with the associating grid points $\{t_0, t_1, t_2, \ldots, t_N\}$ with $t_j = jh$, $j \in \{0, 1, \ldots, N\}$, will be chosen, where h is the step size of the grid. The stepsize is chosen to be uniform and defined as $h = \frac{T}{N}$. The controls $u_1(\cdot)$ and $u_2(\cdot)$ will be discretised to N-dimensional vectors $\mathbf{u}_i = (u_i^0, u_i^1, \ldots, u_i^{N-1}) \in \mathbb{R}^N$, $i \in \{1, 2\}$ with

$$u_i^j = u_i(t_j), \quad i \in \{1, 2\}, \quad u_i(t_j) \in U.$$

These control vectors can be interpreted as piecewise constant functions which approximate the original controls $u_1(\cdot)$ and $u_2(\cdot)$ as good as desired with

$$u_i(t) \approx \sum_{j=0}^{N-1} \chi_{[t_j, t_j+h)}(t) u_i^j, \quad i \in \{1, 2\},$$

where for $\mathcal{G} \subseteq \mathbb{R}$, $\chi_{\mathcal{G}}$ denotes the indicator function, where $\chi_{\mathcal{G}}(x) = 1$ for $x \in \mathcal{G}$ and $\chi_G(x) = 0$ for $x \notin \mathcal{G}$.

The state variables $K(\cdot)$ and $R(\cdot)$ will be discretised to $(N+1)$-dimensional vectors, in the same manner as the two controls, where these two state vectors can be interpreted as a piecewise linear approximation of the original state variables,

$$\mathbf{K} = (K^0, K^1 \ldots, K^N) \in \mathbb{R}^{N+1}, \quad K^j = K(t_j),$$
$$\mathbf{R} = (R^0, R^1 \ldots, R^N) \in \mathbb{R}^{N+1}, \quad R^j = R(t_j).$$

Now, the discretised problem (using Section 1.3) can be formulated as

$$\textit{Maximise} \qquad J(\mathbf{u}_1, \mathbf{u}_2) = \frac{h}{2} \sum_{j=0}^{N-1} \Big(e^{-rt_{j+1}} \left(K^{j+1} u_2^{j+1} R^{j+1} - c_1 u_1^{j+1} - c_2 u_2^{j+1} \right) - e^{-rt_j} \left(K^j u_2^j R^j - c_1 u_1^j - c_2 u_2^j \right) \Big)$$

subject to

$$\begin{aligned} K^{j+1} &= K^j + \frac{h}{2} \left(\left(-\delta K^j + u_1^j \right) + \left(-\delta \tilde{K}^{j+1} + u_1^j \right) \right), \\ \tilde{K}^{j+1} &= K^j + h \left(-\delta K^j + u_1^j \right), \\ R^{j+1} &= R^j + \frac{h}{2} \left(\left(\varrho(R^j) - u_2^j R^j \right) + \left(\varrho(\tilde{R}^{j+1}) - u_2^j \tilde{R}^{j+1} \right) \right), \\ \tilde{R}^{j+1} &= R^j + h \left(\varrho(R^j) - u_2^j R^j \right), \\ & j = 0, 1, \ldots, N-1. \end{aligned}$$

This is a mathematical programming problem that can be solved numerically by the following steps:

(i) Find initial and admissible control vectors $\mathbf{u}_i^0 \in \mathbb{R}^N$, $i \in \{1, 2\}$, set $k := 0$ and set the initial objective value $J^k := 0$.

(ii) Solve the discretised differential equations of the state system (2.22) and (2.23) with the Heun method as described in Section 1.3.1 with input parameters $\mathbf{u}_i^k$, where the solutions of the two differential equations are denoted with $\mathbf{K}^{k+1} \in \mathbb{R}^{N+1}$ and $\mathbf{R}^{k+1} \in \mathbb{R}^{N+1}$ respectively.

(iii) Solve the discretised version of the adjoint equations (2.10) with the transversality condition (2.11) as boundary condition: In order to solve these two differential equations with the Heun method described in Chapter 1, this problem first needs to be transformed to an initial value problem (IVP). In general, a differential equation with a fixed end value can be transformed to an initial value problem by solving the prior problem reversed in time, where the fixed end value then becomes the initial value of the problem. Consider the problem

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \\ x(T) &= \bar{x}, \end{aligned}$$

where $f(\cdot)$ is continuous and $x(\cdot)$ is a differentiable function. Introducing an equivalent new problem, namely an initial value problem, where $y(\cdot)$ is differentiable and which is defined as

$$\dot{y}(t) = -\dot{x}(T-t) = -f(y(t)),$$
$$y(t) = x(T-t),$$

and where then $y(0) = \bar{x}$. This equivalent problem can now be solved with the introduced Heun method, where the obtained solution $y(t)$ needs to be transformed back to its original form, that is

$$x(t) = y(T-t).$$

Here, $\mathbf{u}_i^k$, $\mathbf{R}^{k+1}$ and $\mathbf{K}^{k+1}$ are the input parameters and the solutions of the two differential equations are denoted by $\boldsymbol{\eta}_i^{k+1}$, $i \in \{1,2\}$.

(iv) Calculate a new $\mathbf{u}^{k+1}$ according to the Gradient Projection Theorem as introduced in Section 1.3.3: First, the gradient of the Hamiltonian needs to be calculated, which is (2.9), where $\mathbf{K}^{k+1}$, $\mathbf{R}^{k+1}$ and $\boldsymbol{\eta}_i^{k+1}$ are the input parameters. The solution will be denoted by $\nabla_{u_i}\mathbf{H}_i$, $i \in \{1,2\}$. This will give a new feasible control vector

$$\mathbf{u}_i^{k+1} = \pi_{U^N}\left(\mathbf{u}_i^k + s^k\,\nabla_{u_i}\mathbf{H}_i\right), \quad s^k \geq 0, \quad i \in \{1,2\},$$

where

$$s^k = \arg\max_{s\geq 0} J(\pi_{U^N}\left(\mathbf{u}_1^k + s\,\nabla_{u_1}\mathbf{H}_1\right), \pi_{U^N}\left(\mathbf{u}_2^k + s\,\nabla_{u_2}\mathbf{H}_2\right))$$

with $J^{k+1} := J\left(\mathbf{u}_1^{k+1}, \mathbf{u}_2^{k+1}\right)$, where the objective values can be calculated with the Trapezoidal method introduced in Section 1.3.2. In practise, s^k is calculated by choosing an initial step size $s > 0$ big enough. Then, the objective value is calculated with this s:

$$J_s = J(\pi_{U^N}\left(\mathbf{u}_1^k + s\,\nabla_{u_1}\mathbf{H}_1\right), \pi_{U^N}\left(\mathbf{u}_2^k + s\,\nabla_{u_2}\mathbf{H}_2\right)) \tag{2.26}$$

If $J_s > J^k$, set $s^k := s$ to the above chosen step size. Otherwise, halve s to $s := \frac{s}{2}$ and calculate J_s again, according to (2.26). If no improvement of the objective value can be found as s is going to zero, then s^k is chosen to be zero as well, and there is no possible local improvement around the previously optimal objective value.

(v) If $\|J^{k+1} - J^k\| < \varepsilon$, stop this algorithm since the optimal solution was found. Otherwise increase $k := k + 1$ and proceed at (ii).

Discussions regarding a finite time horizon

Calculating the optimal solution for different time horizons $T \in \{200, 300, 400\}$ will in general give different optimal controls $u^*(t)$. These optimal controls are now compared by calculating the Euclidean norm of the difference between two different control vectors in the time period $[0, 100]$ and are listed in Table 2.1. The grid size N is chosen to be $N = 10 \cdot T + 1$. Upper indices indicate the chosen time horizon when this optimal control was computed.

	no growth	exp growth	log growth
$\left\|\mathbf{u}^{200}_{[0,100]} - \mathbf{u}^{300}_{[0,100]}\right\|_2$	0.0001	0.0015	0.5545
$\left\|\mathbf{u}^{200}_{[0,100]} - \mathbf{u}^{400}_{[0,100]}\right\|_2$	0.0001	0.0023	0.6192
$\left\|\mathbf{u}^{300}_{[0,100]} - \mathbf{u}^{400}_{[0,100]}\right\|_2$	≈ 0	0.0008	0.0674

Table 2.1: Comparison of controls, computed on different time horizons T.
$\delta_R = 0.002$, $a = 0.025$, $b = 0.05$, $\delta = 0.02$, $r = 0.01$, $c_1 = c_2 = 1$

As can be seen in Table 2.1, a longer time horizon than $T = 200$ in the cases with no growth and exponential growth, and $T = 300$ in the case with a logistic growth, has almost no influence on the optimal controls on $[0, 100]$. Therefore, in all calculations presented here, the time horizon is always set to $T = 200$ if the growth of the resource is either zero or exponential. In the case of a logistic resource growth, the time horizon is increased to $T = 300$. The reason that in the case of a logistic growth a bigger time horizon is necessary can be explained with the fact that in the logistic case the resource is extracted in a way that is sustainable, whereas in the other two cases the resource is always extinct when time goes to infinity (a detailed analysis on this behaviour will be given in Section 2.3.1). Therefore, if the resource becomes extinct, it is of no importance to the agent of how long the time horizon actually is, as long as it is long enough and $T = 200$ seems to satisfy this sufficiently. On the other hand, if the resource can be held at a sustainable level, the longer the time horizon is, the more gain the agent will have, and therefore the optimal controls tend to change slightly. This obviously depends highly on r, the discount rate.

In order to gain a better insight on how the optimal solution actually depends on r and other parameters such as c_1, c_2, δ, δ_R, a and b, a sensitivity analysis will be discussed next.

2.3.1 Sensitivity analysis and interpretation of results

Non-renewable resource

In this subsection, the objective value can only be influenced by the discount rate r and the depreciation rate of the physical capital δ, together with the two constants c_1 and c_2 and the initial values for the state equations $x_0(t)$, see (2.12)-(2.14). Setting the initial conditions for $x_0 = (K_0, R_0)^T = (1, 1)^T$ and choosing $c_1 = c_2 = 1$, the influence of the discount rate r with different values for δ can be seen in Figure 2.1. Obviously, when

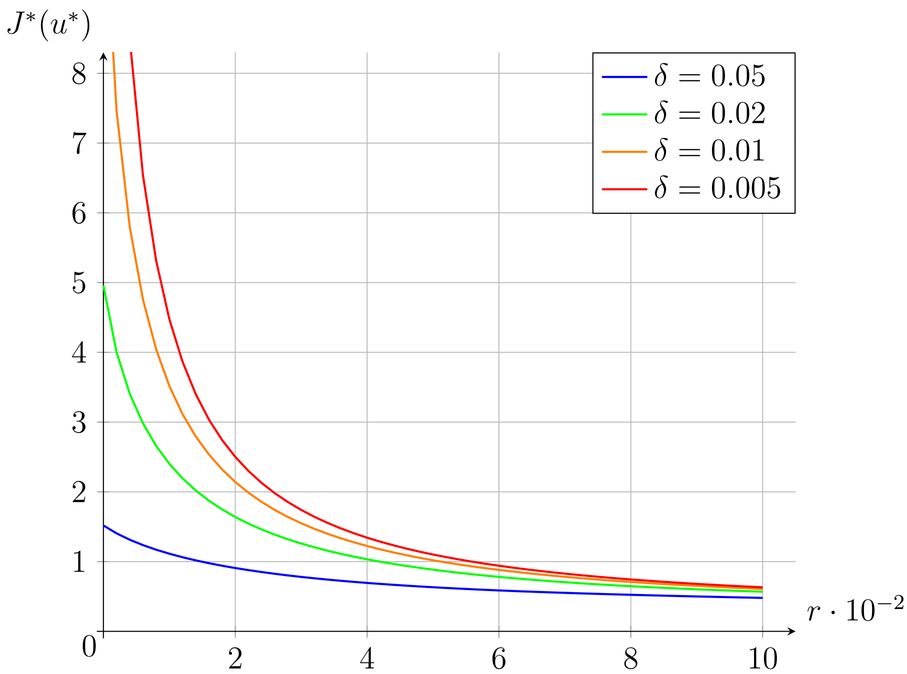


Figure 2.1: Optimal objective value for non-growing resource, depending on r.

leaving the control $u(\cdot)$ and all variables the same, the smaller the discount rate r, which represents a far-sighted agent, the higher is J^*, and vice versa. Another quite natural observation is, that a higher depreciation rate of technology δ results in a lower J^*, since with a bigger δ, technology loses value faster over time. This leads directly to a lower production function.

For further investigations, the depreciation rate of technology δ is chosen to be fixed at a value of 0.02.

For now, let the two constants be $c_1 = c_2 = 1$. When the optimal solution is computed, the behaviour of the optimal controls $u^*(t)$ together with the change of technology K and the resource R are plotted in Figure 2.2. The dashed lines indicate the time when the agent starts the extraction.

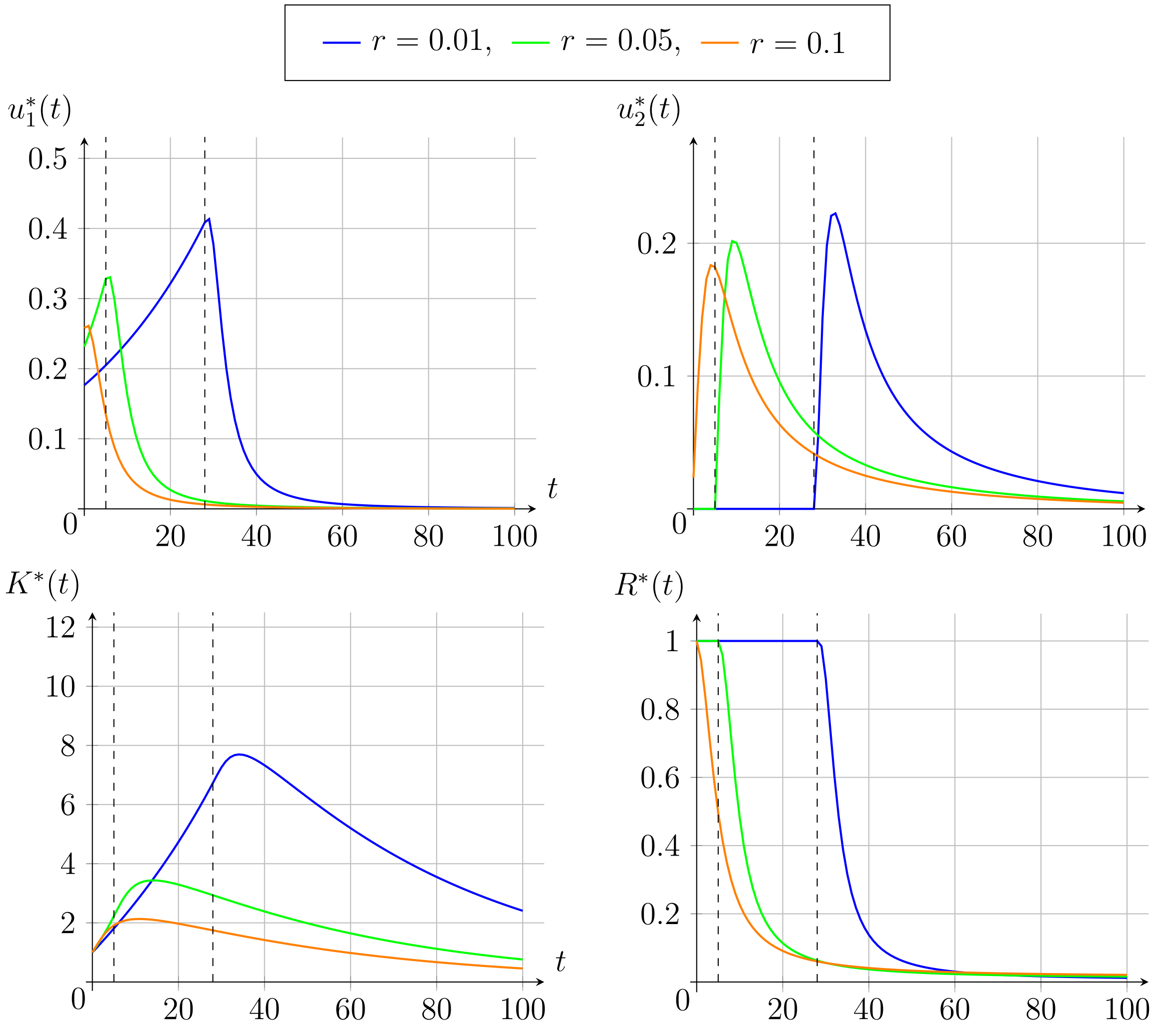


Figure 2.2: Optimal controls and states $u^*(t)$, $K^*(t)$ and $R^*(t)$ for a non-growing resource.

If in this scenario the discount rate r is small enough (i.e. $r < 0.09$) and therefore the agent is at least slightly far-sighted, then Figure 2.2 shows that the agent will postpone the start of extraction of the resource and the smaller r gets, the longer the agent will wait. In this case with no growth rate of the resource, this kind of result could sound odd at first since there cannot be any gain by waiting for the resource to grow to a higher stock. But since a higher value of technology K of the agent also results in a higher efficiency of resource utilisation, it is natural, that the far-sighted agent is tempted to postpone extraction and invest in technology at first by applying control $u_1(t)$.

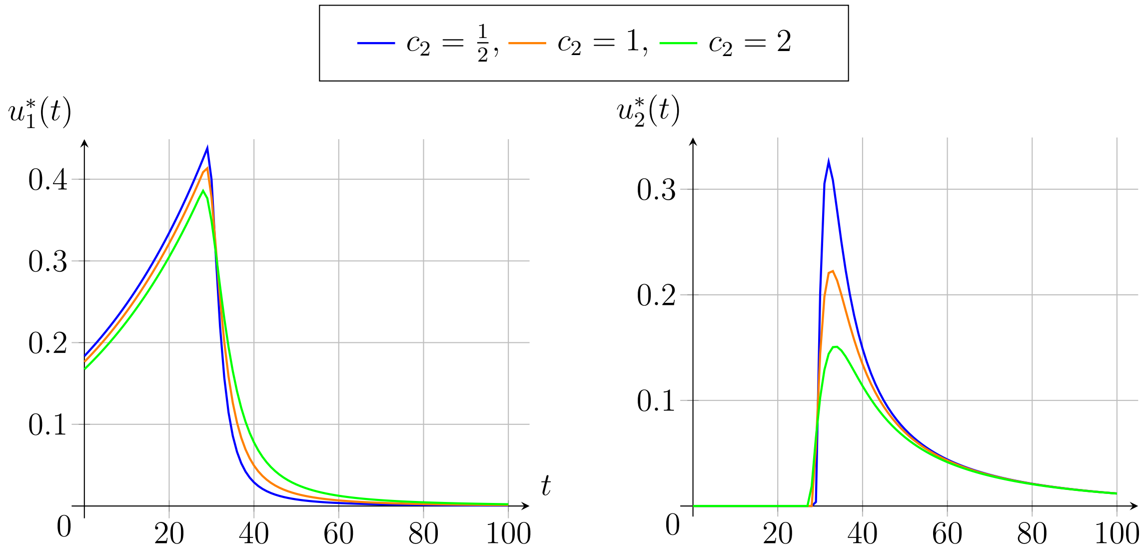
As can be seen in Figure 2.2, as soon as extraction is started for the first time, investing in K slows down immediately and $u_1(t)$ falls down to almost zero soon afterwards. This can be explained by the fact that the agent obviously wants to keep costs low. Costs are represented with the term $c_1 u_1(t)^2 + c_2 u_2(t)^2$. At that point when extraction starts, it starts immediately at a very high level and falls back to a more moderate extraction shortly afterwards. When the amount of resource becomes smaller, investment in further improvement of the technology becomes less reasonable, given that the overall costs stay low.

In order to understand how the two constants c_1 and c_2 actually influence J^*, Table 2.2 shows J^* depending on these two constants, where for now the discount rate and the depreciation rate of technology are fixed at $r = 0.01$ and $\delta = 0.02$.

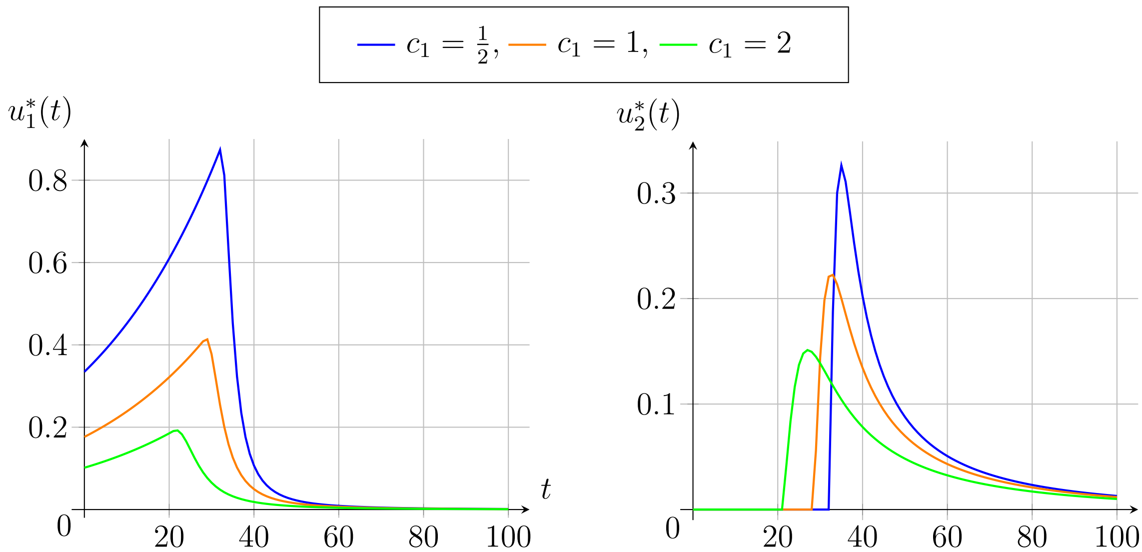
c_1 \ c_2	$\frac{1}{2}$	1	2
$\frac{1}{2}$	5.22512	4.89282	4.46901
1	2.62128	2.40248	2.13068
2	1.39499	1.25029	1.07629

Table 2.2: J^* depending on c_1 and c_2 for a non-growing resource.
$\delta = 0.02$, $r = 0.01$

The fact that an increasing c_1 has a much stronger influence on J^* than an increasing c_2 becomes clear once it is observed, that the control $u_1^*(t)$ is bigger than control $u_2^*(t)$ for almost the whole time horizon. Since there is no upper limit on technology and an increase in technology is possible in any state of the system (whereas the resource will be extinct at some level and cannot be extracted without any limit), it is reasonable that an agent invests in her technology in order to maximise her output. So, with a change of c_1 and c_2, which represent the marginal costs per unit of technology and the marginal costs per unit of extraction, the behaviour of the agent does change as well. This change of controls can be seen in Figure 2.3.



(a) $c_1 = 1$ fixed.



(b) $c_2 = 1$ fixed.

Figure 2.3: Optimal controls $u^*(t)$ depending on c_1 and c_2 for a non-growing resource.

Changing the price per unit of extraction c_2 does not change the behaviour that much: The only major influence on a higher c_2 is that in order to keep extraction costs per unit low, the resource extraction is more modest. Investment in technology has nearly a negligible change, but the higher costs c_2 motivate to keep investments in technology at a slightly higher level for a longer time than otherwise, in order to substitute the loss of less extraction with better sufficiency on how to use the very limited resource.

Changing the price per unit of technology c_1 does change the behaviour quite a lot: If the price per unit of technology is very cheap, then the agent tends to start with the extraction of the resource significantly later. As already mentioned above, a higher value of technology K results in a higher efficiency of resource utilisation. Therefore, if the price to increase technology is low, the agent has a higher incentive to invest in technology for a longer period of time. This is then a motivation to start extraction of the resource also at a later time, which will eventually result in a high value of the production function.

If, on the other hand, technology is expensive, then the agent tends to start with the extraction of the resource significantly earlier. In this case, the agent loses too much when investing in technology for a long period of time or when investing very extensively. The accumulated loss cannot be compensated by a higher efficiency of resource utilisation. This leaves the agent with the only option to start extraction as soon as technology has reached a reasonable level to gain most of the high technology.

Resource with exponential growth

In this case, the objective value can be influenced by the same parameters as in the case with a non-renewable resource and of course additionally by the growth rate of the resource δ_R, see (2.15)-(2.17). Again, the depreciation rate of technology is fixed at $\delta = 0.02$ and the two constants are chosen to be $c_1 = c_2 = 1$ for now.

The initial conditions are chosen as in the case with a non-renewable resource. When the optimal solution is computed with the chosen parameters, the behaviour of the optimal controls $u^*(t)$, and as a result of this controls, the change of technology K and the resource R, are plotted in Figures 2.4. The dashed lines indicate the time when the agent starts the extraction.

In this scenario, δ_R was chosen to be 0.002, which means that the resource would double itself approximately every $t = 347$, if there is no extraction at all. This seems to be a very small reproduction rate, but it is clear when observing the problem formulation as stated in (2.21)-(2.24) in detail, that the objective value of this optimal control problem easily tends to be unbounded if δ_R is big enough since there is no upper bound on the resource R. Seeing that the growth rate of the resource is exponential, the resource has no upper bound and the costs of extraction are only quadratic, this result cannot be avoided.

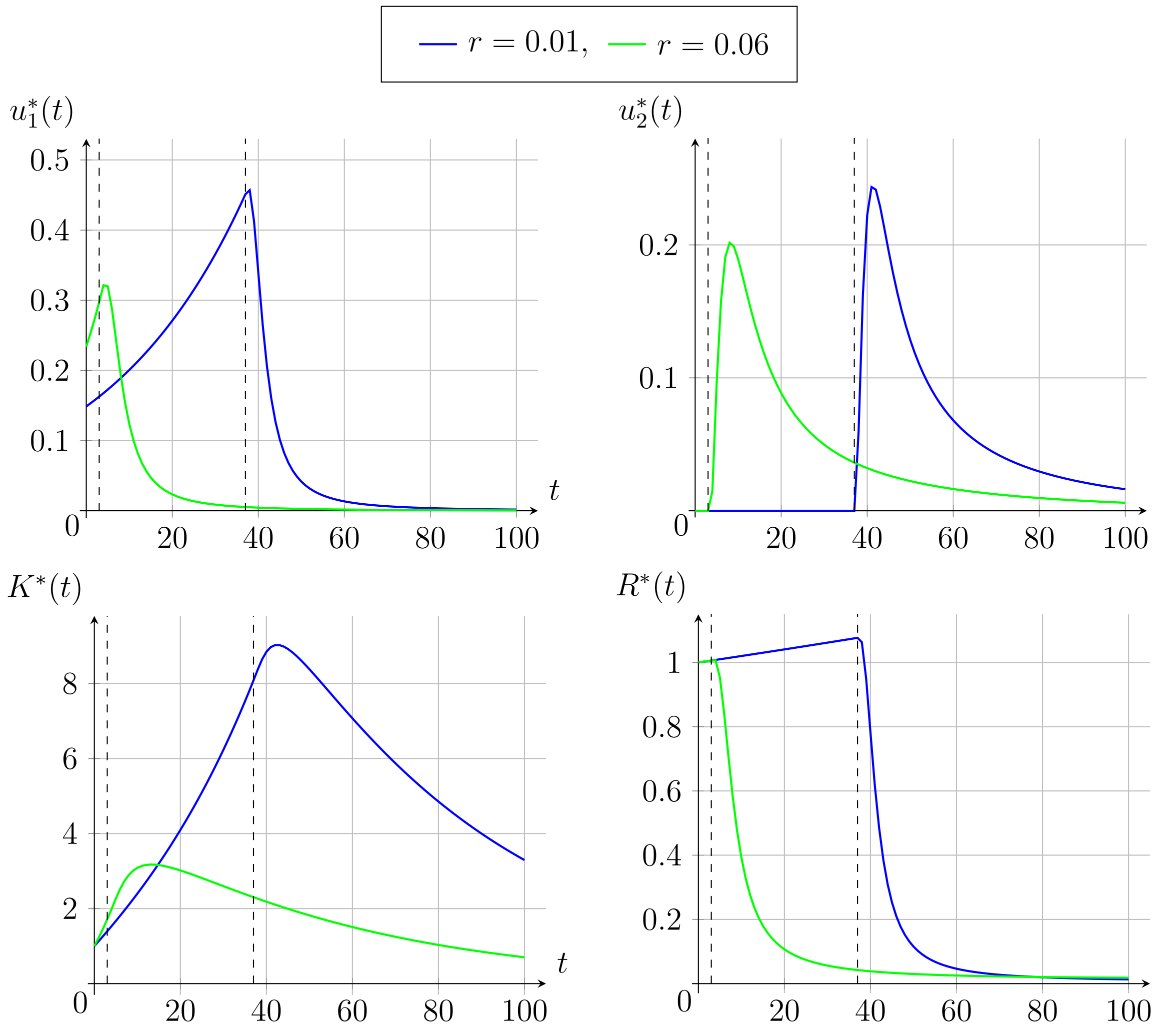


Figure 2.4: Optimal controls $u^*(t)$ and optimal states $K^*(t)$ and $R^*(t)$ for an exponential growing resource.

The agent would postpone extraction of the resource for the longest possible amount of time, since in the production function increasing returns of scale are assumed. Therefore, the resource gets the chance to grow indefinitely large. Just in the last period $T-1$, the agent extracts the whole resource at once, and the objective value grows indefinitely large as well. This is why in this analysis only solutions are presented where δ_R is small enough, which results in a situation where the resource has no chance to ever grow to infinity and will always tend to zero.

In this case, the qualitative behaviour of the player is similar to the case with no resource growth. A further motivation to postpone the start of extraction is now that the resource is growing to a higher level, which will result in a higher optimal objective value eventually. In the case of no growth and $r = 0.01$, the player waits until $t = 28$ to start with extraction. Now, with an exponential growth of the resource, the player waits nearly a third of the time longer until $t = 37$ to start with extraction. As already mentioned, the only reason for this further postponing is, that the resource can now grow to a higher level. This higher stock of the resource will yield a higher objective value. Also, the peaks of the two controls $u_1(t)$ and $u_2(t)$ are both higher in this case than in the one with no growth. Since the resource is growing, the player can obviously extract more.

Now, an analysis on how the two constants c_1 and c_2 influence J^* is given in Table 2.3. The parameters for the discount rate, the depreciation rate of technology and the resource

$c_1 \backslash c_2$	$\frac{1}{2}$	1	2
$\frac{1}{2}$	6.16194	5.81993	5.38029
1	3.05703	2.83070	2.54682
2	1.58083	1.43066	1.24803

Table 2.3: J^* depending on c_1 and c_2 with exponential resource.
$\delta = 0.02$, $r = 0.01$, $\delta_R = 0.002$

growth rate are again fixed at $r = 0.01$, $\delta = 0.02$ and $\delta_R = 0.002$.

For a discussion on these calculations, see the discussion to Table 2.2 in the case of non-growing resource, since the results are to be interpreted in the same fashion. The plots of the actual change of the optimal controls is of course slightly different in these two cases, but the qualitative results stay exactly the same.

Resource with logistic growth

In the case of a resource with a logistic growth behaviour, the depreciation rate of technology is again fixed at $\delta = 0.02$. The parameters to determine the growth dynamics of the resource are chosen to be $a = 0.025$ and $b = 0.05$, see (2.18)-(2.20). These parameters imply that the resource would double in approximately 150 years and will stay at this level afterwards, if there is no depletion at all. The two cost parameters are again chosen to be $c_1 = c_2 = 1$ for now and the initial conditions are chosen as in the case with a non-renewable resource. When in this scenario the optimal solution is computed, the behaviour of the optimal controls $u^*(t)$, and the resulting technology K and resource R are plotted in Figures 2.5. The dashed lines indicate the time when the agent starts the extraction.

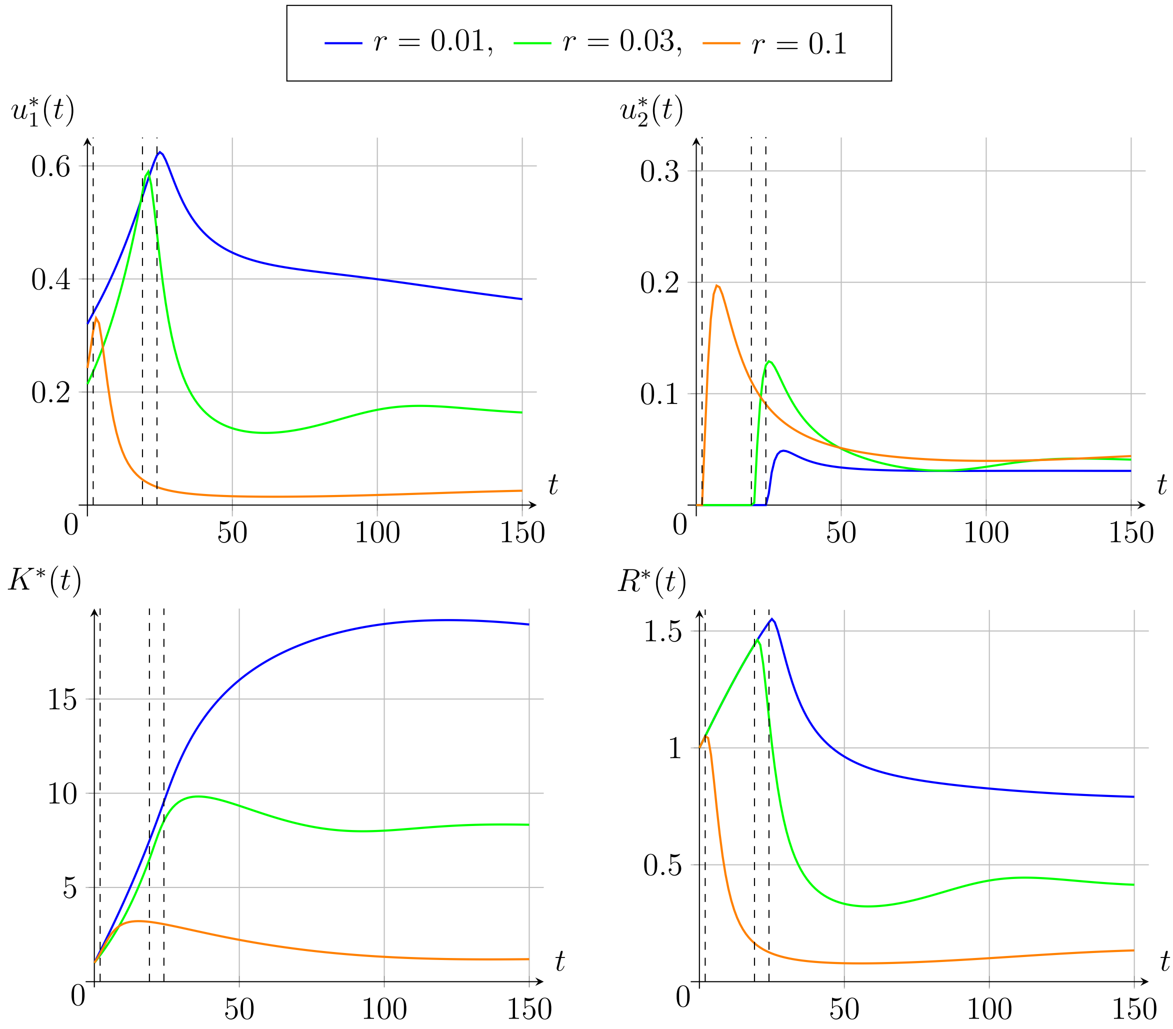


Figure 2.5: Optimal controls $u^*(t)$ and optimal states $K^*(t)$ and $R^*(t)$ for a logistic growing resource.

In contrast to the prior two cases, it is possible that the resource can grow to a sustainable level throughout time, since the logistic growth rate implies a natural upper bound on the resource and the effect, that the resource grows indefinitely large which results in an unbounded optimisation problem as it could happen in the case with an exponential growth, cannot occur anymore. The possibility to have a sustainable resource at hand will result in a much higher J^* if the agent is far-sighted enough and does not start extraction with a rate too high too soon.

A direct comparison of values with previous results is not reasonable anymore since the behaviour of the resource is completely different now, but it can be seen in Figure 2.5, that the start of extraction is nearly the same for discount rates in the range of $r \in \{0.01, 0.03\}$. But the control $u_2(t)$ does differ on the initial extraction rate, which is more than double the amount for $r = 0.03$ than it is for $r = 0.01$. In all cases, even the ones with $r > 0.03$, $u_2(t)$, the rate of extraction, does get very stable shortly after the initial extraction.

Another big difference between the different optimal strategies for different r is the level at which the resource is kept sustainable. In Figure 2.5 it can be seen very clearly that a more far-sighted problem formulation results in a much higher sustainable level of the resource. The behaviour of control $u_1(t)$ and therefore also of technology K is completely different to the prior two cases: Instead of the immediate backdrop of the control $u_1(t)$ as soon as extraction has set in, in the logistic growth case a more or less stable level of investment in technology is applied. Since extraction is spread over the whole time horizon, and not focused on one short time period where the extraction rate is huge compared to other time intervals as it happens in the first two cases, the agent has a motivation to keep technology at a high level all the time.

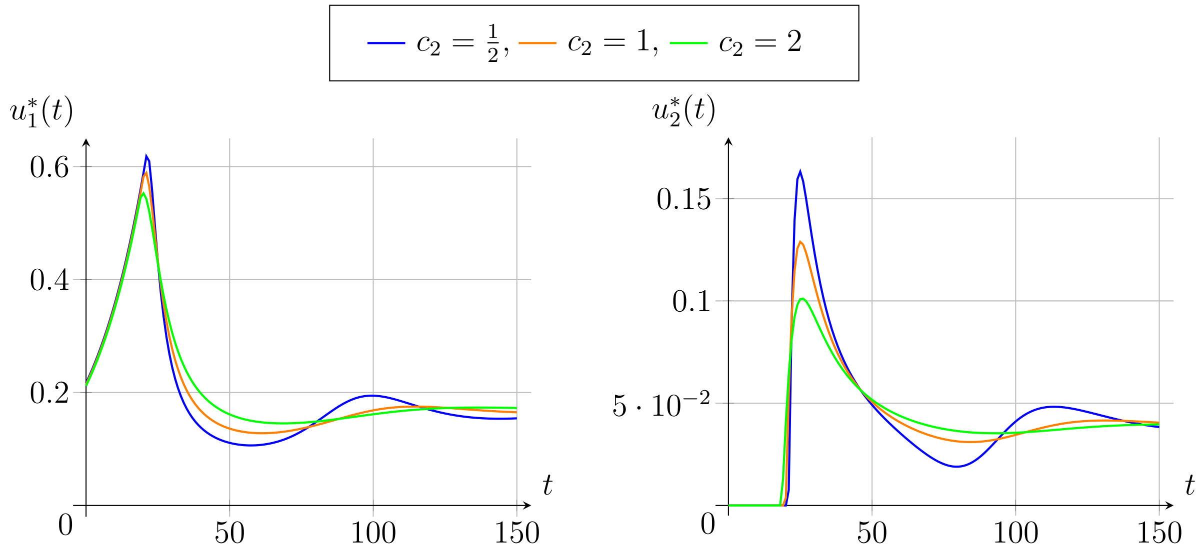
An analysis on how the two constants c_1 and c_2 influence J^* is given in Table 2.4. The parameters for the discount rate, the depreciation rate of technology and the resource growth rate are now fixed at $r = 0.03$, $\delta = 0.02$, $a = 0.025$ and $b = 0.05$.

c_1 \ c_2	$\frac{1}{2}$	1	2
$\frac{1}{2}$	7.16386	7.10134	6.99459
1	3.77251	3.71483	3.62332
2	2.08266	2.03480	1.95871

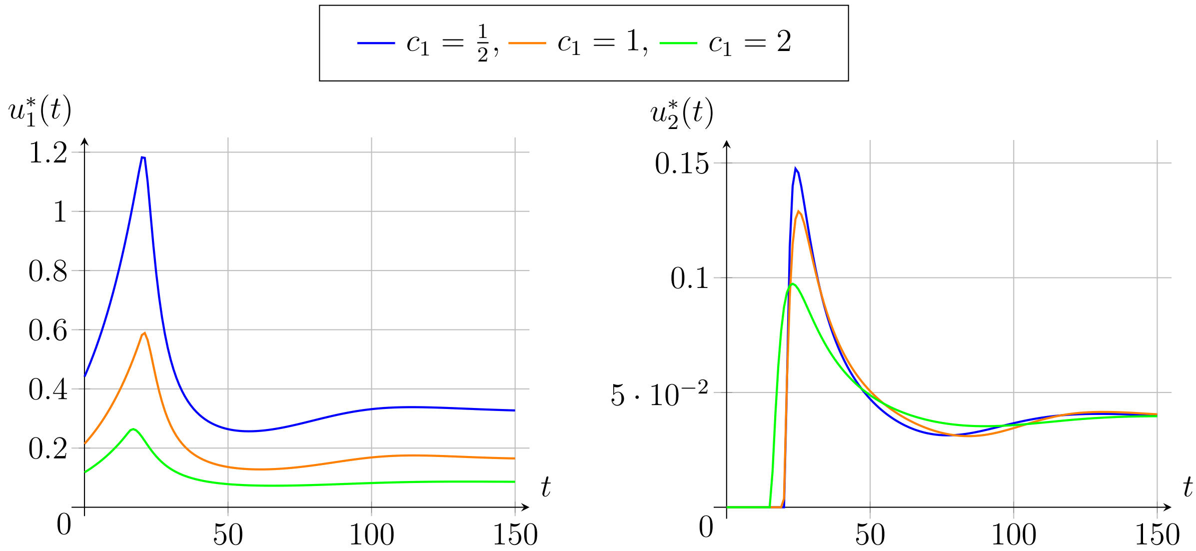
Table 2.4: J^* depending on c_1 and c_2 with logistic resource.
$\delta = 0.02$, $r = 0.03$, $a = 0.025$, $b = 0.05$

The fact that an increasing c_1 has a much stronger influence on J^* than an increasing c_2, holds for the same reason as already discussed in the case with a non-growing resource, see page 25. The actual change of controls can be seen in Figure 2.6. Increasing the costs per unit of resource extraction c_2 does not really change the behaviour of both controls that much. Both controls tend to get more stable, instead of a more dynamic structure if

c_2 is small. Needless to say, a high c_2 results in $u_2^*(t)$ being much lower when extraction starts than if c_2 was small.



(a) $c_1 = 1$ fixed.



(b) $c_2 = 1$ fixed.

Figure 2.6: Optimal controls $u^*(t)$ depending on c_1 and c_2 with a logistic resource.

The case of increasing costs per unit technology c_1 is of more interest. Here the influence on J^* is more extreme, which can be explained by the agents' need to keep technology K always at a very high level. If obtaining technology gets more expensive, the objective value will suffer and the investment towards technology is reduced to a minimum in order to keep the high costs down. It is of interest, that the time the agent starts extraction is only changed marginally to an earlier moment once technology gets more expensive, in sharp contrast to the two prior cases with a non-renewable resource and an exponential resource, where the time shift was very significant. It is also of relevance that the rate of extraction stabilises, starting at around $t = 100$, and is the same in all three cases and does not depend on c_1 at all.

Only the peak extraction gets lower, which is reasonable, considering that technology is at a much lower level and therefore the loss in the production function resulting from a lower technology cannot be balanced out by a higher extraction rate, since extraction imposes quadratic costs.

When the resource is kept at a sustainable level, a further remark is, that in a finite control problem the agent will extract the whole resource completely for time towards T, but only a time horizon $[0, 150]$ is plotted here.

One agent alone in the economy is interested in letting the resource stock grow to a sustainable level. As the above results show, the optimal set of controls guarantees such a sustainable level only in the case where the resource is modelled with a logistic growth, which is a more realistic model than that of exponential growth. Also, the more far-sighted an agent is, the longer will the initial extraction be postponed to the future and the level of technology can rise to an optimum, such that resource extraction is most efficient.

Chapter 3

An optimal growth model of multiple agents utilising a common resource: Nash equilibria

In this chapter, the optimal control problem introduced in Chapter 2 will be utilised in a game-theoretic approach, where now $N > 1$ agents are sharing a common resource. All N agents are in competition with each other and every single agents' goal is to maximise her own output. An analysis depending on the nature of the resource will be given in detail. The focus will be on a two-agent game, finding their open-loop Nash equilibria and understanding the different behaviour which arises if an agent is far-sighted or tends to be short-sighted.

3.1 Problem formulation

Let the economy consist of N agents, $\ell = 1, 2, \dots, N$, where one agent could be a company, a country or a group of countries. All N agents have access to one and the same resource (with the stock of the resource denoted by $R(t)$ at time t), which will be, as in Chapter 2, either of non-renewable or of renewable nature. All N agents can again invest in their technology (with the level of the technology denoted by $K(t)$ at time t). The aim of each agent is to maximise her own economic output by extracting the resource, where the extraction of each agent depends on the richness of the resource and the extraction efforts. The model for one individual agent i is exactly the same as in Chapter 2. The new and interesting feature in this chapter is the fact, that all N agents share the same resource which leads to a competition between the agents and requires a game-theoretic approach. Each agent has to decide for her own on the best strategy of extraction and investment in technology in order to maximise her economic output, whilst knowing that $N-1$ other agents have access to the same resource and face the same problem.

The optimal control problem for the i-th agent is very similar to the optimal control problem (2.1)-(2.4) with only one agent in the economy, and can be formulated as follows:

$$\textit{Maximise} \qquad J^i(u^i(t)) = \int_0^\infty e^{-r^i t}\Big(K^i(t)u_2^i(t)R(t) - c^i(u_1^i(t)) - d^i(u_2^i(t))\Big)dt \tag{3.1}$$

subject to

$$\dot{K}^i(t) = -\delta^i K^i(t) + u_1^i(t), \quad K^i(0) = K_0^i, \tag{3.2}$$

$$\dot{R}(t) = \varrho(t) - \sum_{j=1}^{N} u_2^j(t)R(t), \quad R(0) = R_0, \tag{3.3}$$

$$u_1^i(t), u_2^i(t) \geq 0 \quad \forall t, \tag{3.4}$$

with the substantial difference that here also the controls of the other agents appear in (3.3). Here a dot over variables again denotes the derivative with respect to time t, $r^i \geq 0$ is the discount rate and $\delta^i > 0$ represents the depreciation rate of technology. Player-specific variables, functions and parameters contain an upper index ℓ, which indicates the individual agent's number. As in Chapter 2, the two terms $c^i(u_1^i(t))$ and $d^i(u_2^i(t))$ represent the costs for the i-th agent when investing in technology and when the resource is extracted, respectively. Both terms are defined in the same way again, $c^i(u_1^i) = c_1^i {u_1^i}^2$ and $d^i(u_2^i) = c_2^i {u_2^i}^2$ with c_1^i and c_2^i two constants, representing the price per unit of technology and price per unit of extraction. The term $\varrho(R)$ indicates the resource growth as introduced in Chapter 2, (2.5)-(2.7).

3.2 Mathematical analysis

The focus in this section will be on a non-cooperative game involving two agents $N = 2$, where the first agent is modelled as a far-sighted agent and the second agent as a short-sighted agent. Thus, the two optimal control problems, as stated in (3.1)-(3.4) with

$i \in \{1,2\}$, need to be solved simultaneously. In order to formulate the problem as a 2-agent differential game, Definition 1.3 will be applied.

Let $x(t) \in \mathbb{R}^3, x(t) = (K^1(t), K^2(t), R(t))^T$ be the 3-dimensional state vector and $u(t) \in \mathbb{R}^4, u(t) = (u_1^1(t), u_2^1(t), u_1^2(t), u_2^2(t))^T$ the 4-dimensional control vector of the system. The dynamics of the system is described by three differential equations

$$\begin{aligned} \dot{K}^i(t) &= -\delta^i K^i(t) + u_1^i(t), \quad \text{for} \quad i = 1,2, \\ \dot{R}(t) &= \varrho(R) - \left(u_2^1(t) + u_2^2(t)\right) R(t), \end{aligned} \tag{3.5}$$

with initial states

$$\begin{aligned} K^i(0) &= K_0^i, \quad \text{for} \quad i = 1,2, \\ R(0) &= R_0. \end{aligned} \tag{3.6}$$

The controls $u^i : [0,T] \mapsto \mathbb{R}^2$ satisfy $u^i(t) \in [0,\infty) \times [0,\infty)$. The goal of both agents is to maximise her own payoff, which is

$$J^i(u^1(t), u^2(t)) = \int_0^T e^{-r^i t} \left(K^i(t) u_2^i(t) R(t) - c_1^i u_1^i(t)^2 - c_2^i u_2^i(t)^2\right) dt. \tag{3.7}$$

Both agents have full knowledge of the system dynamics, the admissible set of controls for both players, the payoff functions of both players and the initial state of the system.

The necessary conditions for an open-loop Nash equilibrium $(\tilde{u}^{1*}, \tilde{u}^{2*})$ of the two-agent game (3.5)-(3.7), as introduced in Theorem 1.6, are then for this problem, $i \in \{1,2\}$:

$$\textit{(i)} \quad \frac{\partial}{\partial t} \tilde{K}^{i*}(t) = -\delta^i \tilde{K}^{i*}(t) + \tilde{u}_1^{i*}(t), \tag{3.8}$$

$$\frac{\partial}{\partial t} \tilde{R}^*(t) = \varrho(\tilde{R}^*) - \left(\tilde{u}_2^{1*}(t) + \tilde{u}_2^{2*}(t)\right) \tilde{R}(t), \tag{3.9}$$

$$\textit{(ii)} \quad \tilde{u}^{i*}(t) = \arg\max_{u^i \in U^i} H^i(t, \tilde{\eta}^i(t), \tilde{x}^*(t), u^{1*}(t), u^{2*}(t)), \tag{3.10}$$

$$\textit{(iii)} \quad \frac{\partial}{\partial t} \tilde{\eta}^i(t) = r^i \tilde{\eta}^i(t) - \frac{\partial}{\partial x} H^i(t, \tilde{\eta}^i(t), \tilde{x}^*, \tilde{u}^{1*}, \tilde{u}^{2*}), \tag{3.11}$$

$$\tilde{\eta}^i(T) = 0, \tag{3.12}$$

where

$$\begin{aligned} H^i(t, \eta, x, u^1, u^2) := {} & K^i u_2^i R - c_1^i {u_1^i}^2 - c_2^i {u_2^i}^2 \\ & - \eta_1^i (u_1^i - \delta K^i) + \eta_2^i (\varrho(R) - u_2^1 R - u_2^2 R) \end{aligned}$$

denotes the current-value Hamiltonian of the i-th agent, where $\eta^i(t) = (\eta_1^i, \eta_2^i)^T$ is the 2-dimensional co-state vector of agent i.

Obtaining an analytical solution is not part of this thesis, but a numerical approach is stated in the following section.

3.3 A numerical solution

To obtain an open-loop Nash equilibrium $(\tilde{u}^{1*}, \tilde{u}^{2*})$ of the two-agent game (3.5)-(3.7), the necessary conditions (3.8)-(3.12) need to be satisfied. To find this open-loop Nash equilibrium numerically, both agents' optimal control problem, that arises from the two-agent game, need to be solved simultaneously. The optimal control problem that both agents need to solve, can be written as follows:

$$\text{\textit{Maximise}} \qquad J^i(u^i(t)) = \int_0^T e^{-r^i t}\Big(K^i(t)u_2^i(t)R(t) - c_1^i u_1^i(t)^2 - c_2^i u_2^i(t)^2\Big)dt \tag{3.13}$$

subject to

$$\dot{K}^i(t) = -\delta^i K^i(t) + u_1^i(t), \quad K^i(0) = K_0^i, \tag{3.14}$$

$$\dot{R}(t) = \varrho(R) - \Big(u_2^1(t) + u_2^2(t)\Big)R(t), \quad R(0) = R_0, \tag{3.15}$$

$$u_1^i(t), u_2^i(t) \in U, \quad t \in [0, T], \tag{3.16}$$

where $U = [0, \infty)$.

To obtain an open-loop Nash equilibrium one needs to solve a fixed point problem, as explained below: A pair of admissible controls (u^{1*}, u^{2*}), $u^{i*} = (u_1^{i*}, u_2^{i*})$, needs to be found, such that u^{1*} is an optimal control for the problem of agent 1 if agent 2 chooses her control as $u^2 = u^{2*}$, and u^{2*} is an optimal control of agent 2 if agent 1 chooses her control as $u^1 = u^{1*}$. That is, if $\hat{u}^1[u^2]$ is an optimal control for agent 1, given that agent 2 chooses a control u^2, and if $\hat{u}^2[u^1]$ is an optimal control for agent 2, given that agent 1 chooses a control u^1. Then the fixed point problem that is to be solved is to find controls u^{1*} and u^{2*} such that

$$\hat{u}^1[u^{2*}] = u^{1*}, \qquad \hat{u}^2[u^{1*}] = u^{2*}.$$

The iterative procedure that is applied is described in the next paragraphs.

If agent 1 calculates her optimal controls u^{1*} for her optimal control problem, an assumption of agent 2's optimal control u^{2*} must first be made. If this assumption differs from the open-loop Nash equilibrium $\tilde{u}^{2*}$, then this implies that the stock of the resource $R^*(t)$ differs from $\tilde{R}^*(t)$, which is the resource stock in the open-loop Nash equilibrium. The common resource stock will change through the optimisation process of agent 1. Since the assumed optimal control of agent 2 was optimal for the common resource stock before the resource stock was changed through agent 1's optimisation procedure, the assumed control for agent 2 may not still be optimal anymore. Therefore, agent 2 will calculate her optimal controls u^{2*} again, taking agent 1's optimal control and the new common resource stock into account. This optimisation process will again lead to a change in the common resource stock, as long as the optimal control of agent 1, which is an exogenous variable now for agent 2, differs from the open-loop Nash equilibrium.

This process will eventually converge to a fixed point, which is exactly the open-loop Nash equilibrium $(\tilde{u}^{1*}, \tilde{u}^{2*})$. First, this problem needs to be discretised again, as already stated in Chapter 2. The discretised problem can then be solved numerically by the following steps:

(i) Find an initial and admissible set of controls $\mathbf{u}_j^{i,k} \in \mathbb{R}^N$, for both agents $i \in \{1,2\}$ and both controls $j \in \{1,2\}$ for each agent. Set $k := 0$ and set the initial objective values of both agents to $J^{i,k} = J^i(\mathbf{u}^{i,k})$.

(ii) Solve the optimal control problem (3.13)-(3.16) for agent 1 as provided in Chapter 2, using the control $\mathbf{u}^{2,k}$ of agent 2 as an input parameter. The optimal objective value of this control problem will be denoted by $J^{1,k+1}$ and the optimal control will be denoted as $\mathbf{u}^{1,k+1}$.

(iii) Solve the optimal control problem (3.13)-(3.16) for agent 2 as provided in Chapter 2, using the control $\mathbf{u}^{1,k+1}$ of agent 1 as an input parameter. The optimal objective value of this control problem will be denoted by $J^{2,k+1}$ and the optimal control will be denoted by $\mathbf{u}^{2,k+1}$.

(iv) If $\|J^{1,k+1} - J^{1,k}\| < \varepsilon$ and $\|J^{2,k+1} - J^{2,k}\| < \varepsilon$ holds, stop this algorithm and the Nash equilibrium is obtained by the pair $(\mathbf{u}^{1,k+1}, \mathbf{u}^{2,k+1})$, which is the fixed point of this problem. Otherwise increase $k := k + 1$ and proceed at (ii).

The implementation of this numerical algorithm is done in Matlab, where the convergence to the Nash equilibrium is established normally in less than 10 iterations, depending on the chosen parameter set.

In order to see how the optimal solutions of both agents change when different parameters of the optimal control problems are adapted, a sensitivity analysis will be discussed next.

3.3.1 Sensitivity analysis and interpretation of results

As in Chapter 2, the sensitivity analysis will be discussed for the three different cases of resource growth $\varrho(R)$.

Non-renewable resource

Let the two cost parameters be $c_1^i = c_2^i = 1$. The initial conditions are chosen to be $u_0^i = (0,1)^T$ and $x_0^i = (K_0^i, R_0)^T = (1,1)^T$, $i \in \{1,2\}$. This implies that both agents start with the same amount of physical capital stock K_0^i. An analysis of how the optimal solutions and optimal controls change for both agents, when the discount rates r^i and the depreciation rates of technology δ^i change, is given first. If the parameters are chosen as $\delta^1 = \delta^2$ and $r^1 = r^2$, then the optimal solution of the 2-agent game is completely symmetric and both agents will adopt the same strategy. In this symmetric case, both agents will gain the same maximum possible objective value.

r^2	0.01	0.03	0.06	0.1	0.14
J^{1*}	0.3497	0.3393	0.3287	0.3194	0.3131
J^{2*}	0.3497	0.3266	0.3004	0.2744	0.2542

Table 3.1: Comparison of J^{i*}, r^1 fixed, r^2 changing, with non-growing resource.
$r^1 = 0.01$, $\delta^1 = \delta^2 = 0.02$, $c_1^i = c_2^i = 1$, $i \in \{1,2\}$

As soon as one agent is replaced by a slightly more short-sighted agent, the first agent will be worse off! On the other hand, if one agent is replaced by a slightly more far-sighted agent, then the first agent will be better off. This is why this scenario will not be discussed any further. For detailed calculations of the optimal objective values and how the optimal controls change of the first agent, when the second agent is changing to a more and more short-sighted agent, see Table 3.1 and Figure 3.1. In Figure 3.2 it can be seen how the optimal states K^* and R^* change, if the discount rates change. In these figures, the symmetric case with $r^1 = r^2 = 0.01$, is compared to the following problem formulation S, that agent 1 keeps her far-sighted formulation ($r^1 = 0.01$), but agent 2 is replaced by a more short-sighted agent as opposed to the symmetric case ($r^2 = 0.06$). In both figures, the dashed black line represents the symmetric case, the blue line represents agent 1 and the green line agent 2 in the problem formulation S. The orange line represents the optimal resource stock in problem formulation S.

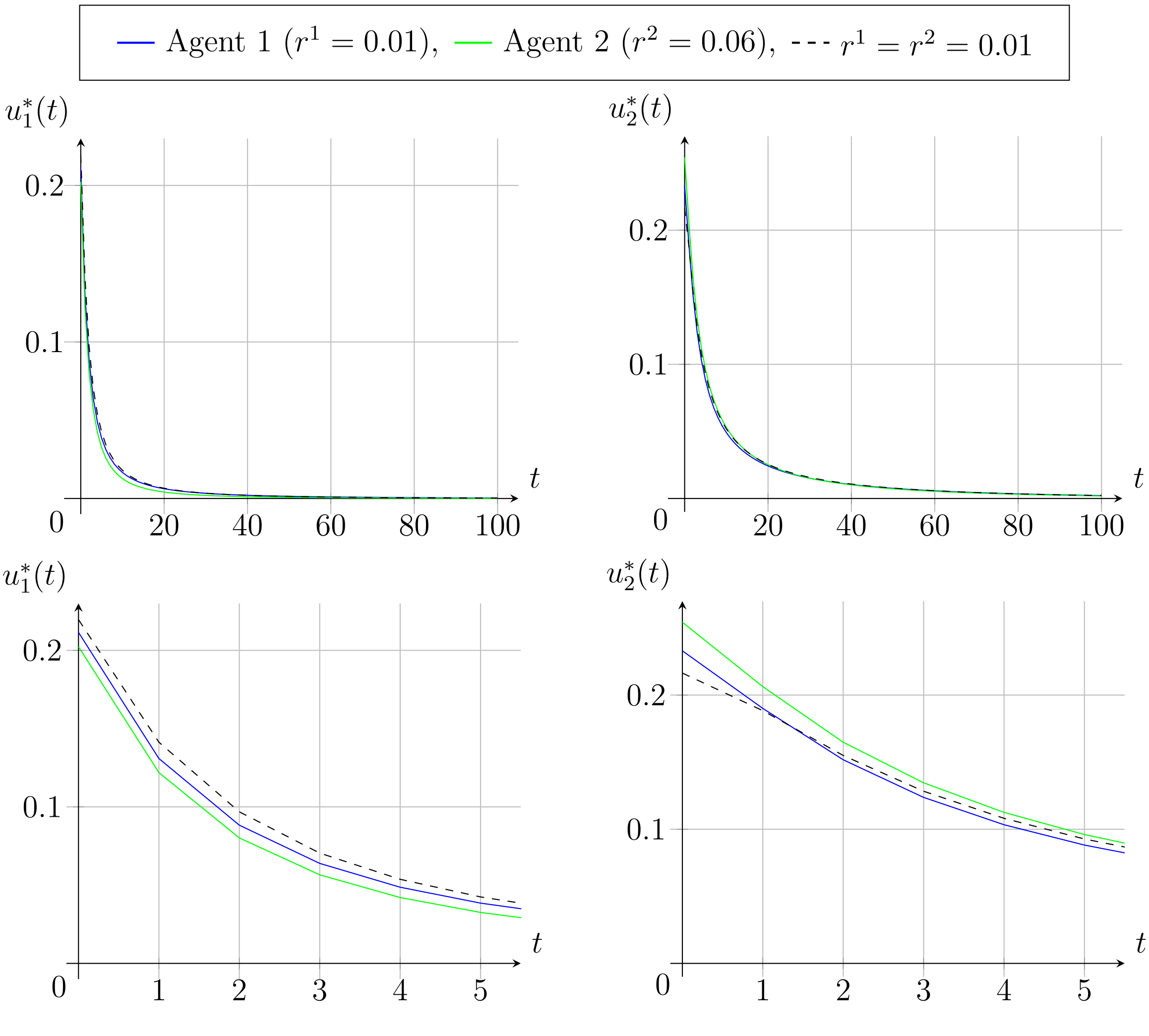


Figure 3.1: Optimal controls $u^*(t)$ depending on r^2 with a non-growing resource.
$r^1 = 0.01$ fixed

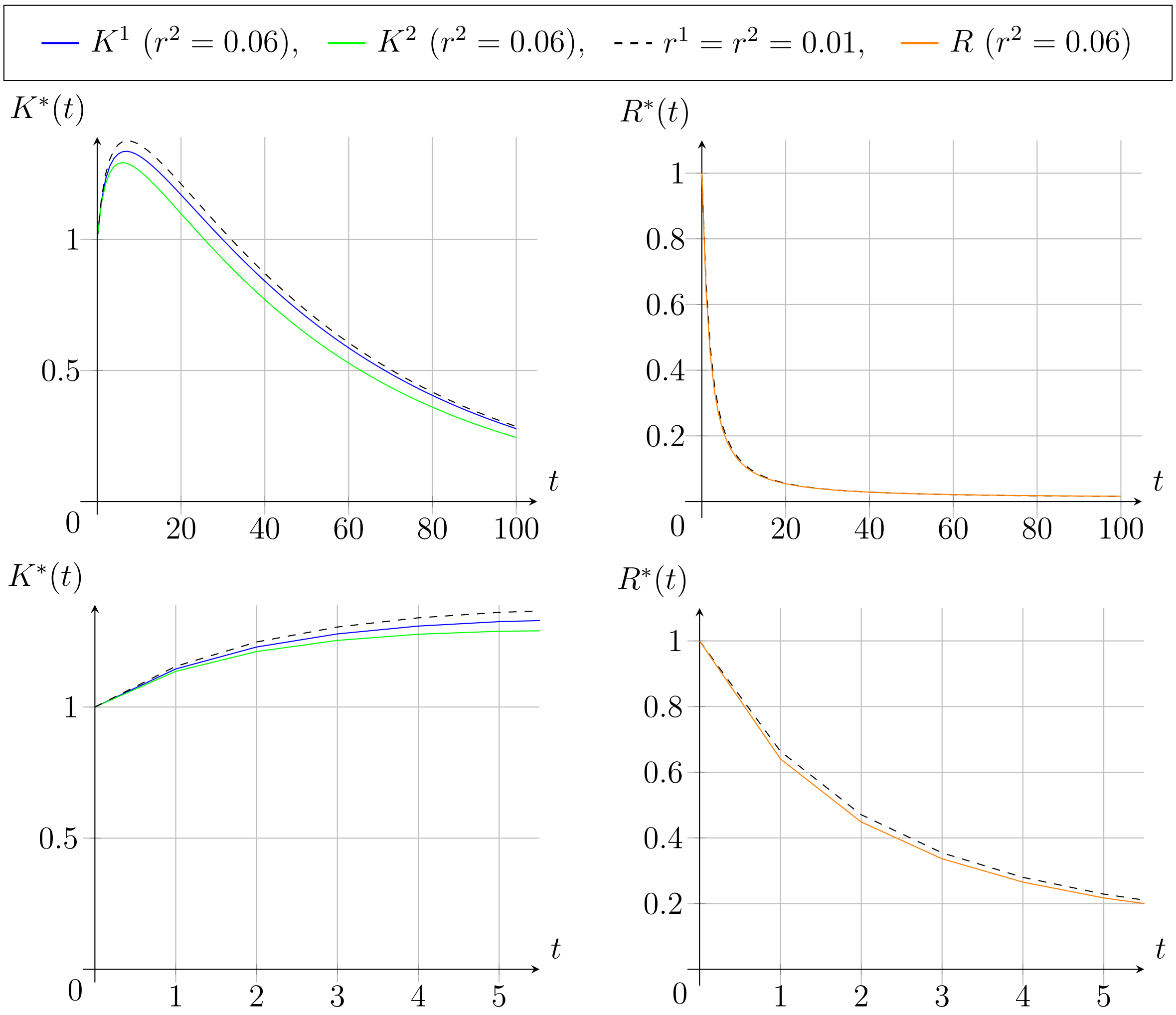


Figure 3.2: Optimal states $K^*(t)$ and $R^*(t)$ depending on r^2 with a non-growing resource. $r^1 = 0.01$ fixed

When the solutions are compared to the one-agent scenario, the difference that catches the eye immediately is that extraction of the resource is not postponed at all. Even for two very far-sighted agents it is not optimal to wait for the technology to reach a higher level, such that extraction of the resource will be more efficient. Game-theoretically, if one agent decides to postpone extraction for only an instance, it would give the other agent an advantage on the available resource, on which the first agent could never catch up again. Therefore, postponing the extraction is no part of the optimal solution.

Let from now on denote the far-sighted agent as agent 1 and the short-sighted agent as agent 2. It is striking, that even agent 1, who maintains the same level of far-sighted strategy will be worse off, only because agent 2 maintains a shorter-sighted strategy instead of the same level of far-sighted strategy. This behaviour can be explained as follows. Since agent 2 sustains a short-sighted strategy, her optimal objective value will increase if extraction starts at a very high level. This high rate of extraction reduces the resource stock drastically. Therefore, agent 1 will also adopt a strategy where her extraction starts at a very high level, in order not to miss out on the resource and not leaving everything to agent 2. In this behaviour there is significantly shorter time to gain a high level of technology K^i, which, in return, would yield in a much bigger optimal objective value. So, the loss both agents are facing is the result of the lower rate of technology when the resource is extracted. This result and behaviour does not depend on the initial values of x^{i0}, which is interesting enough on its own but the same considerations as mentioned above do apply, regardless of the difference of the initial level of technology of the two agents.

The depreciation rate of technology will be assumed to be equal for both agents at $\delta^1 = \delta^2 = 0.02$. The influence of the constants c_1^i and c_2^i will only be discussed briefly, since the structural behaviour is similar to the behaviour with only one agent in the economy. It is clear that higher values of c_1^i and c_2^i result in a less productive and efficient production function, since the costs to increase technology and to extract the resource have risen. Therefore, for instance in the case where $c_1^1 > c_1^2$ and $c_2^1 > c_2^2$, the optimal objective value of agent 1 will decrease and the objective value of agent 2 will increase, the bigger the difference between the constants get. Furthermore, the percental decrease agent 1 will be confronted with will be bigger than the percental increase agent 2 will enjoy, since agent 1 has as a far-sighted agent a smaller rate of discount r and is therefore more volatile to changes in costs. In the case that $c_1^1 < c_1^2$ and $c_2^1 < c_2^2$, the behaviour is vice versa to the one just described above, with the percental increase of agent 1 to be bigger than the percental decrease of agent 2, as the difference between the constants increases.

Resource with exponential growth

Again, let the two constants be $c_1^i = c_2^i = 1$ for now, leave the depreciation rates of technology at $\delta^1 = \delta^2 = 0.02$ and let the two agents be as described above. With the initial conditions as chosen in the case with a non-renewable resource, the optimal solutions for both players depending on the resource growth rate δ_R can be seen in Table 3.2.

δ_R	0	0.001	0.003	0.006	0.01	0.02
J^{1*}	0.3287	0.3307	0.3346	0.3406	0.3496	0.3763
J^{2*}	0.3004	0.3019	0.3053	0.3104	0.3179	0.3406

Table 3.2: Comparison of J^{i*}, δ_R changing with an exponential resource growth.
$\delta^1 = \delta^2 = 0.02$, $r^1 = 0.01$, $r^2 = 0.06$, $c_1^i = c_2^i = 1$, $i \in \{1, 2\}$

With $\delta_R = 0.006$, the behaviour of the optimal controls u^{i*} and the behaviour of the optimal states K^{i*} and R^* can be seen in Figure 3.3.

The optimal solutions in the case with an exponential growth rate compared to the case with no growth rate of the resource are very similar in the scenario where nearly all parameters are equal for both agents, apart from the discount rates r^i. However, the optimal controls do change significantly, as soon as the constants c_1^i and c_2^i are changed, see Figure 3.4.

Here, the constants where chosen as $c_1^1 = c_2^1 = 0.5$ and $c_1^2 = c_2^2 = 2$. This means, if the already far-sighted agent 1 gets an advantage on the costs of investment in technology as well as on the costs of resource extraction over agent 2, then agent 1's initial investment in technology is more than double the amount of the original value (where $c_i^i = 1, \forall i \in \{1, 2\}$), and extraction of the resource is postponed. However, this delay of extraction of the resource by agent 1 is only for a very short period. Extraction cannot be postponed too long, since otherwise agent 2 would take advantage by extracting a very big amount of the resource, despite her higher costs. But the very short period agent 1 uses to postpone extraction is very important to gain a better objective value for her, since her extraction of the resource then is more efficient since the rate of technology will be already higher once extraction starts. The optimal objective values in this scenario then are for agent 1

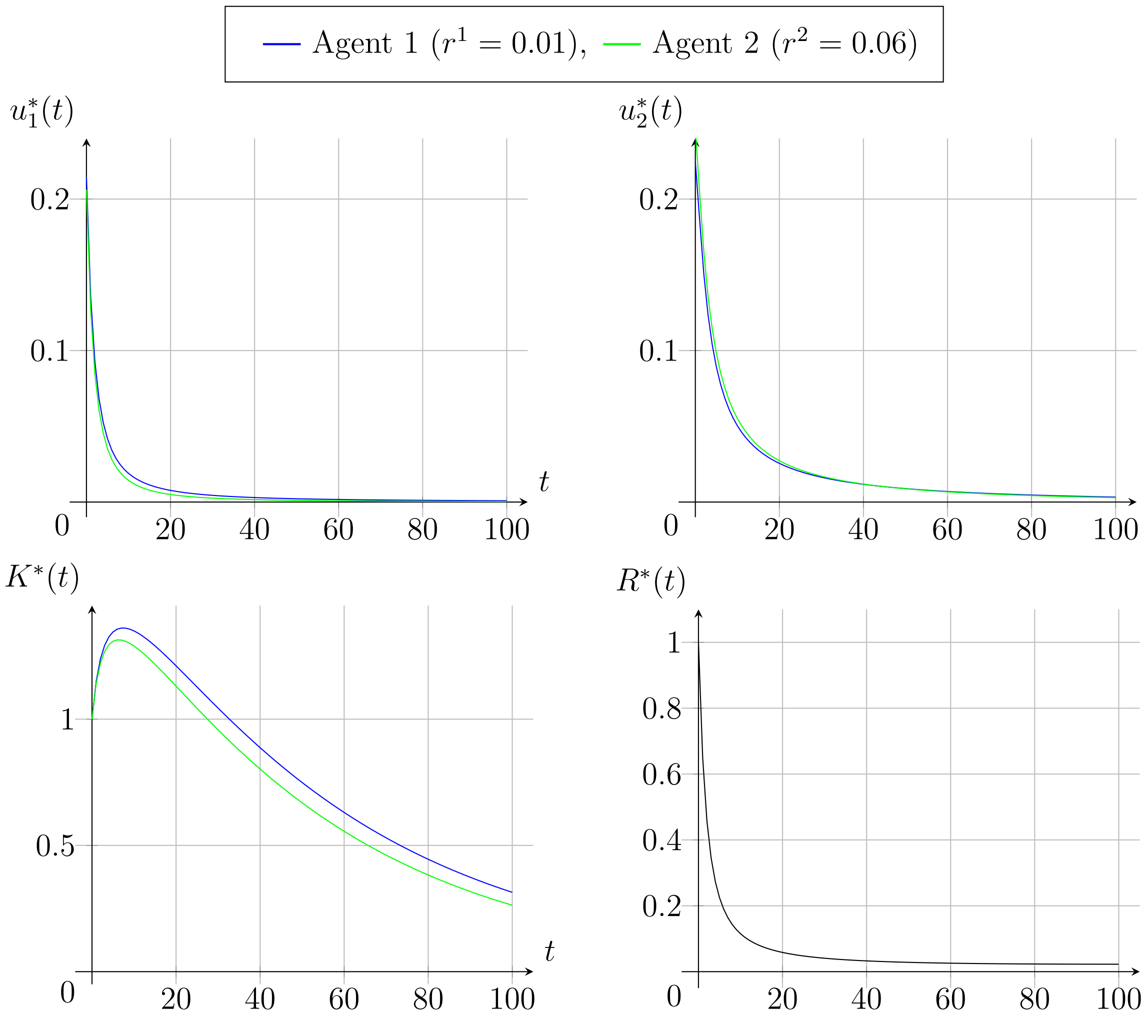


Figure 3.3: Optimal controls and states $u^{i*}(t)$, $K^*(t)$ and $R^*(t)$ with an exponential resource growth.

$\delta^1 = \delta^2 = 0.02$, $r^1 = 0.01$, $r^2 = 0.06$, $c_1^i = c_2^i = 1$, $i \in \{1, 2\}$

$J^{1*} = 0.6972$ and for agent 2 $J^{2*} = 0.2571$. The objective value for agent 1 increased by more than 100% whereas the objective value of agent 2 declined by mere 17%, albeit agent 2's costs being in total four times higher. This results already indicate that a potential policy measure could be reasonable as it will be discussed in Chapter 4, since the loss agent 2 has to take into account is by far outscored by the gain agent 1 obtains.

Resource with logistic growth

In the case where the resource has a logistic growth, again let the two cost parameters be $c_1^i = c_2^i = 1$ for now, the depreciation rates of technology $\delta^i = 0.02$ and the initial

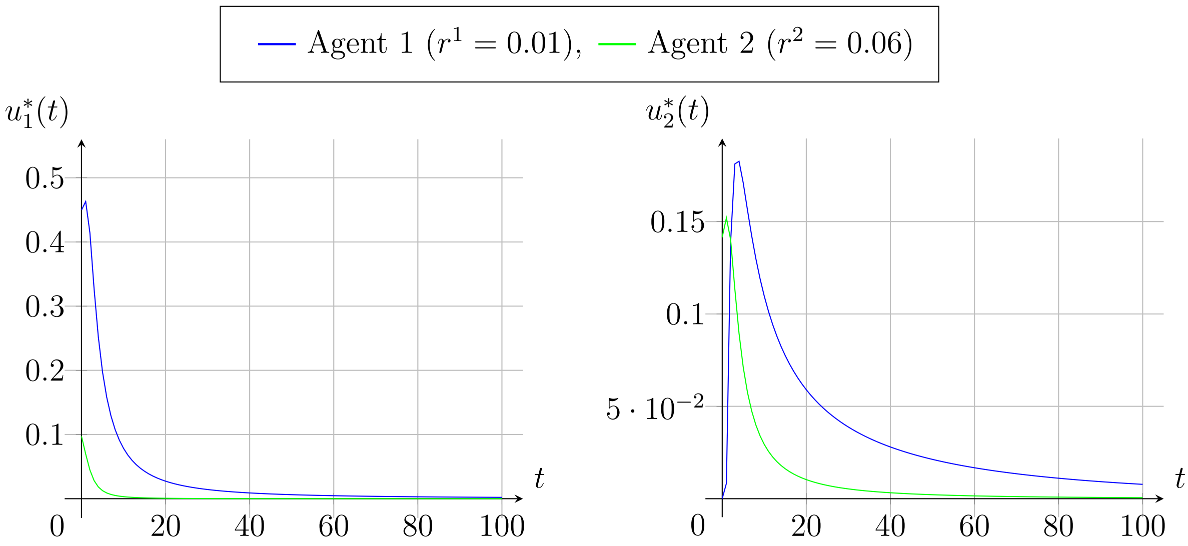


Figure 3.4: Optimal controls $u^{i*}(t)$ with exponential resource growth.
$c_1^1 = c_2^1 = 0.5$, $c_1^2 = c_2^2 = 2$, $\delta^1 = \delta^2 = 0.02$, $r^1 = 0.01$, $r^2 = 0.06$

conditions are chosen as in the case with a non-renewable resource. The parameters which determine the dynamics of the resource are chosen to be $a = 0.01$ and $b = 0.02$, see (2.7). To see how the discount rates of the agents influence the optimal objective values J^{i*}, see Table 3.3.

r^2	0.01	0.03	0.06	0.1	0.14
J^{1*}	0.4136	0.3858	0.3666	0.3520	0.3428
J^{2*}	0.4136	0.3692	0.3306	0.2964	0.2716

Table 3.3: Comparison of J^{i*}, r^1 fixed, r^2 changing, with a logistic resource growth.
$r^1 = 0.01$, $\delta^1 = \delta^2 = 0.02$, $a = 0.01$, $b = 0.02$, $c_1^i = c_2^i = 1$, $i \in \{1, 2\}$

When the optimal solution is computed, the behaviour of the optimal controls $u^{i*}(t)$ can be seen in Figure 3.5 and as a result of these controls, the optimal states $K^{i*}(t)$ and $R^*(t)$ can be seen in Figure 3.6.

It can be observed that the same behaviour as in the case with a non-renewable resource and an exponential resource occurs: As soon as one agent is replaced by another agent with a higher discount rate compared to the case where both agents apply the same discount rate, the first agent is worse off. The gap between the two optimal objective values does get bigger as the discount rate of agent 2 increases. The reason for this

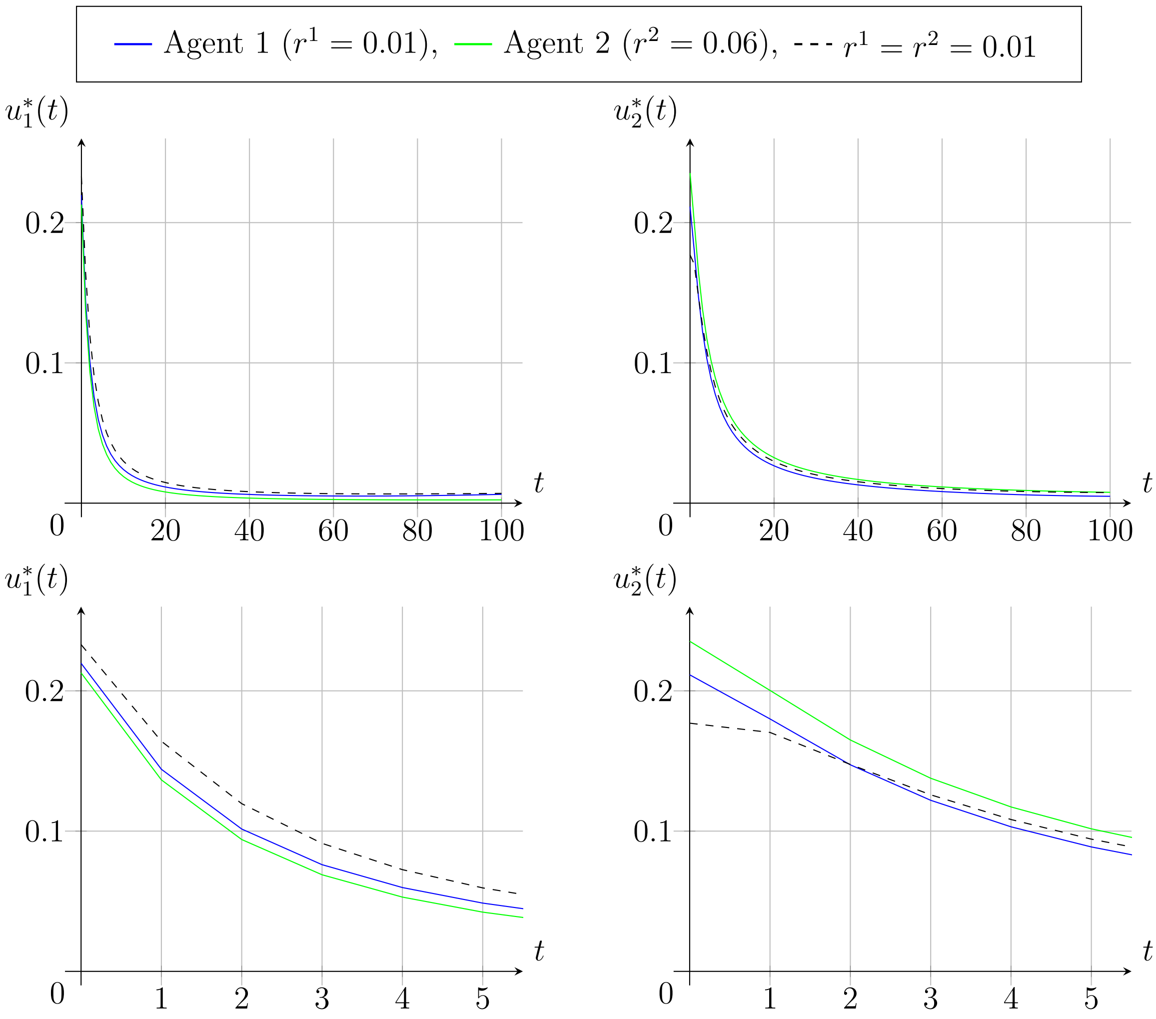


Figure 3.5: Optimal controls $u^{i*}(t)$ with logistic resource growth.
$\delta^1 = \delta^2 = 0.02$, $c_1^i = c_2^i = 1$, $i \in \{1, 2\}$, $r^1 = 0.01$, $a = 0.01$, $b = 0.02$

behaviour was already discussed in the case with a non-renewable resource.

As can be seen in Figure 3.6, the stock of the resource in the case with more than one agent in the economy is always nearly extinct and does not reach a sustainable level, as it happened in the case with only one agent in the economy. This happens because none of the two agents is willing to postpone extraction of the resource. This initial delay of extraction is most crucial in order to let the resource grow to a level where it can be kept sustainable. With the lack of this initial delay, the resource never gets the chance to grow. As soon as one agent postpones extraction, the other agent has an immediate advantage and will take hold of that. Therefore, it cannot be optimal for either agent to delay the extraction at all.

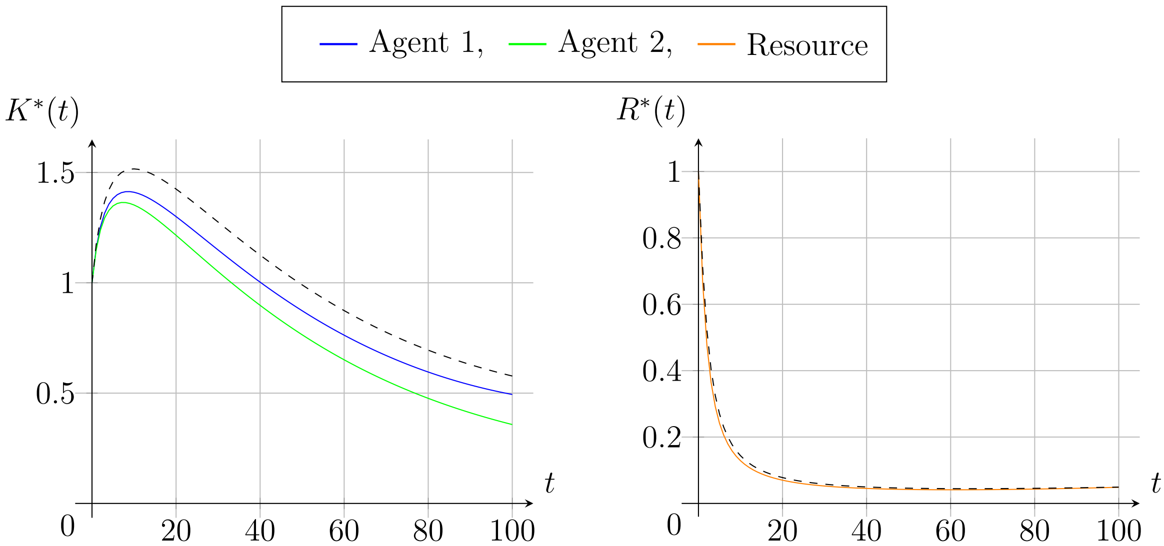


Figure 3.6: Optimal states $K^{i*}(t)$ and R^* with logistic resource growth.
$\delta^1 = \delta^2 = 0.02$, $c_1^i = c_2^i = 1$, $i \in \{1, 2\}$, $r^1 = 0.01$, $a = 0.01$, $b = 0.02$

Now, to understand the influence the constants c_1^i and c_2^i impose on the optimal objective values of both agents, J^{i*} was calculated for changing constants, see Table 3.4. Here, both constants for one agent are always changed equally, which gives $\bar{c}^i = c_1^i = c_2^i$, $i \in \{1, 2\}$.

		$\bar{c}^2 = 0.5$	$\bar{c}^2 = 1$	$\bar{c}^2 = 2$
$\bar{c}^1 = 0.5$	J^{1*}	0.3804	0.5227	0.8044
	J^{2*}	0.3605	0.2754	0.2254
$\bar{c}^1 = 1$	J^{1*}	0.2629	0.3453	0.4779
	J^{2*}	0.4593	0.3139	0.2255
$\bar{c}^1 = 2$	J^{1*}	0.1994	0.2241	0.3043
	J^{2*}	0.5856	0.3914	0.2585

Table 3.4: Comparison of J^{i*} for different constants $\bar{c}^i$, with logistic resource growth.
$\delta^1 = \delta^2 = 0.02$, $a = 0.005$, $b = 0.01$, $r^1 = 0.01$, $r^2 = 0.06$

From Table 3.4 the logical consequence of a change in the costs of investment in technology as well as in the costs of resource extraction can be observed. Rising costs for one agent imply that this agent will face a smaller optimal objective value J^*, whereas the other agent, who can maintain the same level of costs, will improve her optimal objective value (reading Table 3.4 row-wise or column-wise). This is quite an interesting effect: Denote the agent, whose costs did not change as agent A, and the other agent, whose costs did rise as agent B. Agent A faces the exact same optimal control problem as in the prior case, where agent B's costs were equal to the costs of agent A. So the only benefit that agent A can obtain is just the effect that agent B is not extracting the resource as firmly as before, since her costs have risen. This leaves agent A with a bigger amount of the resource to extract, which yields a higher optimal objective value. Needless to say, agent B will lose for two reasons. One the one hand, she faces higher costs and on the other hand she applies a lower rate of extraction of the resource.

For now let the focus be on the case that agent A is our agent 1 and agent B is our agent 2 (this means Table 3.4 is read row-wise). The increase of J^{1*} of agent 1 is much higher than the decrease of J^{2*} of agent 2. The reason for this interesting occurrence is simply that agent 1 is more far-sighted than agent 2 and therefore has a smaller discount rate. This makes agent 1 more sensitive to parameter changes. In the now discussed case, this leads to the behaviour that agent 1's percental increase is higher than agent 2's percental decrease. This effect can also be observed the other way round, when agent A is our agent 2 and agent B is our agent 1. Then, agent 1's percental decrease is higher than agent 2's percental increase of the optimal objective value.

The above presented results clearly indicate that as soon as there is more than one agent in the economy, the competition between these agents will always lead to a complete extinction of the common resource. Even the case where the resource is modelled with a logistic growth is no exception. Another interesting observation is that a short-sighted agent always drags down her far-sighted counterpart with her, affecting and reducing both agents' objective values. The far-sighted agent has no chance at all to overcome this downwards drag which is only a result of the other agents' short-sighted policy. An interesting aspect would be to see an influence on both agents' optimal controls and their optimal objective values, if they can agree on any kind of commitment. This will be discussed in the next chapter.

Chapter 4

Commitment between agents

In this chapter, commitment between agents will be discussed. Based on the 2-agent model introduced and described in Chapter 3, an analysis will now be given, whether it is reasonable for both agents to agree on a specific commitment which will improve both players' optimal objective values. As it became clear in Chapter 3, as soon as one of the two agents discounts the future less than the other agent, the agent who establishes the same far-sighted policy will be worse off in comparison to the case where both agents are equally far-sighted (this case was denoted as the symmetrical case). The agent who is more short-sighted drags the future-oriented agent down with her. This obviously leaves room for suggestions of a possible agreement. Since both agents lose in the free market game where no commitments or restrictions are laid upon the agents, it is well worth to investigate, whether both agents would be better off, if they agree on a commitment. Such an agreement could have a structure where the short-sighted agent (denoted as agent 2 from now on) agrees to reduce her extraction of the resource by a certain percentage, such that the far-sighted agent (denoted as agent 1) will have more resources for herself. Since agent 1 is the far-sighted one, her total output is normally higher than the output of agent 2. So if agent 2 would agree to this kind of commitment, agent 1 is able to gain more output with the higher amount of resource left for her sake, and then re-compensates agent 2 for her loss which she had to take because of the reduced amount of resource she extracted. Clearly, if the maximum gain agent 1 can obtain with the higher amount of resource is higher than the loss agent 2 will face, this commitment is more than likely to be introduced. Of course, it needs to be supposed that both agents are rational in their actions (see 1.3 *(i) - (v)*). On the other hand, if agent 1 is not able to gain more output with the extra resource left to her through the commitment than agent 2 loses, then this commitment would be a loss for agent 1, assuming agent 2 is fully re-compensated. Therefore, it leaves either agent 1 worse off if she fully re-compensates agent 2, or it will leave agent 2 worse off if agent 1 is only willing to re-compensate up to the extra gained output which was generated with the extra resource. Logically, such a commitment will never be approved by both sides.

In a scenario with a commitment between the agents, there are two possible extremes. The one would be that no commitment will take place and the whole scenario is exactly as the game described in Chapter 3 with full competitiveness between the two agents. The second would be a full commitment, where agent 2 agrees not to extract the resource at all and leaving everything to agent 1. There is a variety of possible commitment strategies between these two extreme ones. Whether any of these strategies do indeed benefit both agents will be discussed in this chapter.

4.1 Problem formulation

Consider the two agent differential game (3.5)-(3.7). In order for both agents to agree on a certain commitment, the full competitive game without any commitment needs to be known at first. Therefore the differential game (3.5)-(3.7) is solved which gives the open-loop Nash equilibrium $(\tilde{u}^{1*}, \tilde{u}^{2*})$. Now, a parameter $\alpha \in [0, 1]$ is introduced, which determines how much of the resource agent 2 is actually allowed to extract, if a commitment is introduced. This creates the following new problem: Determine the optimal objective values when the commitment is executed. Given the open-loop Nash equilibrium from the full competitive game $\tilde{u}^{1*}$ and $\tilde{u}^{2*}$, the new optimal objective value for agent 1 can be calculated by solving the following optimal control problem as stated and discussed in Chapter 2, with the optimal control of agent 2 as an exogenous vector which is reduced only to a proportion α of the optimal extraction:

$$\textit{Maximise} \qquad J^1(u^1(t)) = \int_0^T e^{-r^1 t}\Big(K^1(t)u_2^1(t)R(t) - c_1^1 u_1^1(t)^2 - c_2^1 u_2^1(t)^2\Big)dt \tag{4.1}$$

subject to

$$\dot{K}^1(t) = -\delta^1 K^1(t) + u_1^1(t), \quad K^i(0) = K_0^i, \tag{4.2}$$

$$\dot{R}(t) = \varrho(R) - \Big(u_2^1(t) + \alpha\tilde{u}_2^{2*}(t)\Big)R(t), \quad R(0) = R_0, \tag{4.3}$$

$$u_1^1(t), u_2^1(t) \geq 0 \quad \forall t. \tag{4.4}$$

Let $\hat{\cdot}$ denote the optimal values for a commitment strategy with a parameter α. Since agent 2 agreed to extract only $\hat{u}_2^2 = \alpha\tilde{u}_2^{2*}$ instead of the full $\tilde{u}_2^{2*}$, her new objective value can be calculated simply by maximising the following functional:

$$\hat{J}^2(\alpha) = \int_0^T e^{-r^2 t}\Big(K^{2*}(t)\alpha\tilde{u}_2^{2*}(t)R^*(t) - c_1^2\tilde{u}_1^{2*}(t)^2 - c_2^2(\alpha\tilde{u}_2^{2*}(t))^2\Big)dt. \tag{4.5}$$

It is obvious that as soon as $\alpha = 0$, the optimal control problem (4.1)-(4.4) is exactly the optimal control problem (2.21)-(2.24) from Chapter 2 where there was only one agent in the economy. As soon as $\alpha = 1$, the optimal control problem (4.1)-(4.4) is exactly the optimal control problem (3.13)-(3.16), which resulted from the differential game from Chapter 3 where there were two agents in the economy, but no commitment between these two agents whatsoever.

Introducing any $\alpha \in [0, 1)$ will most likely result in $\hat{J}^2(\alpha) < J^{2*}(\alpha)$. If this holds true obviously still depends on how the stock of the resource changes because of the commitment compared to R^*. It is also necessary to observe if $\hat{J}^1(\alpha)$ - the optimal value in (4.1) for a given α - is actually bigger than $J^{1*}(\alpha)$. Nonetheless, if either of these two hypotheses turn out to hold true or not, it is crucial to examine if

$$\hat{J}^1(\alpha) - J^{1*} \geq J^{2*} - \hat{J}^2(\alpha), \tag{4.6}$$

which is the necessary condition for that the commitment can be accepted by both agents, assuming both agents act in a rational way. If (4.6) does hold true, it means that agent 1 has a bigger gain resulting from the commitment than agent 2 loses. Therefore, agent 1 can re-compensate agent 2 sufficiently, so that both agents will be better off than in the scenario where no commitment was introduced.

It could also be the case that both $\hat{J}^1(\alpha) > J^{1*}$ and $\hat{J}^2(\alpha) > J^{2*}$ hold true. In this case agent 1 may not see any reason for re-compensating agent 2 since both are better off anyway. But one needs to consider that agent 2 agreed to extract only $\hat{u}_2^2$ instead of $\tilde{u}_2^{2*}$, whereas she actually needs the full $\tilde{u}_2^{2*}$ and cannot spare to leave the difference to agent 1. So, agent 1 will have to compensate agent 2 for the loss of the extracted resource. This leaves agent 2 with not only the same amount of resource for herself than in the full competitive scenario but she also gains more because of the higher stock of the resource, which is a consequence of agent 1's far-sighted policy.

4.2 A numerical solution

In order to find out whether a commitment between the two agents is possible for a given α, first the full model as described in 3.3 needs to be solved. Now, to find a parameter $\alpha \in [0,1]$ such that both agents gain from a commitment with a parameter α, the optimal control problem (4.1)-(4.4) needs to be solved for a fixed α. The optimal α^* can either be defined as

$$\text{(i)} \qquad \begin{aligned} &\alpha^i = \arg\max\nolimits_{\alpha^i \in [0,1]} \hat{J}^i(\alpha^i), \quad i \in \{1,2\}, \\ &\alpha^* = \max\{\alpha^1, \alpha^2\}, \end{aligned} \tag{4.7}$$

or

$$\text{(ii)} \qquad \alpha^* = \arg\max\nolimits_{\alpha \in [0,1]} \Big(\hat{J}^1(\alpha) + \hat{J}^2(\alpha) \Big). \tag{4.8}$$

Let $\hat{J}^{i*} = \hat{J}^i(\alpha^*)$ denote the objective values for each agent with the optimal α^*. This means, α^* represents the optimal possible commitment strategy, where optimality in this case is defined above.

Definition (4.7) can be interpreted in the following way: Every agent is only willing to agree on a commitment as long as her own optimal objective value increases. As soon as one of the two agents' optimal objective value declines, even if their combined optimal objective values would be higher, no commitment can be established. This case would occur, if for instance

(i) $\hat{J}^1(\alpha) \geq \hat{J}^1(\alpha^*)$,

(ii) $\hat{J}^2(\alpha) < \hat{J}^2(\alpha^*)$ and

(iii) $\hat{J}^1(\alpha) + \hat{J}^2(\alpha) > \hat{J}^1(\alpha^*) + \hat{J}^2(\alpha^*)$,

for $\alpha \in [0, \alpha^*)$. One might argue, since $\hat{J}^1(\alpha) + \hat{J}^2(\alpha) > \hat{J}^1(\alpha^*) + \hat{J}^2(\alpha^*)$ still holds true and therefore agent 1 is able to compensate agent 2 sufficiently enough, that a commitment should still be established nonetheless.

This argument is used for the second definition (4.8), where only the combined optimal objective values are crucial to determine the optimal possible commitment. This definition can of course only be executed if both agents trust each other – especially if agent 2 trusts agent 1, that she will be fully re-compensated and if agent 1 trusts agent 2 not to extract more from the resource than agreed in the commitment. As soon as this level of confidence disappears and one agent has fears that the other agent has the potential to break the agreement, only Definition (4.7) is left to be implemented. Here, agent 1 still needs to trust that agent 2 does not break the commitment of extracting more than agreed upon, but since agent 2 benefits of this commitment straight away anyway, it can be assumed that agent 2 has no incentive whatsoever to break this commitment.

Now, for the calculations, the usual assumptions are made again, leaving the constants $c_1^i = c_2^i = 1$, $r^1 = 0.01$, $r^2 = 0.06$, $\delta^1 = \delta^2 = 0.02$. These assumptions will hold from now on for all three different types of possible resource growth, unless stated otherwise.

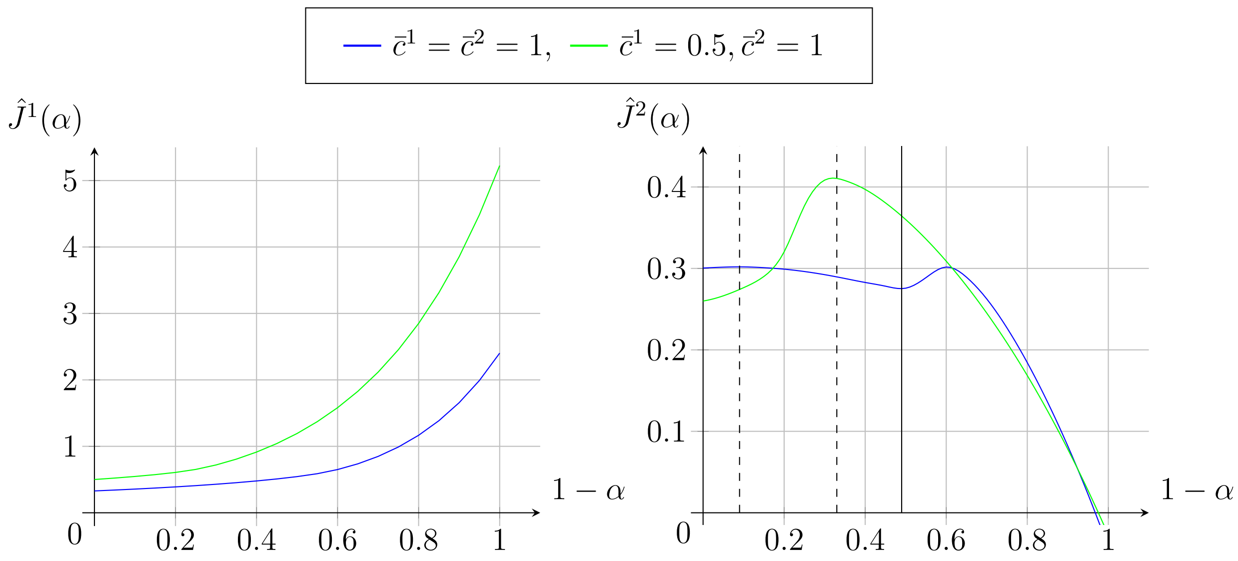
In order to be able to determine if agent 1 is able and willing to compensate agent 2, define

$$\begin{aligned} J_c(\alpha) &= \hat{J}^1(\alpha) - J^{1*} - J^{2*} + \hat{J}^2(\alpha), \\ J_c^* &= J_c(\alpha^*) \end{aligned} \tag{4.9}$$

which is just the difference, agent 1 still gains (or probably already loses), if agent 2 is fully compensated for her loss resulting from the commitment. It is obvious that $J_c \geq 0$ in case (i), see (4.7), since both $\hat{J}^{i*} \geq J^{i*}(\alpha), i \in \{1, 2\}$ hold true. In case (ii), see (4.8), $J_c \geq 0$ must not hold true. If $J_c > 0$, agent 1 will be willing to compensate agent 2 entirely. If $J_c = 0$, there is no motivation to agree on a commitment strategy, since there is no potential gain to make. Furthermore the agents only get vulnerable since the mutual trust is needed in order to gain the full potential of the commitment. If $J_c < 0$, either agent 1 or agent 2 will be worse off than in the fully competitive scenario and a commitment will never be agreed upon.

4.2.1 Non-renewable resource

To see the actual influence of different commitment strategies α on the optimal objective values of both agents, see Figure 4.1, where the black solid line in Figure 4.1b indicates the very first time, that agent 1 postpones her resource extraction in the case with $\bar{c}^1 = \bar{c}^2 = 1$, where $\bar{c}^i := c_1^i = c_2^i$, $i \in \{1, 2\}$.



(a) Agent 1 (b) Agent 2

Figure 4.1: Optimal objective values $\hat{J}^{i*}$ depending on the commitment strategy α with a non-renewable resource.

As expected, agent 1 will always gain from any commitment strategy $\alpha \in [0, 1]$. The optimal objective value of agent 2 has a short increase which is a result of the higher level of the resource, but then decreases rapidly and will even get negative, since agent 2's investment in technology is kept the same for any α. Investigating agent 2 more closely, it can be seen that the gain from a commitment is much higher when agent 1 has an advantage on the costs of investing in technology and extraction of the resource (modelled with $\bar{c}^1 = 0.5$ and $\bar{c}^2 = 1$), than in the case where both agents' costs are equal. This behaviour occurs, because agent 1's costs to increase technology are low, and therefore it is optimal, to invest more in technology and postpone extraction or at least extract the resource with a smaller rate as long as possible. In contrast to this case, when costs are higher, then agent 1 has less incentive to postpone extraction. Postponing extraction, or extracting with a lower rate, has also a benefit for agent 2. This is because her costs for extraction are automatically decreased by the lower rate of extraction, and the higher resource stock for a long initial period increases her objective value, despite the loss of less extraction. This effect is positive until the stock of the resource gets too small and the loss of the lower extraction starts to dominate this effect.

Calculating the optimal values α^* as in (4.7) the two cases are displayed in Figure 4.1, and gives

$$\alpha^* = 0.91 \quad \text{for} \quad \bar{c}^1 = \bar{c}^2 = 1,$$
$$\alpha^* = 0.67 \quad \text{for} \quad \bar{c}^1 = 0.5,\ \bar{c}^2 = 1,$$

where the two dashed lines in Figure 4.1 indicate the two α^* for both cases. In the case $\bar{c}^1 = \bar{c}^2 = 1$, both agents gain from a commitment up to $\alpha = 0.91$, which means that agent 2 is willing to extract only 91% of her original amount as in the free competitive market. If α decreases further, the objective value of agent 2 decreases as well, and, using Definition (4.7), α cannot be optimal anymore. With this commitment, the optimal objective values for the two agents increase from $J^{1*} = 0.3287$ and $J^{2*} = 0.3004$ to $\hat{J}^{1*} = 0.3564$ and $\hat{J}^{2*} = 0.3019$. In the case $\bar{c}^1 = 0.5$, $\bar{c}^2 = 1$, agent 2 is even willing to reduce her extraction to 67% of her original amount, which increases the optimal objective values of the two agents from $J^{1*} = 0.5006$ and $J^{2*} = 0.2599$ to $\hat{J}^{1*} = 0.7521$ and $\hat{J}^{2*} = 0.4109$.

Calculating the optimal values α^* as in (4.8) for both cases gives $\alpha^* = 0$ for both $\bar{c}^1 = \bar{c}^2 = 1$ and $\bar{c}^1 = 0.5$, $\bar{c}^2 = 1$, see Figure 4.2. This result is not surprising at all, since agent 1 is far-sighted and therefore will always gain a higher objective value than agent 2 if each individual agent were the only one in the economy. Thus, in both cases, agent 2 does not extract anything at all and leaves the whole resource to agent 1. In the case $\bar{c}^1 = \bar{c}^2 = 1$, the optimal objective values for the two agents change from

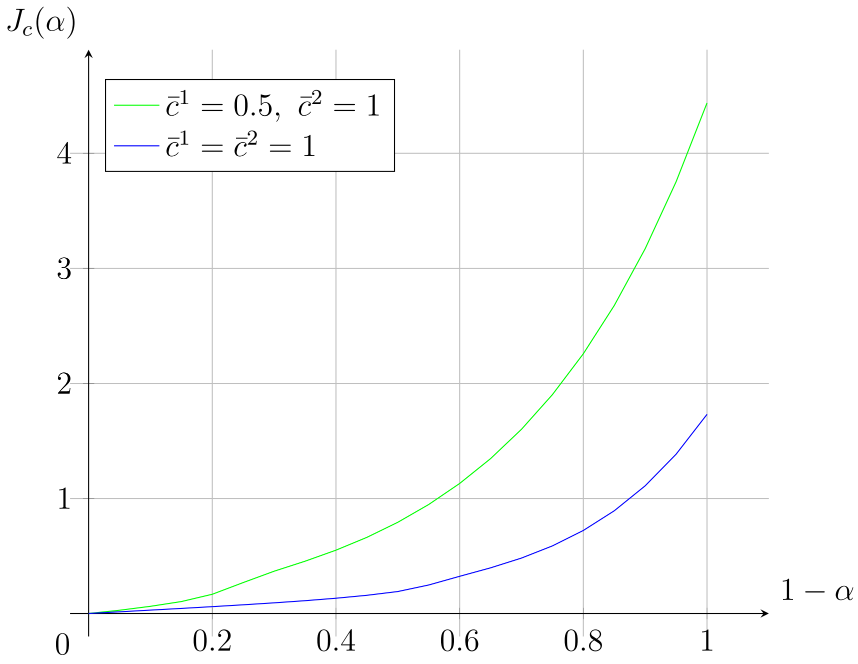


Figure 4.2: $J_c(\alpha)$ depending on the commitment strategy α with a non-renewable resource.

$J^{1*} = 0.3287$ and $J^{2*} = 0.3004$ to $\hat{J}^{1*} = 2.4025$ and $\hat{J}^{2*} = -0.0426$, and their joint objective value increases from $J^{1*} + J^{2*} = 0.6291$ to $\hat{J}^{1*} + \hat{J}^{2*} = 2.3599$. In the case $\bar{c}^1 = 0.5,\ \bar{c}^2 = 1$, the optimal objective values for the two agents change from $J^{1*} = 0.5006$ and $J^{2*} = 0.2599$ to $\hat{J}^{1*} = 5.2251$ and $\hat{J}^{2*} = -0.0273$, and their joint objective value increases from $J^{1*} + J^{2*} = 0.7605$ to $\hat{J}^{1*} + \hat{J}^{2*} = 5.1978$. This gives a joint gain of the objective values of $J_c^* = 1.7306$ and $J_c^* = 4.4272$ in these two cases, respectively, which means that a commitment is also with Definition (4.8) possible, since both agents will be better off.

To understand how the optimal controls, and therefore the optimal states of both agents change with respect to α, some figures will be discussed next. The original case with $\bar{c}^1 = \bar{c}^2 = 1$ is again considered. In the following figures, a plot in bright green represents the optimal path of the controls $u_j^{i*}(t)$ with $\alpha = 1$ (which is the fully competitive scenario), whereas a plot in bright red represents the optimal path with $\alpha = 0$ (which is the scenario, where agent 2 does not extract the resource at all).

As can be seen in Figure 4.3 and Figure 4.4, if α decreases, and therefore agent 2 agrees to extract less, then the optimal behaviour of agent 1 is as follows: At first, with α

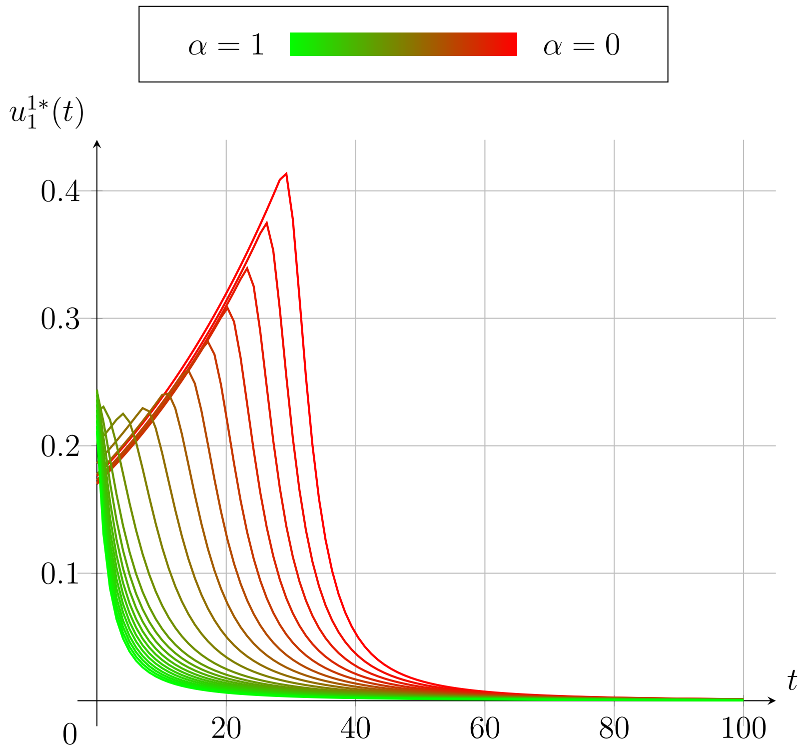


Figure 4.3: Projection of optimal control $u_1^{1*}(t)$ depending on the commitment strategy α with a non-renewable resource.

decreasing, her investment in technology increases and extraction of the resource slowes down. If α then gets small enough, extraction is even postponed at the beginning, in order to let the technology rise to a higher level before extraction starts. The first time extraction does not start immediately but is postponed for a moment, occurs at $\alpha = 0.47$. At exactly this value of α, a short upwards dip can be observed of the objective value of agent 2, see Figure 4.1b, where the black solid line indicates the very first occurrence that agent 1 postpones her resource extraction. This possibility for agent 1 to postpone extraction is of the fact that agent 1 controls already a significant part of the resource. The fear the other agent could extinct the resource whilst agent 1 is absent and not extracting is not that urgent anymore. A short delay of the initial extraction will not cause more harm than the gain this delay will generate later on. The short observable upwards trend of agent 2's objective value is resulting exactly from the fact that agent 1's resource extraction is postponed and agent 2's objective value has a gain with this higher stock of resource available. Nevertheless, this positive trend for agent 2 is only of short

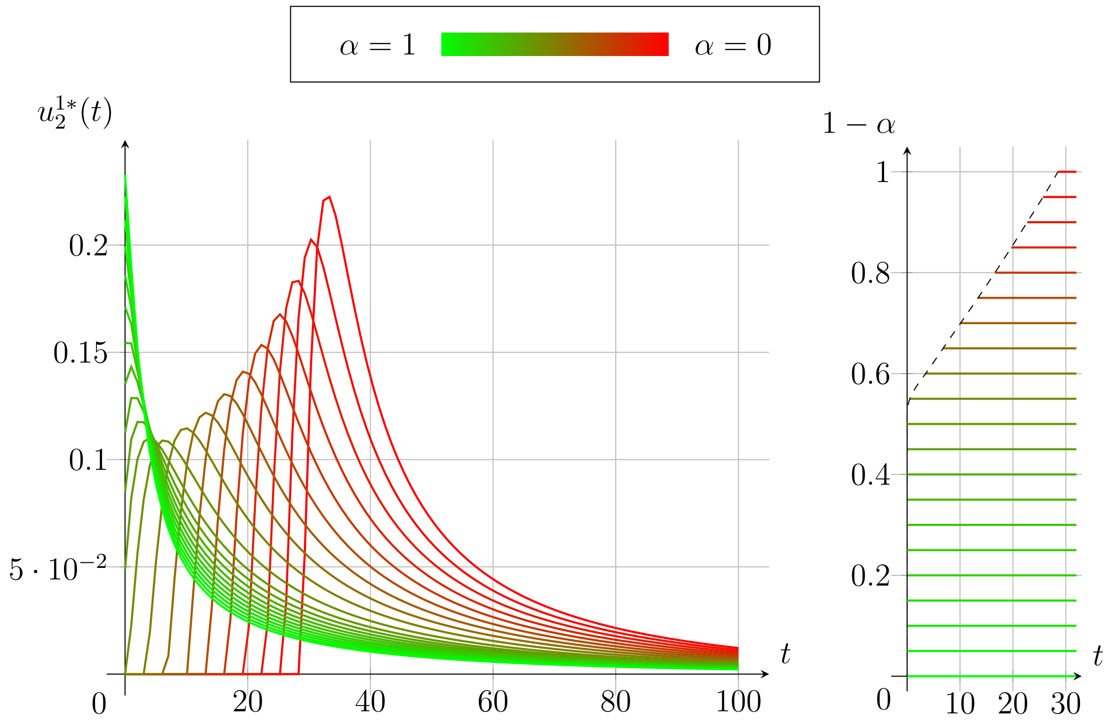


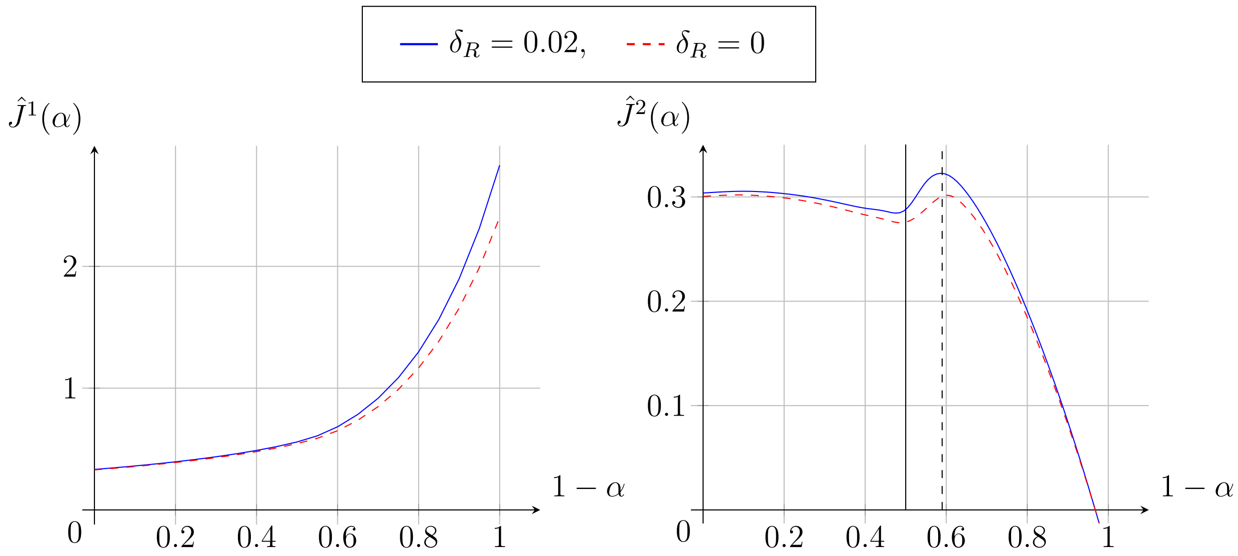
Figure 4.4: Projection of optimal control $u_2^{1*}(t)$ depending on the commitment strategy α with a non-renewable resource.

nature, since with α decreasing further, the loss of the lower rate of extraction cannot be compensated anymore just by using a higher resource stock.

The plots for agent 2 are omitted since her investment in technology $u_1^{2*}(t)$ is chosen to be constant for any α, and her rate of extraction $u_2^{2*}(t)$ is only reduced to α% of the original rate of extraction as in the fully competitive game discussed in Chapter 3, see Figure 3.1. Therefore the change in this control is just linear with α changing.

4.2.2 Resource with exponential growth

To see how different commitment strategies α influence the optimal objective values of both agents, see Figure 4.5, where the black solid line in Figure 4.5b indicates the very first occurrence that agent 1 postpones her resource extraction. The behaviour in the case with a non-renewable resource (red dashed plot) is very similar. The sharp upwards trend starting at $\alpha \approx 0.5$ in both plots is the result of agent 1 postponing the resource extraction. In the case with an exponential resource growth, this upwards trend is even stronger.



(a) Agent 1

(b) Agent 2

Figure 4.5: Optimal objective values $\hat{J}^{i*}$ depending on the commitment strategy α with an exponential resource growth.

Calculating the optimal value α^* as in (4.7), which is displayed as the black dashed line in Figure 4.5b, results in $\alpha^* = 0.41$. Comparing this commitment, where agent 2 is willing to extract just 41% of her original amount of the resource, to the commitment which was agreed upon in the case where the resource is not renewable, and agent 2 is only willing to extract up to 91%, it is obvious that how much the agents are far-sighted or not influences the objective value heavily.

The far-sighted agent is motivated to let the resource grow at first and again agent 2 benefits from this higher stock of resource. With $\alpha^* = 0.41$, the optimal objective values of the two agents increase from $J^{1*} = 0.3326$ and $J^{2*} = 0.3037$ to $\hat{J}^{1*} = 0.6827$ and $\hat{J}^{2*} = 0.3217$.

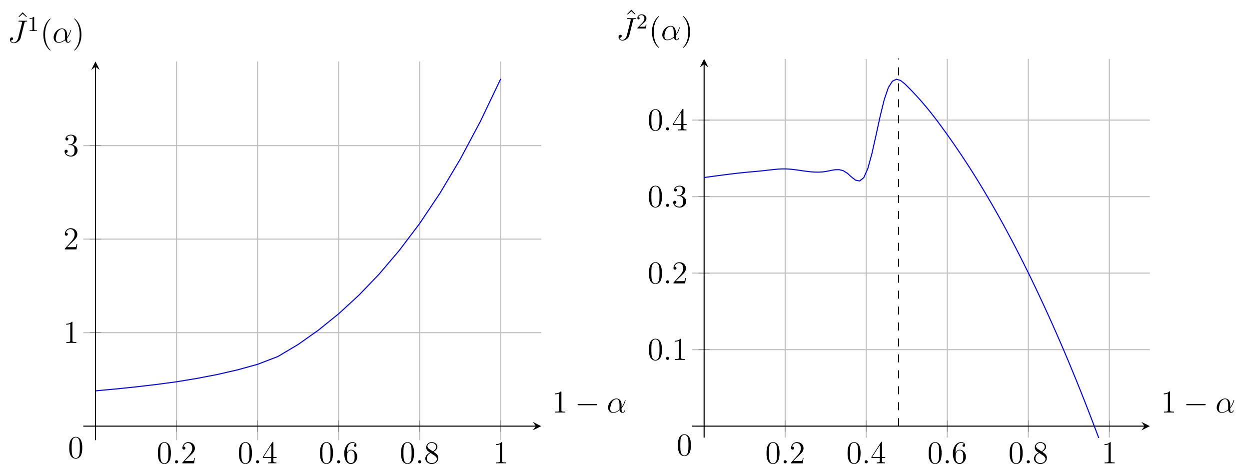
Calculating the optimal value α^* as in (4.8), results in $\alpha^* = 0$, as in the case with a non-renewable resource. Here, the optimal objective values for the two agents changes from $J^{1*} = 0.3326$ and $J^{2*} = 0.3037$ to $\hat{J}^{1*} = 2.8307$ and $\hat{J}^{2*} = -0.0438$, but their joint objective value increases from $J^{1*} + J^{2*} = 0.6363$ to $\hat{J}^{1*} + \hat{J}^{2*} = 2.7869$, which gives a joint gain of $J_c^* = 2.1506$ and a commitment is possible once again.

The projections of the optimal controls look very much like the projections in the case with a non-renewable resource and will therefore not be presented explicitly in this thesis, since the essential behaviour can also be seen in these projections already presented above, see Figure 4.3 and Figure 4.4.

4.2.3 Resource with logistic growth

In this case, where the resource is modelled to have a logistic growth, an interesting result is expected to occur. If there is only one single agent in the economy, as it is the case in Chapter 2, it is part of the optimal controls to let the resource reach a sustainable level. On the other hand, if there is a full competitive market with two agents, as it is the case in Chapter 3, the resource is always nearly extinct. Therefore, if now a commitment is introduced, and then $\alpha = 1$ (representing the full competitive market) gets reduced down to $\alpha = 0$ (representing the one agent market), there has to happen the following threshold. At some $\bar{\alpha}$, the optimal controls will switch from the near extinction of the resource to completely different optimal controls, that let the resource reach a sustainable level.

The two parameters that determine the dynamics of the resource are chosen to be $a = 0.025$ and $b = 0.05$, see (2.7), and the discount rates of the two agents are $r^1 = 0.03$ and $r^2 = 0.1$. First, to see how different commitment strategies α influence the optimal objective values of both agents, see Figure 4.6.



(a) Agent 1 (b) Agent 2

Figure 4.6: Optimal objective values $\hat{J}^i$ depending on the commitment strategy α with a logistic resource growth.

As in the two cases with an exponential resource growth and a non-renewable resource, the optimal objective values $\hat{J}^1(\alpha)$ of agent 1 are strictly monotonously increasing, since a decreasing α implies that agent 1 has access to a higher rate of the resource stock, which can only lead to a higher objective value. The behaviour of the optimal objective values $\hat{J}^2(\alpha)$ of agent 2 are also quite similar to the prior two cases. Agent 2 is able to keep her objective value at or above the original value for $\alpha > 0.63$, because the stock of the resource increases. This again results from the fact that agent 1 extracts at a smaller rate than originally without a commitment.

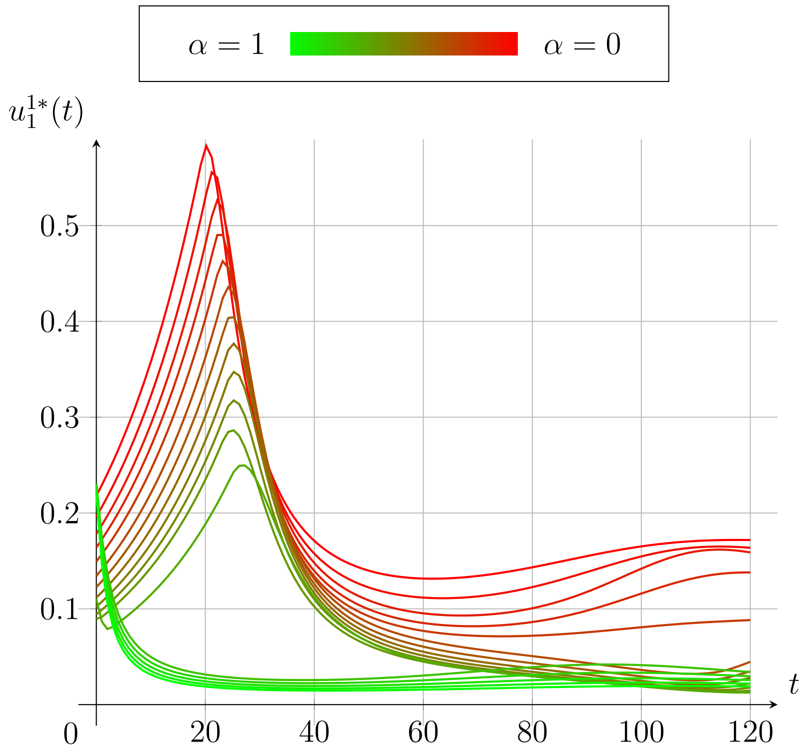


Figure 4.7: Projection of optimal control $u_1^{1*}(t)$ depending on the commitment strategy α with a logistic resource growth.

Calculating α^* as defined in (4.7) results in $\alpha^* = 0.52$. Agent 2 is therefore willing to reduce her rate of extraction to 52% of her original extraction rate, and both agents are still better off. If α decreases further, then the objective value of agent 2 would decline and the solution cannot be optimal anymore, in the sense as defined in (4.7). This increases the optimal objective values of both agents from $J^{1*} = 0.3775$ and $J^{2*} = 0.3249$ to $\hat{J}^{1*} = 0.8672$ and $\hat{J}^{2*} = 0.4534$. If agent 1 is willing to compensate agent 2 for an even lower α than $\alpha = 0.52$, and if agent 2 trusts agent 1, that compensation is actually paid, then α^* can be calculated as defined in (4.8). Using that definition, this gives $\alpha^* = 0$, and agent 2 stops extraction as a whole which leaves only agent 1 alone in the economy. This increases the optimal objective values of both agents from $J^{1*} = 0.3775$ and $J^{2*} = 0.3249$ to $\hat{J}^{1*} = 3.7148$ and $\hat{J}^{2*} = -0.0532$ and their joint objective value increases from $J^{1*} + J^{2*} = 0.7024$ to $\hat{J}^{1*} + \hat{J}^{2*} = 3.6616$ which implies a joint gain of $J_c^* = 2.9592$.

To understand how optimal controls, and therefore optimal states of both agents change with respect to α, see Figure 4.7 and Figure 4.8.

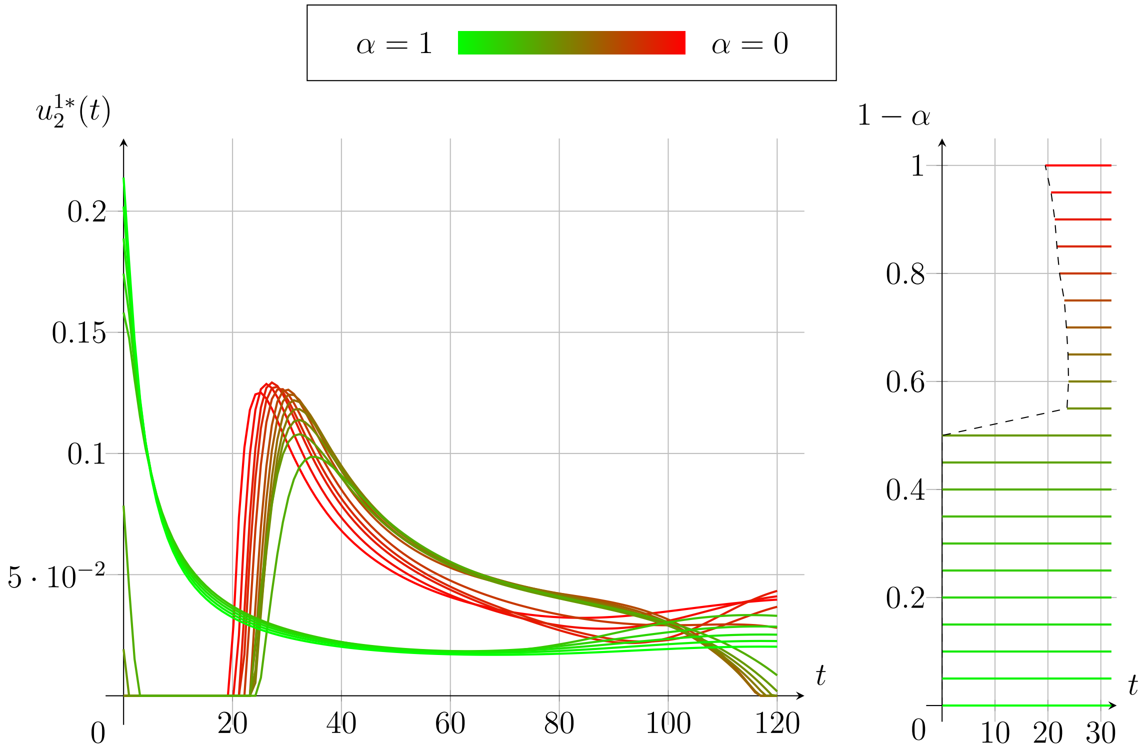


Figure 4.8: Projection of optimal control $u_2^{1*}(t)$ depending on the commitment strategy α with a logistic resource growth.

It is very interesting to see in both figures that the switch from optimal controls where α is big to optimal controls where α is small happens very drastically and is not as smooth as in the prior two cases. This switch happens at around $\alpha = 0.5$, and the first time agent 1 postpones the start of her extraction happens at $\alpha = 0.52$. This threshold will for now on be denoted with $\bar{\alpha}$. Now, a big difference to the prior two cases is that agent 2 gains her maximal objective value at the switching value $\bar{\alpha}$ exactly, and not as before - slightly afterwards. This difference can be explained as follows: Investigating Figure 4.8 in detail, it can be seen that for $\bar{\alpha}$ the resource extraction is slightly higher than e.g. for α near zero, and that at the end of the time horizon, extraction stops completely. This happens because the resource is extracted completely and there is nothing left to be extracted, whereas for α near zero, extraction is more moderate over time. This allows that extraction can be kept at that level for the entire time horizon. This behaviour, that resource extraction is constant at a certain positive level indicates that the resource reached a sustainable level and is kept at this level. The more control agent 1 gets over the resource with decreasing α, the better is the resource management. Agent 1 can start extraction already slightly earlier since she is in power of most, or the entire, resource and can control it in such way that it will reach a sustainable level. The fact that agent 1 starts with extraction slightly earlier once α gets beyond $\bar{\alpha}$ (see the right projection into the $(t, 1-\alpha)$-space of Figure 4.8), means that the objective value of agent 2 cannot increase anymore beyond $\bar{\alpha}$. Agent 2's maximal objective value will therefore be at $\bar{\alpha}$. The resource stock is the only possible way to compensate agent 2's loss from a lower extraction rate. Because the resource stock will now be decreasing with an earlier start of extraction, agent 2 will be worse off.

Figure 4.7 is more intuitive to interpret, and it is very clear that a decreasing α results in a higher investment in technology. The switch between the optimal control falling rapidly to a very low level at first and an increasing control (and reaching the peak at exactly that time t when extraction of the resource starts) which then falls to a more moderate level, happens at $\bar{\alpha}$ as well.

If $\alpha = 0$, then the joint objective value is equal to the objective value a single agent in the economy would have gained, with one reservation: This objective value of the single-agent problem is only reduced by the investment in technology, which agent 2 is still paying. This is an assumption made earlier by the model that the investment in technology stays constant, independent of the rate α, see Section 4.1.

The above presented results indicate that agreeing on a commitment will result that both agents benefit of that commitment.

Chapter 5

Conclusion and Extensions

In this thesis, a dynamic game between multiple agents exploiting a common resource is considered, where the dynamics of the common resource is modelled in three different approaches. It is shown that the resource growth plays a significant role for the strategies of the agents. The depreciation rate of technology also has an impact on how the optimal strategies of each agent develop, but the most crucial parameter is found to be the discount rate, which indicates whether an agent is far-sighted or an agent is short-sighted.

Slight modifications of the discount rate lead to a very significant change in the behaviour of the agents. This is seen very clear in Chapter 2 where there is only a single agent in the economy. An increasing discount rate results in a change of behaviour. The higher the discount rate, the sooner extraction is initiated, leaving less time for technology to increase as well as the resource to grow to a higher stock level (if the resource growth rate was positive). Both these effects imply that the optimal objective value of the production outcome is significantly lower. Even in the case where the resource is modelled with a logistic growth and a sustainable resource can be established, an increase in the discount rate has a direct influence on the resource. The resource is still sustainable, but the stock level at which it is sustainable decreases directly as the discount rate increases. So not only is the outcome of production unnecessarily cut, but also the resource stock is much lower than it would need to be, always with the possible fear of extinction.

As soon as a second agent enters the economy, a differential game is established in Chapter 3. Here, the resource has no chance whatsoever in ever reaching a sustainable stock level. In the competitive market, both agents try to gain as much of the resource as fast as possible, only with some restrictions that technology should rise to a higher level, which takes at least a little pressure out of the game. A very interesting result in this chapter is that the symmetric solution, where both agents are modelled with the exact same parameters, is always better for both agents than any other solution which is only slightly deviated from the symmetric solution[1]. So even if the first agent keeps her far-sighted strategy unchanged and the second agent is replaced with a very slightly more short-sighted agent, it would result that the first agent is worse off.

This naturally results in thinking about introducing a commitment between the two agents, where the short-sighted agent agrees to extract less, leaving more of the resource to the far-sighted agent. A mutually agreed commitment is considered in Chapter 4, with the idea that the more effective agent will get more of the resource and will have to re-compensate the other agent for her losses she has to take into account. As it turns out, any commitment will always be better for both agents than having no commitment at all. This is true as long as both agents trust each other and none of the two agents tries to breach the commitment. If this basic trust is not given, then commitments up to a certain point are still possible to take place, since both agents are able to increase their production outcome by themselves, without any re-compensation payments necessary.

A possible extension to the model considered in this thesis could be, that in the game scenario, the one agent does not know everything about the other agent up to now, but that there is some delay of information. In this setup, adopting the other agents' strategy will be more difficult. Also, in the case where a commitment between the two agents is agreed upon, it would make it possible for any of the two agents to break this commitment and gaining from that breach for as long as the delay lasts. Only when the other agent can detect the deviation of the commitment, a punishment can be imposed. This would lead to a possible new set of solutions, if the punishment, once the deviation is detected, is not high enough and breaking the commitment would actually pay off.

A very interesting extension would be to introduce the concept of a *model predictive Nash equilibrium*, as it is introduced and considered in Bréchet et al. (2012).

Another possible extension could be to introduce an alternative resource to the given

[1]Here of course it is assumed that neither of the two agents is replaced by a more far-sighted agent, but only by a less far-sighted agent compared to the symmetric case.

resource, which is more expensive to extract but has the advantage that it is inexhaustible, as it is considered in Harris et al. (2010).

Bibliography

K. Atkinson. *An Introduction to Numerical Analysis.* Wiley, 2 edition, 1 1989. ISBN 0-47162-489-9.

T. Basar and G. J. Olsder, editors. *Dynamic noncooperative game theory, Volume 160 (Mathematics in Science and Engineering).* Academic Press, 2 1982. ISBN 0-12080-220-3.

H. Benchekroun and N. V. Long. Transboundary fishery: A differential game model. *Economia*, 69:207–221, 2002.

J. Benhabib and R. Radner. The joint exploitation of a productive asset: a game-theoretic approach. *Economic Theory*, 2(2):155–190, 1992.

T. Bréchet, C. Camacho, and V. M. Veliov. *Adaptive Model-Predictive Climate Policies in a Multi-Country Setting.* Research Report 2012-02, Institute of Mathematical Methods in Economics (ORCOS), Vienna University of Technology, 2012.

A. Bressan. *Noncooperative Differential Games. A Tutorial.* Department of Mathematics, Penn State University, 2010.

P. J. Brockwell and R. A. Davis. *Time Series: Theory and Methods.* Springer series in statistics. Springer, New York, 2 edition, 1991.

L. Cesari. *Optimization - Theory and Applications.* Springer, 1983. Vol. 17 of the Series 'Stochastic Modelling and Applied Probability'.

C. W. Clark. *Mathematical Bioeconomics: The Optimal Management of Renewable Resources.* Wiley, New York, 1990.

S. Clemhout and H. Y. Wan. Dynamic common property resources and environmental problems. *Journal of Optimization Theory and Applications*, 46(4):471–481, 1985.

P. S. Dasgupta and G. M. Heal. *Economic Theory and Exhaustible Resources.* James Nisbet, Cambridge University Press, London, 1979.

H. Dawid and M. Kopel. On the economically optimal exploitation of a renewable resource: The case of a convex environment and a convex return function. *Journal of Economic Theory*, 76(2):272–297, 1997.

E. J. Dockner and G. Sorger. Existence and properties of equilibria for a dynamic game on productive assets. *Journal of Economic Theory*, 71(1):209–227, 1996.

E. J. Dockner, S. Jorgensen, N. van Long, and G. Sorger. *Differential Games in Economics and Management Science.* Cambridge University Press, 1 edition, 12 2000. ISBN 0-52163-125-9.

A. L. Dontchev, W. W. Hager, and V. M. Veliov. Second-order runge-kutta approximations in control constrained optimal control. *Society for Industrial and Applied Mathematics*, 38(1):202–226, 2000.

P. K. Dutta and R. K. Sundaram. The tragedy of the commons? *Economic Theory*, 3 (3):413–426, 1993a.

P. K. Dutta and R. K. Sundaram. How different can strategic models be? *Journal of Economic Theory*, 60(1):42–61, 1993b.

K. Erdlenbruch, M. Tidball, and D. van Soest. Renewable resource management, user heterogeneity, and the scope for cooperation. *Ecological Economics*, 64(3):597–602, 2008.

C. Fershtman. Identification of classes of differential games for which the open loop is a degenerate feedback nash equilibrium. *Journal of Optimization Theory and Applications*, 55(2):217–231, 1987.

M. Gerdts. *Mathematische Optimierungsverfahren des Operations Research (De Gruyter Studium) (German Edition).* de Gruyter, 5 2011. ISBN 3-11024-994-1.

H. S. Gordon. The economic theory of a common-property resource: The fishery. *Journal of Political Economy*, 62:124–142, 1954.

D. Grass, J. P.Caulkins, G. Feichtinger, G. Tragler, and D. A. Behrens. *Optimal Control of Nonlinear Processes. With Applications in Drugs, Corruption and Terror*. Springer, Berlin, 2008.

G. Hardin. The tragedy of the commons. *Science*, 162:1243–1248, 1968.

C. Harris, S. Howison, and R. Sircar. Games with exhaustible resources. *Journal of the Society for Industrial and Applied Mathematics*, 70(7):2556–2581, 2010.

E. Isaacson and H. B. Keller. *Analysis of Numerical Methods (Dover Books on Mathematics)*. Dover Publications, reprint edition, 6 1994. ISBN 0-48668-029-3.

E. Kreyszig. *Advanced Engineering Mathematics*. Wiley, 10 edition, 8 2011. ISBN 0-47045-836-5.

A. Krupnick and Z. Wang. *A Retrospective Review of Shale Gas Development in the United States: What Led to the Boom?* Resources for the Future, Washington, 4 2013.

K. Lancaster. The dynamic inefficiency of capitalism. *Journal of Political Economy*, 81 (5):1092–1109, 1973.

D. Levhari and L. Mirman. The great fish war: An example using a dynamic cournot-nash solution. *Bell Journal of Economics*, 11:322–334, 1980.

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *Matematicheskaya teoriya optimal' nykh protsessov*. Gosudarstvennoe izdatelstvo ziko-matematicheskoi literatury, Moscow, 1961. reprint of translation: Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko (1986).

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. Classics of Soviet Mathematics Gordon & Breach Science Publishers, New York, 1986. ISBN 2-88124-134-4. reprint of 1962 translation from the Russian by K. N. Trirogoff.

J. Rosen. The gradient projection method for nonlinear programming. part i. linear constraints. *Journal of the Society for Industrial and Applied Mathematics*, 8(1):181–217, 1960.

J. Rosen. The gradient projection method for nonlinear programming. part ii. nonlinear constraints. *Journal of the Society for Industrial and Applied Mathematics*, 9(4):514–532, 1961.

W. Rudin. *Real and Complex Analysis (International Series in Pure and Applied Mathematics)*. McGraw-Hill Science/Engineering/Math, 3 edition, 5 1986. ISBN 0-07054-234-1.

E. P. Ryan. *Nonlinear and Optimal Control Theory.* Lecture Notes on course MA40061, Department of Mathematical Sciences, University of Bath, 2012. http://www.people.bath.ac.uk/~masepr.

G. Sorger. Markov-perfect nash equilibria in a class of resource games. *Journal of Economic Theory*, 11(1):79–100, 1998.

P. Stevens. *The 'Shale Gas Revolution': Developments and Changes.* Chatham House, London, 8 2012. EERG BP 2012/04.