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Operations Research and Control Systems

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Research Report 2012-05
September 2012

Operations Research and Control Systems
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Optimal age-specific election policies in two-level organizations with fixed size

October 5, 2012

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Abstract

Many organizations — faculties, firms, political bodies, societies, national academies — have recently been faced with the problem of aging. These trends are caused by increasing longevity of the members of these organizations as well as a lower intake at younger ages. The aging problem is particularly pronounced in organizations of fixed size, where no dismissal due to age (up to a statutory "retirement" age) is acceptable. Attenuating the aging process in a fixed-size organization by recruiting more young people leads to another adverse effect: the number of recruitments will decline, so the chances to be recruited decrease. For multi-level organizations transition flows between the levels and recruitment at different levels complicate the aforementioned problems. In this paper we present

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a methodology that can help design election policies based on different objective functions related to the age structure and size of *two-level organization*, without compromising too much the already established election criteria. Technically, this methodology is based on multi-objective optimization making use of the optimal control theory. The current election policies for both full and corresponding members of the Austrian Academy of Sciences constitute the benchmark with which we compare our results based on alternative objective functions.

Keywords: OR in manpower planning, Control, Human resources, Age dynamics of learned societies, Pontryagin's maximum principle

1 Introduction

The ageing of human populations constitutes a major demographic challenge. The increase in the share of older age groups is due to declining fertility rates as well as diminishing mortality, especially, in older age groups. The impact of a top-heavy age structure is tremendous and multiple. Indeed, it also touches *personnel planning*. The dynamics of small sub-populations such as organizations, universities or societies will be affected by population ageing as caused by rising life expectancy of their members.

The control of age structures of such subpopulations is important in avoiding recruitment difficulties and promotion bottlenecks. A classical reference for Markovian person-flow models is Bartholomew (1982). In particular, the question on the sequence of recruiting members that is capable of generating a given stock trajectory is dealt there in Chap. 3. Using an inhomogeneous Markov chain approach, Feichtinger and Mehlmann (1976) studied the asymptotic behavior of the recruitment trajectory corresponding to given stock sequences, particularly stationary ones. Note that this is the inverse problem to the basic problem of person-flow models in which the impact of recruitment, promotion and attrition rates on the structure and growth of the population is studied. The present paper deals with such an inverse problem. Its purpose is to illustrate how intertemporal optimization can be applied to personnel planning. Since age and duration are important variables in hierarchical organizations we use an optimal control approach with distributed parameters (see Grass et al. 2008, Chap. 8.3, for a brief introduction). Another approach would be the framework of Overlapping Generations (OLG). We prefer, however, the continuous time approach using the McKendrick partial differential equation (compare Keyfitz and Keyfitz 1997) as it allows analytical solutions independently from the number of generations considered. There are few examples where optimal control has been applied to problems of population dynamics and economics, i.e.: among them are Arthur and McNicoll (1977), Feichtinger et al. (2003), Feichtinger et al. (2004), Feichtinger and Veliov (2007), and Prskawetz and Veliov (2007). While all these papers use a continuous-time approach, Chung and Park (1996) apply the discrete maximum principle to study the optimal recruitment and promotion (in a hierarchically graded manpower system).

The study of the demographic structure of organizations of fixed size, such as learned societies and political or administrative institutions, recently attracted considerable attention (see e.g. Leridon 2004; van de Kaa and de Roo 2007; Cohen 2003; Cohen 2009; Feichtinger and Veliov 2007; Dawid et al. 2009). The reason is that several adverse effects of the demographic development have been observed reflecting directly or indirectly the increasing longevity of the human life: higher average age of members, lower numbers of new elections, longer waiting time till (respectively lower probability of) promotion (Leridon 2004). These tendencies are most strongly exhibited for organizations in which premature dismissal of members is not allowed (as is the case in most of the academies of sciences or for government employees). The attempt to counteract the above mentioned effects faces several difficulties:

(i) The demographic parameters of the organizations are only of a secondary importance compared with the professional abilities of the members, the traditions in the election procedures, etc. The latter are difficult to be explicitly formalized in a model.

(ii) The relevant demographic indicators in the fixed-size organizations are conflicting with each other. For example, electing more young members would decrease the average age of the organization, but would lead to fewer elections, since the young members would stay longer in the organization.

(iii) In the case of a multi-level organization many reasonable (some of them conflicting) demographic indicators may be meaningful (six such indicators are involved in the analysis in this paper, although we focus on two-level organizations only).

The goal of the present paper is to suggest a methodology which can cope with the above difficulties, and to demonstrate this methodology in the case of the Austrian Academy of Sciences.

A characteristic feature of any organization with a fixed size is that the total inflow of members should equal the total outflow. Since the outflow from an organization depends on the mortality rate (which is age-dependent) and on the retirement age, a relevant model should explicitly involve the age structure of the members. This is also needed to evaluate the demographic indicators of the organization.

We formulate such a model in terms of controlled differential equations along the age, following the consideration in Feichtinger and Veliov (2007). It is to be mentioned that in the present paper we focus on the steady-state version of the dynamic model in Feichtinger and Veliov (2007). The reason is twofold: first, we investigate a two-level organization, which is substantially more complicated than the single-level society studied in the above mentioned paper; second, Feichtinger and Veliov (2007) proved a fundamental result: the optimal election policies (with respect to reasonable demographic indicators) are time-invariant, independent of the initial state of the organization, and can therefore be obtained by analysis of the steady-state model. However, this claim applies to single-level organizations. It is an open question whether this result is valid for multi-level organizations as well but we have some evidence that this may be true.

Since, according to (ii), (iii) and the above paragraph, we deal essentially with multi-objective optimization with constraints given in terms of differential equations, we employ systematically, although sometimes implicitly, ideas from optimal control theory, in particular the Pontryagin maximum principle (Pontryagin et al. 1962).

To cope with the difficulty in modeling the election procedure indicated in (i), we employ the reasoning introduced in Dawid et al. (2009). More specifically, we assume that the effect of all non-demographic objectives (abilities in one or another area, subjective preferences of the selection committee, ets.), which play a role in the traditional election procedure of new members at some of the levels of the organization, is already represented by the observed age distribution of the new recruitments. Taking additionally into account the objectives of demographic nature (average age, number of elections, tenure before election, etc., which may be of secondary importance) should not lead to a substantial deviation of the age distribution of the new recruitments from the observed one. Hence, the corrections in the age structure of the practiced election policy due to the demographic objectives should not be large. This idea is formalized in what follows by the notion of *tolerance*, representing the "weight" of the demographic objectives compared with the traditional ones.

To summarize, the main novelty in this paper is the consideration of a *two-level* fixed-size organization where the practiced election policy should be enhanced in a rational (optimal) way taking into account in a "mild" way the demographic concerns.

In the exposition below we use the terminology inherited from our original motivation: a two-level academy of sciences, where the two levels consist of *full members* (`F`) and *corresponding members* (`C`). The case study of the Austrian Academy of Sciences (ÖAW) uses statistical data provided by the Vienna Institute for Demography (Feichtinger et al. 2007).

The paper is organized as follows. In Section 2 the model structure is presented and the demographic indicators of interest are defined. Section 3 consists of the description of statistical data used in the case study of the Austrian Academy of Sciences. The results of an analysis of single-objective optimization problems are given in Section 4 of the paper. Based on the results of single-objective optimization tasks multi-objective problems are considered in Section 5. Concluding remarks are presented in Section 6. Several technical issues are outlined in the Appendices 7.1–7.4.

2 The Model Structure

Our model is based on the structure of the Austrian Academy of Sciences. We restrict the membership types to corresponding and full members ignoring the affiliation to the two alternative "classes"[5]. Since the number of corresponding and of full members is fixed, elections to either group take place when

[5]The Academy is structured into two sections: mathematical and natural sciences, and social sciences and humanities.

positions fall vacant. The latter happens due to premature out-migration (including mortality), due to transition to the upper level of the Academy, and due to retirement.

Formally: we consider a stationary age-structured population of full members $F(a)$ and corresponding members $C(a)$, where the variable a denotes the age. The dynamics of $C(a)$ and $F(a)$ is described by the following equations[6]:

$$\dot{C}(a) = -\mu(a)C(a) - R_F u_F(a) + R_C u_C(a), \quad C(0) = 0, \tag{1}$$
$$\dot{F}(a) = -\mu(a)F(a) + R_F u_F(a), \quad F(0) = 0, \quad a \in [0, \omega], \tag{2}$$

where ω is the mandatory retirement age, $\mu(a)$ indicates the age-specific quit (mortality) rate, assumed the same for F and C, R_F and R_C denote the number of elections to full and corresponding members per year, respectively, the functions $u_F(a)$ and $u_C(a)$ represent the age density of elections to full and corresponding members.

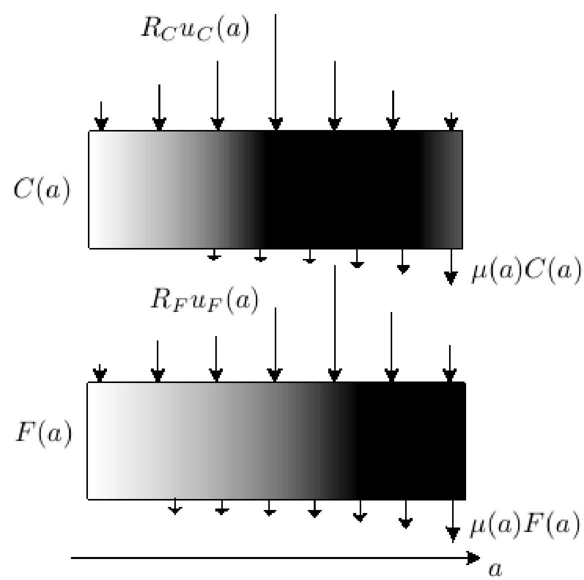

Figure 1: The model structure: the rectangles represent the populations of corresponding and full members, the vertical arrows above the rectangles represent the in-flow of members, the vertical arrows below rectangles – the respective outflows, the spectrum in rectangles corresponds to age densities.

Equation (2) is well known in the demographic literature: for a stationary population the change of the number of members at age a equals the number of recruits, $R_F u_F(a)$, minus the number of quits, $\mu(a)F(a)$. Equation (1)

[6] We use Newton's notation for the differentiation of the function with respect to a.

is constructed analogously. The number of corresponding members at age a is augmented by new elections $R_C u_C(a)$ and reduced by those who quit, $\mu(a)C(a)$, and those who are elected to full members, $R_F u_F(a)$. The age density of elections to corresponding members, $u_C(a)$, and the age density of elections to full members, $u_F(a)$, are considered as decision variables.

Figure 1 displays the model structure in terms of a flow diagram. The rectangles depict the age densities of corresponding, C, and full members, F. Higher densities at specific ages are indicated by darker shades in the rectangles. Flows of elections and quits are indicated by vertical arrows.

The solutions of the first-order inhomogeneous linear differential equations (1)-(2) are given by[7]

$$\begin{aligned} C(a) &= \int_0^a \psi(b,a)(R_C u_C(b) - R_F u_F(b))\,\mathrm{d}b, \\ F(a) &= R_F \int_0^a \psi(b,a) u_F(b)\,\mathrm{d}b, \end{aligned} \tag{3}$$

where the function $\psi(b,a)$ indicates the survival probability from age b to age a and is given by

$$\psi(b,a) = e^{-\int_b^a \mu(\alpha)\,\mathrm{d}\alpha}.$$

The size of the population of full and corresponding members is assumed to be fixed at $\bar{C}$ and $\bar{F}$, respectively:

$$\int_0^\omega C(a)\,\mathrm{d}a = \bar{C}, \quad \int_0^\omega F(a)\,\mathrm{d}a = \bar{M}. \tag{4}$$

Based on equations (3) and (4) one can uniquely determine the number R_M of elections to corresponding members per year (as a function of the control variables and the exogenous parameters):

$$R_C = \frac{\bar{C} + \bar{F}}{\int_0^\omega e(a)\,u_C(a)\,\mathrm{d}a}, \tag{5}$$

and the number R_F of elections to full members per year:

$$R_F = \frac{\bar{F}}{\int_0^\omega e(a)\,u_F(a)\,\mathrm{d}a}, \tag{6}$$

where the function $e(a)$ denotes the average remaining membership time at age a, and is given by

$$e(a) = \int_a^\omega \psi(a,b)\,\mathrm{d}b. \tag{7}$$

Together with (7) we introduce the substitution:

[7] For example, one can find this classical formula in Arnold (1992).

$$X(a)e(a) = \int_a^\omega b\psi(a,b)\,\mathrm{d}b, \tag{8}$$

where $X(a)$ denotes the average age of academy members who are at least a years old and $e(a)$ is given in (7).

Furthermore we can obtain the following expressions for the average age and tenure of corresponding and full members. Note that a change of the order of integration is applied whenever appropriate to obtain the first four formulae below.[8] In the derivation of the various expressions, we assume that the transition rate from corresponding to full membership is independent of the duration spent as a corresponding member. The formulae for tenure are obtained in Appendix 7.1. Note that average duration of stay and tenure are synonyms in this paper.

Average age of corresponding members:

$$\mathcal{A}_C = \frac{(\bar{C}+\bar{F})\int_0^\omega X(a)e(a)u_F(a)\,\mathrm{d}a}{\bar{C}\int_0^\omega e(a)\,u_C(a)\,\mathrm{d}a} - \frac{\bar{F}\int_0^\omega X(a)e(a)\,u_F(a)\,\mathrm{d}a}{\bar{C}\int_0^\omega e(a)\,u_F(a)\,\mathrm{d}a}. \tag{9}$$

Average age of full members:

$$\mathcal{A}_F = \frac{\int_0^\omega X(a)e(a)\,u_F(a)\,\mathrm{d}a}{\int_0^\omega e(a)\,u_F(a)\,\mathrm{d}a}. \tag{10}$$

Average tenure of corresponding members of those who retire as corresponding members:

$$\Theta_C = \frac{R_C}{\bar{C}}\int_0^\omega\int_0^a (a-b)e^{-\int_b^a\left(\mu(\alpha)+\frac{R_F u_F(\alpha)}{C(\alpha)}\right)\mathrm{d}\alpha}u_C(b)\,\mathrm{d}b\,\mathrm{d}a. \tag{11}$$

Average duration of stay as corresponding members of those who are elected to full members:

$$\Theta_{CF} = R_C\int_0^\omega\int_0^a (a-b)e^{-\int_b^a\left(\mu(\alpha)+\frac{R_F u_F(\alpha)}{C(\alpha)}\right)\mathrm{d}\alpha}u_C(b)\,\mathrm{d}b\frac{u_F(a)}{C(a)}\,\mathrm{d}a. \tag{12}$$

Following the ideas in Feichtinger and Veliov (2007) we assume that a reference stationary solution $(C^*, F^*, u_F^*, u_C^*, R_F^*, R_C^*)$ is given and consider how to optimally modify the decision variables $u_F(\cdot)$ and $u_C(\cdot)$ subject to

$$\int_0^\omega u_F(a)\,\mathrm{d}a = 1, \quad \int_0^\omega u_C(a)\,\mathrm{d}a = 1, \tag{13}$$

taking into account particular objective function that may include a combination of the above demographic expressions.

[8] The order of integration can be changed easily: the region of integration of $\int_0^\omega\int_0^a dbda$ is defined as $0 \le a \le \omega, 0 \le b \le a$. The same region is described by $0 \le b \le \omega, b \le a \le \omega$ so that the double integration can be rewritten as $\int_0^\omega\int_b^\omega dadb$.

In addition, following the argumentation in Feichtinger and Veliov (2007), we pose the constraint

$$\underline{u}_F(a) \leq u_F(a) \leq \bar{u}_F(a), \qquad \underline{u}_C(a) \leq u_C(a) \leq \bar{u}_C(a) \tag{14}$$

for the values of u_F and u_C, with

$$\begin{aligned} \underline{u}_F(a) &= (1-\varepsilon)u_F^*(a), \qquad & \bar{u}_F(a) &= (1+\varepsilon)u_F^*(a), \\ \underline{u}_C(a) &= (1-\varepsilon)u_C^*(a), \qquad & \bar{u}_C(a) &= (1+\varepsilon)u_C^*(a). \end{aligned}$$

where $\varepsilon > 0$ is a "tolerance" parameter specifying how much (in relative terms) the election densities u_F and u_C may differ from the reference election densities u_F^* and u_C^*. The argument is that the change in the age density of elections should be small to avoid significant deviations from the election criteria that is currently used (see Dawid et al. 2009).

3 Case Study for the Austrian Academy of Sciences

We base our study on the experience of the Austrian Academy of Sciences (cf. Feichtinger et al. 2007, for a thorough discussion of the historical evolution of the Austrian Academy of Sciences). The Academy is limited to 90 full and 110 corresponding members. The current full members elect newcomers when positions fall vacant. The main opportunities for the election of new members are the mortality of current members, out-migration of members or their leaving the Academy for other reasons, and gaining emeritus status at age 70 from when members no longer count toward the limit of 90 members. Figures 2–3 plot the smoothed age densities of elections as `C` and `F` over the period 1996-2005 approximated by a normal distribution density function. Note, that the age range of elections is between 40 and 70 years, i.e. covering $w = 30$ years in total. From these figures we can deduce that corresponding members are mostly elected at young middle ages while full members are mostly elected at old middle ages. In addition to the election densities we also plot the boundaries of the election variables (see formulas (14)) where we set the tolerance parameter to $\varepsilon = 0.4$.

We furthermore approximate the mortality rate μ based on the historical data of the Academy (see Winkler-Dworak 2008). Figure 4 plots the resulting, monotonically increasing function of mortality. The role of mortality is discussed in Appendix 7.2.

Based on the age-specific schedule of the elections and mortality ($u_F(a)$, $u_C(a)$, and $\mu(a)$), we next calculate the corresponding age densities of corresponding and full members as derived in the previous section. As Figure 5 indicates, the age schedule of corresponding members is tilted toward young middle ages while the age schedule of full members is tilted toward older ages.

Based on the empirical data presented, we can evaluate the expressions of interest as given in the previous section (Table 1). These include the number

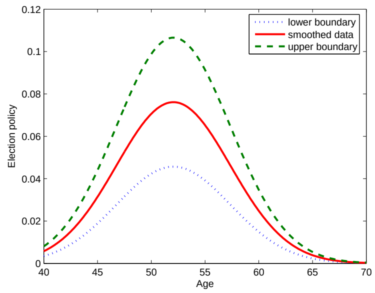

Figure 2: Election policy $u_C(a)$ of corresponding members

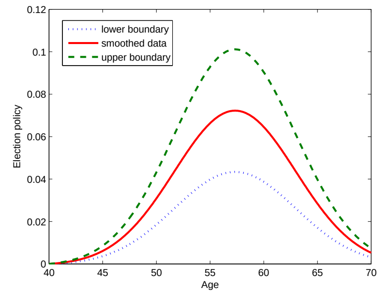

Figure 3: Election policy $u_F(a)$ of full members

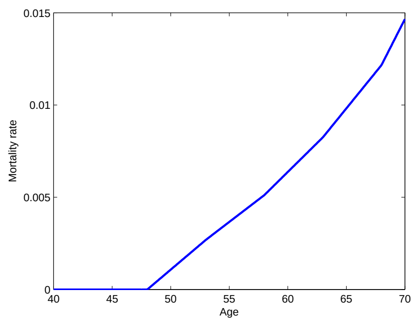

Figure 4: Mortality rate $\mu(a)$

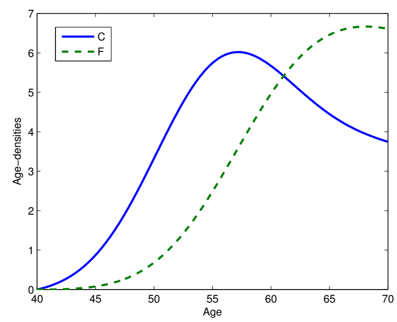

Figure 5: Graphs of $C(a)$ and $F(a)$

of elections to corresponding and full members (R_C, R_F), the average ages of corresponding and full members ($\mathcal{A}_C$, $\mathcal{A}_F$) as well as the duration variables (Θ_C,Θ_{CF}).

Table 1: Recruitment, average age and duration of full and corresponding members. Numbers are based on smoothed data from 1996-2005 of the Austrian Academy of Sciences.

R_C	R_F	$\mathcal{A}_C$	$\mathcal{A}_F$	θ_C	θ_{CF}
11.8	7.4	58.4	62.4	7.0	6.1

The parameters in Table 1 are based on a positive model that describes the current demographic structure of the members of the Academy. We will next discuss optimal election policies for several different objective functions (i.e. we move to the normative framework) and compare the resulting recruitment, average age and duration of full and corresponding members, with the numbers presented in Table 1.

4 Single-Objective Optimization

This section is devoted to optimization problems for a single objective function. More specifically we consider optimization problems of maximizing the elections of corresponding or alternatively full members, and optimization problems of minimizing the average age of full and alternatively corresponding members.

The arising optimization problems are solved numerically. For numerical integration a version of the Simpson method is implemented in the elaborated "Matlab" program (see Yang et al. 2005). In optimization problems the mathematical programming method is applied. In the "Matlab" software this method is provided by the "Optimization toolbox". It performs minimization of the objective function subject to constraints.

The tolerance parameter ε is set equal to $\varepsilon = 0.4$. A sensitivity analysis with respect to the tolerance parameter is given in Appendix 7.4.

4.1 Maximizing the Elections of Corresponding Members

We start with maximizing the number of the elections of corresponding members R_C. From (5) it follows that R_C only depends on the decision variable u_C, i.e. the age density of elections of corresponding members. Election policy u_F is not changed and is equal to current reference policy shown on Figure 3, and the problem is to modify election policy u_C (Figure 2) in such a way that the number of elections R_C is maximized. Technically, the problem is equivalent to minimization of the denominator in (5):

$$\int_0^\omega e(a)\, u_C(a)\, \mathrm{d}a \underset{u_C(a)}{\longrightarrow} \min, \tag{15}$$

subject to constraints

$$\int_0^\omega u_C(a)\, \mathrm{d}a = 1, \tag{16}$$

$$\underline{u}_C(a) \leq u_C(a) \leq \bar{u}_C(a) \tag{17}$$

for the values of u_C, with

$$\underline{u}_C(a) = (1-\varepsilon)u_C^*(a), \qquad \bar{u}_C(a) = (1+\varepsilon)u_C^*(a). \tag{18}$$

Expression (15) can be easily understood as the expected time spent as an academy member after recruitment, averaged over the chosen distribution $u_C(a)$. Thus, if $e(a)$ (cf. equation (7)) is non-increasing in a, an optimal election policy $u_C(a)$ for maximizing the number of recruits will, subject to constraints on election policy and on total number of members, place the maximum possible weight on election of older members with low $e(a)$.

Let us note that in this problem conditions of fixed size for populations of `C` and `F` are fulfilled automatically as the expression of R_C in (5) is obtained from formulae (3) using conditions (4).

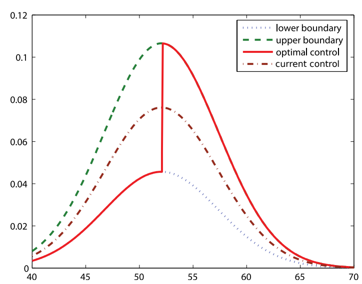

Figure 6: Maximizing R_C: Election policy $u_C(a)$

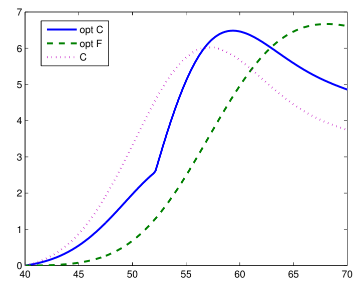

Figure 7: Maximizing R_C: Age densities

In Figure 6 we plot the optimal election policy of corresponding members that imply a maximum value of R_C. The figure also shows the current reference control which is practiced at the Austrian Academy of Sciences and the lower and upper boundaries of the optimization problem (the tolerance area). In addition, on Figure 7 we present the age densities of full and corresponding members before and after optimization. One can note that after optimization the age-density of full members does not change due to its independence from the control variable u_C. The effect of this policy on other demographic indicators is presented in Table 2 at the end of the section. We observe an increase in R_C and A_C accompanied by a decrease in θ_C and θ_{CF}. Parameters R_F and $\mathcal{A}_F$ stay constant as they do not depend on the election policy u_C. The optimal election policy of corresponding members has a strong right tail which corresponds to electing more members at middle and old ages. These ages correspond to high mortality rates μ as well as high probabilities of being elected to full members. From the results so far we can conclude that the policy to increase the recruitment of corresponding members will increase the average age of corresponding members. The resulting election policy significantly decreases both tenure indicators: the duration of stay as corresponding members of those who are retired θ_C and the duration of stay as corresponding members of those who are elected to full members θ_{CF}.

4.2 Maximizing the Elections of Full Members

The problem of maximization of the number of elections R_F to full members per year is similar to the aforementioned problem of maximization of R_C. The control variable to be optimally chosen is now u_F, i.e. the age density of elections from corresponding to full members, and the election policy u_C is fixed equal to the current reference policy. The problem consists in maximization of R_F in (6) subject to the constraints for u_F in (13)–(14).

Let us show that the fixed size condition for C in (4) is fulfilled for an optimal solution u_F. Using expressions (3), (5)–(6) one can obtain the following chain of equalities

$$\int_0^\omega C(a)\,\mathrm{d}a = \int_0^\omega \int_0^a \psi(b,a)(R_C u_C(b) - R_F u_F(b))\,\mathrm{d}b\,\mathrm{d}a =$$
$$R_C \int_0^\omega \int_0^a \psi(b,a)\,\mathrm{d}b u_C(b)\,\mathrm{d}a - R_F \int_0^\omega \int_0^a \psi(b,a)\,\mathrm{d}b u_F(b)\,\mathrm{d}a =$$
$$R_C \int_0^\omega e(a) u_C(a)\,\mathrm{d}a - R_F \int_0^\omega e(a) u_F(a)\,\mathrm{d}a = \bar{C} + \bar{F} - \bar{F} = \bar{C}, \quad (19)$$

which proves that the size of C stays constant in the optimization problems with respect to control parameter u_F.

As Figure 8 shows, a bang-bang solution of the optimal age-specific density of elections is again the best way. First the lower boundary control, and thereafter the upper boundary control is applied (see Figure 8). Age densities C and F corresponding to the optimal election policy are depicted in Figure 9. The graph of C indicates that the policy of maximizing the election of full members leads to a lower density of corresponding members at old middle and older ages in comparison to the currently prevailing situation.

As indicated in Table 2, this policy does not affect the number of elections of corresponding members, R_C. However, compared to the prevailing situation, the optimal policy increases the average age of full members $\mathcal{A}_F$ and therefore decreases the average age of corresponding members $\mathcal{A}_C$. This directly follows from the expression of $\mathcal{A}_C$ (cf. equation (9)). The first term depends only on the elections to corresponding members, u_C, while the second negative term is proportional to $\mathcal{A}_F$. Thus, for a given density of elections u_C, an increase in $\mathcal{A}_F$ decreases $\mathcal{A}_C$.

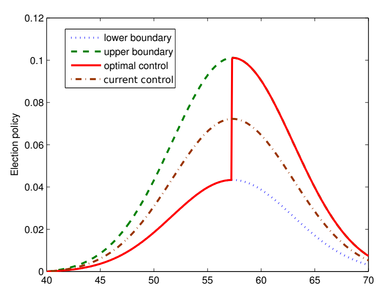

Figure 8: Maximizing R_F: Election policy $u_F(a)$

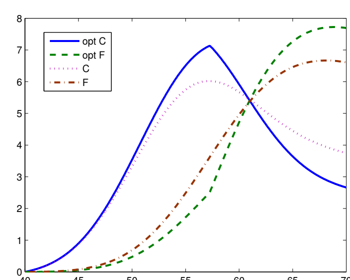

Figure 9: Maximizing R_F: Age densities

Furthermore, the numerical results in Table 2 show that the optimal election policy implies a decrease in the duration of staying a corresponding member (without a transition to a full member), θ_C, and increases the duration of staying a corresponding member among those who are elected as full members, θ_{CF}.

These changes can be explained by the shape of the election policy shown in Figure 8 which suggests electing full members at older ages.

Remark 1 *The optimal policies for maximizing recruitments in Figure 6 and Figure 8 have a special property. Namely, the age (call it z) at which the optimal policy shifts from $(1-\varepsilon)$ to $(1+\varepsilon)$ times the reference distribution is the median of that reference distribution.*

In other words, the optimal policy for the recruitment objective, under the ε constraint, can be expressed as "maximize elections at ages above the current median, and minimize elections at ages below". To see this, notice that in the "bang-bang" solution in the Figure 8 the following chain of relations takes place:

$$\begin{aligned} \int_0^\omega u_F(a)\,\mathrm{d}a &= \int_0^z (1-\varepsilon)u_F^*(a)\,\mathrm{d}a + \int_z^\omega (1+\varepsilon)u_F^*(a)\,\mathrm{d}a = \\ 1 &+ \varepsilon(\int_0^z -u_F^*(a)\,\mathrm{d}a + \int_z^\omega u_F^*(a)\,\mathrm{d}a), \end{aligned} \tag{20}$$

which equals one, if only z is the median of the $u_F^*(a)$ distribution.

4.3 Minimizing the Average Age of Full Members

Let us consider the problem of minimization of the average age $\mathcal{A}_F$ of full members. In this case, the election of full members, u_F, constitutes the control variable. The shape of the optimal election policy is bi-polar (see Appendix 7.3 for the formal derivation).

Expression for A_M (10) is interpretable as a weighted average of $X(a)$ values, where $X(a)$ is an increasing function of a. Thus, if $e(a)$ (cf. equation (7)) is a non-increasing function, policy $u_F(a)$ for keeping A_F low, subject to constraints on election policy and on the total number of members, will place the maximum possible weight on recruitment of young members.

As Figure 10 shows, the policy is to elect more members at young and young middle ages, a few at very old ages, and even fewer at old middle ages. The optimal election policy does not influence the number of elections of corresponding members, R_C, but increases the average age of corresponding members as well as their duration of staying a corresponding member. These latter results follow from the fact that older corresponding members have lower chances to be elected as full members. Moreover, the optimal election policy decreases the number of elections to full membership, R_F, as well as the duration of staying a corresponding member (among those ever gaining full membership status during their lifetime, θ_{CF}). The age densities of C and F are depicted in Figure 11. Compared to the currently prevailing situation of the Academy, the number of corresponding and full members in old middle and old ages increases.

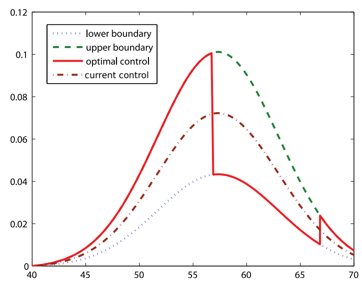

Figure 10: Minimizing $\mathcal{A}_F$: Election policy $u_F(a)$

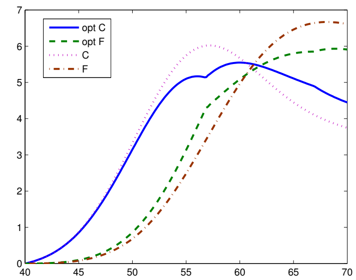

Figure 11: Minimizing $\mathcal{A}_F$: Age densities

4.4 Minimizing the Average Age of Corresponding Members

In case of minimizing the average age of corresponding members, $\mathcal{A}_C$, we should simultaneously consider both decision variables, u_F and u_C.

We need to modify the numerical optimization program accordingly. More formally, the problem is to minimize:

$$\mathcal{A}_C \underset{u_F(a),u_C(a)}{\longrightarrow} \min, \tag{21}$$

with $\mathcal{A}_C$ given by (9), subject to constraints (4), (13)–(14).

In Figures 12 and 13 we plot the optimal election policies. Compared to the objective of minimizing the average age of full members, the minimization of the average age of corresponding members implies an optimal election policy, u_F, that is the opposite. It is optimal to elect corresponding members at old middle and old ages, and avoid electing members at young and very old ages. Thus, more corresponding members will be elected to members at older ages. At the same time, the optimal policy of election of corresponding members, u_C, (see Figure 13) suggests electing people at young and young middle ages. The outcome is a significant increase in the average age of full members, $\mathcal{A}_F$.

What also follows from Table 2 is that the optimal election policies decrease the number of elections to corresponding members, R_C, while they increase elections to full membership, R_M. The latter effect is due to the older age structure of full members which clears some vacancies for new elections once they are above the age limit of 70 years. Since the age structure of the corresponding members is becoming younger, the durations of staying a corresponding member, θ_C and θ_{CF}, increase.

Age densities of corresponding and full members, C and F, referring to optimal election policies are shown in Figure 14. The results clearly demonstrate that the number of older people in the population of corresponding members decreases and at the same time, the number of younger people in the population of full members decreases.

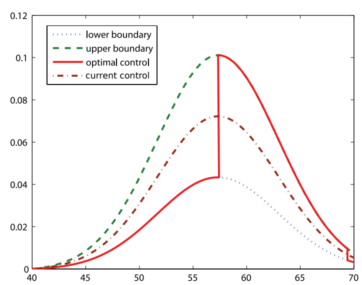

Figure 12: Minimizing $\mathcal{A}_C$: Election policy $u_F(a)$

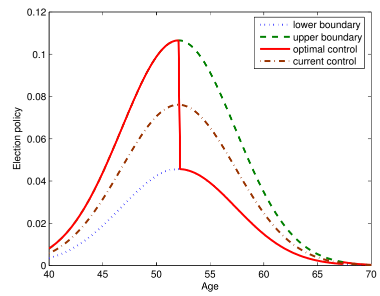

Figure 13: Minimizing $\mathcal{A}_C$: Election policy $u_C(a)$

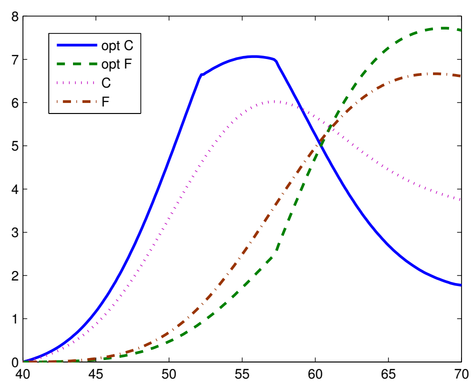

Figure 14: Minimizing $\mathcal{A}_C$: Age densities

4.5 Summary

In this section we studied optimal election policies based on single-objective functions. In Table 2 we compare the resulting demographic structure for each objective function chosen. As these results obviously indicate, the maximization of number of elections to `C` does not effect the average age of full members and the number of elections to full members. It is caused by the fact the function R_C in (5) depends only on a parameter u_F, while functions R_F in (6) and A_F in (10) are only subject to control u_F. It is interesting to note that due to the heterogenous structure of the model both fixed-size conditions on populations of `C` and `F` are fulfilled automatically even when only one election policy is optimized.

It is also found that the maximization of elections and minimization of average ages are conflicting objectives. Optimization of the elections of full and corresponding members, R_F and R_C, implies an increase in the average ages,

$\mathcal{A}_F$ and $\mathcal{A}_C$, respectively. In contrast, minimization of the average ages of corresponding and full members, $\mathcal{A}_C$ and $\mathcal{A}_F$, leads to a decrease in the number of elections of corresponding and full members, R_C and R_F, respectively. These results can be explained using analytical expressions of the indicated objectives. One can note, that the expression of R_F in (6) is proportional to the positive multiplier in expression of $\mathcal{A}_F$ in (10). Similarly, formula of R_C in (5) is proportional to the multiplier in the positive term of $\mathcal{A}_C$ in (9).

Furthermore, the results in Table 2 indicate, that minimization of the average ages of corresponding and respectively full members, are contradicting objectives as well. One can see that expression of $\mathcal{A}_C$ in (9) is proportional to the negative term in the formula of $\mathcal{A}_F$ in (10).

It is interesting to also note the change in the tenure variables Θ_C and Θ_{CF} for different objective functions. Compared to the current prevailing practice in the Austrian Academy of Sciences, the tenure of corresponding members who are not elected as full members, decreases when the objective is to maximize the size of full or corresponding members, but increases if the objective is to minimize the average age of full or corresponding members. The results are less clear regarding the tenure of corresponding members that are eventually elected as full members, Θ_{CF}. Compared to the current situation of the Austrian Academy of Sciences, the tenure variable Θ_{CF} decreases if the objective is to maximize the number of corresponding members or to minimize the average age of full members. Alternatively, the tenure Θ_{CF} increases if the objective is to maximize the number of full members or minimize the average age of corresponding members. Though we did not choose the tenure of corresponding members as an objective function, our results indicate that we can implicitly affect these variables by way of the proposed objective functions.

To account for the tradeoff between different objectives, we consider multi-objective problems in the following sections.

Table 2: Results of single-objective optimization problems, $\varepsilon = 0.4$.

	R_C	R_F	$\mathcal{A}_C$	$\mathcal{A}_F$	θ_C	θ_{CF}
Reference values	11.8	7.4	58.4	62.4	7.0	6.1
Model						
Maximization of R_C	13.0	7.4	59.9	62.4	6.5	4.9
Maximization of R_F	11.8	8.5	57.7	63.2	6.8	7.3
Minimization of $\mathcal{A}_F$	11.8	6.6	58.9	61.7	7.2	5.3
Minimization of $\mathcal{A}_C$	10.8	8.5	56.4	63.2	7.1	8.7

5 Multi-Objective Optimization

The results of the single-objective optimization obtained in the previous section show that an improvement of some of the demographic indicators may result in worse values of other indicators. Thus the question arises to find an optimal trade-off between several objectives. This issue leads to multi-objective optimization problems that we treat by optimizing weighted sums of the respective demographic indicators. The present section provides some numerical results in this direction which are of particularly interest for academies of sciences.

5.1 Keeping the Full Members Young and Elections to Full Members High

The results in Table 2 show that minimization of the average age of full members, $\mathcal{A}_F$, leads to lower number of elections R_F and, vice versa, maximization of R_F increases $\mathcal{A}_F$. To find an optimal trade-off between these two indicators we consider the problem of minimization of the weighted combination

$$\alpha \mathcal{A}_F - \beta R_F, \tag{22}$$

where $\alpha > 0$, $\beta > 0$, with $\alpha + \beta = 1$, are the respective weights. Here the election policy for corresponding members, $u_C(a)$, is kept fixed. A qualitative characterization of the solution of this problem is obtained in Feichtinger and Veliov (2007), claiming that the optimal election policy $u_F(a)$ has a bi-polar shape. Results of numerical optimization for various combinations of α and β are given in Table 3.[9] In parentheses we give the changes of indicators with respect to the current reference values.

Table 3: Minimization of $\alpha \mathcal{A}_F - \beta R_F$.

α	β	R_F	$\mathcal{A}_C$	$\mathcal{A}_F$	θ_C	θ_{CF}
0.4	0.6	8.4 (1.0)	57.7 (-0.7)	63.1 (0.7)	6.9 (-0.1)	7.3 (1.2)
0.5	0.5	7.8 (0.4)	58.4 (0)	62.3 (-0.1)	7.0 (0)	6.7 (0.6)
0.6	0.4	7.3 (-0.1)	58.7 (0.3)	61.9 (-0.5)	7.0 (0)	6.1 (0)
0.8	0.2	6.8 (-0.6)	58.9 (0.5)	61.7 (-0.7)	7.1 (0.1)	5.5 (-0.6)

Table 3 shows that increasing the weight α of the average age of full members leads to less elections, R_F, as it was expected. The interesting observation is that attributing more weight to the average age of full members leads to higher average age of corresponding members, $\mathcal{A}_C$, higher tenure of corresponding members, θ_C, but lower tenure of the corresponding members who become elected to full members, θ_{CF}.

The optimal election policies with various intermediate weights α and β are shown on Figures 15–16. The "extreme" cases of $\alpha = 0$ and $\alpha = 1$ are given by Figure 10 and Figure 13, respectively.

[9] The case, $0 < \alpha < 0.4$, i.e., $(\alpha, \beta) = (0.2, 0.8)$ is not included in the table as it is almost identical to the case, $(\alpha, \beta) = (0, 1)$ given in the section 4.2.

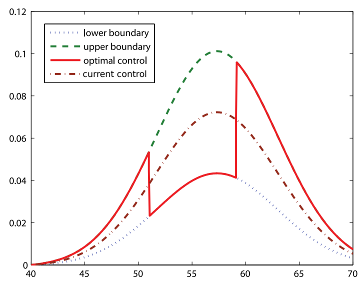

Figure 15: Minimizing (22), $(\alpha, \beta) = (0.5, 0.5)$: Election policy $u_F(a)$

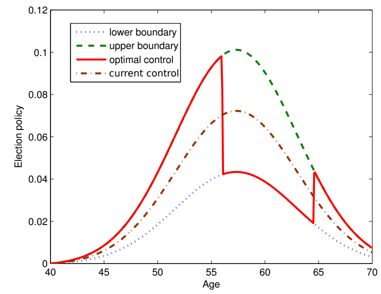

Figure 16: Minimizing (22), $(\alpha, \beta) = (0.8, 0.2)$: Election policy $u_F(a)$

5.2 Keeping the Full Members and Corresponding Members Young

Formulas (9) and (10) clearly show that the demographic indicators $\mathcal{A}_F$ and $\mathcal{A}_C$ are conflicting. Namely, in the expression of $\mathcal{A}_C$ there is a negative term which is proportional to $\mathcal{A}_F$. Therefore we consider the problem of simultaneous minimization of the average ages $\mathcal{A}_C$ and $\mathcal{A}_F$ by introducing corresponding weights, $\alpha \geq 0$, $\beta \geq 0$. That is, we minimize with respect to the election policies u_F and u_C the function

$$\alpha \mathcal{A}_C + \beta \mathcal{A}_F = \alpha \frac{(\bar{C} + \bar{F}) \int_0^\omega X(a) e(a) u_C(a)\, \mathrm{d}a}{\bar{C} \int_0^\omega e(a) u_C(a)\, \mathrm{d}a} + (\alpha \frac{\bar{F}}{\bar{C}} - \beta) \frac{\int_0^\omega X(a) e(a) u_F(a)\, \mathrm{d}a}{\int_0^\omega e(a) u_F(a)\, \mathrm{d}a}, \tag{23}$$

where $\alpha + \beta = 1$.

It is clear that for $\alpha < \dfrac{\bar{C}}{\bar{C} + \bar{F}}$ (for these α the coefficient of the second term in (23) is positive), the above problem splits to the pair of problems: minimization of the first term in (23) in u_C and minimization of the second term in u_F. In the case $\alpha > \dfrac{\bar{C}}{\bar{C} + \bar{F}}$ minimization of (23) is equivalent to minimization of the first term and maximization of the second term. Thus the solution u_C of the problem of simultaneous minimization of the two average ages $\mathcal{A}_F$ and $\mathcal{A}_C$ is independent of the weight α, while the solution u_F is independent of α left and right of the threshold $\alpha = \dfrac{\bar{C}}{\bar{C} + \bar{F}}$ but may change at the threshold.

The results of simultaneous minimization of average ages for various weights α and β are presented in Table 4. In parentheses we give the changes of indicators with respect to the current reference values.

The results of optimization problem (23) show that for all three cases in accordance with the value of parameter α the solution u_C is the same. Namely,

Table 4: Minimization of $\alpha\mathcal{A}_C + \beta\mathcal{A}_F$.

α	β	R_F	$\mathcal{A}_C$	$\mathcal{A}_F$	θ_C	θ_{CF}
0.1	0.9	6.6 (-0.8)	57.7 (-0.7)	61.7 (-0.7)	7.7 (0.7)	6.3 (0.2)
0.55	0.45	threshold				
0.6	0.4	8.5 (1.1)	56.4 (-2.0)	63.2 (0.8)	7.1 (0.1)	8.7 (2.6)

the optimal election policy u_C is the same as in the problem of minimization of the average age $\mathcal{A}_C$ of the corresponding member. Its shape is depicted on Figure 13. The election policy u_F takes three different values depending on parameter α.

The first case, $\alpha < \dfrac{\bar{C}}{\bar{C} + \bar{F}}$, corresponds to the situation when the number of corresponding members is not very high. In this case it is reasonable to care more about the average age of the full members, and the optimal solution u_F corresponds to election policy minimizing the average age $\mathcal{A}_F$. Its shape is depicted on Figure 10. From Table 4 one can see that in this case both indicators of average ages, $\mathcal{A}_C$ and $\mathcal{A}_F$, improve their values compared to the current situation.

The value $\alpha = 0.55$ in Table 4 corresponds to the threshold calculated for the considered case-study. In this case the second term vanishes in formula (23). Therefore, only the control u_C is optimized, and the indicators take various values depending on the election policy u_F.

The last case, $\alpha > \dfrac{\bar{C}}{\bar{C} + \bar{F}}$, corresponds to the situation where the number of full members is small. Then their average age is not taken into account and minimization of the average age of corresponding members is the main problem. The optimal solution u_F is the same as in the problem of minimizing $\mathcal{A}_C$. Its shape is given on Figure 12.

It can also be observed in Table 4 that attributing the higher weight, α to the average age $\mathcal{A}_C$ leads to the decrease in the duration θ_C of stay as corresponding member, and to the increase in the duration of θ_{CF} of stay as corresponding member of those who become a member.

6 Conclusion

In this paper we elaborate a methodology for analysis of age-specific election policies in two-level organizations with fixed size, building on previous results in Feichtinger and Veliov (2007) which concern a single-level organization. This methodology allows us to optimize weighted combinations of demographic indicators (such as the number of elections, the average age, or the tenure of members in each of the levels), by modifying appropriately the election policies practiced by the organization in the past. To study the optimal trade-off between the average age of full members and the number of elections to full members we use the optimal control theory. Particular numerical results of a case study dealing with the Austrian Academy of Sciences are presented, which provide a basis for possible policy implementations.

The approach to optimization of demographic indicators of organizations with fixed size presented in the paper could be extended to multi-level organizations. Several other extensions are interesting and challenging:

(i) The analysis in the present paper is static – regarding the steady-state of the organization. It is important to include explicitly the time as an additional variable (which would bring into consideration a distributed control system), since the mortality rates and the size of some organizations may change exogenously. The latter is the case in many academies of sciences. Moreover, not only the long run (steady-state) behavior may be of interest, but even more – the short run effects of changes in the election policies.

(ii) An additional constraint on the duration of membership at certain levels exists in many organizations, for example the non-tenure or the tenure track positions in universities. In addition, in some universities there exist age-limitations for some positions (for postdocs or assistants, for example). The modifications in the model needed to incorporate such additional constraints are substantial and deserve further investigations.

(iii) In this paper we consider a two-level case of the model. It is planned, however, to apply the same methodology to multi-level hierarchical organizations (with the number of levels more than two). Modeling the tenure of members at different levels is of particular interest, especially, with respect to the carrier prospects of individuals. This can be done by considering distributed dynamics with respect to the variables age and tenure.

Let us conclude with two further applications of the framework developed in the present paper. The first topic deals with personnel management in fixed-size or even shrinking organizations. Similar problems as the two-stage optimization problem of recruitment (promotion) arise in over-tenured universities. In a time of retrenchment, university administrators are pondering ways to issue less tenures. In his remarkable paper Vaupel (1981) compared strategies to reduce over tenure, e.g. by increasing the attrition of tenured faculty members versus lengthening the average time to tenure.

It should be noted that such problems can be studied both, from the point of view of the organization as a whole and in the context of individual career planning. More precisely, in the latter case, the question is to what extent the career prospects for members of the organization decrease in stagnation periods of the growth phase (compare Keyfitz 1977; and Henry 1975). The partial differential equations (PDE) framework has several advantages in comparison with the overlapping generations (OLG) approach commonly used in population economics: it allows a more profound application of analytic tools, and, in some cases, the derivation of analytic solutions; by means of appropriate high-order approximation and adaptive multigrid discretization methods one can achieve high accuracy in a more economical way, which is important in view of the high computational complexity of the problems.

While all these problems can be seen in the context of *Operations research in manpower planning*, the second issue, mainly, migration is a demographic and socio-economic one. Since, however, it formally fits into the framework of our approach, let us briefly refer to it. Models of age-specific recruitment with prescribed population sequences and structure play an important role in the design of efficient age-oriented immigration policies. How many, and at which ages, would migrants have to immigrate in order to achieve a certain goal (e.g., zero growth or fiscal balance in the pension system of the receiving country) or how would age-specific immigration in a sub-replacement population influence its overall dependency ratio. In this context we refer to work of Mitra (1990), Feichtinger and Steinmann (1992), Schmertmann (1992), Wu and Li (2003). The unifying tie consists in the methodology which is applicable to renewable aggregates, both to large populations as well as small subpopulations as organizations, learned societies or firms. The tool to cope with efficiency problems arising in the context of population dynamics is *intertemporal optimization.* And since age and duration are the key control variables, distributed parameter control provides a powerful tool.

Acknowledgements

The second author was supported by the Austrian Science Fund (FWF) Project "Age Structured Populations with Fixed Size", No. P20408-G14, and by the Program of the Presidium of Russian Academy of Sciences "Mathematical Control Theory", No. 29.

The authors would like to thank Werner Richter for the English proof reading of the paper, participants of the seminar at Vienna Institute of Demography, Austrian Academy of Sciences for their valuable comments, and three anonymous referees for their comprehensive reviews and valuable remarks.

7 Appendix

7.1 Average tenure

A formula for the average tenure in a stationary population with inflow $u_C(a)$ and outflow $R_F u_F(a)$ is derived below. Here a is the age, the tenure of a individual in the population is denoted by $\tau \leq a$. If the distribution of u_C across different durations of stay in the population is not given, the average tenure is not uniquely defined. Therefore we assume the following:

Assumption. The τ-density, $\gamma(a, \cdot)$, of the outflow $R_F u_F(a)$ has the form

$$\gamma(a, \tau) = \frac{C(a, \tau)}{C(a)},$$

where $C(a, \tau)$ is the size of the sub-population of age a and tenure τ, $C(a) = \int_0^\omega C(a, \tau)\, \mathrm{d}\tau$ is the size of the sub-population of age a, ω is the maximal age.

In other words, it is assumed that the $Fu_F(a)$ individuals of age a who leave the population are randomly chosen among the individuals of age a (not taking into account their tenure).

The age-profile of the static population satisfies the equation

$$\dot{C}(a) = -\mu(a)C(a) + R_C u_C(a) - R_F u_F(a), \quad C(0) = 0,$$

where $\mu(a)$ is the mortality rate. The initial condition means that the population is not self-reproductive: all individuals come from outside with time-rate $u_C(a)$.

In order to derive the formula for the average tenure of corresponding members, we consider the distributed dynamics (with outflow $\gamma(a,\tau)R_F u_F(a) = \frac{R_F u_F(a)}{C(a)} C(a,\tau)$)

$$C_a(a,\tau) + C_\tau(a,\tau) = -\mu(a)C(a,\tau) - \frac{R_F u_F(a)}{C(a)} C(a,\tau), \quad C(a,0) = R_C u_C(a).$$

With the assumed particular form of γ the solution is known to be

$$C(a,\tau) = R_C e^{-\int_{a-\tau}^{a} \left(\mu(\alpha) + \frac{R_F u_F(\alpha)}{C(\alpha)}\right) \mathrm{d}\alpha} u_C(a-\tau).$$

For the average tenure of the sub-population of age a we have

$$\begin{aligned} T_C(a) &= \frac{1}{C(a)} \int_0^a \tau C(a,\tau)\,\mathrm{d}\tau = \frac{R_C}{C(a)} \int_0^a \tau e^{-\int_{a-\tau}^{a} \left(\mu(\alpha) + \frac{R_F u_F(\alpha)}{C(\alpha)}\right) \mathrm{d}\alpha} u_C(a-\tau)\,\mathrm{d}\tau \\ &= \frac{R_C}{C(a)} \int_0^a (a-b) e^{-\int_{b}^{a} \left(\mu(\alpha) + \frac{R_F u_F(\alpha)}{C(\alpha)}\right) \mathrm{d}\alpha} u_C(b)\,\mathrm{d}b. \end{aligned}$$

Then for the average tenure of the whole population we obtain

$$\begin{aligned} \Theta_C &= \frac{1}{\bar{C}} \int_0^\omega T_C(a) C(a)\,\mathrm{d}a \\ &= \frac{R_C}{\bar{C}} \int_0^\omega \int_0^a (a-b) e^{-\int_{b}^{a} \left(\mu(\alpha) + \frac{R_F u_F(\alpha)}{C(\alpha)}\right) \mathrm{d}\alpha} u_C(b)\,\mathrm{d}b\,\mathrm{d}a. \end{aligned}$$

The duration Θ_{CF} is defined by the expression

$$\begin{aligned} \Theta_{CF} &= \int_0^\omega T_C(a) u_F(a)\,\mathrm{d}a \\ &= R_C \int_0^\omega \int_0^a (a-b) e^{-\int_{b}^{a} \left(\mu(\alpha) + \frac{R_F u_F(\alpha)}{C(\alpha)}\right) \mathrm{d}\alpha} u_C(b)\,\mathrm{d}b \frac{u_F(a)}{C(a)}\,\mathrm{d}a. \end{aligned}$$

7.2 The Role of Mortality

To what extend is the longer life expectancy responsible for the demographic parameters of the academy, and in which direction it changes them?

We compare three scenarios: (i) zero mortality; (ii) present mortality rate $\mu(a)$; (iii) doubled present mortality, $2\mu(a)$.

The conclusion of the numerical evaluation of the demographic indicators of interest is given in Table 5:

(i) when mortality decreases, the indicators R_F and R_C decrease, while $\mathcal{A}_C$, $\mathcal{A}_F$, Θ_C, Θ_{CF} all increase.

(ii) the indicators are not very sensitive with respect to the mortality rate: 100% change in the mortality rate leads to only $4-5\%$ change in R_C and R_F, 3% change in Θ_C, and about 1% or less for the rest of the parameters.

Table 5: Sensitivity analisys with respect to the mortality rate.

Mortality rate	R_C	R_F	$\mathcal{A}_C$	$\mathcal{A}_F$	θ_C	θ_{CF}
zero	11.2	7.0	58.6	62.5	6.7	6.1
2μ	12.3	7.7	58.1	62.2	7.3	6.1

It would be interesting to test the effect of the present election pattern with mortality data from the beginning of 20-th century, and to see if the increased average age is due to the lower mortality, or due to other reasons (different election pattern in the past, for example).

7.3 Minimizing the Average Age $\mathcal{A}_F$

In this section analytical approach to problem of minimization of average age $\mathcal{A}_F$ is presented.

The problem is to minimize

$$\mathcal{A}_F = \frac{1}{\bar{F}} \int_0^\omega aF(a)\,\mathrm{d}a \underset{u_C(a)}{\longrightarrow} \min \tag{24}$$

under conditions

$$\begin{aligned} \dot{C}(a) &= -\mu(a)C(a) - R_F u_F(a) + R_C u_C(a), \quad C(0)=0, && (25)\\ \dot{F}(a) &= -\mu(a)F(a) + R_F u_F(a), \quad F(0)=0, \quad a \in [0,\omega], && (26) \end{aligned}$$

and constraints

$$\int_0^\omega u_F(a)\,\mathrm{d}a = 1, \qquad \int_0^\omega C(a)\,\mathrm{d}a = \bar{C}, \qquad \int_0^\omega F(a)\,\mathrm{d}a = \bar{F}. \tag{27}$$

Applying Lagrange multipliers λ, β and $S\gamma$ to problem (24) with constraints (27) we obtain the optimal control problem

$$\int_0^\omega [aF(a) + \lambda C(a) + \eta F(a) + R_F \gamma u_F(a)]\,\mathrm{d}a \underset{u_F(a)}{\longrightarrow} \min, \tag{28}$$

with dynamics (25). To solve the problem one can apply maximum principle of Pontryagin (Pontryagin et al. 1962). Introducing adjoint variables ψ_1 and ψ_2 we define the Hamiltonian:

$$H = \psi_1(-\mu(a)C(a) - R_F u_F(a) + R_C u_C(a)) + \psi_2(-\mu(a)F(a) + R_F u_F(a)) - \quad (29)$$
$$(aF(a) + \lambda C(a) + \eta F(a) + R_F \gamma u_F(a)). \quad (30)$$

The necessary optimality conditions lead to the following expression

$$\bar{H} = \max_{u_F} H = \max_{u_F} (\psi_2 - \psi_1 - \gamma) R_F u_F(a), \quad (31)$$

which indicates the bang-bang type solution to a problem. The system of adjoint equations is presented by equations

$$\dot{\psi}_1 = -\frac{\partial H}{\partial C} = \mu(a)\psi_1 + \lambda, \quad (32)$$
$$\dot{\psi}_2 = -\frac{\partial H}{\partial F} = \mu(a)\psi_2 + a + \eta. \quad (33)$$

By introducing the variable ψ as

$$\psi = \psi_2 - \psi_1, \quad (34)$$

we rewrite expression (31)

$$\bar{H} = \max_{u_F} H = \max_{u_F} (\psi - \gamma) R_F u_F(a), \quad (35)$$

where ψ is the solution of equation

$$\dot{\psi} = \mu(a)\psi + a + \eta - \lambda. \quad (36)$$

The parameters γ, λ and β should be calculated in such a way that conditions (27) are satisfied for the resulting solution. Following the analysis in Dawid et al. (2009), one obtains that the problem has a bi-polar solution.

7.4 The Role of Tolerance Parameter ε

In this section we perform sensitivity analysis with respect to parameter ε which defines the possible variance in election policies. For this reason we analyze numerically the problem of minimization of average age $\mathcal{A}_M$ for three values of tolerance parameter: $\varepsilon = 0.1$, $\varepsilon = 0.4$, and $\varepsilon = 0.8$. The results for intermediate value, $\varepsilon = 0.4$, are presented in the Section 4.3.

Sensitivity analysis shows that increase in parameter ε leads to numerical increase in the variances of indicators after optimization in comparison to current values.

In Figures 17–18 we present optimal election policies corresponding to parameters $\varepsilon = 0.1$ and $\varepsilon = 0.8$. Figures demonstrate that qualitative features of the optimal election policy is conserved with the variance of parameter ε.

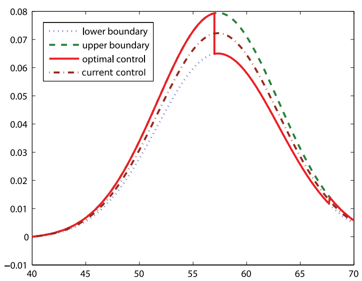

Figure 17: Minimizing $\mathcal{A}_M$, $\varepsilon = 0.1$: Election policy $u_F(a)$

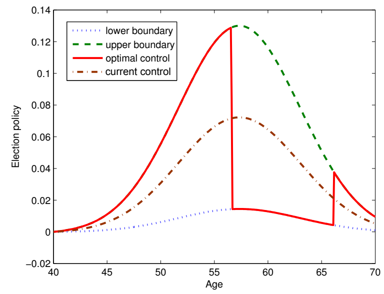

Figure 18: Minimizing $\mathcal{A}_M$, $\varepsilon = 0.8$: Election policy $u_F(a)$

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