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# **Annuities versus Bonds**

## Optimal Consumption and Investment Decisions in a Continuous Life Cycle Model

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DIPLOMARBEIT

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in a Continuous Life Cycle Model

Ausgeführt am Institut für  
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unter der Leitung von  
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durch

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Finally, the thesis is finished and I write these lines.

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Berni

## Abstract

If people behave rationally, why do they invest so little money in annuities? What are the reasons that people consume most when they are approximately 50 years old? In existing models the optimal behaviour is complete annuitisation and the optimal consumption exhibits a peak very late in life. The continuous life-cycle models proposed in the present thesis differ from the existing ones in that either annuities generate lower return than the actuarially fair rate, or the utility of bequest depends on the number of dependent family members. Each of these features leads to an age-dependent portfolio of investments in bonds and annuities. In addition, it is numerically shown that the risk aversion towards death leads to a consumption profile which is qualitatively consistent with the real data. In all considerations the rent of the capital stock and the wages are endogenously determined by the rational expectation framework, in contrast to most of the existing studies, where both are exogenous.

**Keywords:** annuities, continuous life cycle model, dynamic optimisation, investment, rational behaviour, optimal control

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# Introduction

Bonds or Annuities? Whenever people have money left and want to save it for the future, there are many different options: buying stocks, deposit it in bank accounts, hiding it in the mattress, or buying annuities and bonds. Typically not all money is invested the same way: some money is deposited in the bank, for some of the money stocks or bonds are bought and maybe some annuities are contracted. But how is this decision made? What factors determine which fraction of the money is invested for example in bonds? Does it depend on the number of children or only on the interest rate?

This question is not only relevant to any of us, in order to make the “best” choice but also for governments and insurance companies. If it is known what is the most favourable investment for people, the government can try to take measures in order to improve the situation of the society. Insurance and investment companies can, based on this knowledge, enhance their products and improve their quality of counselling.

Of particular interest is the decision whether to invest in bonds or annuities. The annuities market is considerably smaller than the bonds market (Sommer, 2005), so bonds are more attractive to people than annuities. However, theoretically annuities give a higher return because the real interest rate is higher than on bonds (Koller, 2010). There are two possible conclusions. Either people do not behave rationally and buy a product which is not optimal for them, or, all models presented so far do not incorporate the correct motivational factors which determine the optimal decision. The latter is presumed in this thesis.

The investment decision shall not be isolated from another very important question, namely how much money is consumed and how much money is saved? What factors determine the saving rate?

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The consumption over lifetime is hump shaped. In young ages it increases, rises to its maximum at about 50 years and decreases afterwards (Gourinchas and Parker, 2002). The share of money invested in insurance contracts are some 20% (Sommer, 2005). In many life cycle models, the optimal consumption is also hump shaped, but the hump occurs later in life. The amount of money invested in annuities is mostly higher than observed. The aims for a good life cycle model, which explains the investment decision between bonds and annuities, are that the hump can be reproduced and the optimal investment in annuities is close to the 20%. The task is to identify the important factors, to incorporate them in the model and reproduce the empirical behaviour observed.

Bonds or Annuities? This question was first addressed in a continuous life cycle model by Yaari (1965) and became popular towards the end of the 20th century. The subject was examined from different perspectives motivated by different questions: “What is the optimal decision for a 65 year old?” (Brown and Poterba, 2000; Davidoff et al., 2005; Friedman and Warshawsky, 1990), “What may be relevant factors that determine the decision?” (Sommer, 2007; Hubbard et al., 1994; Kotlikoff, 1989) and “What is the long term effect of such decisions?” (Feigenbaum and Gahramanov, 2010; Yaari, 1965; Feigenbaum and Caliendo, 2010).

Part of this thesis is a thorough summary of the basic of annuities and a detailed literature review, which includes all of the above mentioned works and more in Chapter 1.

Identifying one option as “best” requires that it is possible to measure the effects of one decision. Chapter 2 deals with different methods for that and discusses assumptions which are implicitly made in many papers.

The core of this thesis are the models discussed in Chapter 3. In a continuous life cycle model, the effects of a load on annuities (Section 3.2), of a non expected utility (Section 3.2.3), of a heterogeneous population (Section 3.2.4) and the influence of changing motives for bequests over lifetime (Section 3.3) are examined. The model was also implemented in Matlab (Section 3.2.1 and Appendix A).

# Chapter 1

## Annuities

In this chapter the basic facts about annuities are summarised (Section 1.1), the empirical investment behaviour described (Section 1.2) a review about existing literature is given (Section 1.3) and finally the Austrian insurance market is characterised in Section 1.4.

### 1.1 Annuity Basics

Originally, an *annuity* is a contract between two parties, the annuity buyer and the annuity seller, which guarantees the *annuity buyer* an annual payment, which gave the contract the name, for the rest of his life in exchange for a one-time payment in money or assets, the *annuity consideration*, in advance. Therefore, an annuity contract exchanges a series of uncertain future payments (called *instalments*) for one certain payment.

A contract which guarantees an annuitant a yearly payout of \$100 at the end of each year is called an *annuity of* \$100. If the payment is payable every half year, it is called a half yearly annuity of \$100, the single instalments are each \$50. Contracts which guarantee payments in advance (at the beginning of the year) are called *annuity due*. The instalments are also called annuities, so the payouts and the contract are referred to by the same word.

Nowadays a wide range of options is available for annuities, so that today the above definition is too strict for modern annuities. Especially the border to life insurances is blurred and often the words are used synonymously. For examples,

the contract may not depend on the life of the annuity buyer but of another person, the *nominee*. The instalments, also called annuity payments, may occur not annually but also quarterly or monthly and the maximal number of annuity payments can be restricted. Available options are described in more details below. In this chapter, the term annuity refers to the modern interpretation of this contract form. However, in the models in Chapter 3 the annuities of the original form are considered.

The first reported annuity contracts date back to the Middle Ages when it was possible to provide a regular income for descendants, for example a wife, by paying a lump sum to a reliable source such as the King. Today annuities with considerations in form of assets are still very popular in agriculture. A farmer bequeaths or sells the farm in exchange for an annuity payment until the end of his life.<sup>1</sup>

Besides asset backed annuities such as described above, in most of the world only insurance companies are allowed to sell annuities and the level of regulation of the risk management is very high. Therefore annuities can be considered as insurance contracts. In analogy to other insurance contracts, the considerations of the annuitant is also called premium.

Modern annuity contracts have two phases. During the *accumulation phase*, the annuitant pays once or regularly the consideration(s) and the capital often participates in the market. In the *distribution phase*, the insurance company distributes the premiums and the interest on it to the annuitant. If the premium is paid at once this payment is called *lump sum* and the corresponding annuity a single-payment-annuity in comparison to regular-payment-annuities.<sup>2</sup>

<sup>3</sup>Annuities can be characterised according to the length of the accumulation phase, the payout options, or which payments are fixed. In the next paragraphs these details will be explained before the basic premium calculation methods are

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<sup>1</sup><http://www.basis-rente-vergleich.de/>

<sup>2</sup>see US Securities and Exchange Commission, [www.sec.gov/answers/annuity.htm](http://www.sec.gov/answers/annuity.htm), for details

<sup>3</sup>see for example <http://annuities-explained.net/> or Koller (2010)

shown.

Annuities differ according to the length of the accumulation phase. *Immediate annuities* are contracts in which the time span between the last premium payment and the first payout is less than between the payments. Such contracts are primarily bought by older people with less risk tolerance in order to ensure their future consumption. The paid premium usually is invested in risk-free investment products such as treasury bonds and does not participate in the stock market, so there is no market risk, the risk that the value of the portfolio will decrease due to changes in market factors, in these contracts. Therefore, these annuities have a low but mostly guaranteed return. The conservative investment choice by the insurance companies is often also due to investment regulations by law. The main risk in such contracts is inflation risk, the risk that inflation rises and the value of the guaranteed payments decreases. Another risk in any contract is counterparty credit risk, the risk that the contract counterparty cannot fulfil its obligations. For annuity contracts, this is only a minor risk because insurance companies in most countries have to be accredited and meet very strict regulatory requirements. Furthermore, there are deposit guarantees for private annuitants when entering an annuity contract with an insurance company.

*Deferred annuities* are contracts in which there lie a couple of years between the premium payments or between the premium payment and the payout. The premiums usually are invested not as conservative as for immediate annuities and participate in the share or stock market. Deferred annuities are often vehicles to accumulate savings and consume them later in life. It is possible to accumulate money in such vehicles because in many countries there is no or reduced capital gain tax on dividends in the accumulation phase. The invested amount is furthermore in many countries such as UK, USA and also Austria at least partly eligible for an income tax discount. Tax is only due when withdrawing the money later in life, which allows a significant increase in capital. Only when withdrawing the money earlier, there are high fees and taxes on it. Deferred annuities are often

purchased by rich people who benefit from the tax discount more strongly.<sup>4</sup>

Modern annuity contracts can ensure a payment in case of survival and in case of death.

If they ensure a payment in case of survival, they are called *life annuities* and are like a pension. The annuitant receives a regular payment as long as he is alive. The annuitant gives sort of a loan to the insurance company, for which the company only has to pay interest as long as the nominee is alive. Such contracts provide longevity insurance, that is insurance against a higher life expectation than the average. This is because if the annuitant dies young, he finances the payout for the others, but he is rewarded if he lives longer than the average and is provided with an income until the end of life. Therefore he cannot outlive his resources. The annuity payments may be restricted to a certain period, e.g. 20 years. Such contracts do not provide longevity insurance any more because if the buyer lives longer than the period, he can outlive his financial resources. Many life annuity contracts also offer to pay out a lump sum at a certain age instead of a yearly pension if the nominee is still alive. These contracts can be used to accumulate wealth for large investments such as buying a house or financing education of the children.

The opposite of an annuity is a *term life insurance*, which corresponds to short selling of annuities and ensures a payment in case of death. The annuitant pays a regular payment or a lump sum, and if he dies during the contract period he (or rather his descendants) receives a lump sum or rent. A term life insurance can reduce the impact on the financial situation of the death of the only earner in a family. For an illustration of short selling annuities consider a loan which has to be repaid only if the agent survives, in return higher interest has to be paid. A *whole life insurance* is a term life insurance with infinite contract period. In many cases such contracts are required to apply for a loan to ensure the payments of the contract even in case of death.

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<sup>4</sup>for the UK see <http://www.annuityarrow.com/purchasing-guide.html> (Financial Service Authority, the UK regulator for financial services), for the USA: <http://www.sec.gov/answers/annuity.htm> (US Securities and Exchange Commission), for Austria: [http://www.bmf.gv.at/Finanzmarkt/Altersvorsorge/\\_start.htm](http://www.bmf.gv.at/Finanzmarkt/Altersvorsorge/_start.htm) (Austrian Ministry of Finance)

Mixed contracts between annuities and life insurances, which guarantee a high payment in case of death and a small payment or rent in case of survival, are also available and called *death and endowment insurances*.

There is a variety of other options purchasable on annuity contracts. Couples can enter the contract together (*joint annuities*), so the annuity is paid as long as either of the partners is alive. The payouts may decrease after the death of the first partner in order to account for the reduced need of money and make the annuity cheaper. Another alternative is that the payments are ensured for a certain period, regardless of the survival of the nominee, and only after this period they depend on the life of the nominee. For deferred annuities, *escalation options*, an adaptation to inflation, are also available. If stocks are purchased from the premiums, then there is the risk that the stock loses its value when the money shall be withdrawn. *Step up* options protect certain levels (e.g. yearly) the capital stock has reached and *rising floor* options guarantee at least a certain level of capital stock at the beginning of the distribution phase.

The different contract types and options described here, as well as a thorough introduction in actuarial mathematics, can be found in Koller (2010).

In this thesis life insurance always refers to term or whole life insurance, so a contract which guarantees payments in case of death. Annuities refer to contracts which ensures annuity payments until the end of the life. Please note that in some literature, e.g. Sommer (2005), annuity contracts are also covered by the term “life insurance”.

Many annuity contracts are not of a single annuitant but are paid from a company, which provides its workers with a company pension scheme. Such contracts are called *group contracts*. In the beginning of the 20th century when the first group annuity contracts were signed, they were mostly *defined benefit* plans, in which the amount paid out to employees was fixed and the payment of the company was uncertain and changed with the market factors such as interest rates. Traditionally, these plans paid out benefits in form of life annuities and provided the annuitant with an insurance against living longer than average in two ways. He is provided with a certain income until death and does not have to fear to outlive his resources (Brown and Warshawsky, 2001).

In the 1990s the trend went to *defined contribution* plans, which accounted for 92% of all plans in 1996 (Brown et al., 2000). In these plans the regular payment by the annuitant or company is fixed and the annuity depends on the current value of the premiums. Interestingly, the change went hand in hand with an increase in the accessibility of lump sums as an alternative to yearly annuity considerations. In 1998 only 25% of all annuity buyers had access to annuities whereas a lump sum was available in more than 90% of all contracts (Brown et al., 2000). Defined contribution plans allow to reallocate wealth and increase return on it in case of survival, but do not offer life insurance as the flow of payments is not constant but ends after some years. So towards the end of life, people again have to face the danger of outliving their resources.

Return on annuity premiums is generated by two sources. Primarily, the premiums are reinvested in the market (usually in products with no or very low risk such as government bonds, also called treasury bonds) and thus generate a return. Secondly, if an agent dies, his paid premiums belong to the insurance company and can be redistributed to the other contractors in order to increase the payout.

Annuities, as all insurances, depend on the law of large numbers. Whereas a single contract would be very risky, a large number reduces the risk if the death of different people is independent from each other. Calculating the value of annuities depends mainly on a table of mortality rates (life table) which reflects reality as good as possible. The first modern life table was published in 1693 by Edmond Halley (1656-1742) who monitored births and deaths in Breslau, Poland (Halley, 1693). The mathematics was developed few years earlier by Johan de Witt (1625-1672) who first calculated the value of annuities based on estimated life tables in 1671. This was shortly after Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662) set up the mathematical methods of calculus of probability in the 1650s. Abraham de Moivre (1667-1754) developed the formulas further in his “Treatise on Annuities” in 1752 (de Moivre, 1752).<sup>5</sup>

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<sup>5</sup><http://www.annuity-insurers.org>

$$V(A) = \sum_{j=T_{ret}}^T \frac{Ap_j}{\prod_{k=1}^j (1 + i_k)}. \quad (1.1)$$

The present value of an annuity can be calculated according to formula (1.1), which discounts expected future payments. Here,  $A$  is the nominal annual premium,  $p_j$  is the survival probability until the  $j$ -th period,  $i_k$  is the interest rate of the  $k$ -th period,  $T_{ret}$  is the retirement age and  $T$  is the maximal expected lifetime. To adapt this formula for inflation, take  $A$  as the real payout and  $i_k$  as the real growth rate (Koller, 2010).

The first institution which dedicated its work to this topic was the Institute of Actuaries<sup>6</sup>, founded in England in 1848. Its journal (Journal of the Institute of Actuaries) has been one of the most important on the topic of annuities until today.

Annuities came in the focus of the economical literature especially towards the end of the twentieth century. Better data was available and bigger insurance companies and politics became interested in it, especially as many papers promised a benefit from annuities for society (e.g. Brown and Poterba (2000) - see Section 1.3). The increase in utility for an individual might be a reason for the government to promote annuities. A second, probably more important reason, is that with annuities it is less likely that people fall short in their savings and need social security payments when growing old. If a government wants to promote annuities there are different regulatory options.<sup>7</sup> Mandatory annuities, such as social security, are considered as the least preferable because they do not leave the choice to the people but rather incapacitates them. However, the impact of such measures is the most severe and administrative costs are comparatively low. An alternative is that by default people participate in such programs but they can leave the program (opt out) if they want to. This leaves the decision to the people and increases average annuitisation but at higher administration costs because the management of the opting outs is more expensive than a compulsory

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<sup>6</sup>In 2010 it merged with the Faculty of Actuaries and is now named Institute and Faculty of Actuaries, see <http://www.actuaries.org.uk>

<sup>7</sup>see the report of Prof. Poterba on <http://www.annuity-insurers.org>

annuity program. Tax incentives, which were introduced in Austria in 2003, reduce the revenue for the government but leave the free choice to the people and have relatively low costs of administration (Brown and Warshawsky, 2001).

As pointed out above, return on annuities is generated by two sources, interest and mortality. Therefore, the return on annuities should always be strictly bigger than solely the interest on risk-free assets such as treasury bonds. Annuities are also considered as risk-free, because most countries have deposit guarantees for annuitants. A rational choice is therefore to buy annuities instead of investing money in bonds as it yields a higher return (Yaari, 1965).

This is why many papers (see Section 1.3) predict a significant welfare gain when giving individuals access to the annuity market. As the magnitude of utility is not quite comparable (see Section 2 for more on that), a typical measure is the Annuity Equivalent Wealth (AEW). It is defined as the fraction of their wealth individuals need to receive additionally in order to be indifferent between the access to the annuity market and the money at the moment of their retirement. Estimations for the AEW in a single household with log-utility (see Section 2) function are approximately 50% (Brown and Poterba, 2000). In the theoretical models it is optimal to buy annuities for a large share of the income and invest only a small share in bonds. This is in sharp contrast to reality, where only a small amount of wealth is held in annuity contracts as will be pointed out below. This may be partly due to the reason that especially in Europe many people have a guaranteed state pension and therefore already some of their wealth annuitised, but also when accounting for these guaranteed payments, annuitisation is lower than the models predict.

The question why people do not buy annuities, or invest only a relatively low fraction of their wealth, if annuities offer a gain in welfare is subject to many papers and called *Annuity Puzzle*. Possible reasons often stated are bequest motives, hidden costs which reduce the return and the aspect of freedom of choice (see Section 1.3).

## 1.2 Empirical Savings Behaviour

Data on consumer expenditure is scarce because studies and surveys are very cost intensive. The most popular data is perhaps provided by the US Consumer Expenditure Survey (CES), as used in Gourinchas and Parker (2002). Most papers on annuity markets therefore concentrate on the US market. Some, such as Brown et al. (2000), also consider the market in the United Kingdom. In Germany there is similar data as the US-CES available.

The German Central Bank compiles annually statistics on the national accounts based on banking statistics, reports of insurance companies and the private sector. Sommer (2005) analyses this data as well as data from the German Income and Expenditure Survey (EVS) which provides every five years data from 40,000 households, and points out the main changes in German households' portfolios. The gross financial wealth (the sum of the money invested in banks, stocks, building society saving contracts, fixed interest securities and in insurance or annuity contracts) of each household in real terms, meaning adapted for inflation, rose from 5,000\$ in 1960 to 43,000\$ in 2001. Until that point, it grew cyclically and only declined during the stock market downturn of 2001. Fixed interest securities and life insurances experienced a long steady growth.<sup>8</sup> Deposits with banks had the lowest growth rates, while the fraction of money invested in insurance contracts doubled from 12% to 24% in this time frame. The demand for life insurances is increasing in age with a maximum at about 50 years and decreasing afterwards. Direct participation in the stock market increased by absolute numbers, the portfolio share decreased. Recently, since 1990, investment in mutual funds increased strongly (Sommer, 2007).

The investment behaviour of a person depends on his age and the cohort, that is, all people born within the same period, he is born in. At different stages in life, different motives play a role in the investment behaviour, and the investments available depend on the current year. Sommer (2005) tries to separate the effects. The age effect on the participation rate is that the demand for secure investments, such as saving accounts and treasury bonds, increases throughout

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<sup>8</sup>In Sommer (2005) the term life insurance covers also annuity products.

life whereas the demand for life insurances peaks somewhat around 50. The age effect on the portfolio share is similar to the effect on the participation rate: an increase in securities and a hump shaped demand for life insurances can be observed. The effect on saving contracts in absolute numbers is neutral but the portfolio share of saving contracts is also hump shaped.

The cohort effect, that is the effect that people are born in a different time and therefore have different experiences, results in a decreasing demand in life insurance. The fraction of money held in life insurances when retiring dropped from 25% in 1975 to 10% today. On securities the cohort effect is positive and on savings neutral when considering absolute values. The portfolio share of securities is increasing under the cohort effect, therefore the share of life insurances and savings decreasing.

According to Sommer (2007) the average life insurance owner is richer than average, is married and has children. However, the number of children does not have a significant effect on the possession of life insurances and the amount invested in life insurances is higher without children. According to the author, neither the deductibility of contributions from income tax nor tax-free returns have a significant or clear implication on the participation rate or portfolio share.

The participation rate in whole life insurance across the overall population is 58%, that means that 58% of the people have life insurance contracts. The proportion among people older than 65, the official retirement age in Austria, is 35% (Sommer, 2007). The overall portfolio share of whole life insurance, the fraction of total wealth invested in such products, is 28%. In 2003, 6.5% of the population held death benefit insurance, especially among the older it is widely spread with 15%. 2.7% of all households hold both life insurance and annuity products. 21% of households with term life insurance also have annuity contracts. The overall ownership rate of annuity contracts, in Sommer (2007) referred to by “private pension plans”, was 13.3% in 2003 with a portfolio share of 15.9%. The possession of both life insurance and annuity products is especially difficult to explain under the assumption of rational behaviour because either buying or selling annuities will be optimal, but not both.

It has to be mentioned, that the data lacks sufficiency. People who save assets to buy an annuity before retirement will never show up as an investor in the data.

Only people who have a long-term saving plan will show up until retirement, regardless whether the payment is received as a lump sum or regular annuity, so the data might have deficiencies in describing the actual savings behaviour of people (Sommer, 2005).

The saving motives of people are in the focus of many studies, e.g. Hubbard et al. (1994) and Kotlikoff (1989). Mostly, old age provision, uncertainty, support of children and bequests are cited as reasons.

Börsch-Supan et al. (2006) analyse the motives based on the German SAVE data. The SAVE panel study is a longitudinal study by the Mannheim Research Institute for the Economics of Ageing (MEA) on household's financial behaviour in Germany. It has been conducted the eighth time in 2010.<sup>9</sup> Old age provision and uncertainty are the most important reasons for saving. They are very important for about 40% of the respondents whereas less than five percent state this reason as unimportant. The importance of supporting children as a saving motive is generally high: 25% of the people younger than thirty, 35% of those aged 30-40, 30% of the 40-50 year old and 25% of those older than 50 state it as an important reason. It is unimportant for 20% of the people older than 30. Only among the younger generation (< 30 years) this statement has more follower, 35%.

In contrast to this, bequest motives are stated by less than 15% across all age groups as an important reason for saving. They are unimportant as a saving reason for more than 50% of the people.

### 1.3 Literature Review on Life Cycle Models

In this section, an overview of the most important works in the field of life cycle models and investment decisions is given.<sup>10</sup>

In general, so called *overlapping generations* models are used to describe the economy and the effects of the agents' decisions. A cohort of people is born at

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<sup>9</sup>see <http://www.mea.uni-mannheim.de/index.php?id=26&L=2>

<sup>10</sup>If the reader is not at all familiar with life cycle models, reading Section 3.1 before the Literature Review is recommended.

each instant of time and lives for a finite time but long enough, so that for some period a part of the cohort is alive together with the next generation. The classification of the models with respect to the modelling of the time is as follows: two period models, multi-period models and continuous time models.

In a *two period model*, agents typically are endowed with initial money and can choose their consumption and investment strategy for period one. The first period therefore is the accumulation phase and can be identified with the agent's working life. The agent is assumed to survive the first period with a probability  $p \in [0, 1]$  and can choose the consumption and therefore the size of the bequest, all the money that is not consumed, in the second phase. The second period corresponds to retirement in which the agents income depends on the investment decisions in period one. The consumption and investment strategies for both periods are chosen at the beginning of the first stage with the aim of maximising utility from consumption (Lockwood, 2010; d'Albis and Thibault, 2010).

*Multi-period models* extend the above mentioned. A simple version is a three period model, in which the decisions of consumption and investment behaviour are made for three periods. In more detailed models, each phase corresponds to a period in the agents' life, for example a year. The amounts invested in bonds or annuities are modelled as state variables, in contrast to the decision variables, such as consumption or how much of the savings is invested in bonds, which the agent chooses. The change in the state variables depends on the decision variables and is modelled by difference equations. In each period before retirement agents have a, in general fixed, income, and can choose consumption and investment for this period to maximise utility (Kotlikoff and Spivak, 1981; Ponzetto, 2003).

Lastly, there are *continuous time models* in which all functions are chosen as (at least piecewise) continuous functions. The dynamics of the state variables are described by differential equations. These models can be seen as limits of multi-period models, in which the length of the period tends to zero, and are adopted in numerous papers (Yaari, 1965; Feigenbaum and Gahramanov, 2010; Feigenbaum et al., 2009).

Examinations of models can be made *ceteris paribus* or in a steady state. In *ceteris paribus* examinations, except for a change in one parameter everything else

remains constant (see for example Brown et al. (2000) and Lockwood (2010)). In a *steady state* the system does not change by inner procedures, so the question behind steady state examinations is: “What are the effects of certain changes on the complete system?” In order to calculate the effects, a commonly used assumption is the rational expectation, so that an agent individually chooses his consumption while assuming that externalities such as the interest rate are constant. (Yaari, 1965; Feigenbaum and Caliendo, 2010; Feigenbaum et al., 2009). A different method is to optimise from the point of view of a social planner who is able to predict the effects of his decisions on externalities such as interest and wage rate (Feigenbaum and Gahramanov, 2010).

Brown et al. (2000) examine the real prices of annuities in comparison to actuarially fair prices and try to investigate the reasons for the difference. The annuities considered here as well as in the other models are basic annuities without any of the additional options described in Section 1.1.

When analysing the US market in the year 1998, the authors observe a change in the payouts of dividends. Whereas in 1995 for a \$100,000 lump sum the payout rate was \$794 per month, in 1998 it was down by roughly 10%. They try to find the reason for this change by valuating available annuities in the US. Valuation by formula (1.1) needs data from two sources: the mortality rate, and the discount rate. The discount rate is usually taken equal to the interest rate of long term treasury bonds, which are considered as default free. This is partly required by law, and partly due to the fact that a great share of the money is invested in long term treasury bonds.

The mortality rate is hard to estimate due to the fact that any mortality table consists of past data. It does not include the progress in medication and sanitation which results in a higher life expectancy of the younger generation over the older and so overestimates the mortality rate. Furthermore, research has shown that the death rate of annuitants is not equal to the average death rate. Especially in the age group 65 to 75, the mortality rate is only half as high among possessors of annuities as in average. At higher ages the differences are smaller. Due to the discounting and the mortality rate, the magnitude of early payouts influence the value of a contract more heavily than payouts later in life.

The higher life expectancy might be due to the selection process. People with a lower life expectancy than average are less likely to buy annuities because the expected payout does not match their expectation. With a rising life expectancy of annuitants, annuities become more and more expensive and therefore less and less attractive for a large group of society. This upward spiral leads to a high selection of annuitants in the general society. Only people with a high life expectancy buy annuities.

Additionally, as Friedman and Warshawsky (1990) report, available annuity contracts on the market do not offer fair premiums. The authors estimate the effect of hidden management costs to 3-5% of the premiums. Contracts are mostly sold not by the insurance companies themselves but by banks or financial professionals who receive a commission from the insurance companies and not from the customers. There exist different schemes with commissions paid up front or ongoing (trail commissions) which account to about 1% to 5% of the premiums. These payments are often not clear to the customer when he signs a contract, and together with the costs of managing the funds, account for the mentioned 3-5%.

The effect of population selection on the price is estimated to about 10% by Friedman and Warshawsky. This is what Brown et al. observe, too. When calculating the net worth of a dollar, that is the expected discounted payout of a dollar invested in an annuity contract, with life tables of the whole society, the worth of a dollar is about 90 cents. However, using the mortality table for average annuitants give a larger value of approximately 1. In other words only roughly 90 cents of the paid premiums are actually invested and paid out in the contract. In the US, there is hardly any market to inflation indexed products. Only a small fraction of contracts provide insurance against inflation risk. Valuating these with past data as well as predictions for the interest and inflation rate and the population life table, the authors obtain that a dollar is only worth about 75 cents in an inflation indexed annuity contract. So the cost of insuring against inflation risk, the price spread of \$0.75 to the calculated net worth above \$0.90, is about 15%. The low demand for annuities in the US may be a result of these high costs.

Another possible reason is that people do not fully understand inflation and

the magnitude of its impact on future payouts. The target for inflation of the European Central Bank (ECB)<sup>11</sup> and of the FED in the USA<sup>12</sup> is about 2%, although many economists propose an even higher inflation. An inflation of 3% reduces the value of annuities in 22 years by half. Especially the contract period of deferred annuity covers at least two decades and many involve even a longer accumulation period. Inflation risk as a reason for or against investment decision in annuities is rather unlikely as most other investment products do not offer inflation protection either. Inflation and the returns on stocks are correlated but there is no certainty or inflation indexing which secures the returns.

Brown et al. (2000) try to compare annuities all over the world, with a particular focus on the UK market. The authors observe that the charge on a real annuity in comparison with a nominal annuity is not as high as in the US. In the UK the difference is only about 5%. This may be the reason why in the UK the share of annuities is higher than in the US.

The first paper which includes annuities in a life cycle model was Yaari (1965). He describes a continuous life cycle model with uncertain lifetime, in which agents maximise expected utility. He considers four cases of market conditions and compares the consumption in each of them to consumption in case of certain lifetime. The four states are: the presence and absence of bequest motives combined with or without insurance. The interest rate and the wage are known functions and depend only on time. The utility function  $g(c)$  fulfils  $\frac{d}{dc}g(c) =: g'(c) > 0$  and  $g''(c) < 0$  (for a justification of these properties see Chapter 2). In both cases without bequests, so with and without life insurance, a constraint is imposed that the assets have to be positive all the time, to ensure that there are no negative bequests. In the cases when utility is derived from bequests, the utility from endowment ensures that the assets are always positive by penalising negative assets. The utility from leaving a heritage is weighted by a function which Yaari proposes to be hump shaped in time, in order to account for the changing importance of bequests over the life cycle, see Section 3.3 for more on that.

Under these assumptions he derives the differential equations for the optimal

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<sup>11</sup><http://www.ecb.int>

<sup>12</sup><http://www.federalreserve.gov>

consumption path and for the optimal asset allocation. It is shown that with uncertain lifetime, without bequest motives and with available life insurance, the differential equation (Euler equation) for the consumption is the same as for consumption with certain lifetime. However, the initial consumption  $c(0)$  is different in the two cases, therefore also the consumption paths. The optimal investment in this model is to buy only annuities and not to invest in bonds. This result justifies the statement that annuities are a possible way of insuring against uncertain lifetime. In absence of insurance the future consumption is effectively discounted more heavily due to the uncertainty of survival.

In a market with bequest motives as well as insurance, Yaari is able to derive separate differential equations for the optimal consumption and the optimal stock of bonds. The differential equation for consumption is again the same as in absence of mortality. The equality of marginal utility requires that in optimum, the marginal utility from discounted future consumption has to equal the marginal utility from savings times the weighting function of bequests. In any equilibrium the marginal utility of two alternatives has to be equal. If e.g. marginal utility from savings were lower than marginal utility from future consumption, a decrease in savings reduces utility less than the increase in future consumption increases it. Therefore, the state can be improved upon and it cannot be optimal. The author also proves that the optimal consumption is continuous whenever the saving constraint is not binding.

The solution of complete annuitisation in absence of bequest motives in Yaari (1965) raised the Annuity Puzzle. The results are in complete contrast to the actual behaviour of people as pointed out above. At the time the paper was written, there was no common understanding of utility functions, so under these general assumptions on utility further calculations were not possible. This model is extended in Section 3.3.

Many papers consider bequest motives as a possible reason why people do not annuitise. Then annuities are rejected because they do not allow to bequeath one's wealth. This non-altruistic behaviour is controversially discussed, as no evidence in the data was found by Hurd (1989). People with and people without children behave similarly, neither of them bequeaths more or less than average

people and they do not annuitise differently. Furthermore, in many surveys bequest motives are not stated as an important reason for saving (Sommer, 2007). Another difficulty is that, even when dynastic behaviour or bequest motives are included in the model, typically the effect is not strong enough to explain such a low annuitisation rate as observed in reality (Altonji et al., 1992).

Still, people do bequeath quite a lot of money. The distribution of bequests has an infinitely long right tail, although most of its mass is concentrated at the left end. This means that the bequeathed amount is relatively low in most cases but there are very rare bequests which are very large. The 30th percentile of the bequest distribution is just \$2,000, the median \$42,000 and the mean \$82,000 (Hurd and Smith, 2002). If altruistic behaviour is adopted and there is no utility derived from bequests, then leaving bequests happens incidentally when trying to save money for retirement and they are an unwanted consequence of mortality risk. Simulations show that due to mortality risk only some 75% of initial wealth are consumed by pensioners during their pension and the other 25% are left incidentally as bequests at time of death (Davidoff et al., 2005). In Sinha (1987) it is shown, that there exists an upper bound for bequests under the assumption of selfish behaviour in a overlapping generations model and this fact decreases the credibility of mortality risk as solely reason for bequests.

Lockwood (2010) investigates the effect of bequest motives on the saving decisions not in a perfect competitive annuities market but takes into account the real prices of annuities on the market. So the assumption is that annuities earn a larger gross net return (after management fees) than bonds ( $r_A > r_B$ ) but less than the actuarially fair rate ( $r_A < r_B + h(a)$ , where  $h(a)$  is the mortality rate at age  $a$ ). An explanation, why  $r + h(a)$  is the actuarially fair interest rate for annuities can be found in Section 3.1.

In a two period model without annuities and with uncertainty about the time of death, the decision whether to bequeath cannot be separated from the investment decision in period one. Annuities provide the tools to separate this choice. The fraction (or amount) of wealth that shall be bequeathed is invested in bonds, the rest is invested in annuities. The optimality condition requires that the marginal utility of bequests equals the marginal utility of wealth in period two. Without bequest motives, complete annuitisation is always optimal as shown

in Davidoff et al. (2005), if the return on annuities is greater than the return on bonds. They use the method of minimising the costs for a certain utility level, instead of maximising utility to prove this.

To account for the difference between actuarially fair and real annuities, Lockwood (2010) introduces in the considered model a load  $\lambda \in [0, 1]$  on annuities. This charge makes up for the difference between actuarially fair and real annuities. Hence, the return on annuities is  $R_a = (1 - \lambda) \frac{R_b}{p}$  with  $R_b$  being the return on bonds,  $p$  the probability of dying and  $\lambda$  the load. He further assumes a 10% load in accordance with Friedman and Warshawsky (1990). According to his calculations, people can increase their utility even if they have 15% less wealth but access to the annuity market. Consequently, people should be willing to pay still 15% for annuitisation. However, when considering only a small bequest motive, the willingness to pay for annuitisation decreases sharply.

The simulations of a periodical model without borrowing in Lockwood (2010) range from age 65 to 110 with the mortality rate modified to match the maximum age. Utility from bequests is dictated by the present value of future bequests at age 65 instead by the real value at time of death. The real value is the face amount paid on a certain date whereas the present amount is the face amount paid discounted to today if the payment lies in the future and is, with a positive interest rate, always smaller than the real value. The utility from bequests is equal to the utility a head of a dynasty derives from bequests, so the utility is equal to the increase in utility all later generations have through the bequeathed wealth. He further assumes that the agents have a half of their wealth already annuitised in state pensions and only make their decision on the remaining part.

Lockwood calculates the willingness to pay (WTP), that is the percentage people are willing to pay for annuitising, in case of access to fair annuities and to real annuities with a 10% load. The willingness to give bequests is then measured in two ways. The first measure is the fraction  $b \in [0, 1]$  of the wealth people would bequeath if they had access to actuarially fair annuities. The authors simulations indicate that for real annuities the WTP equals only some 2 percent in case of  $b = 0$ , and decreases for increasing  $b$ , reaching zero for  $b \approx 0.25$ . For fair annuities the WTP, so the maximal fee agents are willing to pay for annuities, is higher but also decreasing from 10% ( $b = 0$ ) to some 3% ( $b = 0.25$ ). The second method to

measure the willingness to give bequests is by variation of a parameter  $a$ , which is a multiplier of the utility from bequest. For  $a = 0$ , so without a bequest motive, WTP equals 15% for real annuities and 25% for fair annuities but steeply decreases for increasing  $a$ .

In a next step, Lockwood decomposes the utility from annuities into three parts: the trading of bequests in consumption, the smoothing of the consumption and the insurance of bequests. In absence of bequest motives, trading bequests for higher consumption increases utility most. With stronger bequest motives, the possibility of insuring the bequest plays the most important role. The smoothing of consumption plays never a leading role in the demand for annuities.

Lockwood further examines the effects of other restrictions of the annuity market besides a load on annuities. Many contracts require a minimal amount invested because otherwise the conclusion costs of the contract are too high. However, according to the calculations, this should not play an important role on the demand because most of the desired contracts in the model exceed this minimal amount.

More important is the imposition of fixed costs on conclusion of a contract: this heavily decreases the demand. These fixed costs can be interpreted either as actual costs such as contracting-fees or also as the costs of acquiring information about annuities and the different offers. This might explain the reduced interest on annuities observed in the data.

So concluding, Lockwood implements a multi-period model to evaluate the effect of bequests on the annuitisation rate with a broad variation of the importance of bequests for the agents and is able to prove that for real annuities and only small bequest motives, the optimal fraction annuitised is very low.

Kotlikoff and Spivak (1981) follow the idea of altruistic behaviour. This means that there is no utility in leaving a bequest to others in their investigations. Focus of their investigations is whether there might be annuity-like contracts in society, particularly in families. Families offer a higher degree of trust and shared information, so the authors propose that this familiarity offers the chance to enter annuity contracts without risks that exist in the public market such as adverse selection, counterparty risk, moral hazard and deception. Leaving bequests to

others may be an unwritten contract with the partner in marriage or with children in exchange for the insurance that the children provide support if the parents accidentally outlive their resources.

The authors work with a multi-period model and iso-elastic utility functions (see Section 2) in which agents are provided with an initial wealth but do not receive any income after it. For a 55 year old male they estimate that he leaves 25% of his initial wealth as unintended bequests. The additional utility obtained by having access to a fair annuity market is measured by the Annuity Equivalent Wealth (AEW). It is equal to the percentage of additional initial money which needs to be given in order to obtain the same level of utility as with annuities. With a coefficient of relative risk aversion  $\sigma$  (see Chapter 2) equal to 0.75 they obtain 47% AEW which is similar to the findings of Brown and Poterba (2000). The older the people the higher the AEW due to the higher mortality rate.

The homotheticity property of the utility function (see Chapter 2) allows the authors to separate utility gains from substitution effects. The hypothetical contract between an agent and a company which gives the agent additional 25% of initial wealth in exchange for bequeathing his money to the insurance company is fair in expectation. The optimal consumption profile of this additional money is, due to the homotheticity, the same consumption rule as before. Again, 25% of these 25% are left as bequests and a new contract can be entered over this value. For the additional bequest, a new contract can be entered etc. Adding up this geometric series (with the exact value of .2447 instead of the approximate 0.25) equals 32.4%. By this calculation Kotlikoff and Spivak show that the income effect amounts to 69% ( $= \frac{32.4}{47}$ ) of the total AEW of 47% for access to the annuity market. The intertemporal substitution accounts for the remaining 31% of the gain from annuities.

Families permit pooling of resources in the sense that the optimal consumption path is higher if an agent can trust on inheritance from the other family members in case of death and takes this into account. Kotlikoff and Spivak calculate the AEW of a marriage of two persons aged 55 with  $\sigma = 0.75$  and obtain some 20%. A marriage can therefore offset almost half of the gain obtained from an annuity market. They further calculate that a polygamy with three people, or three brothers or sisters joining a similar agreement, would result in a utility gain

of 28% AEW. By increasing the number of people in such an agreement (with all having the same age, initial wealth and probability of dying) the authors show that this converges towards the state with an active annuity market. So a family, or rather a tribe, can to a certain extent provide longevity insurance and increase expected utility of each member.

All partners in such a family contract have high incentives deceive the others, that means to leave their optimal consumption path and increase their consumption. If they die, they have consumed more and leave less bequests and if they outlive their partner they can live from the inheritance. In both cases, utility is increased. However, if both partner live for a long time the early excessive consumption requires reduction later in life and therefore reduces utility. Kotlikoff and Spivak consider this as a competitive game and investigate the stability of such a Nash equilibrium.<sup>13</sup> In that game, agents decrease their utility if they are not able to estimate the effect of their cheating on the behaviour of their partner and therefore on expected bequests.

Besides having shown that the income effect of annuities dominates the substitution effect, the authors have also shown that intact families increase the utility from their members and provide some compensation for annuities.

Concerning the annuity puzzle, d'Albis and Thibault (2010) add a different perspective on the problem. People exhibit a high risk aversion towards uncertain survival probabilities. The standard deviation of lifetime according to life tables is 15 years and a high heterogeneity remains even after controlling for different variables such as sex or social status. Furthermore, life tables are based on mortality rates of earlier generations, and it is not easy to tell the quality of predictions for present generations. Due to this lack of knowledge, the authors propose the use of the maxmin utility (also called Wald criterion) in which the agent seeks to maximise the utility of the worst plausible case.

They apply these preferences to a two period model, in which the survival to the second period is uncertain. The agent is in complete ignorance about his survival probability. Agents are endowed with an initial income in the first period,

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<sup>13</sup>John Nash (1928-). In a Nash equilibrium neither side can improve upon the equilibrium when taking the action of the other into account.

but they do not have an income in the second period. The return on annuities  $r_A$  is bigger than the return on bonds  $r_B$ . The amount invested in bonds and annuities are given by  $B$  and  $A$  respectively. It is not possible to leave negative wealth in bonds, so the constraint  $B \geq 0$  is imposed. Negative wealth in annuities is possible, that means that a life insurance is contracted. Bequests are given in period three, independent from time of death. The utility function is additive in the consumption of the first period, the second period and the bequest, given the survival, and additive in the consumption of the first period and the bequest in the case of early death. One optimality condition is that the utility from consumption in the second period and the bequest have to equal the utility from bequest in case of early death. The authors prove that the consumption in the second period always exceeds the annuities plus interest, so agents always own bonds. Even more than that, the optimal strategy is to sell annuities short, that is to buy whole life insurance.

The assumption of a maxmin utility is very strong as agents usually do not assume that they have a survival probability close to zero and therefore put more emphasis on consumption in case of survival. Agents estimate an interval  $[p_0, p_1]$  in which their survival probability lies. For these probabilities the agent calculates the expected utility and maximises the worst case utility. The authors prove that there exists a positive probability  $\hat{p}$  such that  $0 < \hat{p} < \frac{r_B}{r_A}$ , and that for all  $p \leq \hat{p}$  it is optimal to sell annuities short. If  $p_0$  is small enough, agents will sell annuities in order to eliminate the mortality risk. For older people it is also very plausible that their survival probability is smaller than  $\hat{p}$  so at some point in life it becomes optimal to sell annuities short.

Most of the maximisation models with a time additive expected utility function cannot distinguish between risk aversion and the elasticity of intertemporal substitution (see Chapter 2). The Epstein-Zin recursive intertemporal utility function, introduced in Epstein and Zin (1989), separates these two parameters. It calculates the certainty equivalent of stochastic future consumption and derives utility from it as well as from the deterministic present consumption. Under the assumption that the certainty equivalent is a momentum of the distribution, the

utility function can be determined in the following way:

$$U_t = W(c_t, E_t \tilde{U}_{t+1}) = (c_t^\phi + \beta(E_t \tilde{U}_{t+1}^\alpha)^\frac{\phi}{\alpha})^\frac{1}{\phi}. \quad (1.2)$$

Thereby is  $U_t$  the utility in period  $t$  which depends on the consumption  $c_t$  and a prediction of the future consumption  $E_t \tilde{U}_{t+1}$ . The parameter  $0 < \alpha < 1$  may be interpreted as the inverse measure of relative risk aversion,  $\eta := (1 - \phi)^{-1}$  is the elasticity of intertemporal substitution and  $\beta$  discounts the expected future consumption. For  $\alpha = \phi$ , this coincides with expected utility (Epstein and Zin, 1989).

Under the general assumption that the optimal consumption path is never increasing in time, Ponzetto (2003) shows in a multi-stage period model that only for the special case  $\alpha = \phi$ , annuities provide real insurance against mortality risk, in the sense that the differential equations for optimal consumption are identical to the model with certain lifetime. Relatively risk-averse agents with  $\alpha < \phi$  have decreasing consumption profiles even if all their wealth is converted into annuities. He even shows that the utility gain from access to an annuity market is a negative function of risk aversion. So annuities help to reallocate the income over time but do not make people indifferent with respect to their time of death in the sense that the optimal consumption path equals the path in absence of mortality risk.

Yaari (1965) only analyses the competitive equilibrium. The agents maximise their utility and take factor prices (interest rate, wage) as given. In most literature, shutting down the annuities market is considered as hurting people. However, Pecchenino and Pollard (1997) examine the external effect of the aggregated capital stock in an overlapping generations model. The model is similar to Eichenbaum and Peled (1987), in which the authors proved that saving is in general suboptimally high in steady-state equilibrium, and to Sheshinski and Weiss (1981) in which the introduction of a mandatory social security scheme improved upon the steady state outcome. However, this does not imply that the availability of actuarially fair annuities will always increase social welfare, as shown in the paper.

Pecchenino and Pollard describe a two period model, the first period repre-

sents the capital accumulation phase and the second the retirement, the social security provided in retirement is completely funded by an income tax paid by the younger generation. People can invest in bonds or in actuarially not fair annuities, so there exists a load on annuities. The government, however, has the possibility to grant the people access to a fair annuity market. It can make access voluntarily, but capped at a certain rate, or mandatory with a fixed participation rate. The authors examine the effects of the measurements on the people and are able to show the following results.

A tax on income has a negative income effect for the young but a positive income effect on the old. For both the incentive to save shrinks because there is less money to save and at old ages less money is needed to maintain the consumption level. So a reduction of income tax increases savings and thus the growth rate of the economy. The maximal growth of economy happens in absence of any tax. It holds for any tax rate that any increase in taxation of income increases the demand for annuities, reduces savings and thus leads to a reduced growth.

If all wealth is annuitised, an increase in longevity, the expected length of retirement, increases wealth. If not everything is invested in annuities, the reduced bequests and the decreasing marginal revenue of annuities decrease utility, however the shrinking tax revenue per citizen increases saving motives. Thus, the effect is not clear.

From the point of view of a central planner, Pecchenino and Pollard try to establish in simulations whether Pareto optimal improvements are possible in comparison to an equilibrium-balanced growth path. Pareto optimal improvements increase the utility of at least one person without harming (decreasing utility) any other. They find that by increasing annuitisation and decreasing the tax and therefore social security, Pareto efficient improvements are possible.

While these possible improvements hold for the partial equilibrium for a single individual, James Feigenbaum and Emin Gahramanov study this question from the viewpoint of a society in a general equilibrium (Feigenbaum and Gahramanov, 2010). The behaviour of a single individual does not affect the interest rate and wage because the society is too big so that the decision of a single person would change the factor prices. The aggregated behaviour of the whole society,

however, influences the rate as it changes the capital stock. Therefore, a social planner might be able to improve significantly over the rational equilibrium by taking the effects of saving decisions on the interest rate and wage into account.

In the model of Feigenbaum and Gahramanov, the factor prices, interest and wage rate, are the first derivative of a Cobb-Douglas production function with respect to capital and labour supply, respectively. So an increase in savings across the whole population decreases the interest rate but increases wage if labour supply remains constant.<sup>14</sup>

Another aspect which is incorporated in their paper are the effects of bequests. The access to annuity products reduces the amount of money bequeathed to children. Consequently, if this intergenerational money transfer is taken into account, the income at younger ages is reduced by higher annuitisation.

The authors extend the partial equilibrium solution of Yaari (1965) with CRRA (constant relative risk aversion - see Chapter 2 for details) preferences and derive a closed form solution for optimal consumption. The main focus of their work, however, lies on the general equilibrium. In the main proposition they can show that for any allocation of resources in which annuities are held in a positive amount (except on a subset with measure zero), the social planner can improve. However, this is only true if the constraints for annuities and bonds are not active at the same time, that means that total wealth is always positive. That did not happen in any of their simulations. It is worth to point out that Feigenbaum and Gahramanov, contrary to the papers cited before, explicitly take bequests into account as a source of income for the living people.

Furthermore, the authors examine the situation in which only annuities are available and are able to derive an optimal consumption profile which equals the one in Feigenbaum and Caliendo (2010), where a general equilibrium with bonds and without mortality risk is considered. So the fact, that annuities eliminate the mortality risk from the consumption profile holds also in general equilibrium.

In the rational equilibrium, they consider only uniform bequests, so everyone alive receives the same share. In a general equilibrium, the benevolent social

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<sup>14</sup>There are also papers which allow people the choice how much labour they supply or how much leisure time they have. They derive utility from leisure and so leisure time and consumption are substitutes. See e.g. Bodie et al. (1992) and Heckman (1976).

planner can diversify and consider different bequest distribution schemes in order to improve utility. In case of a constant hazard rate the authors prove that the solution is to bequeath always the youngest generation alive, because the present value of the bequest is higher than the present value for an older agent.

Comparing the optimal consumption profiles of rational competitive equilibrium with/without annuities and irrational optimal behaviour with/without annuities, one can see that the higher the expected utility of the equilibrium, the steeper is the decrease in consumption over time. In the rational competitive equilibrium with annuities the consumption is only slightly decreasing over time, in case of a social planner with annuities only, consumption is decreasing until some age above retirement age but then strongly increasing to consume the rest of the wealth. This is unexpected because it cannot be observed in reality that old people consume all their wealth at the very end and increase consumption before death strongly. A reason is possibly the fixed lifetime and so this strange effect could be delayed by increasing the maximal lifetime in the model. As it only affects a very small fraction of the population who survive that long and this is discounted very heavily, it does not affect the equilibrium much. The same shape, but even more emphasised, has the consumption curve for the optimal irrational behaviour with bequests. It starts at a very high level, decreases immensely but boosts up again shortly before death.

Feigenbaum and Gahramanov have been able to prove that full annuitisation is not optimal when considered from the point of view of a central planner or the society as a whole. The positive effects from bequests outperform the additional return on annuities. Nevertheless, this is not a solution to the annuity puzzle, because it means that people do not act selfish but include the effect of their own action on the factor prices. It is not likely that people have enough insight in economics to understand the effects and consider them when planning their consumption.

## 1.4 The Austrian Insurance Market

<sup>15</sup>In Austria there were 142 insurance companies with roughly 27,000 employees in 2009. The total amount of premiums was EUR 16.435 Billion in 2009. Between 2003 and 2009, the total premiums increased by approximately 30%. The fraction of premiums for life insurances is approximately constant with some 45%.

Insurance companies invest approximately 45% of their money in bonds, 16% in mutual funds and 16% in shares. Less than 6% are invested in other forms such as real estate, bank accounts and loans account. Insurance companies require liquidity in order to maintain their business, so not all of the money can be invested in the market as pointed out above but a few percent are not capitalised and do not yield interest. New regulations, such as Solvency II<sup>16</sup> impose further restrictions on insurance companies. The effects of these are not yet known.

Comparing bonds to annuities, they are supposed to have the same basic interest rate, because insurance companies invest the money in the capital market and gain the same return on it. However, tax treatment might be different.

In Austria, there is a 25% capital gain tax (KESt, short for German “Kapitalertragssteuer”) on interest from bonds except for very rare papers such as bonds of building societies. The share on the market of these bonds is very small, so that they are negligible. Furthermore, this exclusion is in order to enable cheaper loans for building, so that the effective interest rate after tax should not differ greatly to other bonds.

Until July 2011, there is no tax on capital gain in Austria. Zero coupon bonds, gains from capital on mixed bonds and shares are not taxed at all, increasing their return compared to a bond. Beginning in July 2011 also these capital gains are taxed by 25% KESt. The taxation shall reduce speculation in the stock market but will most likely have effects on the investment decisions of most of the population too.

Costs of buying bonds account for less than 0.5% (the exact value strongly de-

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<sup>15</sup>All data from the reports of the Austrian Underwriting Association (Versicherungsverband Österreich) available at <http://www.vvo.at>

<sup>16</sup><http://www.fma.gv.at/en/special-topics/solvency-ii/general-information.html>, even newer regulations, Solvency III, are being discussed currently

pend on the bank/investment company) and holding them approximately 0.1% per year. Internet banks provide deposits for securities even cheaper than regular banks.<sup>17</sup>

For private agents, returns from insurances are (under regular circumstances) not being taxed so it seems as they are favoured compared to bonds. However, insurance companies in Austria have to pay themselves KESt on any dividends or interest from bonds. A difference to personal engagement in the market is, that insurances are excluded from speculative tax on gains on capital when holding shares less than a year. As insurance companies do not produce any goods but their business is to pool money and invest it, they have to invest the deposits in the capital market and pay capital gain tax on this interest. So from that point of view, interest from bonds and insurances are treated equally by means of tax on capital.

However, there is a 4% insurance tax on insurance premiums (11% on insurance contracts of less than 10 years) which is immediately due and not only when selling zero coupon bonds. Additionally, laws require that a certain percentage of the premiums has to be invested in especially safe and liquid assets, and these regulations decrease the return on the invested capital.<sup>18</sup>

Fees by the insurance companies differ only slightly. There is an administration fee of 6% of all premiums and furthermore a contraction fee which is a 5% charge on premiums in the first 15 years, but payable in the first five years (each year 15%).<sup>19</sup>

In 2003 a publicly sponsored private pension plan was introduced in Austria. A certain amount (2011: EUR 2313.36) of premiums is not only tax free but

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<sup>17</sup>see e.g. [www.bawag.at](http://www.bawag.at), [www.raiffeisenbank.at](http://www.raiffeisenbank.at), [www.erste.at](http://www.erste.at), [www.ba-ca.at](http://www.ba-ca.at) for regular Austrian banks and [www.direktbank.at](http://www.direktbank.at) for an example of an internet bank

<sup>18</sup>The effects of stricter requirements by Solvency II on the insurance market are not yet predictable but are likely to increase the costs of annuity contracts in the future.

<sup>19</sup>Figures presented here are from the financial product: BAWAG Zukunftsvorsorge - [www.bawagpskvers.at](http://www.bawagpskvers.at) - other insurance companies have slightly different premium schemes but the overall charge is similar.

also sponsored by some 5-9% (2011: 8.5%) by government. Since the data for comparison is a couple of years old, this recent offer cannot yet have a significant effect on savings behaviour. The possible effect on a similar product, the “Riester Rente” in Germany is analysed by researchers from the Mannheim Research Institute for the Economics of Ageing in Börsch-Supan and Gasche (2010) and Essig and Reil-Held (2004) theoretically and in Börsch-Supan et al. (2006) and Coppola and Reil-Held (2009) on the basis of some recent market survey data. The number of contracts is increasing but the total money saved is not yet significantly higher.

For these private pension plans, there is by law a minimum for the quota invested in shares, which declines throughout the life from 30% to 15% before pension. This is to enable the funds to participate in the market but reduce fluctuations in the value of the portfolio before retirement.<sup>20</sup>

After this introduction in annuities and the review of the literature, the next Chapter focuses on the concept of utility. In Chapter 3, two continuous life cycle models, which will be used in the subsequent sections for determination of optimal consumption and investment decisions, are presented.

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<sup>20</sup>[www.bawagpskvers.at](http://www.bawagpskvers.at)

# Chapter 2

## Utilitarianism

A central point of life cycle models is to measure the utility an agent derives from consumption or, in extended versions, also from leisure or other factors. Nowadays, constant relative risk aversion (CRRA) utility functions seem to be standard across a wide range of papers, e.g. Feigenbaum and Gahramanov (2010), Feigenbaum and Caliendo (2010), Laibson (1998), Gourinchas and Parker (2002) and Evans et al. (2009). The use of this particular family of functions and the general concept is of course not without criticism and so it is probably worth spending a few lines in this thesis about it.<sup>1</sup>

An agent is a representative person from a society considered in a model. If the society is considered homogeneous, then all agents are identical. Utilitarianism (also utilism) is the concept that agents determine their actions by maximising their utility. Positive utility can be regarded as happiness and pleasure whereas pain and suffering can be seen as negative utility. Happiness can be derived from anything - money, spare time, a house, a football game, etc. This also gives a concept for morality: the moral worth of an action is defined by its outcome on all human beings. Jeremy Bentham (1748-1832) and John Stuart Mill (1806-1876) are two of the earliest and most important advocates of this theory. According to them, utilitarianism can be a moral criterion for the right organisation of society.

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<sup>1</sup>This Chapter is based on the lecture “Operations Research” of Professor Topritzhofer at the Vienna University of Economics and Business in 2010 as well as the additional literature given.

In order to base decisions on utility, there is the need to measure utility. Utility functions are cardinal preference functions and map possible outcomes to real numbers, the utility of this outcome. Positive utility can be seen as the price an agent is willing to pay for a certain good or outcome. Negative utility is the price an agent would pay in order to avoid this action. However, there is the danger of a circular definition which was criticised by Joan Robinson (1903-1983), an economist from Cambridge: “Utility is the quality in commodities that makes individuals want to buy them - and the fact that they want to buy them shows that they have utility.” (Robinson, 1962)

Utility functions  $u$  are cardinal preference functions which map all possible actions to the real numbers  $u : \mathcal{A} \rightarrow \mathbb{R}$ ,  $\mathcal{A}$  denotes the set of all possible actions. Preference functions in general fulfil the axiom of completeness ( $\forall (X, Y) \in \mathcal{A}^2 : X \succ Y$  or  $Y \succ X$  or  $X \sim Y$ ) and the axiom of transitivity ( $\forall (X, Y, Z) \in \mathcal{A}^3 : X \succ Y \wedge Y \succ Z \Rightarrow X \succ Z$ ). The first means that any two actions are comparable (either  $X$  is preferred, or  $Y$ , or the agent is indifferent between the actions), whereas the second states that if one action is preferred to another, and this one is preferred to a third one, then also the first action is preferable over the third. This, however, is only true for simultaneous decisions so that the reason for choice does not change. This accounts for the fact that decisions are dependent from the circumstances and in a different setting, choices are made based upon different facts. In general this can be modelled that the utility depends not only on the action but also on the time, as a representative for the conditions and circumstances at that time.

Cardinal preference functions in contrast to preference functions state not only the preference of action  $A$  over action  $B$  ( $A \succ B$ , represented through  $u(A) \geq u(B)$ ), but also a quantification, e.g.  $u(A) = 2$  and  $u(B) = 1$ . If we identify eating an orange with action  $A$ , and eating a sandwich with action  $B$ , then we could say eating an orange is two times as good as a sandwich. Although the exact utility for actions are hard to estimate, it is a powerful method to deal with decisions. Furthermore, the quantification of utility, the mapping from a choice to a real number, is not unique for a certain set of preferences but invariant under linear transformation. So any multiplication or addition of a constant

gives another utility function.

One of the most common use of utility functions is to measure the utility of money. The utility of money is equal to the maximal utility that can be achieved by purchasing goods. This allows to uncouple detailed preferences (oranges over sandwiches) by assuming that the agent uses this money in the personally best way. Utility functions which map wealth to utility usually are considered as a function of the (positive, if debts are not allowed) real axis in the real numbers  $u : \mathbb{R}^{(+)} \rightarrow \mathbb{R}$ . The function shall be strictly monotonous ( $u$  is in general assumed to be differentiable, therefore this is equal to  $u' > 0$ ) because an additional unit of money cannot decrease utility as it would be possible to simply throw the money away. It shall also be concave ( $u'' < 0$ ) because one additional dollar is more valuable to a poor person than to a rich one. In many cases when non-monetary aspects have to be judged, the alternatives are at first expressed in monetary equivalents which then is measured by the utility function.

The concept of utility is quite straight but so far only valid for a single individual because two different people will almost exclusively have different preferences. Furthermore, utility is bound to the existence of at least two alternatives. There is no absolute measure of utility and therefore only works for comparison between alternatives. Lastly, the utility of an alternative is not necessarily constant over time. The next paragraphs deal with the problem of aggregation of utility of more than one person.

In a society, how can the single utilities be aggregated to a utility from the society or a social welfare function? This aggregation is especially essential if the moral worth of an action is defined by its outcome on all human beings. John Rawls (1921-2002) suggested that the utility which had to be maximised should not be an aggregated value but the utility of the individual with the lowest utility in the beginning. Contrary to that, in total utilitarianism, the sum of the utilities of individuals is taken as the utility function of choice, whereas average utilitarianism considers only the average utility of the population. When considering total utilitarianism, adding any people to the society increases utility, even if their utility-level is way less than the average population. However in average utilitarianism it is desirable to get rid of people whose happiness is less

than average, as this action increases the average.

One concept to overcome these problems is the Pareto<sup>2</sup> efficiency. A state in an economic system is Pareto efficient if and only if no individual can be better off without harming another one and it is not Pareto efficient if one individual can increase its own utility without harming the others. Although the Pareto criterion is a reasonable choice, it only allows changes which do not harm anybody. As long as someone feels himself harmed by an alternative, this alternative shall not be considered. This heavily restricts the possibilities for regulations by politics. It strengthens current injustices in the system and might prohibit a “better” society by means of aggregated or average utilitarianism or any other measure. There may exist many Pareto optimal states, but neither of them might be desirable by a society. For example, in Feigenbaum et al. (2009), the authors show that there is no Pareto efficient transition from the rational equilibrium to the optimal irrational steady state. The transition will always harm the first generation which is alive when the changed consumption rules firstly are adopted.

In this thesis, the models are always from the viewpoint of a single individual, but for example in Section 3.2.4, a heterogeneous population is examined and aggregation of utility is essential in order to identify the “better” alternative. And especially in possible future continuations of this work, when the models are extended to infinite time horizon models and transitions instead of the steady state are in the focus, it will be necessary to compress all the information of the individual utility functions in one aggregated utility function in some way, despite all the ethical problems arising. It is, of course, worth considering alternatives and continue discussing the different aggregations of utility. As often, there probably is no “right” aggregate utility but alternatives which suit different situations.

The discussion of these difficulties when trying to find non-subjective measures for decision is necessary and affects other parts of our life, such as politics: When politicians say they set the optimal actions for their nation, which objective function do they maximise? How do they aggregate utility? Are they average utilitarianists and average the utility or total utilitarianists and maximise the total? Which constraints do they have? Which constraints do they impose for

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<sup>2</sup>Vilfredo Pareto (1848-1923)

example while reducing the unemployment rate?<sup>3</sup>

Another difficulty is to measure utility under uncertainty. What is the utility of a bag of food, which contains either an orange or a sandwich with the same probability? The average?

The easiest model with choices under uncertainty and utility is a lottery. In case A, which happens with a probability of  $\pi$ , the agent wins, e.g. 10 000 dollars. In case B, with probability  $1 - \pi$ , he does not win anything. What is a fair price for this lottery? How much is one willing to pay for a lottery ticket? This value is called certainty equivalent. If the certainty equivalence is equal to the expected payoff, which is  $10,000 \pi$  dollars, we call the agent risk-neutral. If the certainty equivalence is lower than the expectational value, the agent is risk-averse, if it is higher, risk-loving.

This leads to the St. Petersburg Paradox, which was first mentioned by Nikolaus Bernoulli (1687-1759) in 1713 and is named after the location where Daniel Bernoulli published his solution in 1738. The St. Petersburg Paradox is a lottery, in which a coin is tossed. If at the  $k$ -th toss the coin shows head, the payoff is  $2^{k-1}$ , otherwise the game continues. What is a fair price to buy a ticket for this lottery? The expected payoff is  $\sum_{k=0}^{\infty} p(X = k)2^{k-1} = \sum_{k=0}^{\infty} 2^{-k}2^{k-1} = \sum_{k=0}^{\infty} \frac{1}{2} = \infty$ . Actually, one should be willing to invest everything in such a lottery ticket, contrary to the behaviour of most people, who are not willing to pay more than a few dollars for this game. The solution given by Bernoulli is to consider a non-linear utility function, so that the utility from very high earnings is less heavily weighted. With a utility function  $u(x) = \ln(x)$ , the expected utility is a finite value and the paradox solved. However, it is possible to create a lottery with a higher payout in which the paradox holds again.

The Bernoulli principle, or expected utility hypothesis, was proposed as part of the solution for the St. Petersburg Paradox and includes concrete actions of how to solve a problem of choice with a finite set of actions. First, transform the possible outcomes with a utility function into utilities. Then calculate for each action the expected utility value and choose the action with the highest expected

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<sup>3</sup>Exaggerating this could lead also to questions like: “If a single life is priceless, why are we only willing to pay a finite sum of money to help?”

utility.

John von Neumann (1903-1957) and Oskar Morgenstern (1902-1977) developed the idea of Bernoulli further. According to von Neumann and Morgenstern (von Neumann and Morgenstern, 1944), a rational behaviour is defined by four axioms. Additionally to the axioms of completeness and transitivity, they add the axioms of independence and continuity. The independence axiom is fulfilled, if the order of two outcomes mixed with a third one remains the same, mathematically:  $X \succ Y \wedge t \in (0, 1] \Rightarrow tX + (1 - t)Z \succ tY + (1 - t)Z$ . Continuity requires that for any three outcomes  $A \succ B \succ C$  it is possible to find a probability  $\pi \in (0, 1)$  such that the gambler is indifferent between  $B$  and the lottery of winning  $A$  with a probability of  $\pi$  and  $C$  with  $(1 - \pi)$ . The Bernoulli principle can be logically deduced from these axioms. These axioms are a little controversially. For example, many people prefer 5,000,000 dollars over a lottery with a possible win of 7,000,000 dollars but a (even so slight) chance of losing everything. In this case, the axiom of continuity is not fulfilled.

There are also other systems of axioms from which it is possible to deduct the Bernoulli principle. One of the most common, although also confronted with criticism, is the one of Luce and Howard (1957) which includes five axioms of rational choice. Additionally to the axiom of ordinality (which includes both completeness and transitivity) and the axiom of continuity, they state an axiom of substitution, of reduction and an axiom of stochastic dominance. The deduction of the Bernoulli principle is fairly easy and can (as well as the axioms) be found in Laux (2005).

In the example with  $u(\textit{orange}) = 2$  and  $u(\textit{sandwich}) = 1$  it might be ridiculous to try to compare the utility of one orange with that of two sandwiches. The second sandwich does not necessarily give me the same utility as the first. But still, it is possible to make comparisons based on these axioms involving lotteries according to the Bernoulli principle. The question is whether the agent prefers a sandwich or a lottery in which he can win an orange with a probability of  $\pi$  and loses (doesn't win anything) with the probability  $1 - \pi$ . By adjusting  $\pi$  it is possible to find a ratio of the utilities of two options. In a finite set of alternatives it is possible to construct a utility function by setting  $u(z) = 1$ , when  $z$  is the most desirable alternative, and  $u(a) = 0$ , with  $a$  the least desirable alternative.

The utility of any other action  $x$  is defined by the highest probability  $\pi < 1$  (which exists due to the axiom of continuity), so that the agent prefers  $x$  over the lottery to win  $z$  with a probability of  $\pi$ . Utility functions created in such a way are called von Neumann-Morgenstern (vNM) utility functions and are functions from the set of choices in the real numbers. Under the assumptions made, agents prefer  $B \succ A$  if and only if  $u(B) > u(A)$ .

It is possible to show that von Neumann-Morgenstern utility functions have the following property: the reciprocal of the absolute risk aversion ( $ARA = -\frac{u''(x)}{u'(x)}$ ) is a linear function of wealth  $x$ ,  $\frac{1}{ARA} = ax + b$ . Any utility function with this property exhibits “hyperbolic absolute risk aversion” (HARA). All HARA utility functions can be obtained by solving the differential equation and transforming it linearly to get rid of unnecessary constants. They are of the form  $u(x) = \frac{1-\sigma}{\sigma} (\frac{ax}{1-\sigma} + b)^\sigma$ ,  $a, b, \sigma \in \mathbb{R}$ . An example which is often used and is easy to handle is the exponential utility function  $u(x) = 1 - e^{-ax}$ , which comes from the HARA utility with  $b = 1$  and  $\sigma \rightarrow -\infty$ .

What is a good measure for the willingness to accept risk, also called risk tolerance? Since the utility function is only unique up to a linear transformation, the measure shall exhibit the property, that it is invariant under such linear transformations. Therefore, the first derivative cannot be a measure of risk aversion. Better suitable is the absolute risk aversion  $ARA(x) = -\frac{u''(x)}{u'(x)} = a$ , also called the “Arrow-Pratt measure” named after Kenneth Arrow and John Pratt, who proposed it in 1970 (see e.g. Arrow (1970)). The Arrow-Pratt measure comes from the consideration of a lottery  $L$  (which is a random variable) with an expected return  $\mathbb{E}[L] = 0$  and a variance  $\text{Var}[L] > 0$ . The gambler has a fortune of  $X$  dollars and can buy his way out of the lottery for  $P$  dollars, so for paying  $P$  dollars he does not have to play the lottery. When considering expected utilities, the amount  $P$  should be fixed by the equation  $u(X - P) = \mathbb{E}[u(X + L)]$ . If  $P$  were smaller, buying out of the lottery would be better than playing the lottery. If  $P$  were bigger than the solution of the equation, it would be preferable to play the lottery. Approximating the utility functions in the equation with a Taylor polynomial we obtain the the solution  $P \approx -\frac{u''(X) \text{Var}[L]}{u'(X) 2}$ . So the agent is willing to pay the Arrow-Pratt measure times a constant, which depends on the lottery,

to get rid of the lottery ticket.

For the exponential utility, the risk aversion is constant regardless of the wealth. This is not very plausible as with increasing wealth the risk aversion usually declines and the investment in riskier assets increases, whereas poorer people tend to save their money in their savings account and not invest in shares. This is in contrast to constant ARA. More plausible, and therefore adopted in many papers (e.g. Feigenbaum and Caliendo (2010), Feigenbaum and Gahramanov (2010), Gourinchas and Parker (2002) and Laibson (1998)), are functions with a constant relative risk aversion (CRRA). The relative risk aversion is defined by  $RRA(x) = xARA(x) = -x\frac{u''(x)}{u'(x)}$ . Functions which satisfy this are the HARA functions with  $a = 1 - \sigma$  and  $b = 0$ :

$$u(c) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma} & \sigma \in \mathbb{R}^+, \sigma \neq 1 \\ \ln(c) & \sigma = 1. \end{cases} \quad (2.1)$$

Since a constant term in the utility function does only change the maximal value, but not the maximiser, that is the consumption for which the maximal utility is achieved, the subtraction of one in the case  $\sigma \neq 1$  is usually not taken into account. The transition  $\lim_{\sigma \rightarrow 1} \frac{c^{1-\sigma}-1}{1-\sigma} = \ln(c)$  can be shown with the rule of de l'Hôpital by differentiating numerator and denominator with respect to  $\sigma$ .

Functions with a CRRA (also called isoelastic functions) allow for a linear transformation of wealth without changing the behaviour of the function. So initial wealth or wages can be normalised to any value desirable. CRRA functions, as well as exponential utility functions, belong to the class of von Neumann-Morgenstern utility functions.

An alternative proposed in Epstein and Zin (1989) is the Epstein-Zin utility function, which separates the (in case of CRRA utilities combined) parameters risk aversion and elasticity of intertemporal substitution, see equation (1.2).

People might not always be risk-averse, risk-loving or risk-neutral. For example, each week more than 3,000,000 tickets for the lottery in Austria are bought, where the price for a lottery ticket is higher than the expected return.<sup>4</sup> A conclusion from this fact would be, that most of the people in Austria are risk-loving.

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<sup>4</sup><http://www.lotto.at>

However, premiums on insurances are higher than the expected costs and still most people (also those who buy lottery-tickets) have insurances, which makes them behave risk-averse. As people do not behave strictly risk-loving or risk-averse, Kahneman and Tversky were working since 1979 on the prospect theory. In their model, not a utility but a value function accounts for the different judgement of people. It sets losses or gains in comparison to a reference point and allows risk-averse, risk-loving and risk-neutral behaviour at the same time but for different amounts of expected losses/gains. So for high losses, people may behave risk-averse but for small losses people may be risk-loving. In Kahneman and Tversky (1992), for which Kahneman received the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel in 2002, they overcome problems of prospect theory and even manage to create a continuous model in contrast to prospect theory, which only works for discrete points and violates the axiom of first order stochastic dominance. This new model for descriptive decisions under risk is known as Cumulative Prospect Theory (CPT). This theory has been applied to many situations which seemed not to be in accordance with economic rationality and managed to give explanations e.g. for the equity premium puzzle, the question why equity stocks have to have a higher return compared to government bonds in order to reconcile. Applying this model to the life cycle model seems like a challenging task but with probably very interesting results.

The aggregation of utility from different people is difficult, especially as it involves many people. Similar, but not quite as hard, is the aggregation of utility at different times. How can the overall utility be determined, when today's utility is equal to  $u_0$  and tomorrow it is equal to  $u_1$ ?

When the experience of tomorrow is certain, the standard way is to discount the future utility by a factor  $\beta < 1$ , or equivalently by  $e^{-\rho}$  with  $\rho = -\ln \beta$ . This resembles the fact that people usually prefer, for example, consumption today over consumption tomorrow. The personal discount factor can be estimated by the question: "Do you prefer receiving \$100 today or \$100 +  $x$  tomorrow?" With the knowledge of  $x$  the personal discount factor can be calculated. The discount rate may also vary over time. A discount rate is called time consistent if the discount function is known in advance, or more detailed, the discounting of utility

from a certain time point in the future  $t_2$  to another time point  $t_1$  will not be changed at any time  $t$  in between ( $t_0 \leq t \leq t_1$ ).

The method of time consistent discounting is questioned in Laibson (1998). Laibson proposes a time inconsistent discounting motivated by the fact that people love to procrastinate. If people have to do something displeasing, they do not discount much between tomorrow or the day after, but the difference between today and tomorrow is big. So the discounting changes at different time points and is therefore not time consistent. This poses additional problems in optimising and is therefore rarely used.

The situation is even more complex, when the experience of tomorrow is uncertain. An additional discount factor may be added to the time discounting, depending on whether people are risk-averse, risk-neutral or risk-loving towards death. Most of the work which includes mortality risk (e.g. Feigenbaum and Gahramanov (2010) and Yaari (1965)) consider a risk-neutral agent who maximises the expected utility. There are also examples of different assumptions: Hippolyte d'Albis and Emmanuel Thibault adopt in their (2010) paper the maxmin-preferences, so risk-averse preferences, in a two-period model. A summary of this paper is given in Chapter 1.3 of this thesis. In a continuous life cycle model, however, it is difficult to adopt maxmin-preferences. In the extreme case, agents would consume all the money they have immediately, because they could die and want to have a maximal utility in that case because living longer gives them a higher utility anyways. The maxmin-preferences were first introduced by Abraham Wald (1902-1950) and represent a very pessimistic point of view. According to him, since the probabilities are not known, it is the best choice to maximise the “worst-case” (minimal) utility.

The opposite, a very optimistic choice would be the maximax-preferences, in which the decision is preferred, which maximises the maximal utility. The Hurwicz-rule<sup>5</sup> combines the maximum and the minimum, weighted with  $\lambda$  and  $1 - \lambda$  respectively. The Laplace rule is based on the idea that, if the probabilities are unknown, the only logical assumption can be that all states are equally likely. So the average utility is to be maximised according to this rule. There are a variety of additional rules, which are not presented here, see e.g. Bamberg and

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<sup>5</sup>Leonid Hurwicz 1917-2008

Coenenberg (2008) for more details.

Under the assumption of rational behaviour, decisions should not depend on the way, the questions are posed but only on the facts available. This is comparable to a meeting in which each participant has a voting right. The vote for or against a project should be the same whether the question is “Who is for the project?” and people who support the project have to raise their hand or the question is “Who is against the project?” and people who oppose it have to give a sign. Abstention from voting is not allowed. However, as Thaler and Benartzi (2004) point out, the decision whether to take part in the SMART-plan, in which you dedicate money from future wage raises to savings, depends on the formulation of the question. When workers have to opt-in, (“Do you want to take part?”), the participation rate is reasonably lower compared to when they have to opt-out (“Do you want to leave this program?”). This effect is called “framing” and was first described by Amos Tversky and Daniel Kahneman. Behaviourism is a philosophy of psychology, which still assumes that one can predict the behaviour of human beings, but has to account for the beings current state of mind.

So under the general assumption of behaviourism and utilitarianism, people evaluate different options according to their own rules which may change over time and then decide rationally in order to maximise their individual utility.

There are plenty of possibilities to treat mortality risk, time preference and aggregation of utility which all have their advantages and disadvantages. Especially different value functions such as the CPT might bring a new point of view in the discussion. In this thesis, the expected utility is also adopted in Section 3.2. Furthermore non-expected utility is taken into account in Section 3.2.3. Both risk-averse and risk-loving behaviour towards death are examined. Risk-aversion towards life means that people want to have had a good life especially if they die earlier than the average.

The task is to find a model which incorporates some but not all possible explanations for human behaviour in which the optimal consumption path is close to the consumption profiles of people observed in consumer expenditure surveys (see Section 1.3). If this resemblance can be met with plausible choices of param-

eters, the main influencing factors for people's choices might be revealed.

One last remark: many people doubt, that they behave just like a “statistic” especially since they do not calculate their lifetime income or optimal consumption. So how shall it be possible to describe their behaviour mathematically, if they do not behave according to any rules?

Gladly, diffusion processes help understand mass panics, although not a single human being thinks to behave just like an atom. Or as John von Neumann said: “Whether I play a game or it is a question of life and death, is the same.”<sup>6</sup>

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<sup>6</sup>cited by Prof. Taschner in “Mathematik und die Moral”, <http://www.mathcast.org>

# Chapter 3

## The Model

In this chapter a general continuous life cycle model is formulated in Section 3.1.

In Section 3.2 a load is introduced into this model, so that annuities generate less than the actuarially fair return. This model is solved analytically by using Pontryagin's Maximum Principle and numerically and the effects of such a load or tax examined in Section 3.2.2. People might not maximise expected utility but non-expected utility, which is considered in Section 3.2.3. The effects of a heterogeneous population are in the focus of Section 3.2.4. Whether all these effects and changes can explain reality better is analysed in Section 3.2.5.

In the second model considered in Section 3.3, utility from bequests depends on the number of family members which depend on the agent and on the security provided for them. Dependent family members are an incentive to save in bonds. In Section 3.3.1 the utility function is described in more details and the model solved analytically. The used dependency function is presented in 3.3.2 and the reality check is done in 3.3.3.

### 3.1 Model Formulation

The continuous overlapping generations (OLG) model considered here is similar to Feigenbaum and Caliendo (2010), Feigenbaum and Gahramanov (2010) and Yaari (1965).

Agents enter the society 20 years old at age  $a = 0$  in a shifted scale and live for a maximum of another  $T$  years. The probability of surviving until age

$a \in [0, T]$  is denoted by  $Q(a)$  and shall be a strictly decreasing  $C^1$ -function. The first derivative of  $Q(a)$  is denoted by  $-q(a)$  so that  $q(a)$  is equal to the probability of dying at age  $a$ . The hazard rate  $h(a)$  is given by  $h(a) = -\frac{d}{da} \ln Q(a) = \frac{q(a)}{Q(a)}$  and is equal to the conditional probability of dying at age  $a$ , given the survival until that age. The time of death for an individual is a realisation of the random variable  $T^*$  which is distributed with the density function  $q(a)$ ,  $a \in [0, T]$ .

In all graphs the axes show the actual age, which means that always the function  $f(a - 20)$  is plotted instead of  $f(a)$ . This makes the graphs more comprehensible and the conversion from  $a \in [0, 80]$  to the actual age from 20 to 100 years is not necessary.

Although the time of death cannot be foreseen by a single agent the death rate is assumed to be deterministic for the whole population due to the law of large numbers. The birth rate is set to 1 but some people die before entering the society in childhood or youth. So there are  $Q(0) < 1$  people of age  $a = 0$ , and  $Q(a)$  is equal to the number of people of age  $a \in [0, T]$  who are alive.

All considerations point to the steady state without any changes in population number, birth rate or death rate, that is, a stationary population is considered. This is also the reason why age and time are used interchangeably.

Agents can invest in annuities and bonds. The state variables  $A(a)$  and  $B(a)$  represent the stock of money invested in these asset classes at time  $a \in [0, T]$ , respectively. Bonds yield a constant return rate  $r$ , the return on annuities depends on the hazard rate and is  $r + h(a)$ . This return rate can be derived from the following argument. Let all living people  $Q(a)$  invest one unit of money into annuities. Then the insurance company invests all the money into bonds with a return of  $r$ . The return for the surviving people is  $Q(a)(1 + rs + xs)$ , where  $x$  denotes the yet unknown additional return created by mortality risk and  $s$  is a certain small period of time. During this period  $s$ ,  $Q(a) - Q(a + s)$  people have died and their money,  $(Q(a) - Q(a + s))(1 + rs)$ , has been redistributed among the rest of the population  $Q(a + s)$ . So the fraction  $xs = \frac{(Q(a) - Q(a + s))(1 + rs)}{Q(a + s)}$  is the rate of additional money gained from the annuities in the period  $s$ . Dividing through  $s$  and passing on to the limit  $s \rightarrow 0$  gives  $x = -\frac{Q'(a)}{Q(a)} = h(a)$ .

Not only the effect of taxes on the individual behaviour but also the effects on the factor prices shall be taken in consideration. An economy with a Cobb-Douglas production function  $Y = f(K, N) = K^{\alpha_K} N^{1-\alpha_K}$  is assumed, where  $K$  stands for the aggregated capital of society,  $N$  for the total labour force and  $Y$  for the total output of the economy. The parameter  $\alpha_K$  takes values in  $[0, 1]$  and accounts for the output elasticity. This function was tested against empirical evidence by Cobb and Douglas (1928). There is quite some criticism on this production function, especially for not having meaningful dimensions of the input factors and for not being additive. However, it satisfies the crucial features in connecting the capital and labour supply to the interest rate and wage, such as diminishing returns in each input factor. Furthermore the function is mathematically traceable. This function is also used in Feigenbaum and Gahramanov (2010).

The interest rate in such an economy is equal to the derivative of the production function with respect to capital minus the depreciation rate. A subscript denotes a partial derivative with respect to the indicated variable:

$$r = r(K, N) = f_K(K, N) - \delta = \alpha_K \left( \frac{N}{K} \right)^{1-\alpha_K} - \delta, \quad (3.1a)$$

the wage is given by the derivative of the production function with respect to labour:

$$w = w(K, N) = f_N(K, N) = (1 - \alpha_K) \left( \frac{K}{N} \right)^{\alpha_K}. \quad (3.1b)$$

The total capital and labour supply in the life cycle model for the Cobb-Douglas production function is given by integration over the capital, respectively the labour efficiency, of the whole population.

$$K = \int_0^T Q(a)[A(a) + B(a)]da, \quad (3.2a)$$

$$N = \int_0^T Q(a)e(a)da. \quad (3.2b)$$

The labour efficiency  $e(a)$  depends on the age, and is zero in retirement. Retirement age is fixed at  $\bar{T}$ . The population and therefore labour supply is

constant over time in our model.  $P$  denotes the constant size of the population,  $P = \int_0^T Q(a)da$  and  $N$  denotes the constant labour supply.

Social Security  $S$  is financed by two sources. First, bequests are redistributed evenly among the whole population. The bequests consist of the money which all people who die have invested in bonds. At any time  $a \in [0, T]$ , the people dying are the people living,  $Q(a)$ , times the hazard rate  $h(a)$ . Second, a possible *tax* collected by the government is redistributed also evenly among the population.

$$S = \frac{\int_0^T Q(a)h(a)B(a)da + tax}{P}. \quad (3.3)$$

In the model considered in this work, the utility function  $u(c)$ , which measures the utility an agent derives from consumption  $c$ , will be a member of the constant relative risk aversion (CRRA) family, with  $\sigma$  being the risk aversion coefficient and the reciprocal of the elasticity of intertemporal substitution, that is

$$u(c, \sigma) = \frac{c^{1-\sigma}}{1-\sigma}. \quad (3.4)$$

A discussion on utility functions can be found in Chapter 2. In case  $\sigma = 1$ ,  $u(c, 1) = \ln c$ . Marginal utility is  $\frac{d}{dc}u(c, \sigma) =: u' = c^{-\sigma}$ .

Let  $T^*$  be a realisation of the random variable which represents the time of death. The probability of such a realisation is  $q(T^*)$ . The utility of an agent accumulated before death is described by

$$\int_0^{T^*} e^{-\rho a} u(c(a)) da. \quad (3.5)$$

The exponential discount factor  $e^{-\rho a}$  accounts for the preference of consumption today against consumption tomorrow. This discount rate might also be non-constant, in this thesis it is assumed constant. For a discussion of this and of aggregation of utility, which is postulated for this formula, see Chapter 2.

A standard choice of the objective function for decision making is to maximise

the expected utility. So in our case the expected utility is calculated as

$$\begin{aligned}
\mathbb{E}[u] &= \int_0^T q(T^*) \int_0^{T^*} e^{-\rho a} u(c(a)) da dT^* \\
&= \int_0^T e^{-\rho a} u(c(a)) \int_a^T q(T^*) dT^* da \\
&= \int_0^T e^{-\rho a} u(c(a)) Q(a) da.
\end{aligned} \tag{3.6}$$

The order of integration can be changed according to Fubini's theorem. Agents seek to maximise this expression. In Section 3.2.3 this assumption will be weakened.

The dynamics of the total assets  $A(a) + B(a)$  is given by

$$\dot{A}(a) + \dot{B}(a) = we(a) + rB(a) + (r + h(a))A(a) + S - c(a). \tag{3.7}$$

The total change in assets has to be equal to the labour income,  $we(a)$ , the return on the assets plus the social security minus the consumption at age  $a$ . The agents take the factor prices wage and interest rate as given. In our case they are constant over time, as we consider steady state. In general, agents could observe today's interest rate and make a prediction of future changes. In the steady state, the interest rate is constant over time, so the agent takes the interest rate as given.

In order to treat the two asset classes separately, the control  $d(a)$  is introduced additionally to the control variable of consumption  $c(a)$ . The change in annuities is set equal to  $d(a)$ , so the dynamic is described by  $\dot{A}(a) = d(a)$ .

There is no short-selling of annuities or borrowing of money, so the state constraints  $A(a) \geq 0$  and  $B(a) \geq 0$  have to hold. Agents enter society without any assets, so the initial conditions are  $A(0) = B(0) = 0$ .

The last condition that has to hold is, that consumption cannot be negative, i.e.  $c(a) \geq 0$ . Due to the shape of the utility function, this constraint will never be active that means consumption is a strictly positive function for all ages. This is because the marginal utility for  $c \rightarrow 0$  is  $c^{-\sigma}$  and tends to minus infinity. Therefore, the slightest increase in consumption will boost utility infinitely. To bring up the money for this, consumption at some other age has to be decreased.

The loss of utility will always be less than the increase achieved. So  $c(a) > 0$  will hold for all  $a \in [0, T]$ .

The basic model is now completely specified and the basic properties derived. The next step is to consider models with slight variations.

In Section 3.2 the optimal consumption given a load on annuities will be derived. The plausibility of this load is pointed out in Section 1.4. A part of this work was also to develop a solver in Matlab using data from Statistik Austria. A description of the solver's architecture and the data can be found in Section 3.2.1, the code is given in the Appendix A. Section 3.2.3 deals with the possibility, that agents do not seek to maximise the expected utility but are risk-averse or risk-loving towards an early death. This can also be interpreted as a non-constant discount factor. In Section 3.2.4 a heterogeneous population with respect to the discount factor and the effect on the steady state is considered. Finally, in Section 3.3 changes in the intrinsic motives during life are modelled and in Section 3.3.1 the model is solved for a specific utility function.

## 3.2 A Load on Annuities

As pointed out in the first chapter, there is statistical evidence for a load on annuities which is created by taxes and by contract and administration premiums. It would be very difficult to model all these different tax treatments and premiums. To simplify the models the real interest rate  $r$ , which internalises the effects of capital gain taxes and the costs of buying and holding bonds, is used instead of the nominal interest rate. Bonds generate a return of  $r$  whereas annuities furthermore generate a return from mortality risk which depends on the age and is denoted by  $h(a)$  (see Section 3 for details). The difference between the two asset classes depends on the insurance tax, insurance premiums and the regulations by law. The resulting effect of these taxes and fees is, that only a certain percentage of the premiums paid actually generates return and therefore the return on annuities accounts to  $(1 - \eta)(r + h(a))$  with  $\eta \in [0, 1]$ . Realistic values are, depending on the assumed impacts of regulations and the duration of contracts,  $\eta \in (0.04, 0.2)$ .

Our model represents a closed economy, any money which is deducted has to be repaid in another way. The effects are combined and it is assumed that the whole load results from taxes and the government redistributes this tax money evenly to the whole population as some sort of government aid. Under the assumption of this load, the optimal investments in bonds and annuities are examined.

Agents maximise<sup>1</sup>

$$\max_{c,d} \int_0^T e^{-\rho a} Q(a) u(c(a)) da \quad (3.8)$$

subject to

$$\dot{A}(a) = d(a), \quad (3.9a)$$

$$\dot{B}(a) = we(a) + rB(a) + (1 - \eta)(r + h(a))A(a) - c(a) - d(a) + S, \quad (3.9b)$$

$$A(a) \geq 0, A(0) = 0, \quad (3.9c)$$

$$B(a) \geq 0, B(0) = 0, \quad (3.9d)$$

$$c(a) \geq 0. \quad (3.9e)$$

Here  $S$  is the money from bequests and taxes on annuities redistributed to the whole living population. So it satisfies the equation

$$S \int_0^T Q(a) da = \int_0^T [Q(a)h(a)B(a) + Q(a)\eta A(a)] da. \quad (3.10)$$

Bequests given are equal to the share of the living population  $Q(a)$  multiplied by the fraction of population which dies multiplied by the amount of bonds  $B(a)$  they possess. Tax on annuities depends on the living population  $Q(a)$  times the annuities  $A(a)$  possessed.

The solution, that is the optimal consumption  $c(a)$  and the optimal investment in annuities  $d(a)$ , is derived using Pontryagin's Maximums Principle. The corresponding Hamiltonian and Lagrangian are

$$\begin{aligned} \mathcal{H}(a, A, B, c, d, \lambda_1, \lambda_2, \lambda_0) = & \lambda_0 Q(a) u(c(a)) + \lambda_1(a) d(a) + \\ & + \lambda_2(a) [we(a) + rB(a) + (1 - \eta)(r + h(a))A(a) - c(a) - d(a) + S], \end{aligned} \quad (3.11)$$

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<sup>1</sup>See e.g. Grass et al. (2008) for the theory of such a problem.

$$\begin{aligned} \mathcal{L}(a, A, B, c, d, \lambda_1, \lambda_2, \lambda_0, \mu, v_1, v_2) = & \mathcal{H}(a, A, B, c, d, \lambda_1, \lambda_2, \lambda_0) + \\ & + \mu(a)c(a) + v_1(a)A(a) + v_2(a)B(a). \end{aligned} \quad (3.12)$$

The first order necessary optimality conditions are:

$$\mathcal{L}_c = \lambda_0 Q(a)u'(c(a)) - \lambda_2(a) + \mu(a) = 0, \quad (3.13a)$$

$$\mathcal{L}_d = \lambda_1(a) - \lambda_2(a) = 0, \quad (3.13b)$$

$$\dot{\lambda}_1(a) = \rho\lambda_1(a) - \mathcal{L}_A = \rho\lambda_1(a) - \lambda_2(a)(1 - \eta)(r + h(a)) - v_1(a), \quad (3.13c)$$

$$\dot{\lambda}_2(a) = \rho\lambda_2(a) - \mathcal{L}_B = \rho\lambda_2(a) - \lambda_2(a)r - v_2(a), \quad (3.13d)$$

$$v_1(a)A(a) = v_2(a)B(A) = 0, \quad (3.13e)$$

$$\mu(a)c(a) = 0. \quad (3.13f)$$

From (3.13b) we know that  $\lambda_1(a) = \lambda_2(a)$  for all  $a \in [0, T]$  and denote it further by  $\lambda(a)$ . The economic interpretation of this equation is as follows:  $\lambda$ , the costate (or dual) variable, is a measure for the additional utility when provided with an additional unit of this asset. Since selling one asset and buying another can be done instantaneously without any costs, the additional utility derived from both assets has to be equal at any time.

In our case,  $\lambda(a)$  is the current value<sup>2</sup>, so the utility of an additional unit of this asset at time  $a$ . The present value is the current value discounted to time zero  $\lambda_{pv}(a) = e^{-\rho a}\lambda(a)$ .

Given a positive interest rate  $r > 0$ , the present value  $\lambda_{pv}(a)$  is always monotonically decreasing. Indeed, assume that there is a point in time  $a_0 \neq 0$  when  $\lambda_{pv}(a)$  has the maximal present value. We now show that the utility is even larger if the agent receives additional money at an earlier time point  $a_1 < a_0$ . The agent can save this money until  $a_0$ , the time of maximal present value, use it then, and still has some money left from the interest generated (as  $r > 0$ ). He can spend this additional money too, thus increasing his utility even further. Therefore  $\lambda_{pv}$  is monotonically decreasing.

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<sup>2</sup>There are different formulations for the Hamiltonian and Lagrangian, the current and the present, which give the same optimal solution for the system. Here the current Hamiltonian is used, see e.g. Grass et al. (2008) for details.

In the calculations the integral of  $h(a)$  is needed:

$$\int_a^b h(s)ds = \int_a^b -\frac{d}{ds} \ln Q(s)ds = -\ln Q(b) + \ln Q(a) = \ln \frac{Q(a)}{Q(b)}, \quad (3.14)$$

and therefore  $e^{\int_a^b (r+h(s))ds} = e^{r(b-a)} \frac{Q(a)}{Q(b)}$ .

With this result and the property  $\lambda_1(a) = \lambda_2(a) = \lambda(a)$ , two different but equivalent expressions for  $\lambda(a)$  are obtained by solving the differential equations for  $\lambda(a)$ .<sup>3</sup>

$$\lambda(a) = e^{(\rho-(1-\eta)r)a} \left( \frac{Q(a)}{Q(0)} \right)^{(1-\eta)} \left[ \lambda(0) - \int_0^a v_1(s) e^{-(\rho-(1-\eta)r)s} \left( \frac{Q(0)}{Q(s)} \right)^{(1-\eta)} ds \right] \quad (3.15a)$$

$$\lambda(a) = e^{(\rho-r)a} \left[ \lambda(0) - \int_0^a v_2(s) e^{-(\rho-r)s} ds \right]. \quad (3.15b)$$

Still,  $v_1(a)$  and  $v_2(a)$  are unknown. By inserting (3.13c) and (3.13d) in the equation  $\dot{\lambda}_1(a) = \dot{\lambda}_2(a)$ , which is the derivative of (3.13b), the following is obtained:

$$(v_1(a) - v_2(a)) = [r - (1 - \eta)(r + h(a))]\lambda(a). \quad (3.16)$$

Under the assumption  $r > \rho$  the agent holds a positive amount of capital all the time. Since there are no diminishing returns or bequest motives, it will in general not be optimal to hold both assets at the same time. So either  $v_1(a)$  or  $v_2(a)$  is positive at age  $a$  but at the same time  $v_1(a)v_2(a) = 0$  (see Equation (3.13e)). As  $v_1$  and  $v_2$  are greater or equal to zero, the question is, when in the life cycle will  $v_1(a)$  be positive, hence  $A(a) = 0$ , and when  $v_2(a)$ , hence  $B(a) = 0$ . Since  $\lambda(a) \geq 0$ , which  $v$  is positive is defined by the sign of

$$\Delta(a) := r - (1 - \eta)(r + h(a)). \quad (3.17)$$

Further it is assumed that  $\Delta(a)$  has at most one root  $\tilde{a}$ . This assumption is equivalent to  $h(a)$  being a strictly increasing function. This is for example fulfilled by any convex survival function  $Q \in C^2$  because  $h'(a) = \left( \frac{q(a)}{Q(a)} \right)' = \frac{q'Q + q^2}{Q^2}$ . If  $q'(a) > 0$  then  $h'(a) > 0$ . For the further calculations it is assumed that there

<sup>3</sup>The solution of an initial value problem of the form  $\dot{y}(t) = a(t)y(t) + b(t)$ ,  $y(0) = y_0$  is  $y(t) = y_0 e^{\int_0^t a(s)ds} - e^{\int_0^t a(s)ds} \int_0^t b(s) e^{-\int_0^s a(x)dx} ds$ .

exists a root  $\tilde{a} \in (0, T)$ , so  $\eta$  is neither too big nor too small. This is only for convenience so that no distinctions about the number of roots have to be made. If no root exists, depending on the, in this case constant, sign of  $\Delta(a)$ ,  $\tilde{a}$  can be set to either  $\tilde{a} = 0$  ( $\Delta(a) < 0$ ) or  $\tilde{a} = T$  ( $\Delta(a) > 0$ ) without changing the validity of the calculations.

The mortality rate from Statistik Austria (see Section 3.2.1) fulfils that it is monotonically decreasing except for a small time after birth and at age 18. These ages are not included in our model, so the assumptions above are fulfilled by the data.

Since  $\tilde{a}$  is the only time point when  $\Delta(a) = 0$ , we conclude that at any other time it is not optimal to hold both annuities and bonds. There are no switching costs from one asset class to another, therefore selling and buying will be done instantaneously. Because  $\Delta(0) > 0$  is assumed for  $a \in [0, \tilde{a})$ ,  $v_1(a)$  is positive and therefore  $A(a) = 0$ , so the agent holds bonds in positive amount. For  $a \in (\tilde{a}, T]$ ,  $v_2(a) > 0$  and the agent holds annuities.

This result is intuitive, as the agent holds annuities only when the return on it is higher than on bonds, and this is when the additional benefit from the premium on mortality risk is bigger than the costs of the load. From equation (3.16) the following is derived:

$$v_1(a) = \begin{cases} (r - (1 - \eta)(r + h(a)))\lambda(a) & a < \tilde{a}, \\ 0 & \text{for } a > \tilde{a}, \end{cases} \quad (3.18)$$

$$v_2(a) = \begin{cases} 0 & a < \tilde{a}, \\ -(r - (1 - \eta)(r + h(a)))\lambda(a) & \text{for } a > \tilde{a}. \end{cases} \quad (3.19)$$

So far the solution for  $\lambda(a)$  resulting from (3.15a) and (3.15b) includes  $v_1(a)$  and  $v_2(a)$  and at the same time the solutions for  $v_1(a)$  and  $v_2(a)$  involve  $\lambda(a)$ . Solving this nonlinear set of equations could be difficult but actually the exact solution for  $v_2$  is not relevant. The only relevant fact is, that  $v_2(a) = 0$  if  $a \leq \tilde{a}$ . By using the suitable equation of (3.15a) and (3.15b) for the interval  $[0, \tilde{a}]$ , respectively  $[\tilde{a}, T]$ , the solution for  $\lambda$  can be easily calculated: for  $a \in [0, \tilde{a})$ ,  $v_2(a)$  is zero and by equation (3.15b)  $\lambda(a) = \lambda(0)e^{(\rho-r)a}$  holds for  $a \in [0, \tilde{a}]$  since the

integral is independent of the value of  $v_2(\cdot)$  at the single point  $\tilde{a}$ .

For  $a \in (\tilde{a}, T]$  equation (3.15a) is used. Since  $v_1(a) = 0$  in this interval, only the value of the integral in (3.15a) for the upper bound  $\tilde{a}$  is needed. The equality of the two different descriptions for  $\lambda$  (Equations (3.15a) and (3.15b)) and the fact, that  $\int_0^{\tilde{a}} v_2(s) ds = 0$  lead to

$$e^{(\rho-(1-\eta)r)a} \left( \frac{Q(a)}{Q(0)} \right)^{(1-\eta)} \left[ \lambda(0) - \int_0^a v_1(s) e^{-(\rho-(1-\eta)r)s} \left( \frac{Q(0)}{Q(s)} \right)^{(1-\eta)} ds \right] = e^{(\rho-r)a} \lambda(0),$$

$$\lambda(0) - \int_0^{\tilde{a}} v_1(s) e^{-(\rho-(1-\eta)r)s} \left( \frac{Q(0)}{Q(s)} \right)^{(1-\eta)} ds = e^{-\eta r \tilde{a}} \left( \frac{Q(0)}{Q(\tilde{a})} \right)^{(1-\eta)} \lambda(0).$$

Inserting this in (3.15a), a closed form solution for  $\lambda(a)$  is obtained which does not involve  $v_1(a)$  or  $v_2(a)$

$$\lambda(a) = \begin{cases} e^{(\rho-r)a} \lambda(0) & a < \tilde{a}, \\ e^{(\rho-(1-\eta)r)a} \left( \frac{Q(a)}{Q(\tilde{a})} \right)^{1-\eta} e^{-\eta r \tilde{a}} \lambda(0) & a > \tilde{a}. \end{cases} \quad (3.20)$$

As it can be seen,  $\lambda(a)$  is exponentially decreasing (under the assumption  $\rho < r$ ) with the exponent  $(\rho - r)$  up to  $\tilde{a}$ , but after changing the asset class it decreases at an even faster rate because of  $\int_0^a v_2(s) e^{(\rho-r)s} ds$  in (3.15b) (see Figure 3.1). This is more clear if (3.20) is inserted in the equations (3.13c) and (3.13d) for  $\dot{\lambda}$ , and the fact that  $\rho - (1 - \eta)(r + h(a)) < r - (1 - \eta)(r + h(a)) < 0$  for  $a > \tilde{a}$  is used.

$$\dot{\lambda}(a) = \begin{cases} (\rho - r) e^{(\rho-r)a} \lambda(0) & a < \tilde{a}, \\ (\rho - (1 - \eta)(r + h(a))) e^{\rho a - (1-\eta)r a} \left( \frac{Q(a)}{Q(\tilde{a})} \right)^{1-\eta} e^{-\eta r \tilde{a}} \lambda(0) & a > \tilde{a} \end{cases} \quad (3.21)$$

So we can use (3.13a) and the last equation for  $\lambda$  to obtain

$$c(a) = \left( \frac{\lambda(a) - \mu(a)}{\lambda_0 Q(a)} \right)^{\frac{-1}{\sigma}}$$

$$= \left( \frac{\lambda_0 Q(a)}{\lambda(a)} \right)^{\frac{1}{\sigma}}. \quad (3.22)$$

In the second line  $\mu$  is suppressed because it will never be optimal to consume nothing since the derivative of  $u$  with respect to consumption at  $c = 0$  is by assumption infinite and so it follows from (3.13f) that  $\mu(a) = 0$ .

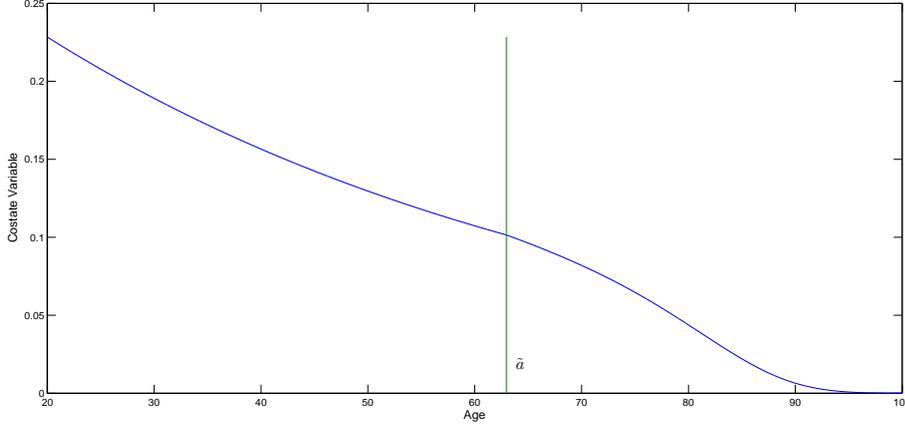


Figure 3.1: Costate Variable  $\lambda(a)$   
 $\rho = 0.04$ ,  $r = 0.0589$ ,  $\eta = 0.1$ ,  $\tilde{a} = 62.98$

Taking the derivative of the consumption with respect to time and using the quotient rule we obtain that

$$\begin{aligned}
 \dot{c}(a) &= \frac{-1}{\sigma} \frac{\lambda(a)^{-\frac{1}{\sigma}-1}}{Q(a)} \left( \frac{\dot{\lambda}(a) + \lambda(a)h(a)}{Q(a)} \right) & (3.23) \\
 &= \frac{-1}{\sigma} \frac{\lambda(a)^{-\frac{1}{\sigma}}}{Q(a)} \left( \frac{\dot{\lambda}(a)}{\lambda(a)} + h(a) \right) \\
 &= \begin{cases} \frac{-1}{\sigma} \left( \frac{\lambda(a)}{Q(a)} \right)^{-\frac{1}{\sigma}} (\rho - r + h(a)), & a < \tilde{a}, \\ \frac{-1}{\sigma} \left( \frac{\lambda(a)}{Q(a)} \right)^{-\frac{1}{\sigma}} (\rho - (1 - \eta)r + \eta h(a)), & a > \tilde{a}. \end{cases}
 \end{aligned}$$

Until that point it is not possible to say whether consumption is increasing or decreasing, because the sign of  $\rho - r + h(a)$  is not determined.

Not only consumption but also its derivative is continuous. At any time except for  $\tilde{a}$  this is obvious because  $h(a)$ ,  $\lambda(a)$  and  $Q(a)$  are continuous functions. For  $a = \tilde{a}$  the continuity is now shown. By the definition of  $\tilde{a}$  (see Equation (3.17)),  $r = (1 - \eta)(r + h(\tilde{a}))$  holds. The continuity of  $h(a)$  gives

$$\begin{aligned}
 \lim_{a \rightarrow \tilde{a}^-} (\rho - r + h(a)) &= \rho - r + h(\tilde{a}) = \rho - (1 - \eta)(r + h(\tilde{a})) + h(\tilde{a}) = \dots \\
 &= \rho - (1 - \eta)r + \eta h(\tilde{a}) = \lim_{a \rightarrow \tilde{a}^+} (\rho - (1 - \eta)r + \eta h(a)),
 \end{aligned}$$

which shows the continuity of  $\dot{c}(a)$ .

At younger ages mortality risk is low so it is likely that  $\rho + h(a) - r < 0$ . Therefore consumption is increasing over time, but with increasing mortality rate  $h(a)$ , at some point (in general it is not possible to determine whether this happens before or after  $\tilde{a}$ ), consumption reaches its maximum and decreases afterwards. It is possible that the consumption is always decreasing, depending on  $\rho$  and  $h(0)$ . In different simulations, the peak in consumption occurs somewhere around 70-80, as it can be seen in Figure 3.2 and 3.7.

The interpretation of the sign of the consumption's derivative is straightforward. If the interest rate is smaller than the discount rate plus the mortality risk, consumption decreases. This is because the discount factor decreases utility more than the interest increases it, and secondly so the risk of dying before consuming one's wealth can be avoided. The only reason to postpone consumption is the shape of the utility function. For large consumption  $c(a)$ , a further increase in today's consumption increases utility less than increasing tomorrow's consumption due to the diminishing marginal utility for increasing consumption. If the interest rate is bigger, then the interest paid on savings increases utility more strongly than a high today's consumption and the consumption increases over time.

For  $a > \tilde{a}$ , the bracket can be rewritten to  $\rho + h(a) - (1 - \eta)(r + h(a))$  which differs to the equation considered above only in the way that now the discount rate equals the return on annuities instead of the plain interest rate  $r$ , the return of bonds. If the return on annuities is bigger than the mortality risk plus the discount factor, then consumption is increasing, otherwise decreasing.

So far  $\lambda(0)$  is not determined. In the optimal state, the aggregated discounted consumption for  $a$  from 0 to  $T$  has to equal the discounted income from work and social security as no money will be left behind at  $a = T$ . The discount factor is  $e^{-ra}$  for  $a < \tilde{a}$  and is  $e^{-r\tilde{a}}e^{-(1-\eta)\int_{\tilde{a}}^a(r+h(s))ds}$  for  $a > \tilde{a}$  as the appropriate discount factor is determined by the investment vehicle used.

$$\int_0^{\tilde{a}} e^{-ra} c(a) da + e^{-r\tilde{a}} \int_{\tilde{a}}^T \left( e^{-ra} \frac{Q(a)}{Q(\tilde{a})} \right)^{(1-\eta)} c(a) da = \int_0^{\tilde{a}} e^{-ra} (we(a) + S) da + e^{-r\tilde{a}} \int_{\tilde{a}}^T \left( e^{-ra} \frac{Q(a)}{Q(\tilde{a})} \right)^{(1-\eta)} (we(a) + S) da. \quad (3.24a)$$

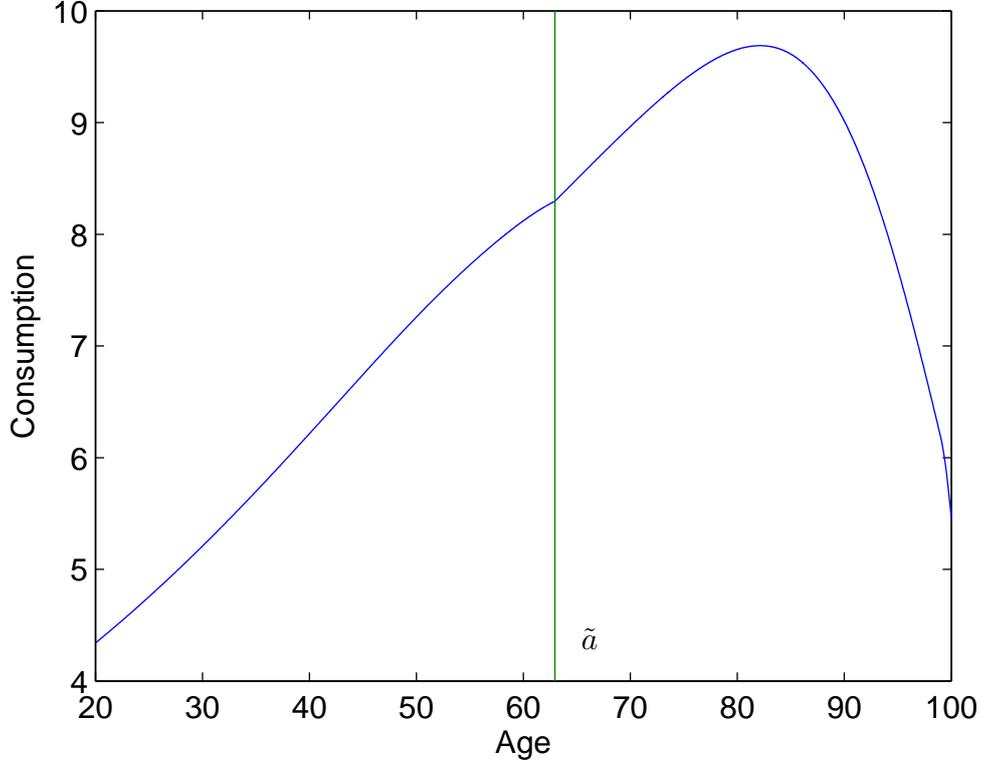


Figure 3.2: Optimal Consumption over Time

$$\rho = 0.04, r = 0.0589, \eta = 0.1, \sigma = 1, \tilde{a} = 62.98, \lambda(0) = 0.2284$$

The optimal consumption path (3.22) is already known, and can therefore be inserted in the equation, which is then solved for  $\lambda(0)$ :

$$\lambda(0) = \left( \frac{\int_0^{\tilde{a}} e^{-ra} (Q(a)e^{(r-\rho)a})^{\frac{1}{\sigma}} da + e^{-ra} \int_{\tilde{a}}^T \left( e^{r(\tilde{a}-a)} \frac{Q(a)}{Q(\tilde{a})} \right)^{(1-\eta)} (Q(a)e^{(r-\rho)a})^{\frac{1}{\sigma}} da}{\int_0^{\tilde{a}} e^{-ra} (we(a) + S) da + e^{-ra} \int_{\tilde{a}}^T \left( e^{r(\tilde{a}-a)} \frac{Q(a)}{Q(\tilde{a})} \right)^{(1-\eta)} (we(a) + S) da} \right)^{\sigma}. \quad (3.24b)$$

The discounting described above is equivalent to solving the differential equation (3.9b) from 0 to  $\tilde{a}$ , then solving (3.9a) from  $\tilde{a}$  to  $T$  and solving the equation  $A(T) = 0$  for  $\lambda(0)$ .

So the optimal solution for a given interest rate  $r$ , wage  $w$  and bequest distribution  $S$ , or stated differently for given  $K$  and  $S$ , is derived.

### 3.2.1 Numerical Solution

A numerical solver was programmed in Matlab in order to solve the model described above. The basic architecture and data are described below while the Matlab code can be found in the Appendix A.

The data for  $h(a)$  is from Statistik Austria<sup>4</sup>, precisely the life table from the year 2000. The available annual mortality rate is linearly interpolated to obtain a continuous function  $h(a)$ .  $Q(0)$  represents the probability of surviving until the 20th birthday in the life table, so it is equal to the fraction of the initial population which actually enters working life.  $Q$  is calculated from  $Q(0)$  and  $h(a)$  by solving the differential equation  $\dot{Q}(a) = -h(a)Q(a)$  with initial condition  $Q(0) = Q^0$ , the solution is given by  $Q(a) = Q(0)e^{-\int_0^a h(a)}.$  Due to the linear interpolation, the calculated survival probability does not completely resemble the true survival probability. It slightly overestimates the chance of surviving, in particular  $Q(T)$  is positive but very small ( $Q(T) = 0.0001$ ), see Figure 3.3.

Especially problematic is the estimation of  $h(a)$ . Statistik Austria only gives yearly rates, which are surviving probabilities of periods, conditional survival until the beginning of the period. As a probability, this function is bounded by one. When passing on to the limit with small time steps, and thus to a continuous function, any limit function still is bounded by one. However, for a probability function this boundary must not hold any more. Towards the certain end of life at  $a = T$ , any people alive will die, which means the probability 1 in the life-table. For a continuous function, this requires that  $\lim_{a \rightarrow \infty} h(a) = \infty$ . This could be estimated by increasing  $h(T)$  when reducing the step size so that the area underneath the curve remains constant. In this work this is not done, because the effect of this disturbance at the end of life is very small (only 0.01 percent of the people born survive our model) and does not qualitatively change the results.

For the labour efficiency function  $e(a)$  which determines wage in the differential equation (3.9b) the data from Gourinchas and Parker (2002) is fitted with a quadratic function. The peak of the wage is (depending on the population group) around age 50 and is about 1.5 times the initial wage. At retirement age the wage is roughly 1.25 times the initial wage. Fitting this with a quadratic

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<sup>4</sup>[www.statistik.at](http://www.statistik.at)

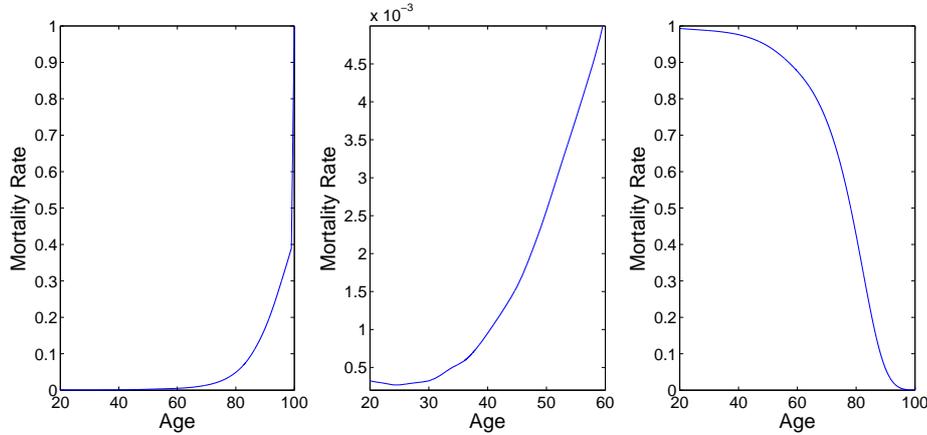


Figure 3.3: Mortality Rate and Survival Function

Data for mortality rate from Statistik Austria,  
Survival Function computed by integrating the mortality rate

function  $e(a) = 1 + \frac{7}{180}a - \frac{1}{1350}a^2$  is obtained, where  $a = 0$  corresponds to the entry in working life when 20 years old. Retirement age is set to 65, so  $a = 45$ , as it is the regular pension age nowadays in Austria<sup>5</sup>, see Figure 3.4.

The parameters  $\rho$  (discount factor),  $\sigma$  (coefficient of relative risk aversion),  $\alpha_K$  (parameter in Cobb-Douglas productivity function),  $\delta$  (capital depreciation in the function) and  $\eta$  are taken as constant. The exact value used for calculation of each graph is mentioned in the caption.

Integrals are calculated according to Simpson's rule and differential equations are solved by Heun's method.

For a given step size the functions described above are discretised and the constant population size and labour supply are calculated, compare (3.2b). For the given constants  $K$  and  $S$  the interest rate  $r$  and wage  $w$  are calculated according to (3.1a). Under the assumption of rational expectations the switching point  $\tilde{a}$  is determined by solving the equation  $\Delta(a) = 0$ , see (3.17). Then  $\lambda(0)$  and  $\lambda(a)$  (see (3.24b) and (3.20)) and with this knowledge  $c(a)$  (according to Equation (3.13a)) are computed. Next, the differential equations (3.9) for  $A(a)$

<sup>5</sup>[www.help.gv.at](http://www.help.gv.at)

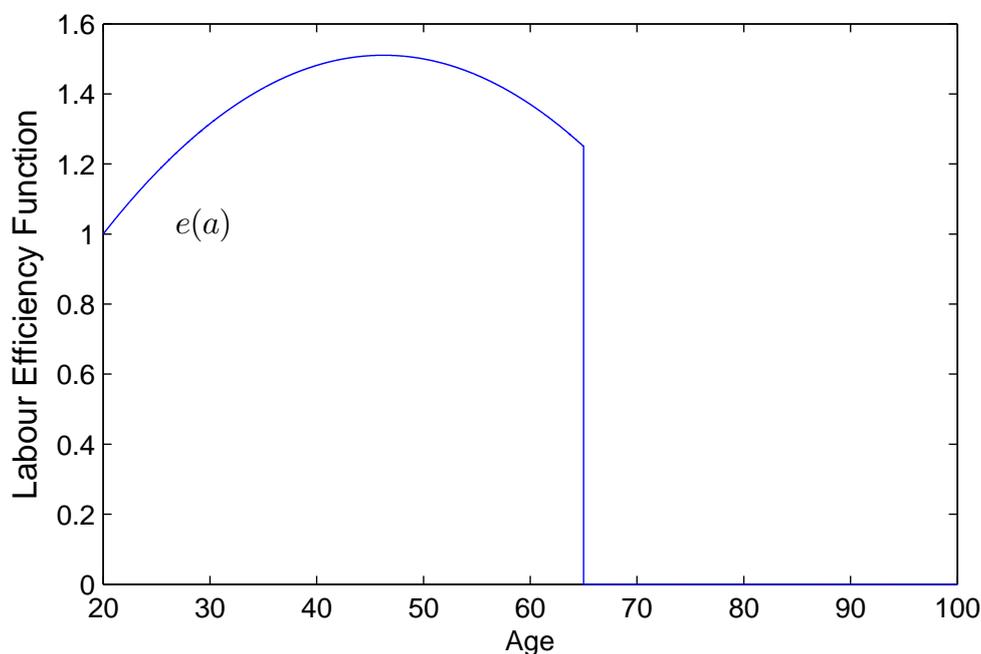


Figure 3.4: Labour efficiency function

and  $B(a)$  are solved backwards starting with  $B(T) = A(T) = 0$  due to the better numerical stability. Finally,  $K^{new}$  and  $S^{new}$  are calculated (see Equation (3.2a) and (3.3)).

The equilibrium capital  $K^{eq}$  and social security  $S^{eq}$  are the values of  $K$  and  $S$  for which  $K = K^{new}$  and  $S = S^{new}$  are satisfied. For  $K > K^{eq}$  the inequality  $K^{new} < K^{eq}$  is obtained and vice versa, so with a binary search  $K^{eq}$  and  $S^{eq}$  can be easily determined.

### 3.2.2 Effects of taxation

What is the effect of taxes on the utility of the people? How does it influence the investment decision? A taxation of  $\eta = 1$  corresponds to investment only in bonds, whereas the other extreme,  $\eta = 0$ , coincides with the investment only in annuities. The model examined here is a generalisation of other existing models.

Whereas Feigenbaum and Gahramanov (2010) obtain that a social planner can improve upon annuitisation by investment in bonds, simulations show, that

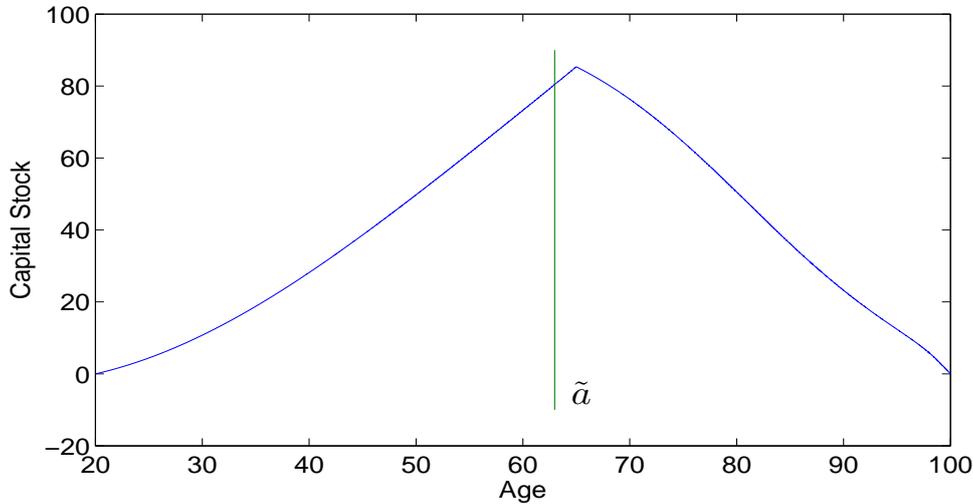


Figure 3.5: Optimal Capital Stock over Time, given survival

$$\rho = 0.04, \eta = 0.1, \sigma = 1, \alpha_K = 0.5, \delta_K = 0.02,$$

this is only true if the social planner also can choose the consumption profile for the agents. When the consumption profile is chosen individually, the utility is lower (although only slightly) with bonds than with annuities, as shown in table 3.6. The difference between bonds and annuities is very small in the steady state.

As described in Section 1.3, some authors (e.g. Brown and Poterba (2000) and Kotlikoff and Spivak (1981)) calculate a high increase in utility when access to annuity markets is granted. However, they only consider changes *ceteris paribus*. Changes in the interest rate and wage, and changes in bequest sizes are not taken into account. In the steady state, when incorporating effects on the interest rate and on bequests, the utility gain from access to annuities is only very small.

If the social planner can only choose taxes on annuities, but not the consumption profile, a low tax rate of approximately 10% is optimal. The increased capital through higher savings in younger ages leads to higher bequests and the taxes paid on annuities lead to a higher social security. So the capital transfer from the older to the younger increases utility (see Figure 3.6 and 3.7). The increased social security and higher wage make up for the loss of utility by the lower interest rate. In the simulations, it can be seen that the higher social security, the higher utility. The maximal utility is achieved for  $\eta \approx 0.1$ . This result is robust in the simulations with different parameters of  $\rho$ . A realistic value for  $\eta$  is, as pointed

out in 1.4,  $\eta \in [0.04, 0.2]$ .

|             | $\eta = 0$ | $\eta = 0.05$ | $\eta = 0.1$ | $\eta = 0.15$ | $\eta = 0.2$ | $\eta = 1$ |
|-------------|------------|---------------|--------------|---------------|--------------|------------|
| $K$         | 2993       | 3090          | 3229         | 3148          | 3002         | 2379       |
| $S$         | 0          | 2.13          | 2.61         | 2.46          | 2.17         | 0.34       |
| $\tilde{a}$ | 20         | 49.7          | 60.7         | 65.9          | 69.3         | 100        |
| $u$         | 40.5       | 50.80         | 52.28        | 51.27         | 49.95        | 39.45      |

Figure 3.6: The effect of taxes on capital stock,  $\tilde{a}$  and utility  
 $\sigma = 1, \rho = 0.04, \alpha_K = 0.5, \delta_K = 0.02$

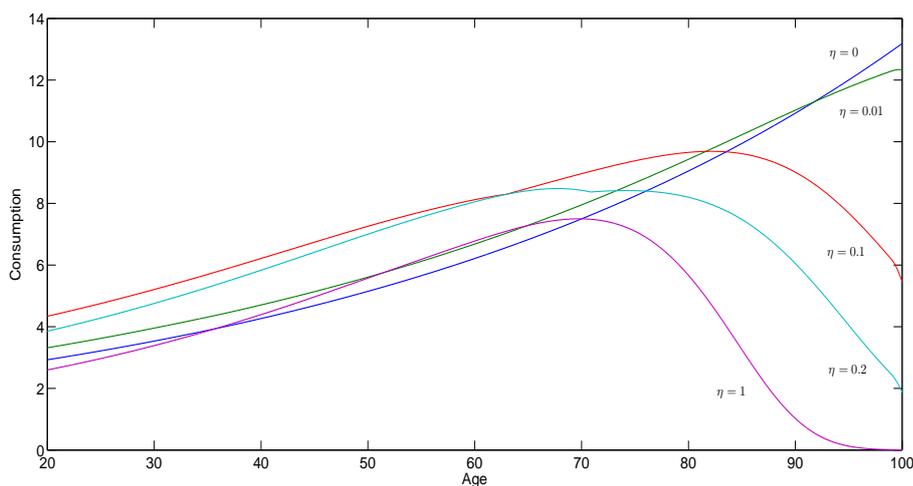


Figure 3.7: Consumption for different tax rates  
 $\sigma = 1, \rho = 0.04, \alpha_K = 0.5, \delta_K = 0.02$

### 3.2.3 Non-expected utility

So far the assumption was that all agents maximise the expected utility. As pointed out in Section 1.3 and Chapter 2, this might not be true.

First, the individual might have a specific estimation of his survival probability. The agent assumes in his target function a different survival function than the insurance company uses for calculating the additional return on annuities.

Second, a risk-averse agent will weigh utility more heavily if he dies young but may weigh it less heavily if he lives long. This reason is adopted in this section. So the expected utility postulated in the previous sections is substituted by a different discount function which is still time consistent, in comparison to e.g. Hyperbolic Discounting (see Chapter 2).

Assume that an agent has a modified weighting of the realisations of life in formula (3.6). A realisation of his life is the utility he acquires when dying at  $T^*$ , where the time of death is a realisation of the random variable. Instead of  $q(T^*)$ , the probability of dying at that age according to the life table, he takes an arbitrary weighting function  $f(T^*)$ . For simplicity only functions of the form  $f(T^*) = \alpha q(T^*)Q^{\alpha-1}(T^*)$ , with a parameter  $\alpha \in (0, \infty)$ , are considered. These functions fulfil for any survival function  $Q(s)$  with  $Q(0) = 1$  and  $Q(T) = 0$  the equation  $\int_0^T \alpha q(T^*)Q^{\alpha-1}(T^*)dT^* = 1$  and can therefore be interpreted as probability densities themselves. Obviously,  $\alpha = 1$  corresponds to expected utility. The derivative of this density at one point  $a$  with respect to  $\alpha$  is  $q(a)Q^{\alpha-1}(a)(1 + \alpha \ln Q(a))$ . If  $\alpha > 1$  then for  $Q(a) > e^{-\alpha}$ , equivalently for  $a$  small enough, the derivative has a positive sign.

The parameter  $\alpha$  can be interpreted as risk aversion coefficient towards length of life. In case  $f$  is interpreted as a probability function,  $\alpha > 1$  means that the agent overestimates his mortality risk. If interpreted as a weighting function, this means that an agent puts more emphasis on those realisations of his life, in which he dies young. He is risk averse towards death. It is more important to the agent, that the lifetime utility is roughly equal independent from time of death. Death in an age which most people survive is more tragic to him, and therefore more heavily weighted, than the rest. The opposite is true for  $\alpha < 1$ : From the perspective of a probability function, an agent underestimates his mortality risk and thinks he lives longer than the average population. From perspective of maximisation, agents with  $\alpha > 1$  seek to maximise the utility in case that they survive long, but care less about what happens if they die young. This is visualised in Figure 3.8.

The effective discount function can be calculated by integration by parts (com-

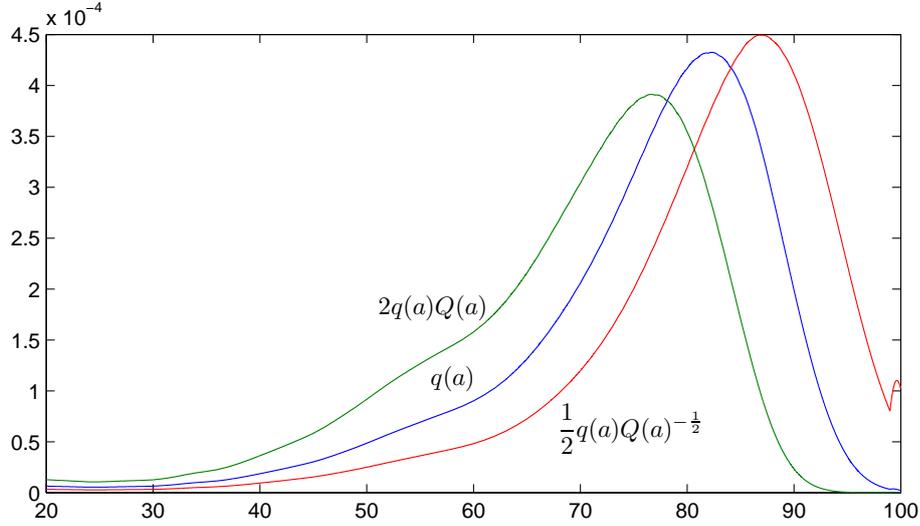


Figure 3.8: Weighting functions for different risk coefficients  $\alpha$

pare (3.6)):

$$\begin{aligned} \int_a^T q(s)Q^{\alpha-1}(s)ds &= -Q(s)Q^{\alpha-1}(s)|_a^T - \int_a^T -Q(s)(\alpha-1)Q^{\alpha-2}(s)(-q(s))ds = \\ &= -Q^\alpha|_a^T - \int_a^T (\alpha-1)Q^{\alpha-1}(s)q(s)ds, \end{aligned}$$

because of  $\dot{Q}(s) = -q(s)$  and  $Q(T) = 0$ . Bringing the integral on the right to the left gives  $\int_a^T \alpha q(s)Q^{\alpha-1}(s)ds = Q^\alpha(a)$ .

Accounting for these modifications in the agent's problem of maximisation the expected utility (see (3.6)), Equation (3.8) changes to

$$\max_{c_{[\alpha],d}} \int_0^T e^{-\rho a} Q^\alpha(a) u(c_{[\alpha]}(a)) da. \tag{3.25}$$

This formulation shows that different weighting functions can also be interpreted as time discounting functions which are not constant over time. Utility from expected consumption at time  $a$ ,  $Q(a)u(c(a))$ , is discounted by  $e^{-\rho a + (\alpha-1)\ln(Q(a))}$ . As  $\ln(Q(a)) < 0$  we get for  $\alpha > 1$  a higher discount rate and for  $\alpha < 1$  a lower discount rate than in Section 3.2.

The differential equations for the costate  $\lambda$  remain unchanged but the initial value  $\lambda(0)$  changes. Optimal consumption under non-expected utility  $c_{[\alpha]}(a)$  and

$\dot{c}_{[\alpha]}(a)$  can be calculated by (compare with formula (3.13a) and (3.23))

$$c_{[\alpha]}(a) = \left( \frac{Q^\alpha(a)}{\lambda(a)} \right)^{\frac{1}{\sigma}}, \quad (3.26)$$

$$\dot{c}_{[\alpha]}(a) = \begin{cases} \frac{-1}{\sigma} \left( \frac{Q^\alpha(a)}{\lambda(a)} \right)^{\frac{1}{\sigma}} (\rho - r + \alpha h(a)), & a < \tilde{a}, \\ \frac{-1}{\sigma} \left( \frac{Q^\alpha(a)}{\lambda(a)} \right)^{\frac{1}{\sigma}} (\rho - (1 - \eta)r + (\eta + \alpha - 1)h(a)), & a > \tilde{a}. \end{cases} \quad (3.27)$$

Analytical comparison of risk aversion towards death ( $\alpha > 1$ , Equation (3.26)) to the risk neutral expected utility ( $\alpha = 1$ , see Equation (3.22) or take  $\alpha = 1$  in (3.26)) reveals that under risk aversion consumption will start decreasing earlier in life. This is because for  $\alpha > 1$ ,  $\rho - r + h(a) < \rho - r + \alpha h(a)$  and therefore the sign of the right side is earlier in life positive, thus  $\dot{c}_{[\alpha]}$  negative. For the same reason, and because  $Q^\alpha(a) < Q(a)$ , any increase in consumption is smaller if the agent is risk averse than if he is risk neutral, when compared at the same age. If at the same age consumption is decreasing, it is not possible to say whether the decrease is smaller or bigger for risk aversion. The additional multiplicative factor  $(\alpha - 1)Q(a)$  makes the decrease smaller for risk aversion, whereas the additional  $(\alpha - 1)h(a)$  increases the derivative. The same property as in (Epstein and Zin, 1989) (see Section 1.3), which incorporates risk aversion too, holds: if the risk aversion is big enough ( $\alpha$  very large), consumption will always decrease.

It is not possible that  $c(a) > c_{[\alpha]}(a)$  for the whole life, because then risk averse agents would leave a positive amount of assets at  $a = T$ . If the inequality were the other way round, risk averse agents would leave debts when dying which is also ruled out by the model. The derivative of risk averse agents is smaller, so it follows  $c(0) < c_{[\alpha]}(0)$  and there is  $a_\alpha \in (0, T)$  when  $c(a_\alpha) = c_{[\alpha]}(a_\alpha)$ . For  $a > a_\alpha$ , it holds that  $c(a) > c_{[\alpha]}(a)$ . So a larger  $\alpha$  leads to a higher consumption in younger ages and lower consumption in old ages, it shifts the consumption hump to the left.

For  $\alpha < 1$ , a risk loving agent, it is the other way round. At first consumption increases more heavily, and the signum-change happens later. For small  $\alpha$  consumption may always increase. Initial consumption is lower than risk neutral consumption.

It is possible to get the same results by using an arbitrary time-varying discount rate  $\rho(a)$  in Equation (3.8). However, in the way done in this section it has a

meaningful representation both in the case when considering different realisations of lifetime (see Equation (3.6)) and in the resulting integral (see (3.8)).

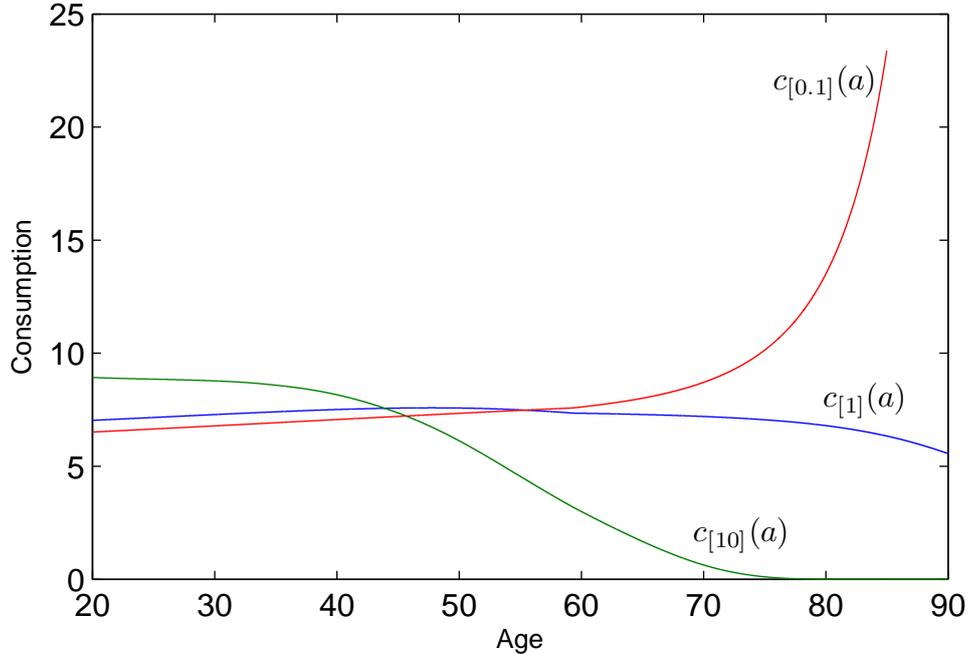


Figure 3.9: Consumption for different risk aversion coefficients  $\alpha$   
 $\rho = 0.04, r = 0.0442, \eta = 0.1, \alpha_K = 0.5, \delta_K = 0.01, \sigma = 1$

### 3.2.4 Heterogeneous Discount Rate

In this section, the focus is on the role of heterogeneity with respect to the personal discount rate  $\rho$  in society. How will the capital stock change if not the whole society has the same time discount factor  $\rho$ , but different people or groups in society have different discount factors  $\rho$ ?

Since it is not possible to include the capital stock  $K$  in the optimality conditions and derive the steady state capital stock analytically as the appearing integral equations are not analytically solvable, the effects are simulated numerically. For each value  $\rho$  in the distribution, the optimal consumption is calculated and the optimal capital stock is computed by iteration.

In order to see the effects clearly an easy example population is assumed: one third of the population incorporates the discount factor  $\rho = 0.02$ , one third

$\rho = 0.03$  and one third  $\rho = 0.04$ . The optimal consumption is computed for a given  $K$ , and the mean resulting capital stock is taken for a new iteration. The same is done for the income from social security (bequests and tax from annuities). Within a few iterations the steady state capital stock and social security income is achieved.

For the simple example, the different consumption profiles are plotted in Figure 3.11. The optimal capital stock and social security for a homogeneous population with  $\rho = 0.03$  are  $K = 3115$  and  $S = 2.62$ , the interest rate is therefore  $r = 4.88\%$  with  $\alpha_K = 0.5$  and  $\delta_K = 0.02$ , see the table in Figure 3.10. For the heterogeneous population, the capital stock increases to  $K = 3193$  and  $S = 2.74$  with an interest rate of  $r = 4.79\%$ .

As can be seen in Figure 3.10, utility for the part of the heterogeneous population with  $\rho = 0.03$  increases slightly (52.81 compared to 52.28). The increasing capital stock reduces the interest rate but the reduction in utility is offset by the higher utility from increased social security because higher savings lead to more “accidental bequests” when holding bonds. Not only for the part of the population with  $\rho = 0.03$  the utility increases, also the average utility is in the heterogeneous population higher than in the homogeneous. and the utility of the least satisfied people increases. A heterogeneous society is better not in a Pareto efficient way, as it is worse for people with  $\rho = 0.02$ , but is better according to average utilitarianism as well as according to John Rawls suggestion that the utility of the people worst off in the beginning is to improve.

A comparison by magnitude of utility is, as outlined in Chapter 2, not the best choice to compare between the states. Similarly to the Annuity Equivalent Wealth mentioned in Chapter 1, a comparison here is implemented by reducing or increasing initial wealth. The optimal solution remains the same, only the calculation of  $\lambda(0)$  changes. In formula (3.24b), the initial capital is added in the denominator. The equivalent initial wealth for the population with  $\rho = 0.02$  is  $-20$ , which is approximately five times the amount earned in the first year at work. For the people with  $\rho = 0.03$  the equivalent initial wealth is 2, about half a years wage, and for the group with  $\rho = 0.04$  it is approximately 18. Another possible measure to compare is how much additional consumption in percent is needed at each time to maintain the same utility level. For the population with

$\rho = 0.03$  the increased utility corresponds to a 2% increase in consumption but for the group with  $\rho = 0.04$  this is equal to an increase in consumption by 23%.

|   | $\rho = 0.02$ | $\rho = 0.03$ | $\rho = 0.04$ | heterogeneous population  |
|---|---------------|---------------|---------------|---------------------------|
| K | 4384          | 3115          | 2353          | 3193                      |
| S | 4.12          | 2.61          | 1.80          | 2.74                      |
| r | 3.78%         | 4.88%         | 5.88%         | 4.79%                     |
| u | 74.02         | 52.28         | 38.36         | 54.38 (67.27/52.81/43.08) |

Figure 3.10: Capital, Social Security, interest rate and utility in a heterogeneous society

Every discount rate is adopted by one third of the population.

$$\eta = 0.1, \sigma = 1, \alpha_K = 0.5, \delta_K = 0.02$$

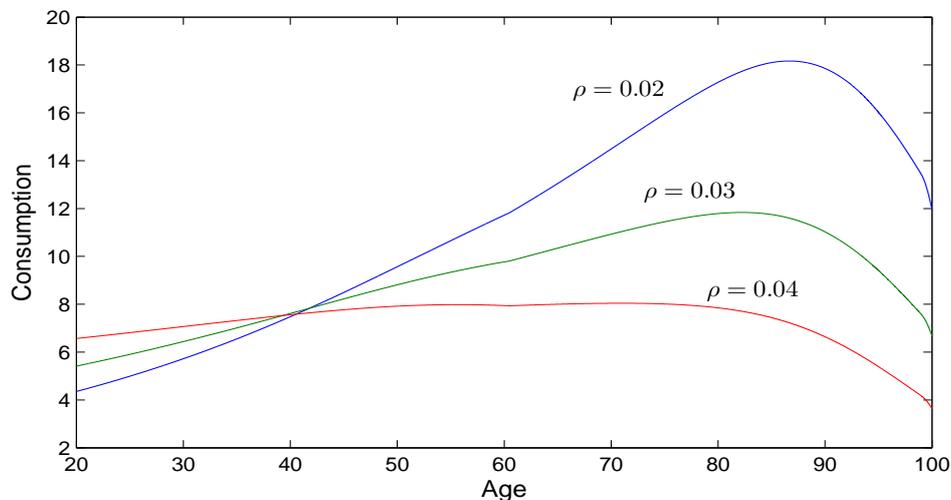


Figure 3.11: Optimal consumption for different time discount rate  $\rho$

$$K_{opt} = 3193, S = 2.74, r = 0.0479, \eta = 0.1, \sigma = 1$$

### 3.2.5 Model versus Reality

There are multiple possible criteria to evaluate how good a model represents reality. From the examination of household expenditure surveys (see Section

1.3), the following questions arise: Does a model resemble the behaviour of people that money is invested in bonds and annuities at the same time (compare Sommer (2005))? Does it reproduce the hump in the consumption profile observed in the data, that means that the peak is correct by magnitude (roughly 1.5 times the initial consumption) and the peak occurs when the agent is approximately 50 years old (compare Gourinchas and Parker (2002))?

The model considered here can explain a low participation rate in annuities in the population. With  $\rho = 0.04$ ,  $\sigma = 1$ ,  $\alpha_K = 0.5$  and  $\delta_K = 0.01$ , which are all reasonable values, the fraction of overall savings invested in annuities equals roughly 40%. The fraction increases when  $\rho$  decreases or  $\eta$  increases. The change from bonds to annuities happens at age 63, and therefore roughly 25% of the population invest in annuities. This fraction exhibits the same behaviour regarding changes in  $\rho$  and  $\eta$  as the savings rate in annuities. However, this model cannot give an explanation why many people invest some of their money in annuities because always either complete or no annuitisation is optimal.

The consumption hump can be reproduced better but it is still not satisfying: With a high discount rate ( $\rho = 0.1$ ) together with non-expected utility ( $\alpha = 10$ ) and  $\sigma = 10$  the hump can be shifted to approximately 60 years and the size of the hump is about 2.5 times the initial consumption. This is a significant improvement over the initial situation in which the hump is later and even more pronounced. However, the values, especially for  $\alpha$  and  $\rho$  are rather unlikely high but especially the non-expected utility plays a significant role for the shape of the hump which may explain the behaviour of people in a broader model together with other factors.

### 3.3 Time-dependent Utility from Bequests

Whereas the model in Section 3.2 assumes a different return on bonds and annuities, in this chapter an intrinsic motive for buying bonds is analysed. The additional return of annuities is created by sacrificing bequests. When buying an annuity, the agent loses any money he has not yet consumed in exchange for a higher return. Bonds have a lower return but when dying, the current amount held is bequeathed to the descendants.

As pointed out in Section 1.3, the importance of bequests as a saving motive is low, however children and especially taking care for their education is an important motive for saving. It is plausible that the importance of savings is time dependent and low when the agent has no family or the children are grown up but high when children are young and depend on the income of the agent. An agent increases his utility when he knows that he has taken care for his descendants in case of death, or vice versa reduces his utility if no precaution measures are taken.

Yaari (1965) investigates this effect already and derives some differential equations which an optimal solution has to fulfil. Here these findings are extended and the differential equations partially solved. In a very general form, the utility derived from holding assets which will be inherited in case of death can be written as  $u = u\left(a, c(a), \frac{B(a)}{K(a)}, D(a)\right)$ . Utility depends not only on the age and on the consumption but also on the fraction of the total wealth held in bonds and on the dependence function  $D(a)$ . Utility depends on the fraction of wealth instead of absolute wealth because a millionaire wants to ensure a higher standard of living for his children than an agent from middle class. The amount saved will correspond to the total income and total wealth. The dependency function  $D : [0, T] \rightarrow [0, 1]$  includes the information how many people depend on the agent. When the agent is young and has not (yet) children, it is zero or close to zero. Social commitments, such as marrying or having or adopting children, increase dependency of others on the agent and therefore the dependency function increases. When children are getting older and start earning their own money, they become less dependent on the money from parents and the dependency function decreases. So the dependence function in this thesis is assumed to be hump

shaped, which is in accordance to the suggestion by Yaari (1965).

For the utility from bequests, it is assumed that  $u_B(a, c, x, D)$  is monotonically decreasing in  $x$ . When little money is saved in bonds, so only a little fraction of total wealth will be passed on to the children, an increase in this fraction pushes utility more heavily than if already much money is saved. Knowing that someone depends on the agent but no money will bequeathed leaves him with a bad conscience all the time, and the possibility to salve ones conscience by increasing the money left behind improves the utility immensely. The derivative with respect to  $B$  shall be zero when all money is already held in bonds ( $u_B(a, c, 1, D) = 0$ ) because when all wealth is held in bonds, that means that the total wealth will be bequeathed, the utility cannot be increased any more.

So with this general form of utility function, the model is as follows (compare (3.6) and (3.8)):

$$\max_{c,d} \int_0^T e^{-\rho a} Q(a) u \left( a, c(a), \frac{B(a)}{K(a)}, D(a) \right) da \quad (3.28)$$

subject to

$$\dot{A}(a) = d(a), \quad (3.29a)$$

$$\dot{B}(a) = we(a) + rB(a) + (r + h(a))A(a) - c(a) - d(a) + S, \quad (3.29b)$$

$$A(a) \geq 0, A(0) = 0, \quad (3.29c)$$

$$B(a) \geq 0, B(0) = 0, \quad (3.29d)$$

$$c(a) \geq 0. \quad (3.29e)$$

As in Section 3.2,  $S$  is the social security. It represents the money transfer from old to young by bequests. In this alternative form of the model, there is no tax money to be distributed.

$$S \int_0^T Q(a) da = \int_0^T Q(a) h(a) B(a) da. \quad (3.30)$$

The corresponding Hamiltonian and Lagrangian are

$$\begin{aligned} \mathcal{H} = \mathcal{H}(a, A, B, c, d, \lambda_1, \lambda_2, \lambda_0) &= \lambda_0 Q(a) u \left( a, c(a), \frac{B(a)}{K(a)}, D(a) \right) \\ &+ \lambda_1(a) d(a) + \lambda_2(a) [we(a) + rB(a) + (r + h(a))A(a) - c(a) - d(a) + S], \end{aligned} \quad (3.31)$$

$$\begin{aligned} \mathcal{L}(a, A, B, c, d, \lambda_1, \lambda_2, \lambda_0, \mu, v_1, v_2) = & \quad (3.32) \\ \mathcal{H}(a, A, B, c, d, \lambda_1, \lambda_2, \lambda_0) + \mu c(a) + v_1(a)A(a) + v_2(a)B(a). \end{aligned}$$

The first order necessary optimality conditions are:

$$\mathcal{L}_c = \lambda_0 Q(a) u_c \left( a, c(a), \frac{B(a)}{K(a)}, D(a) \right) - \lambda_2(a) + \mu(a) = 0, \quad (3.33a)$$

$$\mathcal{L}_d = \lambda_1(a) - \lambda_2(a) = 0, \quad (3.33b)$$

$$\dot{\lambda}_1(a) = \rho \lambda_1(a) - \mathcal{L}_A = \rho \lambda_1(a) - \lambda_2(a)(r + h(a)) - v_1(a), \quad (3.33c)$$

$$\dot{\lambda}_2(a) = \rho \lambda_2(a) - \mathcal{L}_B = \rho \lambda_2(a) - u_B \left( a, c(a), \frac{B(a)}{K(a)}, D(a) \right) - \lambda_2(a)r - v_2(a), \quad (3.33d)$$

Again, as in Section 3.2, we obtain that  $\lambda_1(a) = \lambda_2(a)$  which is due to the missing transaction costs. Therefore we denote it by  $\lambda(a)$  and the differential equations are easier to solve:

$$\lambda(a) = \lambda(0) e^{(\rho-r)a} \frac{Q(a)}{Q(0)} - e^{(\rho-r)a} \frac{Q(a)}{Q(0)} \int_0^a v_1(s) e^{-(\rho-r)s} \frac{Q(0)}{Q(s)} ds, \quad (3.34a)$$

$$\lambda(a) = \lambda(0) e^{(\rho-r)a} - e^{(\rho-r)a} \int_0^a [v_2(s) + u_B(s)] e^{-(\rho-r)s} ds, \quad (3.34b)$$

where  $u_B(s)$  is short for  $u_B \left( s, c(s), \frac{B(s)}{K(s)}, D(s) \right)$ .

Taking the derivative in (3.33b), using (3.34a) and (3.34b) we obtain

$$\begin{aligned} (v_2(a) - v_1(a)) &= -u_B(a) + h(a)\lambda(a) \\ &= -u_B(a) + e^{(\rho-r)a} h(a) (\lambda(0) - \int_0^a e^{-(\rho-r)s} [u_B(s) + v_2(s)] ds). \end{aligned} \quad (3.35)$$

As in Section 3.2,  $\lambda(a) := \lambda_1(a) = \lambda_2(a)$  holds because there are no costs for changes in the portfolio. Therefore the utility of additional unit of money is equal all the time. Unlike in the other model, now it is possible that both assets are held in a positive amount and therefore  $v_1(a)$  and  $v_2(a)$  might be zero at the same time. There are four possible cases:  $v_1 > 0, v_2 > 0$ ;  $v_1 > 0, v_2 = 0$ ;  $v_1 = 0, v_2 > 0$ , or  $v_1 = v_2 = 0$ , which will now be considered in this order.

From  $v_1(a) > 0$  and  $v_2(a) > 0$  it follows that  $A(a) = 0$  and  $B(a) = 0$ . Except for  $a = 0$  and  $a = T$  this will not happen as under the assumption  $r > \rho$  wealth is positive all the time. So this case cannot happen.

If  $v_2(a) = 0$  but  $v_1(a) > 0$ , corresponding to  $A(a) = 0$  and  $B(a) > 0$ , the fraction of money saved in bonds is one and therefore, by the requirements on  $u$ ,  $u_B(a) = 0$ . Inserting this in Equation (3.35) we obtain that  $0 > -v_1(a) = h(a)\lambda(a)$ . However,  $h(a)$  and  $\lambda(a)$  are both positive functions, so there is no solution for this equation. So under the assumption  $u_B(a, c, 1, D) = 0$  it will never be optimal to hold only bonds.

One of the two remaining possibilities is  $v_2(a) > 0$  and  $v_1(a) = 0$ , so  $A(a) > 0$  and  $B(a) = 0$ . The equation above gives  $v_2(a) = -u_B(a, c, 0, D) + h(a)\lambda(a)$ . If the right hand side is positive, which is equivalent to  $u_B(a, c, 0, D) < h(a)\lambda(a)$ , this equation can be fulfilled. When considering the assumptions on  $u$ , this can only happen when dependency at time  $a$  is low enough. As long as the utility from bonds because of the dependency is less than the additional utility derived from the higher return on annuities, agents will only invest in annuities.

The last case is that the agent holds annuities as well as bonds in positive amount, so  $v_1(a) = v_2(a) = 0$ . The equation which has to be fulfilled is

$$u_B(a, c, \frac{B}{K}, D) = h(a)\lambda(a). \quad (3.36)$$

This is an equation in  $B$  which has a solution according to the intermediate value theorem if  $u_B(a, c, 0, D) > h(a)\lambda(a)$ ,  $u_B(a, c, 1, D) = 0$ , and  $u_B$  is continuous. So the marginal utilities from additional interest,  $h(a)\lambda(a)$ , and from higher bequests,  $u_B(a, c, \frac{B}{K}, D)$ , have to be equal.

Since  $v_1(a) = 0$  in all possible cases,  $\lambda$  is easily calculated by (3.34a)

$$\lambda(a) = \lambda(0)e^{(\rho-r)a} \frac{Q(a)}{Q(0)}. \quad (3.37)$$

A plausible shape of the dependence function is, that it is increasing in  $[0, \tilde{a}]$  and decreasing in  $[\tilde{a}, T]$ , with the maximum large enough, so that  $u_B(a, c, 0, D) >$

$h(a)\lambda(a)$  (for an explicit function see Section 3.3.2). In this case, there will be two times  $a_1 \in [0, \tilde{a})$  and  $a_2 \in (\tilde{a}, T]$ , when  $u_B(a, c, 0, D) = h(a)\lambda(a)$ .

The corresponding trajectory satisfies that agents hold at first only annuities since they yield a higher return than bonds and nobody is dependent from them, so that they have a motivation to invest in bonds. As dependency increases over time, there is a point when it is large enough, so that disregarding the bequest-motive would decrease utility more than the additional interest of annuities increases it. Then agents invest a share of their money in bonds, but never all of it, because this case is not optimal as derived above. When growing older, children might leave home and become independent from parents and so the dependence decreases and secondly the additional interest on annuities  $h(a)$  increases, so that investing in annuities is again better than holding bonds.

### 3.3.1 Multiplicative Utility from Dependency

For further calculations, the utility function has to be specified in more details. In this section it is assumed that utility from dependency is a multiplicative factor to the utility from consumption, so

$$u\left(c(a), \frac{B(a)}{K(a)}, D(a)\right) = u(c(a))\hat{u}\left(\frac{B(a)}{K(a)}, D(a)\right). \quad (3.38)$$

The notation above is not clear without ambiguity:  $u(c(a))$  denotes a CRRA utility function as in Section 3.2.5 whereas  $u\left(c(a), \frac{B(a)}{K(a)}, D(a)\right)$  denotes the utility function used in (3.28). The arguments will be explicitly written all the time to exclude confusion.

The factor  $\hat{u}$  shall be close to zero in the case that someone depends on the agent but he does not acquire wealth to bequeath. It shall be approximately one, if the agent acquires enough assets,  $\frac{B(a)}{K(a)} > D(a)$ , to bequeath. There shall be no benefit of possessing too many bonds, so for  $\frac{B(a)}{K(a)} \gg D(a)$ ,  $\hat{u} \approx 1$  shall hold.

The following simple function exhibits all these properties:

$$\hat{u} = 1 - \kappa D(a) \left(1 - \frac{B(a)}{K(a)}\right) \left(D(a) - \frac{B(a)}{K(a)}\right), \quad \kappa \in (0, 1]. \quad (3.39)$$

The reduction of utility is proportional to the dependency  $D$ , to the difference

of  $D$  and  $\frac{B}{K}$  (because if the fraction is close to  $D$ , then the conscience is salved) and proportional to  $A = 1 - \frac{B}{K}$  (because the more annuities the worse).

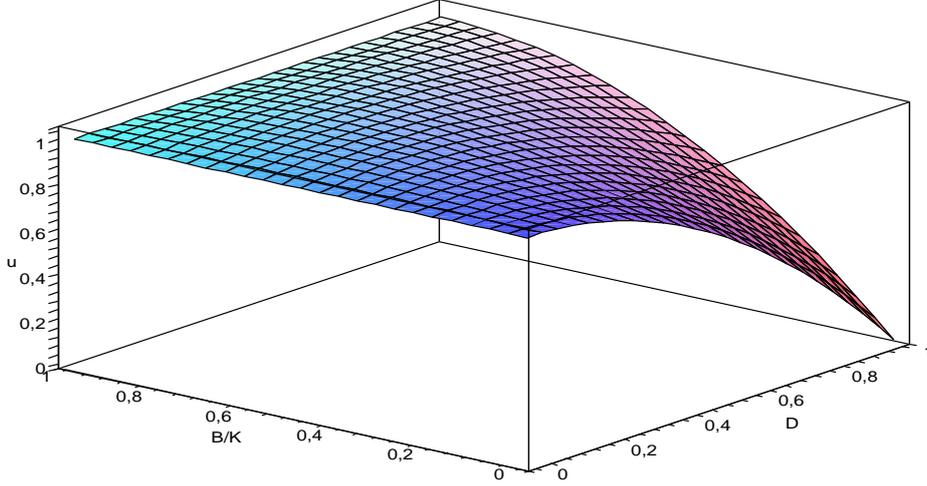


Figure 3.12: Shape of the utility reduction function

$$\hat{u} = 1 - \kappa D(a) \left(1 - \frac{B(a)}{K(a)}\right) \left(D(a) - \frac{B(a)}{K(a)}\right), \quad \kappa = 1$$

The shape of  $\hat{u}$  can be seen in Figure 3.3.1. The derivative with respect to  $B$  is  $\tilde{u}_B = u(c)\hat{u}_B = u(c)\frac{\kappa D}{K}\left[1 + D - \frac{2B}{K}\right]$ . The term  $\hat{u}_B(c(a), 0, D(a))$  determines which of the two cases mentioned in the previous section happens, so whether bonds are held or not. If  $u(c(a))\frac{\kappa D(a)}{K(a)}\left[1 + D(a)\right] > \lambda(a)h(a)$ , then bonds are held in a positive amount at time  $a$ . From equation (3.36) the following optimal fraction of investment in bonds is obtained

$$\frac{B(a)}{K(a)} = \frac{1}{2} \left(1 + D(a) - \frac{h(a)\lambda(a)K(a)}{\kappa D(a)u(c(a))}\right). \quad (3.40)$$

Inserting this into (3.33a), an implicit formula for optimal consumption is obtained:

$$\begin{aligned} \frac{\lambda}{Q} = \tilde{u}_c = u_c & \left(1 - \kappa D \left[1 - \frac{1}{2} \left(1 + D - \frac{h\lambda K}{\kappa D u}\right)\right] \left[D - \frac{1}{2} \left(1 + D - \frac{h\lambda K}{\kappa D u}\right)\right]\right) \\ & = u_c \left(1 - \frac{\kappa D}{4} \left[(1 - D) + \frac{h\lambda K}{\kappa D u}\right] \left[-(1 - D) + \frac{h\lambda K}{\kappa D u}\right]\right) \\ & = u_c \left(1 + \frac{\kappa D(1 - D)^2}{4} - \frac{(h\lambda K)^2}{4\kappa D u^2}\right). \end{aligned}$$

For CRRA utility functions, the equation can be simplified:

$$u^2 u_c \left( 1 + \frac{\kappa D(1-D)^2}{4} \right) - u_c \frac{(h\lambda K)^2}{4\kappa D} - \frac{u^2 \lambda}{Q} = 0$$

$$\frac{c(a)^{2(1-\sigma)}}{(1-\sigma)^2} \left( 1 + \frac{\kappa D(1-D)^2}{4} \right) - \frac{\lambda(a)}{Q(a)} \frac{c(a)^{2-\sigma}}{(1-\sigma)^2} - \frac{(h(a)K(a)\lambda(a))^2}{4\kappa D(a)} = 0. \quad (3.41)$$

The solution of this equation is denoted by  $c(a, K)$ , where  $K$  is emphasised because it is the only remaining unknown, because  $D(a), h(a), Q(a)$  and  $\lambda(a)$  (except for  $\lambda(0)$ ) are known functions. The evolution of the capital stock is described by following the differential equation:

$$\dot{K}(a) = we(a) + S - c(a, K) + rB(a) + (r + h(a))A(a) \quad (3.42)$$

$$= we(a) + S - c(a, K) + rK(a) + h(a)K(a) \left[ \frac{(1-D)}{2} + \frac{h(a)\lambda(a)K(a)}{2\kappa u(c(a, K))D(a)} \right]. \quad (3.43)$$

All money yields the return of  $r$  and the capital invested in annuities ( $K(1 - \frac{B}{K})$ ) gives the additional return  $h(a)$ .

This equation is not explicitly solvable because  $c(a, K)$  is nonlinear in  $K$ . Neither can  $K(T)$  be determined by integration, so an explicit formula for  $\lambda(0)$  is hard or even impossible to derive.

For  $\sigma = 2$  an explicit solution for  $c(a)$  can be derived:

$$c^2(a) = \frac{1 + \frac{D(a)(1-D(a))^2}{4}}{\frac{\lambda(a)}{Q(a)} + \frac{(h(a)K(a)\lambda(a))^2}{4\kappa D(a)}}. \quad (3.44)$$

Next, a realistic dependency function will be determined.

### 3.3.2 A Dependency function

Parents want to take care of their children as long as they are not able to earn their own money. The rate at which children become independent is denoted by  $g(a)$  and it is assumed that children, or rather teenagers, start becoming independent at age 16 and all children are autonomous at age 25. This function can

be interpreted as fraction of children who become independent because they start working and move out, as well as a measure of how dependent children are. If the child is in general able to earn his own money but is still studying at university, leaving bequests is not so important compared to the situation in which a child is not yet able to work. So the dependency function shall be proportional to the number of children times the dependency of these children.

Let  $n(a, b)$  be the number of dependent children at age  $b$  of an  $a$  years old parent. When parents grow older, children grow older with the same rate. One child grows up at the rate  $g(b)$ , so multiplied with the number of children this gives the total change in dependency. This justifies the following equation  $n_a + n_b = -g(b)n(a, b)$ . The initial conditions are that a baby has itself no children ( $n(0, b) = 0$ ), and the number of babies born by people of a certain age ( $n(a, 0) = \phi(a)$ ). A solution for this equation is  $n(a, b) = \phi(a - b)e^{-\int_0^b g(s)ds}$  as can be seen by derivation.

As an estimation of the fertility rate, the German fertility rate in the years 2001-2009<sup>6</sup> was approximated by a piecewise linear function. In order to meet the birth rate in this model, the fertility rate was scaled so that at each time point the total number of births equals 1. People younger than 20 do not give birth to children in this model. Fathers are assumed to be of the same age when the child is born as the mothers, so the dependency rate is equal for both parents.

$$\phi(a) = \begin{cases} \frac{4}{625}a + \frac{2}{125} & a \in [0, 10], \\ \frac{-4}{625}a + \frac{18}{125} & a \in [10, 20], \\ \frac{-2}{625}a + \frac{2}{25} & a \in [20, 25], \\ 0 & a \geq 25. \end{cases} \quad (3.45)$$

For the rate of children becoming independent the following function is assumed:  $g(s) = e^{\frac{1}{25-x}} - e^{\frac{1}{25-16}}$ ,  $s \in (16, 25)$ ,  $g(s) = 0$ , otherwise. Please note that  $s$  is the real age,  $a = s - 20$ . This is because it is the first only time in this thesis that the youth has to be considered. This function has a pole at 25, so  $\int_0^{25} g(s)ds = \infty$  and therefore fulfils that all children are independent at age 25. With this function, no child is independent before 16, at age 20 approximately

<sup>6</sup>Data source: Statistisches Bundesamt Deutschland <http://www.destatis.de/>

15% of the children are independent, at age 22 40% are independent and at age 24 only 16% are still dependent.

The dependency is then assumed to be equal to the number of dependent children at age  $a$ ,  $D(a) = \phi_0 \int_0^a n(a, b) db$ . It fulfils  $D(a) \in [0, 1], \forall a \in [0, T]$  and is plotted together with the fertility function and the rate of becoming independent in Figure 3.3.2.

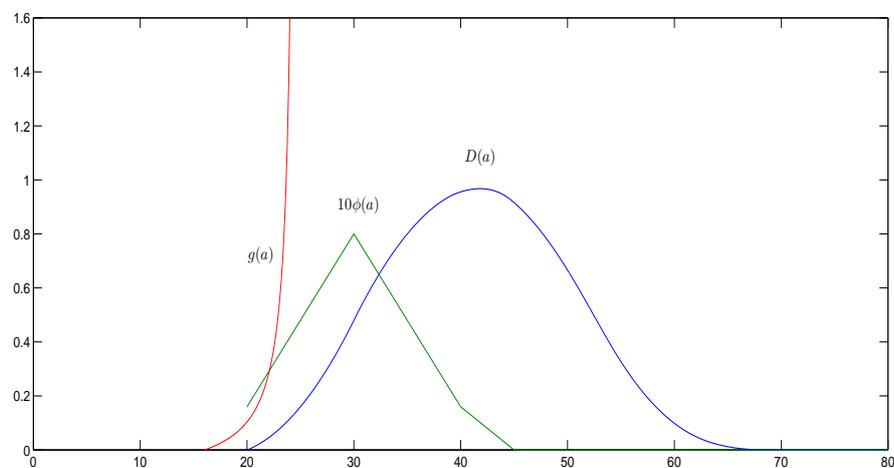


Figure 3.13: Dependency function  $D(a)$ , fertility rate  $\phi(a)$  and rate of becoming independent  $g(a)$

Real age on x-axis, so the plotted functions are  $D(x - 20)$ ,  $\phi(x - 20)$  and  $g(x)$

More complex dependency functions are possible, for example grandchildren or partners might be included. However this is beyond the scope of this work. Already with this simple function the optimal behaviour is to possess bonds and annuities for most of the lifetime, which is in focus of the next section.

### 3.3.3 Model versus Reality

The dependency function is hump shaped and is a crucial factor for the determination of the optimal fraction invested in bonds. Therefore the optimal fraction over time is also supposed to be hump shaped and decreasing in  $\kappa$ . When solving the model numerically with  $\kappa = 1$  the maximal fraction is invested in bonds ap-

proximately at the same age as dependency function reaches its maximum. The value at its peak is almost 1. The percentage invested in bonds increases with age until the peak of the dependency function and decreases afterwards, however after 60 with an even faster rate and at about 62 years no money is invested in bonds any more although the dependency is not yet zero. The effect of increasing capital  $K(a)$  and the mortality rate  $h(a)$  is more severe than the increase of consumption  $c(a)$  and decrease of  $\lambda(a)$ , so that Equation (3.36) cannot be fulfilled and the optimal investment is complete annuitisation, see Figure 3.3.3. This means that the additional interest from annuities is bigger than the increase in utility by salving one's bad conscience.

This does not yet explain the low annuitisation rate in society but one assumption in the model is that there are no switching costs, any money invested in annuities can be without any costs transferred to bonds. It is realistic, that bonds can be sold anytime with hardly any costs on the stock market and additional contributions can be made to an existing annuity contract, however the other way round it is not true. Withdrawing money from an annuity contract is in general very costly if it is not explicitly included in the contract. Towards the end of life, the assumption to withdraw money is more realistic because then some part of the savings can be invested in contracts which pay the annuity only for some years and not until the end of the life and so the total amount invested in annuities is reduced.

The model cannot give an answer to the question what is optimal under the condition that withdrawing money from an annuity contract is costly, but it is plausible that before the peak in the dependency function occurs the annuitisation would be somewhere in between the annuitisation at the peak and the annuitisation before. With  $\kappa \in [0.01, 0.1]$  it is possible to obtain an optimal fraction invested in annuities in younger years (agents younger than approximately 50 years) similar to the approximately 20% observed by Sommer (2005). This is a step towards an explanation why young people invest only a small fraction of their wealth in annuities. Further it has to be mentioned that state pensions are not in all surveys considered as annuities. This would result in a higher annuitisation rate before pension age.

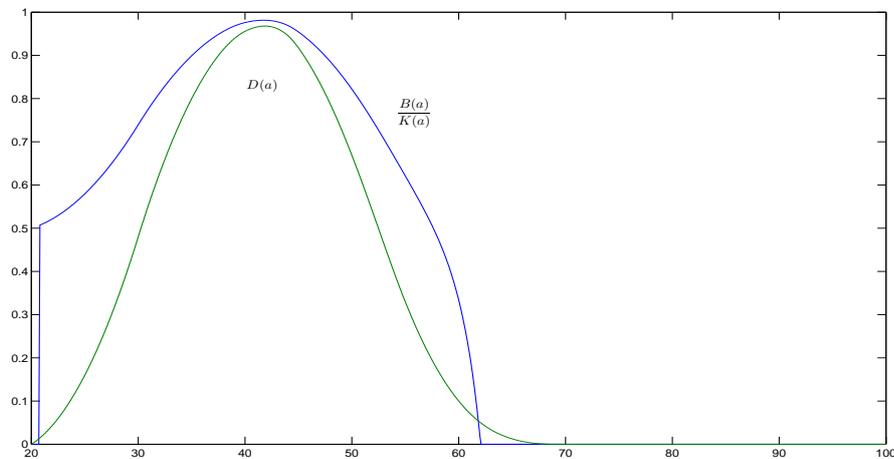


Figure 3.14: Dependency function  $D(a)$ , and optimal fraction invested in bonds  $\frac{B(a)}{K(a)}$ ,  $\kappa = 1$

The result is not very sensitive to  $\kappa$ . The optimal fraction for  $\kappa = 0.1$  is plotted in Figure 3.3.3. A decrease in  $\kappa$  shifts the peak slightly to the left and complete annuitisation is adopted earlier than compared to  $\kappa = 1$ . The peak is also slightly less pronounced but the difference is small. Even for  $\kappa = 0.01$ , the peak fulfils ( $a = 18$  corresponds to 38 years)  $\frac{B(18)}{K(18)} \approx 0.8$  and  $\frac{B(a)}{K(a)} = 0$  for  $a \geq 28$  (48 years).

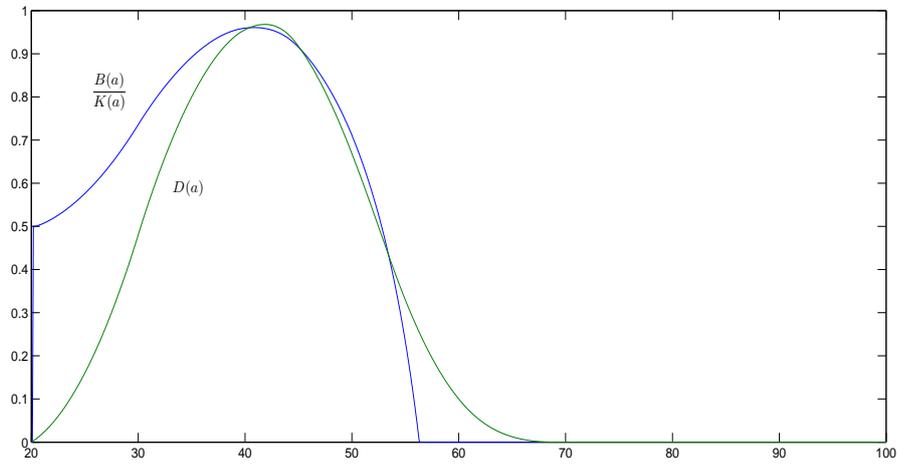


Figure 3.15: Dependency function  $D(a)$ , and optimal fraction invested in bonds  $\frac{B(a)}{K(a)}$ ,  $\kappa = 0.1$

# Appendix A

## Matlab Code

Part of the thesis was to implement a solver in Matlab. The basic structure of the solver is presented in Section 3.2.1, the complete code is printed here. It is also electronically available from the author upon request<sup>1</sup>. In the comments between the code fragments, references to the formulas in this work are given.

The function *start* executed the calculations for a set of parameters. It does a binary search for  $K$  and  $S$  in order to find the steady state as described at the end of Section 3.2.1.

```
function [] = start()

%-----specification of parameters-----
step = 0.01;
S = 0.01;
rho = 0.04;
alphaK = 0.5;
ΔK = 0.02;
sigma = 1;
R = 65-20;
K0 = 0;
eta = 0.1;

K = 10000;
d = 10000;
%-----iterating for Knew = K, Snew = S-----
for i=1:15
```

---

<sup>1</sup>bernhard.skritek@tuwien.ac.at

```

[ r,w,Knew,Snew,Kovertime,e10,c,atilde,util ] = ...
    solvemodel( step,K,S,rho,eta,alphaK,deltaK,sigma,R,K0 );
if Knew < K
    K = K-d/(2^i);
    S = Snew;
else
    K = K+d/(2^i);
    S = Snew;
end
end
end

```

The function *solvemodel* is basically the inner core because it solves the model for a given capital stock and social security as described in Section 3.2.

```

function [ r,w,Knew,Snew,Kovertime,e10,c,atilde,util,L ] = ...
solvemodel( step,K,S,rho,eta,alphaK,deltaK,sigma,R,K0 )

if nargin < 10 %if no initial wealth is given, take it as zero
    K0 = 0;
end

%-----parameter-----
n = 80/step + 1; %number of steps of length stepsize
x = 20:step:100; %vector of the discrete steps
[h,Q] = gethQ(step); %mortality and survival function
e = gete(step,R); %labour efficiency

pop = simpson(step,Q); %population size
N = simpson(step,Q.*e); %total labour efficiency

% solves for a given initial capital stock and social security rate

r = alphaK*(N/K)^(1-alphaK)-deltaK;
w = (1-alphaK)*(K/N)^alphaK;

atilde = getatilde( step,r,eta,h ); %age when investment changes
Lp = getLp( step,atilde,rho,r,eta,Q ); %Lambda discretized
e10 = gete10( step,atilde,r,rho,w,e,S,Q,sigma,eta,h,Lp,K0 );
L = e10.*Lp; %Lambda
c = ( Q./L )^(1/sigma); %consumption

%-----solve differential equations for A and B-----
A = 0; B = 0; T = 80/step + 1; %reset

A(n-atilde+1)=0; %Heun correction backwards due to stability
for i=n:-1:(atilde+1)

```

```

    par = (1-eta)*(r+h(i));
    val1 = w*e(i)-c(i)+S;
    val2 = w*e(i-1)-c(i-1)+S;
    A(i-atilde) = heun( -step ,A(i-atilde+1),val1 , val2 , par );
end

B(atilde)=A(1);      %if a≤atilde Bonds are held
par = r;
for i=atilde:-1:2
    val1 = w*e(i)-c(i)+S;
    val2 = w*e(i-1)-c(i-1)+S;
    B(i-1) = heun( -step ,B(i) , val1 , val2 , par );
end

Kovertime = [B, A(2:length(A))];

%—— calculate Knew——
Aeff=0; Beff=0;
Aeff = Q(atilde:n).*A; Beff = Q(1:atilde).*B;      %living people times wealth
Knew = simpson( step , Beff ) + simpson ( step , Aeff ); %integrate

%—— calculate S——
Bsec = Q(1:atilde).*h(1:atilde).*B; Asec = Q(atilde:n).*A*eta;
Sec = [Bsec , Asec(2:length(Asec))]; Snew = simpson( step , Sec ) / pop ;

%—— calculate utility——
for i = 1:n
    discount(i) = exp(-rho*(i-1)*step);
end
if sigma == 1
    util = simpson(step , discount.*log(c).*Q);
else
    util = simpson(step , discount.*(c^(1-sigma))/(1-sigma)).*Q);

```

The function *gethQ* returns the functions  $h(a)$  and  $Q(a)$ . The first is linearly interpolated from the life table for 2000 from Statistik Austria, the latter by integration of  $h(a)$  (see Sections 3.1 and 3.2.1). The initial value for  $Q(a)$  is also from the life table.

```

function [ h,Q ] = gethQ( step )
hsource=[0.0003234,0.0003082, ...
0.0002973,0.0002831,0.0002698,0.0002695,0.0002799,0.0002906,0.0003004, ...
0.0003092,0.0003225,0.0003531,0.0003966,0.0004510,0.0005000,0.0005394, ...
0.0005850,0.0006533,0.0007412,0.0008412,0.0009502,0.0010633,0.0011789, ...
0.0012999,0.0014259,0.0015650,0.0017304,0.0019207,0.0021238,0.0023362, ...

```

```

0.0025657,0.0028073,0.0030534,0.0032958,0.0035351,0.0037768,0.0040245, ...
0.0042739,0.0045302,0.0048119,0.0051423,0.0055330,0.0059968,0.0065535, ...
0.0072130,0.0079820,0.0088790,0.0099125,0.0110837,0.0124070,0.0139145, ...
0.0156392,0.0176233,0.0199112,0.0225557,0.0256061,0.0291105,0.0331236, ...
0.0377085,0.0429438,0.0489131,0.0557097,0.0634426,0.0722388,0.0822385, ...
0.0936000,0.1065040,0.1204218,0.1353127,0.1517418,0.1698118,0.1895022, ...
0.2108727,0.2338888,0.2583034,0.2836835,0.3095368,0.3357864,0.3624251, ...
0.3893823,1.0000000];

```

```

h(1)=hsource(1); n=80/step+1; h(n)=hsource(81);
for i= 2 : (n-1)
    h(i) = h(i-1) + step*( hsource( ceil((i-1)*step+1) ) ...
        - hsource( ceil((i-1)*step) ) );
end

```

```

Q(1) = 0.993009203;
for i= 2:n
    Q(i) = Q(i-1)* exp( -(h(i-1)+h(i))*step );
end

```

The switching point from bonds to annuities is under the assumptions made the unique time point when equation (3.17) is fulfilled and can therefore be determined by a simple binary search:

```

function [ atilde ] = getatilde( step,r,eta,h )

n = 80/step + 1;
atilde=1;
for i=n:-1:1
    if ( ( r-(1-eta)*(r+h(i)) ) ≥ 0 )
        atilde = i;
        return
    end
end
end

```

At first,  $\lambda$  is calculated independently from  $\lambda(0)$ , so actually the function *getLp* calculates  $L = \frac{\lambda}{\lambda(0)}$ .

```

function [ L ] = getLp( step , atilde , rho , r , eta , Q )

n = 80/step + 1;

```

```

for i = 1:atilde %a=(i-1)*step, L(i=1)=L(a=0)
    L(i) = exp( (rho-r)*(i-1)*step );
end
for i = (atilde+1):n
    L(i) = exp(-eta*r*(atilde-1)*step) * ( Q(i)/Q(atilde) )^(1-eta) ...
        * exp( (rho-(1-eta)*r)*(i-1)*step );
end

```

The next step is to calculate  $\lambda(0)$  by equation (3.24b).

```

function [ e10 ] = gete10( step, atilde, r, rho, w, e, S, Q, sigma, eta, h, Lp, K0 )

n = 80/step + 1;
T = 80/step;
atc = (atilde-1)*step; %atilde continuous value

if atilde ≠ 1 %if at least some investment in bonds
    for i = 1:atilde
        s = (i-1)*step; %i-1)*step=a, e(i=1) = e(a=0)
        D1(i) = exp(r*(atc-s)); %discount factor
        int1(i) = D1(i) * ( exp( (r-rho)*s ) * Q(i) )^(1/sigma);
        int2(i) = D1(i) * ( w*e(i)+S );
    end
    int1cinc = cumtrapz(int1.*step);
    int2cinc = cumtrapz(int2.*step);

    if atilde ≠ n
        for i = atilde:n
            s = (i-1)*step; %see above
            D2(i) = exp( (1-eta)*r*(atc-s) ) * ( Q(i)/Q(atilde) )^(1-eta); %discount factor
            int3(i) = D2(i) * ( (Q(atilde)/Q(i) )^(1-eta) * exp(eta*r*atc) ...
                * exp( ((1-eta)*r-rho)*s ) * Q(i) )^(1/sigma);
            int4(i) = D2(i) * ( w*e(i)+S );
        end
        int3cinc = cumtrapz(int3.*step);
        int4cinc = cumtrapz(int4.*step);
    end
else %if never invested in bonds
    for i = atilde:n
        s = (i-1)*step;
        D2(i) = exp( (1-eta)*r*(atc-s) ) * ( Q(i)/Q(atilde) )^(1-eta);
        int3(i) = D2(i) * ( (Q(atilde)/Q(i) )^(1-eta) * exp(eta*r*atc) * ...
            exp( ((1-eta)*r-rho)*s ) * Q(i) )^(1/sigma);
        int4(i) = D2(i) * ( w*e(i)+S );
    end
    int1int = 0;
    int2int = 0;
end

```

```

    int3int = cumtrapz(int3.*step);
    int4int = cumtrapz(int4.*step);
end

el0= ( (int1int(length(int1int)) + int3int(length(int3int))) / ...
    (exp(r*atc)*K0 + int2int(length(int2int))+int4int(length(int4int))) )^(sigma);

```

Solving the differential equation is done by Heun's method.

```

function [ xneu ] = heun( step ,x, val1 ,val2 ,par )

%—— prediction ——
xp = x + step*(val1 + par*x);

%—— correction ——
xneu = x/2 + 1/2*( xp + step*(val2 + par*xp) );

```

Integration is done according to Simpson's rule.

```

function [ int ] = simpson( step ,f )
%there is a index shift compared with the original formula
n=length(f);
sum1=0; sum2=0;
for i = 2:(n-1)
    if (mod(i,2) == 0)
        sum1 = sum1 + f(i);
    else
        sum2 = sum2 + f(i);
    end
end
int = ( f(1) +f(n) +4*sum1 +2*sum2 ) *step /3;

```

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