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# On the Relationship Between Continuous- and Discrete-Time Control Systems

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*This paper is dedicated to the 70-th anniversary of Gustav Feichtinger*

**Abstract**

Building on previous results of the author this paper presents two new error estimates for the reachable set of an affine control system if only piece-wise constant admissible controls on a uniform mesh are used instead of all measurable admissible controls. It is natural to expect that the resulting "shrinkage" of the reachable set is of the order of the mesh size. In this paper it is proved that under certain reasonable conditions the error is of higher than first order.

**Keywords:** control systems, discretization, error analysis

## 1 Introduction

Control theory plays a substantial role in the mathematical economics and in the investigation of typical dynamic problems of operations research such as optimal production planning, exploitation of renewable resources, labour allocation, ets. Both continuous- and discrete-time control models are used in the literature, and the debates about the advantages and shortcomings of these two types of models are persistently present in meetings and publications.

In many practical problems discrete-time control models seem to be more appropriate at least for the following reasons: (i) state measurements become available at discrete-time instances; (ii) control decisions are taken at discrete times. The main advantage of the continuous-time models is that one can employ more conveniently the technique of the classical and the modern mathematical analysis. In addition, in many areas the present computing and communication facilities are so fast that monitoring and decision-implementation take very short times. The appropriate time-scale is always relative, therefore continuous-time models can be relevant even if the observation/decision time-quantum is large, provided that the dynamics is sufficiently slow.

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The relation between continuous-time dynamic systems described by differential equations and their discrete-time versions (involving a small time-step) is profoundly investigated starting from the beginning of the 18-th century. The approximation theory for differential equations by discrete-time dynamic equations and the corresponding numerical methods are still a hot topic, although they have already been rather well developed. In our opinion this is not the case with discrete-time approximations of control systems. The principle reason is that a good approximation requires good properties of the solution, while in the case of a controlled system such properties often cannot be ensured a priori. Sometimes using discontinuous or highly oscillating controls brings advantage and, respectively, creates troubles.

In this paper we address only the following question: given a continuous-time control system, what, quantitatively, is the disadvantage of using only piece-wise constant controls on a given time-mesh instead of all admissible controls. On one hand using piece-wise constant controls is a common engineering practice, which makes the question meaningful. On the other hand, restricting the class of admissible controls to piece-wise constant ones opens the door for complete discretization of the control system by using single-step discretization schemes and makes the standard error analysis for ODEs applicable.

As it is well-known, using richer finitely parameterized sets of admissible controls (say, ones having one free jump in every mesh interval) may bring qualitative advantage concerning the approximation rate (Doitchinov and Veliov 1993; Baier and Lempio 1994; Krastanov and Veliov 2010). Piece-wise linear controls are also widely used in the context of discretization of optimal control problems (Schwartz and Polak 1996; Dontchev et al. 2000; Hager 2000), although they may be advantageous only under regularity conditions for the optimal control. We do not touch these richer classes of "discrete" admissible controls in this paper. Moreover, to avoid (sometimes substantial) technical complications we restrict the considerations to systems that are affine with respect to the control.

The main message of the paper is that the class of piece-wise constant controls on the uniform time-mesh with step $h$ on a finite time interval is (somewhat surprisingly) capable to provide higher than first order approximation (relative to $h$) of the reachable set of affine commutative control systems, together with the usual first approximation of the trajectories. Moreover, we introduce and investigate the notion of *information pattern* of the approximation and show that the information pattern of the higher than first order approximations is *anticipative*.

The time horizon is finite in the present paper. The important case of infinite horizon (see e.g. Caulkins et al. 2005; Grass et al. 2008) is rather challenging and almost no results are available.

## 2 The approximation problem and some known results

We consider an affine control system

$$\dot{x}(t) = f_0(t, x(t)) + \sum_{i=1}^{m} f_i(t, x(t))u_i(t), \qquad x(0) = x^0 \in \mathbf{R}^n, \qquad t \in [0, T], \tag{1}$$

where $u \in \mathcal{U} = L_\infty([0,T] \mapsto U)$, $U \subset \mathbf{R}^m$ is convex and compact. For $u \in \mathcal{U}$ we denote by $x[u]$ the solution of (1) that corresponds to $u$ (assuming existence and uniqueness). Control theory and set-membership estimation theory raise two main problems: (i) approximate the set of trajectories, $\mathcal{X} = \{x[u] : \; u \in \mathcal{U}\}$, of (1); (ii) approximate the reachable set, $R = \{x[u](T) : \; u \in \mathcal{U}\}$, of (1).

Since the set of admissible controls $\mathcal{U}$ contains rather irregular functions[1] it is natural to split the approximation problems of (1) into two parts:

(P1) Replace the set of admissible controls $\mathcal{U}$ by a finitely parameterized subset $\mathcal{V}_N$ consisting only of functions $u$ for which (1) can be discretized efficiently;
(P2) Apply a discretization scheme for solving (1) for $u \in \mathcal{V}_N$.

The requirement that $\mathcal{V}_N$ is a finitely parameterizable set (say, with a "degree of freedom" proportional to $N$) is needed to make the approximation "computable". Moreover, for each $u \in \mathcal{V}_N$ equation (1) should be well discretizable, that is, the functions from $\mathcal{V}_N$ should be sufficiently regular.[2] Then the error analysis of the discretization can be carried out in the usual way as for differential equations. Therefore, as mentioned in Introduction we focus on the approximation questions in problem (P1): what is the approximation error if the set of admissible controls $\mathcal{U}$ is replaced with the set

$$\mathcal{V}_N = \{u \in \mathcal{U} : \; u(\cdot) \text{ is constant on each } (t_{k-1}, t_k))\}, \tag{2}$$

where $t_k = kh$, $h = T/N$, $N$ is a natural number. That is, we want to estimate the *uniform error*

$$H_C(\mathcal{X}, \mathcal{X}_N) = \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}_N} \|x[v] - x[u]\|_{C[0,T]} \tag{3}$$

and the *terminal error*

$$H(R, R_N) = \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}_N} |x[v](T) - x[u](T)|, \tag{4}$$

[1] The reachable set $R$ is usually not generated by continuous controls. Even more, control functions of unbounded variation or non-integrable in Riemann sense may generate points of $R$ that are not reachable by other controls, as in Fuller's phenomenon or as in Silin (1981). This creates the difficulty of approximating (1) by discrete-time dynamic systems.

[2] Of course, there is a trade-off in choosing $\mathcal{V}_N$: the larger $\mathcal{V}_N$, the better the approximation to $\mathcal{X}$ and $R$ by controls from $\mathcal{V}_N$; on the other hand, the lower is the accuracy of discretization.

where $\mathcal{X}_N$ and $R_N$ are the set of trajectories and the reachable set corresponding to the set $\mathcal{V}_N$ of admissible controls.

We mention that Problem (P1) makes sense and is studied in the literature also for richer finitely-parameterized classes of admissible controls, see e.g. Baier and Lempio (1994), Doitchinov and Veliov (1993), Ferreti (1997), Krastanov and Veliov (2010). This paper addresses only the simplest and most often used case (2).

Under standard assumptions the mapping $u \longrightarrow x[u]$ is continuous in $L_1$ and since $\mathcal{V}_N$ is compact in the same space, the infimum in (3) and (4) is achieved. Then there exists a mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ such that

$$\sup_{u \in \mathcal{U}} \|x[\pi_N(u)] - x[u]\|_{C[0,T]} = H_C(\mathcal{X}, \mathcal{X}_N), \tag{5}$$

or

$$\sup_{u \in \mathcal{U}} |x[\pi_N(u)](T) - x[u](T)| = H(R, R_N), \tag{6}$$

respectively (the mapping $\pi_N$ needs not be the same in the two equalities). Thus the question of accuracy of approximation can formulated in terms of the mapping $\pi_N$: $\mathcal{V}_N$ provides approximation of order $\alpha$ to $\mathcal{X}$ if there exists a mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ (called further *approximation mapping*) such that

$$\|x[\pi_N(u)] - x[u]\|_{C[0,T]} \le \text{const.}\, N^{-\alpha} \quad \text{for every} \;\; u \in \mathcal{U}.$$

Similarly for the reachable set. This reformulation of the approximation problem has an advantage: one can study the information pattern of the mappings $\pi_N$ that provides a given approximation rate. Namely, we can distinguish the following cases:

**Definition 1** *(i) The mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ is called "local" if for every $k = 0, \ldots, N-1$, and for every $u'$, $u'' \in \mathcal{U}$ with $u'(t) = u''(t)$ on $[t_k, t_{k+1}]$ it holds that $\pi_N(u')(t) = \pi_N(u'')(t)$ on $[t_k, t_{k+1}]$;*
*(ii) The mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ is called "non-anticipative" if for every $k = 1, \ldots, N$, and for every $u'$, $u'' \in \mathcal{U}$ with $u'(t) = u''(t)$ on $[0, t_k]$ it holds that $\pi_N(u')(t) = \pi_N(u'')(t)$ on $[0, t_k]$;*
*(iii) The mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ is called "anticipative" if it is not non-anticipative.*

The above notions are adapted from the theory of differential games (Varaiya-Roxin-Elliot-Kalton strategies). As we shall see, it may happen that a certain order of approximation can be achieved by anticipative approximating mappings $\pi_N$ but cannot be achieved by non-anticipative (resp. local) mappings. The information pattern of $\pi_N$ may play a role for the order of the accuracy.

It is to be stressed that in different problems related to the control system (1) one may need to restrict the choice of the approximation mapping to a prescribed information pattern: local or non-anticipative. This is the case, for example, if one has to simulate a real system modeled by (1) only knowing the current, or the past information about the input $u$. For other problems, say for an optimal open-loop control problem one can freely employ anticipative approximation mappings to pass directly to mathematical programming. Below we recall a few known approximation results in the light of the above concept of information pattern.

One commonly used approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ is defined as

$$\pi_N(u)(t) = \frac{1}{h} \int_{t_{k-1}}^{t_k} u(s)\, \mathrm{d}s \text{ for } t \in (t_{k-1}, t_k). \tag{7}$$

Obviously it is local and even more, it is independent of the specific form of the equation (1).

Let us consider first a linear control system in (1): $\dot{x} = Ax + Bu$. From a general result in Dontchev and Farkhi (1989) it follows that for the *local* approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ defined by (7) ensures

$$\|x[\pi_N(u)] - x[u]\|_{C[0,1]} \le ch \qquad \forall\, u \in \mathcal{U}. \tag{8}$$

In the same time the results in Veliov (1992) and Doitchinov and Veliov (1993) imply that there exists an *anticipative* approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ (which is not explicitly defined in these papers) such that

$$|x[\pi_N(u)](T) - x[u](T)| \le ch^2 \qquad \forall\, u \in \mathcal{U}. \tag{9}$$

We mention that the result holds for an arbitrary convex and compact set $U$, therefore it applies also to the "pathological" examples in Silin (1981) mentioned in footnote 1. A second order approximation as in (9) cannot be achieved by using local approximation mappings.

An important extension is proved in Pietrus and Veliov (2009): there exists an *anticipative* approximation mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ that ensures simultaneously (8) and (9). This result opens the door to error estimates for non-linear systems by local linearization. The approach will be followed in Section 4.

There are only few higher than first order approximation results concerning the non-linear case (1).[3] The first is that in Veliov (1989), where a second order approximation of $\mathcal{X}$ is proved (or an approximation of order 3/2 for a more general form of the right-hand side in (1)) assuming, however, that the set $f(t,x)U := (f_1, \ldots, f_m)U$ is uniformly strongly convex, which is a rather strong assumption for many applications. The implicitly involved approximation mapping $\pi_N$ in this paper is *local*.

[3] Higher than first order approximations to optimal control problems are known. However, most of these results are based on a priori assumption that the optimal control is sufficiently regular (i.e. Lipschitz continuous with the first derivative having bounded variation), see e.g. Dontchev et al. (2000) and Hager (2000). The results recalled or obtained in the present paper are applicable in the optimal control context without such assumptions.

Another group of results concerns the case of commutative affine systems, i.e. such that the Lie brackets $[g_i, g_j]$ are all zero for $i, j \geq 1$. A rather general indirect (variational) estimation of $H(R, R_N)$ in the class $\mathcal{V}_N$ is obtained in Veliov (1997). It allows to obtain a second order estimation of $H(R, R_N)$ provided that the sets $R_N$ have (uniformly) the so called *exterior ball property*. This is done in Section 3.

The last issue we briefly recall is that of approximations using the class $\mathcal{V}_N^{\text{extr}}$ of controls, where

$$\mathcal{V}_N^{\text{extr}} := \{u : [0, T] \mapsto U^{\text{extr}} : \ u(t) \text{ is constant on each } (t_{i-1}, t_i))\}$$

and $U^{\text{extr}}$ is the set of all extreme points of $U$. This issue is important for numerical treatment of optimal control problems for switching systems, see e.g. Sager (2009). The following estimation is obtained, essentially, in Donchev (2001) and Grammel (2003): for the approximating class of controls $\mathcal{V}_N^{\text{extr}}$,

$$H_C(\mathcal{X}, \mathcal{X}_N^{\text{extr}}) \leq Ch^{1/2}. \tag{10}$$

This estimation is proved for more general systems than (1) under Lipschitz continuity of $f$. In Veliov (2003) the author of the present paper conjectured that a first order estimation holds in (10) and proved this in several particular classes of systems. The paper by Pietrus and Veliov (2008) also contains a small contribution in this direction. A substantial progress in proving the conjecture is done in Sager (2009), where however, $U$ is assumed to be a polyhedral set. In all the above contributions the (implicitly or explicitly) involved approximation mapping $\pi_N$ is *non-local* and *non-anticipative*. Also, it is quite clear that local approximation mappings cannot provide even (10). In general, the problem of first order approximation is still open.

# 3 A second order approximation result under "exterior ball property"

In this section we prove that the set of piece-wise constant admissible controls $\mathcal{U}_N$ is powerful enough to ensure a second order approximation to the reachable set of the control system (1), provided that a certain condition known as "exterior ball property" holds (see e.g. Nour et al. 2009 for a recent discussion of this property). In particular, this property is trivially satisfied if the approximating reachable sets $R_N$ are convex, thus the next result extends those in Veliov (1992), Doitchinov and Veliov (1993) devoted to linear systems and Theorem 4.1 in Veliov (1997), where the approximating reachable sets are assumed convex.

We start with a list of assumptions for system (1).

*Assumption 1.* There is a convex and compact set $S \subset \mathbf{R}^n$ such that
(i) the functions $f_i : [0, T] \times S$, $i = 0, \ldots, m$ are differentiable and the first derivatives are Lipschitz continuous;

(ii) for every $u \in \mathcal{U}$ the solution $x[u]$ of (1) exists in the interior of $S$ on $[0, T]$;
(iii) the system is *commutative*: the Lie brackets of the controlled vector fields $f_1, \ldots, f_m$,

$$[f_i, f_j] := \frac{\partial f_i}{\partial x} f_j - \frac{\partial f_j}{\partial x} f_i, \tag{11}$$

are identically equal to zero for every $i, j = 1, \ldots, m$.

Below we shall employ the following reformulation of Theorem 2.2 in Veliov (1997).

**Theorem 1** *Let Assumption 1 be fulfilled. Then there exist a constant $C$ such that for every natural number $N$ and for every function $g : S \mapsto \mathbf{R}$ which is differentiable in the interior of $S$, $\partial g/\partial x$ is bounded by a constant $L_g$ and is Lipschitz continuous with a Lipschitz constant $L'_g$ at each point of $R$, the following estimation holds:*

$$0 \le \inf_{x \in R_N} g(x) - \inf_{x \in R} g(x) \le C \frac{L_g + L'_g}{N^2}. \tag{12}$$

Before continuing with the exterior ball property we briefly discuss the above result.

The mapping $\pi_N$ that ensures second order accuracy of approximation in the above theorem is anticipative.

We mention that the constant $C$ does not depend on the properties of the controls at which $\inf_{x \in R} g(x) = \inf_{u \in \mathcal{U}} g(x[u](T))$ is achieved. As shown in Silin (1981), these controls can be of unbounded variation, non-integrable in Riemann sense and discontinuous almost everywhere (Assumptions 1 does not exclude these possibilities) even for linear systems. We also mention that Assumption 1 (iii) is restrictive for many applications. However, the problem of higher than first order approximations of non-commutative control systems (even using larger finitely parameterized classes of admissible controls) is still open. The result in Krastanov and Veliov (2010) concerning a rather specific class of problems shows that the issue is probably complicated. The last two remarks apply also to the result presented in the next section.

**Definition 2** *The compact set $Q \subset \mathbf{R}^n$ has the exterior $r$-ball property (with a positive real number $r$) if for every $x \in \mathbf{R}^n \setminus Q$ and $y \in \mathcal{P}_Q(x)$*

$$\left( y + r\frac{x-y}{|x-y|} + r\mathring{B} \right) \cap Q = \emptyset,$$

*where $\mathcal{P}_Q(x)$ is the projection of $x$ on $Q$ and $\mathring{B}$ is the unit ball in $\mathbf{R}^n$.*

This means that the exterior of $Q$ is a union of balls of radius $r$ (see Nour et al. 2009) for more details).

**Theorem 2** *In addition to Assumptions 1, let there be $r > 0$ such that for all (sufficiently large) $N$ the sets $R_N$ have the exterior $r$-ball property. Then there exists a constant $C$ such that*

$$H(R, R_N) \leq Ch^2.$$

**Proof.** Denote $\rho_N = H(R, R_N)$. From Dontchev and Farkhi (1989) we know that $\rho_N \longrightarrow 0$. Since $R$ is compact, there exists a point $x_N \in R$ such that $\text{dist}(x_N, R_N) = \rho_N$. We may assume $\rho_N > 0$ since the alternative case is trivial. Let $y_N \in \mathcal{P}_{R_N}(x_N)$. According to the exterior ball property it holds that

$$\left(y_N + rl_N + r\overset{\circ}{I\!B}\right) \cap R_N = \emptyset, \qquad \text{where } \; l_N = \frac{x_N - y_N}{|x_N - y_N|}. \tag{13}$$

For a sufficiently large fixed $N$, so that $\rho_N \leq r/2$, we define the function

$$g_N(x) = |y_N + rl_N - x|.$$

According to (13) there are no points of $R_N$ in the open ball with radius $r$ around $y_N + rl_N$. Then the definition of $\rho_N$ implies that there are no points of $R$ in the open ball with radius $r - \rho_N$ centered at the same point. Hence, for $x \in R$ we have that

$$|y_N + rl_N - x| \geq r - \rho_N \geq r/2.$$

Then the derivative

$$g'(x) = \frac{y_N + rl_N - x}{|y_N + rl_N - x|}$$

exists for $x \in R$ and its norm is $L_g = 1$. Moreover, $g'$ is Lipschitz with some constant $L'_g$ at points $x \in R$, where $L'_g$ depends on $|S|$ and $r$ but not on $N$. Thus the conditions in Theorem 1 are fulfilled and it implies that

$$Ch^2 \geq \inf_{x \in R_N} g(x) \; - \; \inf_{x \in R} g(x),$$

where $C$ is independent of $N$. Due to (13) the first infimum is attained at $x = x_N$, while the second infimum is not larger than $g(x_N)$. Hence,

$$Ch^2 \geq |y_N + rl_N - y_N| - |y_N + rl_N - x_N| = r - (r - \rho_N) = \rho_N.$$

This completes the proof. Q.E.D.

The above theorem has the drawback that the exterior ball property is difficult to check in the non-linear case and even more, it may fail to hold. An example where it fails is provided by the following simple two-dimensional bilinear commutative systems:

$$\dot{x} = u_1 A_1 x + u_2 A_2 x, \quad x \in \mathbf{R}^2, \quad u_1 + u_2 \leq 1, \quad u_i \in [0, 1], \tag{14}$$

with

$$A_1 = \begin{pmatrix} 1 & 1.3\pi \\ -1.3\pi & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.7 & \pi \\ -\pi & -0.7 \end{pmatrix}. \tag{15}$$

The reachable set at time $T = 1$ is plotted on Figure 1. The exterior $r$-ball property fails at one point (whatever is $r > 0$). Since $R_N$ converges to $R$, the exterior $r$-ball property fails also for $R_N$ if $N$ is sufficiently large.

Figure 1: The reachable set in the example (14), (15).

In the next section we consider the general case of a commutative affine system, not relying on the exterior ball property.

# 4 Approximation of a single input system

In this section we consider the affine system (1) in the case $m = 1$:

$$\dot{x} = f_0(x) + f_1(x)u, \quad x(0) = x^0 \in \mathbf{R}^n, \quad u \in [0, 1]. \tag{16}$$

*Assumption 2.* There is an open set $S \subset \mathbf{R}^n$ such that
(i) the functions $f_i : S \mapsto \mathbf{R}^n$, $i = 0, 1$ are time-invariant, twice differentiable and the second derivatives are Lipschitz continuous;
(ii) for every $u \in \mathcal{U}$ the solution $x[u]$ of (1) exists in $S$ on $[0, T]$;

The next theorem extends Theorem 2 in Pietrus and Veliov (2009) for non-linear systems and, in addition, adds the statement of simultaneous first-order approximation of the trajectory bundle.

**Theorem 3** *Let Assumption 2 hold. Then there exists a constant $C$ such that for every $N$ there exists (an* anticipative*) mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ for which*

$$\|x[\pi_N(u)] - x[u]\|_{C[0,T]} \le Ch, \tag{17}$$

$$|x[\pi_N(u)](T) - x[u](T)| \le Ch^{1.5}, \tag{18}$$

*where $h = T/N$.*

The key tool for proving the above theorem is provided by the following lemma

**Lemma 1** *(Lemma 2 in Pietrus and Veliov 2009) Let Assumption 2 hold and let $N$ and $M \in [2, N]$ be natural numbers and $h := 1/N$. Then there exists (an anticipative) mapping $\pi_N : \mathcal{U} \mapsto \mathcal{V}_N$ such that for every $u \in \mathcal{U}$*

$$\left| \int_0^t (u(s) - \pi_N(u)(s))\,\mathrm{d}s \right| \le h \quad \forall\, t \in [0, Mh],$$
$$\left| \int_0^{Mh} (u(s) - \pi_N(u)(s))\,\mathrm{d}s \right| \le \frac{1}{2}h^2, \quad \left| \int_0^{Mh} s\,(u(s) - \pi_N(u)(s))\,\mathrm{d}s \right| \le \frac{1}{2}h^2.$$

**Proof of Theorem 3.** The idea of the proof is the same as in Pietrus and Veliov (2009). Without any restriction we assume $T = 1$ (a different finite $T > 0$ affects only the constant in estimations (17), (18)). Let $M$ be the largest natural number such that $M^2 \le N$ and let $t = Mh$.

Let $u_1 \in \mathcal{U}$ be arbitrarily fixed and let $u_1^h \in \mathcal{V}_N$ be defined as $u_1^h = \pi_N(u_1)$, where $\pi_N$ is the mapping defined in Lemma 1. Define, in addition $u_0(t) = u_0^h(t) \equiv 1$. The third-order Volterra expansion of the solution of (1) for initial point $x$ gives a representation of the solution $x[u_1]$ (see e.g. Grüne and Kloden 2006) of the following form:

$$\begin{aligned} x[u_1](t) \;=\; & x^0 + \sum_{i=0}^{1} f_i(x^0) \int_0^t u_i(s)\,\mathrm{d}s + \sum_{i=0}^{1}\sum_{j=0}^{1} \int_0^t\!\!\int_0^s L^j f_i(x_0) u_j(\tau)\,\mathrm{d}\tau u_i(s)\,\mathrm{d}s \qquad (19) \\ & + \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} \int_0^t \int_0^s \int_0^\tau L^k L^j f_i(x_0) u_k(\theta)\,\mathrm{d}\theta u_j(\tau) u_i(s)\,\mathrm{d}s + O(t^4), \end{aligned}$$

where by definition $L^i = f_i \frac{\partial}{\partial x}$ and as usual $O(s)/s$ is bounded, uniformly in $u_1 \in \mathcal{U}$. The above representation holds also for $x[u_1^h]$.

We shall estimate $|x[u_1](t) - x[u_1^h](t)|$ by considering each of the four terms in (19) separately. Obviously the difference of the first term is zero. Also

$$\begin{aligned} & \left| \sum_{i=0}^{1} f_i(x^0) \int_0^t u_i(s)\,\mathrm{d}s - \sum_{i=0}^{1} f_i(x^0) \int_0^t u_i^h(s)\,\mathrm{d}s \right| \\ & = |f_i(x^0)| \left| \sum_{i=0}^{1} \int_0^t (u_i(s) - u_i^h(s))\,\mathrm{d}s \right| \le C_1 h^2 \le C_1 M^2 h^3, \end{aligned}$$

(where $C_1$ is independent of $t$) according to the second inequality in Lemma 1, which obviously holds also for $u_0$ and $u_0^h$.

The terms with $L^0 f_1$ and $L^1 f_0$ in $x[u_1](t) - x[u_1^h](t)$ can be estimated by $C_3 h^2 \le C_3 M^2 h^3$ as above. To estimate the rest of the third term we notice that

$$\int_0^t \int_0^s u_1(\tau)\,\mathrm{d}\tau\, u_1(s)\,\mathrm{d}s = \frac{1}{2}\left(\int_0^t u_1(s)\,\mathrm{d}s\right),$$

which implies that

$$\begin{aligned}&\left|L^1 f_1(x^0)\left[\int_0^t \int_0^s u_1(\tau)\,\mathrm{d}\tau\, u_1(s)\,\mathrm{d}s - \int_0^t \int_0^s u_1^h(\tau)\,\mathrm{d}\tau\, u_1^h(s)\,\mathrm{d}s\right]\right| \\ &\le \frac{|L^1 f_1(x^0)|}{2} h^2 (2Mh) = C_2 M h^3.\end{aligned}$$

To estimate the difference of the triple integrals we use the representations

$$\begin{aligned}
&\int_0^t \int_0^s \int_0^\tau u(\theta)\,\mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}s = t^2 \int_0^t u(s)\,\mathrm{d}s - t\int_0^t s u(s)\,\mathrm{d}s - \int_0^t s \int_0^s u(\tau)\,\mathrm{d}\tau,\\
&\int_0^t \int_0^s u(\tau) \int_0^\tau \mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}s = 2\int_0^t s \int_0^s u(\tau)\,\mathrm{d}\tau + t\int_0^t s u(s)\,\mathrm{d}s - t^2 \int_0^t u(s)\,\mathrm{d}s\\
&\int_0^t u(s) \int_0^s \int_0^\tau \mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}s = \frac{t^2}{2}\int_0^t u(s)\,\mathrm{d}s - \int_0^t s \int_0^s u(\tau)\,\mathrm{d}\tau,\\
&\int_0^t u(s) \int_0^s u(\tau) \int_0^\tau \mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}s = \frac{1}{2}\int_0^t \left(\int_s^t u(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s\\
&\int_0^t u(s) \int_0^s \int_0^\tau u(\theta)\,\mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t u(s)\,\mathrm{d}s \int_0^t (t-s)u(s)\,\mathrm{d}s - \int_0^t \left(\int_s^t u(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s\\
&\int_0^t \int_0^s u(\tau) \int_0^\tau u(\theta)\,\mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}s = \frac{1}{2}\int_0^t \left(\int_0^s u(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s\\
&\int_0^t u(s) \int_0^s u(\tau) \int_0^\tau u(\theta)\,\mathrm{d}\theta\,\mathrm{d}\tau\,\mathrm{d}s = \frac{1}{6}\left(\int_0^t u(s)\,\mathrm{d}s\right)^3.
\end{aligned}$$

Then one can easily estimate the third order integrals in the difference $x[u_1](t) - x[u_1^h](t)$ by $CM^2h^3$. For example,

$$\begin{aligned}
&\left|\int_0^t \left(\int_s^t u_1(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s - \int_0^t \left(\int_s^t u_1^h(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}s\right| \\
&\le \left|\int_0^t \int_s^t (u_1(\tau) - u_1^h(\tau))\,\mathrm{d}\tau \int_s^t (u_1(\tau) + u_1^h(\tau))\,\mathrm{d}\tau\,\mathrm{d}s\right| \le Mh\,(0.5h^2 + h)\,2Mh \\
&\le CM^2h^3,
\end{aligned}$$

since

$$\left|\int_s^t (u_1(\tau) - u_1^h(\tau))\,\mathrm{d}\tau\right| = \left|\int_0^t (u_1(\tau) - u_1^h(\tau))\,\mathrm{d}\tau\right| + \left|\int_0^s (u_1(\tau) - u_1^h(\tau))\,\mathrm{d}\tau\right| \leq 0.5h^2 + h$$

according to Lemma 1.

Thus we obtained that the local error $x[u_1](Mh) - x[u_1^h](Mh)$ is of order $M^2h^3$. Then repeating this on every subinterval $[iMh, (i+1)Mh]$, $i = 1, \ldots, M$ we obtain by a standard propagation of error argument that

$$|x[u_1](T) - x[u_1^h](T)| \leq CM^2h^3.(M+1) \leq C'M^3h^3 \leq C''h^{1.5}.$$

The first order estimation for the trajectories follows in a standard way from the first inequality in Lemma 1. Q.E.D.

Having in mind Theorem 2 one can ask if the above estimation is sharp. This challenging question is still open even without the requirement that the approximation map $\pi_N$ provides a first order approximation of the trajectory bundle, that is, it is not know whether the estimation $H(R_N, R) \leq Ch^2$ does not hold under Assumption 2.

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