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Abstract

This study is motivated by the evidence of global warming, which is caused by human activity but affects the efficiency of the economy. We employ the integrated assessment Nordhaus DICE-2007 model [16]. Generally speaking, the framework is that of dynamic optimization of the discounted inter-temporal utility of consumption, taking into account the economic and the environmental dynamics. The main novelty is that several reasonable types of behavior (policy) of the economic agents, which may be non-optimal from the point of view of the global performance but are reasonable form an individual point of view and exist in reality, are strictly defined and analyzed. These include the concepts of "business as usual", in which an economic agent ignores her impact on the climate change (although adapting to it), and of "free riding with a perfect foresight", where some economic agents optimize in an adaptive way their individual performance expecting that the others would perform in a collectively optimal way. These policies are defined in a formal and unified way modifying ideas from the so-called "model predictive control". The introduced concepts are relevant to many other problems of dynamic optimization, especially in the context of resource economics. However, the numerical analysis in this paper is devoted to the evolution of the world economy and the average temperature in the next 150 years, depending on different scenarios for the behavior of the economic agents. In particular, the results show that the "business as usual", although adaptive to the change of the atmospheric temperature, may lead within 150 years to increase of temperature by 2°C more than the collectively optimal policy.

Keywords: environmental economics, dynamic optimization, optimal control, global warming, model predictive control, integrated assessment

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1 Introduction

In his seminal paper, Nordhaus [13] elaborated the very first integrated assessment model (IAM) of the world economy with global warming, the DICE model.[1] This paper has been followed by a plethora of quite similar computational models. Surprisingly, almost all of them are computed under only two basic runs: an optimal policy (the one that maximizes intertemporal welfare) and a business-as-usual scenario (no emission abatement). It is by comparing these two scenarios that the benefits of a global climate policy are assessed. The typical message provided by IAMs is that slowing down the increase in greenhouse gases is efficient, while stronger emission reductions would impose significant economic costs. Several modeling developments have been carried out to make IAMs more complex or more realistic (backstop technologies, endogenous growth, resource exhaustion...), but the two basic scenarios remain. Our aim is to propose alternative—and arguably more realistic—ways to define business-as-usual scenarios in integrated assessment models. By doing this, we also question the costs and benefits of climate policies.

Let us start by explaining why we consider that the two basic scenarios used in IAMs ("optimal policy" and "business-as-usual") may be subject of concerned. The reasons why the "optimal policy" cannot be seen as realistic are well-established in the literature. The optimal policy scenario consists in maximizing the intertemporal welfare in the economy. It thus assumes a perfect foresight and benevolent policy makers, or a single representative perfect foresight private agent, which is, in both cases, far from realism. For this reason, many authors consider that the optimal scenario just provides a Pareto efficient solution and is not to be considered as a *policy scenario* as such. As the best achievable solution, it is a benchmark, but it has little policy relevance. We follow this interpretation. As far as the business-as-usual (hereafter, BaU) scenario is concerned, the drawbacks are of a different nature. Nordhaus, as well as the following authors, define the BaU scenario as the trajectory in productive investment that maximizes intertemporal welfare net of the damages incurred by global warming, but without emission abatement. In other words, the agent is still perfect foresighted, but she does not see the impacts of her own decisions on climate change and the related damages she will bear. This scenario is not only unrealistic (because of perfect foresight), but it is also rationally inconsistent (because of a combination of perfect foresight and myopia about climate damages). This is the scenario we question in this paper. [2]

Roughly speaking, we model the behavior of an agent doing BaU in the following way: the agent optimizes her economic objective disregarding her influence on the environment and taking the state of the environment as exogenously given. If at some later time the agent encounters changed

[1] The acronym DICE stands for "Dynamic Integrated Climate-Economy". The first version of the DICE model can also be found in [14] and [15]. See [16] for the latest version. The model is publicly available on Nordhaus' web page.

[2] It is only when the model distinguishes many regions or countries that it becomes able to compute alternative scenarios based on coalitions of countries. The non-cooperative Nash equilibrium is the one where each country implements its optimal policy by taking the strategy of the other countries as given. Because of the global externality, it is inefficient. The full cooperative equilibrium is the one where all countries cooperate, and it coincides with the Pareto solution. In between, any coalition can be considered. For such analyses, see e.g. [2, 3, 7, 18].

environmental conditions, then she updates her optimal policy regarding the new environmental state, but still ignoring the impact of her economic activity on the environment. The same approach of the BaU is repeated persistently. We formalize this type of behavior by introducing an idealized version of the BaU-agent which is independent of the time between subsequent updates. This is done in a general framework in which the behavior of the economic agents is based on exogenous predictions for the evolution of the environment instead of predictions regarding the environmental impact of their economic policies. In particular, also a type of agent's behavior that resembles the basic features of *free riding* (FR) is well defined in our framework as corresponding to a particular prediction pattern. Even more, we extend our general concept of *prediction-based optimality* to the case of multiple agents who may implement different decision concepts: some regarding their impact on the environment, others doing business-as-usual or free riding. The same framework seems to be relevant and may find interesting applications in other problems in resource economics.

The modeling technique that we employ in defining the above solution (behavioral) concepts adapts ideas from an area in the engendering-oriented control theory known as *model predictive control* or *receding horizon control* (see e.g. [1, 5, 8]). The above theory originates from problems of stabilization of mechanical systems and its translation to the optimal control context in the present paper rises a number of mathematical problems that are not profoundly studied in the literature. The key one is the issue of Lipschitz dependence of the optimal control in a long-horizon optimal control problem on the initial data and on the optimization horizon (we refer to [4, 6] for a relevant information). In this paper we take a shortcut by formulating as assumptions all the properties needed for the correct definition and results concerning the agent's behavior. These assumptions look at first glance cumbersome, but they are quite natural and essential. The verification of some of the assumed properties is not easy, in general, and provides an agenda for a future research. In addition, the fulfillment of these assumptions in the main case study in this paper—the global warming—is rather conditional, as one can learn from the rather striking results in [9] and [10], but still possible, as argued in Appendix 2.

The paper is organized as follows. Our modification of the Nordhaus integrated assessment model is presented in the next section. In Section 3 we introduce the general concept of prediction-based optimality and the particular cases of BaU and FR. This concepts are extended in Section 4 to the case of co-existing agents with different behavior. Then in Section 5 we present and discuss numerical results for the global warming problem. The two appendixes that follow summarize some technical issues.

2 The world economy facing global warming

Our modeling of the world economy relies on the DICE-2007 model (see [16]). In a nutshell, there exists a policy maker who maximizes discounted welfare, integrating in the analysis the economic activity with its production factors, CO_2 emissions and its consequences on climate change. Our climate block reduces to two equations which describe the dynamics of the concentration of CO_2

(depending on emissions) and the interaction between CO_2 concentration and temperature change. In this sense, our modeling looks like previous DICE versions in which the carbon-cycle was not detailed.

The economy is populated by a constant number of individuals, which we normalize to 1.[3] A single final good is produced. This good can either be consumed, invested in the final good sector or used to abate CO_2 emissions. A representative agent chooses optimal consumption and abatement time-dependent policies aiming at maximization of the total inter-temporal discounted utility from consumption (net of climate damages). The model involves the physical capital, $k(t)$, the CO_2 concentration in the atmosphere, $m(t)$, and the average temperature, $\tau(t)$, as state variables. The decision (control) variables are: $u(t)$ – fraction of the GDP used for consumption, and $a(t)$ – emission abatement rate. The overall model is formulated in the next lines and explained in detail below:

$$\max_{u,a} \left\{ \int_0^\infty e^{-rt} \frac{[u(t)p(t)\varphi(\tau(t))k(t)^\gamma]^{1-\alpha}}{1-\alpha} \,\mathrm{d}t \right\} \tag{1}$$

subject to (the argument t of k, τ and m is suppressed):

$$\dot{k} = -\delta k + [1 - u(t) - c(a(t))]\, p(t)\varphi(\tau)k^\gamma, \quad k(0) = k^0, \tag{2}$$

$$\dot{\tau} = -\lambda(m)\tau + d(m), \quad \tau(0) = \tau^0, \tag{3}$$

$$\dot{m} = -\nu m + (1 - a(t))\, e(t)p(t)\varphi(\tau)k^\gamma + E(\tau), \quad m(0) = m^0, \tag{4}$$

$$u(t),\; a(t) \in [0, 1]. \tag{5}$$

Physical capital accumulation is described by equation (2). Production is realized through a Cobb-Douglas production function with elasticity $\gamma \in (0, 1)$ with respect to physical capital. The depreciation rate is $\delta > 0$. A fraction u of the output is consumed and another part, $c(a)$, is devoted to CO_2 abatement. Here $c(a)$ is the fraction of the output that is used for reducing the emission intensity by a fraction a.[4] The function $p(t)$ stands for productivity level and is assumed exogenous, while $\varphi(\tau)$ represents the impact of the climate on global factor productivity.

The evolution of CO_2 concentration is depicted by equation (4), where ν is the natural absorption rate, $E(\tau)$ is the non-industrial emission at temperature τ, and $e(t)$ is the emission for producing one unit of final good without abatement. Finally, (3) establishes the link between CO_2 concentration and temperature change. The CO_2 concentration increases the atmospheric temperature directly through d but also may affect the cooling rate λ. The initial value k^0, τ^0, m^0 are given.

The intertemporal elasticity of substitution of the utility in (1) with respect to consumption is denoted by $\alpha \in (0, 1)$, while $r > 0$ is the discount rate.

The numerical investigation of this model and its versions developed in the next two sections is postponed to Section 5, where all the above data are specified. However, before turning to numerical

[3] Naturally, considering a changing population size over time would be more realistic and would change the numerical results presented below, but it plays no role for the analytic concepts developed in the next two sections.

[4] Notice that $u + c(a)$ may be greater than one, in principle, in which case existing capital stock is sacrificed for lower emission rate.

analysis, we shall introduce in the next two sections two alternative solution concepts reflecting possible non-optimal, but still rational and realistic agent's behavior.

3 Prediction-based optimality: a general consideration

In this section we consider a more general optimal control framework in which we present the basic concept introduced in this paper—that of prediction-based optimality—and some of its particular cases. In the explanations below we use the economic/environment interpretation of the variables given in the above section, although several different economic interpretations are meaningful.

Consider the optimal control problem

$$\max_{v} \int_0^\infty L(t, v(t), x(t), y(t))\, \mathrm{d}t \tag{6}$$

subject to the dynamic constraints

$$\dot{x}(t) = f(t, v(t), x(t), y(t)), \tag{7}$$
$$\dot{y}(t) = g(t, e(t, v(t), x(t)), y(t)), \tag{8}$$
$$x(0) = x^0, \quad y(0) = y^0, \tag{9}$$

and the control constraint

$$v(t) \in V. \tag{10}$$

The state variable $x \in \mathbf{R}^n$ is interpreted as a vector of stocks of economic factors, while the state variable $y \in \mathbf{R}^m$ represents "environmental" (in general sense) factors whose evolution depends on the economic activity through the function e. The control vector-variable $v \in V \subset \mathbf{R}^r$ may be interpreted as investment/abatement in different sectors and (6) maximizes the aggregated output (or utility). The function $e(t, v, x)$ has values in a finite dimensional space and represents the impact of the economic control, v, and the economic state, x, on the dynamics of the environment.

The measurable functions $v : [0, \infty) \mapsto V$ will be called *admissible controls*. The model (7)–(8) will be considered as relevantly representing the evolution of the environmental-economic system. Thus for any given economic input $v(t)$ we identify the corresponding solution $x(t)$, $y(t)$ with the real economic-environmental state.

The particular case of the world economy facing global warming, presented in the previous section, corresponds to the specifications $x = k$, $y = (m, \tau)$, $v = (u, a)$, $e(t, v, x) = (1 - a(t))\, e(t)p(t)\varphi(\tau)K^\gamma$.[5]

[5]Note that there is an overloading of notations: the dimensions m and r in the general model have nothing to do with the concentration m and the discount r, etc. This could in no way lead to a confusion.

Let $(\hat{v}, \hat{x}, \hat{y})$ be a solution of problem (6)–(10) (this problem is called further OPT). Our basic argument is that, in real life, the motives for the decision-makers are to a large extend self-interest. They are also narrow-minded in the sense where they are unable to grasp the whole picture. The last token means that the agent does not necessarily believe, or is not fully aware of equation (8). As a consequence, agent's decisions need not result in resembling the optimal path $(\hat{v}, \hat{x}, \hat{y})$. The question is now of how to define alternative behavioral patterns? Below we define a concept of optimality which is not directly based on the model (8) of the environment, rather, on a prediction of the future environment obtained otherwise. Namely, we assume that at any time s the representative economic agent obtains a prediction for the environmental variable $y(t)$ on a (presumably large) horizon $[s, s+\theta]$, depending on the history of y on some interval $[s-\kappa, s]$ and on the current economic state $x(s)$. This prediction will be given by a mapping (predictor) $\mathcal{E}_s : C^m(s-\kappa, s) \times \mathbf{R}^n \mapsto C^m(s, \infty)$, that is, $\mathcal{E}_s(y_{|[s-\kappa,s]}, x(s))(t)$, is the prediction of $y(t)$ for $t \geq s$ that results from a history $y_{|[s-\kappa,s]}$ of y and the current economic state $x(s)$. In fact, only the values $y(t)$ for $t \in [s, s+\theta]$ will be taken into account in the construction below.[6]

3.1 Step-wise definition

The starting idea is rather similar to that of the so-called *model predictive control*, or *receding horizon control* (see e.g. [1, 5, 8]). It is that, at time $t = 0$, the agent uses the prediction $y(t) = \mathcal{E}(y_{|[-\kappa,0]}, x^0)(t)$ to solve the problem (6), (7), (9), (10) on the time horizon $[0, \theta]$ (that is, with bounds of integration in (6) set to $[0, \theta]$ instead of $[0, \infty)$). Notice that this problem involves only the economic component of the overall model, while the environment $y(t)$ is taken as exogenous. The optimal control v, although obtained on the horizon $[0, \theta]$, is implemented on a "small" time interval $[0, t_1]$, after which the agent observes that the actual value of the environment, $y(t_1)$, has changed from the predicted one and repeats the same procedure with an updated prediction for y given by $\mathcal{E}(y_{|[t_1-\kappa,t_1]}, x(t_1))(t)$.[7] The formal definition of the respective agent's behaviour is given below.

Assume that the past data $y(t)$ for $t \in [-\kappa, 0]$ are known. Let at times $t_i = ih$ the agent reevaluates the past evolution of the environmental state y by measurements, where $h > 0$ is a positive time-step; presumably $h << \theta$. We denote $\sigma = (h, \theta)$ and define a path $(v^\sigma, x^\sigma, y^\sigma)$ recursively as follows.

Set $y^\sigma = y(t)$ for $t \in [-\kappa, 0]$, $x^\sigma(0) = x^0$. Assume that $(v^\sigma, x^\sigma, y^\sigma)$ is already defined on $[0, t_k]$, $k \geq 0$. Consider the problem

$$\max_{v(t) \in V} \int_{t_k}^{t_k+\theta} L(t, v(t), x(t), y_k(t))\, \mathrm{d}t \tag{11}$$

[6]Further on we skip the subscript s in $\mathcal{E}_s$, since it will be clear from the arguments of $\mathcal{E}$. Moreover, the two particular predictors we shall consider below are shift-invariant, thus the subscript s is, in fact, redundant.

[7]Such a step-wise revision is consistent with the fact that, in climate science, a climate regime is defined by averaging a 30-year time period. So a changing in a climate regime can be statistically demonstrated only after several decades.

$$\dot{x}(t) = f(t, e(t, v(t), x(t)), y_k(t)), \quad x(t_k) = x^\sigma(t_k), \tag{12}$$

where $y_k(t) = \mathcal{E}\left(y^\sigma_{|[t_k-\kappa, t_k]}, x^\sigma(t_k)\right)(t)$ for $t \in [t_k, t_k + \theta]$. Let v_{k+1} be an optimal control of this problem on $[t_k, t_k + \theta]$. We define v^σ on $[t_k, t_{k+1}]$ as equal to v_{k+1}, and extend continuously (x^σ, y^σ) as the respective solution of (7), (8) on $[t_k, t_{k+1}]$.

This recurrent procedure defines $(v^\theta_h, x^\theta_h, y^\theta_h)$ on $[0, \infty)$.

The idea is clear: the agent follows his optimal policy based on the prediction for the environment at time s for a future period of length h, after which she realizes that the real environment has declined from the prediction and re-solves the optimization problem again with an updated prediction. The definition below is a mathematical idealization in which the re-evaluation period h tends to zero. This makes the resulting process independent of the particular choice of step h.

Definition 1 Every limit point of any sequence $(v^\sigma, x^\sigma, y^\sigma)$ (defined as above) in the space $L_1^{\text{loc}}(0, \infty) \times C(0, \infty) \times C(0, \infty)$ when $\sigma = (h, \theta) \longrightarrow (0, +\infty)$ (if such exists) will be called *prediction-based optimal solution*[8].

Remark 1 We stress that neither the existence nor the uniqueness of a prediction-based optimal solution is granted. "Academic" counterexamples can easily be constructed. Even more, for a similar "global warming" model as the one presented in Section 2 a non-uniqueness of the optimal solution for initial data lying on a certain "critical" hyperplane in the state space was established in [10]. For this model the existence and the uniqueness of a prediction-based optimal solution are questionable for "critical" initial data. Therefore, the conditions for existence and uniqueness stated below in this section for a particular prediction map $\mathcal{E}$, although not necessary in general, are essential.

3.2 Two special prediction mappings

Two special cases of prediction-based optimality for two different predictors $\mathcal{E}$ are of special interest: a *business-as-usual* scenario and a *free-rider* scenario.

1. Business as Usual (BaU)

If the environmental quality y is slowly changing compared with the economic factors x, then the representative agent may interpret the observed changes of y as unpredictable or exogenous fluctuations around an equilibrium, not substantially affected by the economic activity. In other words, it reflects some natural exogenous fluctuations (for example, the effect of solar activity on the

[8] A sequence z^σ converges to z in $C(0, \infty)$ if it converges uniformly on every compact interval $[0, T]$. A sequence v^σ converges to v in $L_1^{\text{loc}}(0, \infty)$ if $\int_0^T |v^\sigma(t) - v(t)| \, dt$ converges to zero for every $T > 0$.

global temperature on earth's surface). As a result, the agent may reasonably take a time-invariant value for the expected value of y:

$$\mathcal{E}\left(y_{|[s-\kappa,s]}\right)(t) \equiv \frac{1}{\kappa}\int_{s-\kappa}^{s} y(\theta)\,\mathrm{d}\theta,$$

or merely take the current value at s as a prediction of the future: $\mathcal{E}\left(y_{|[s-\kappa,s]}\right)(t) = y(s)$, $t \in [s, s+\theta]$ (this corresponds to $\kappa = 0$).

This approach of the agents, which disregards the dynamics of the environment although adapting to its changes a posteriori, will be called *Business as Usual (BaU).*

The simplest case of BaU is that with $\kappa = 0$, which is realistic in a slowly changing environment. We shall formulate this particular case more explicitly. Consider the problem

$$\max_{v(t)\in V} \int_{s}^{s+\theta} L(t, x(t), y, v(t))\,\mathrm{d}t$$

$$\dot{x}(t) = f(t, e(t, v(t), x(t)), y), \quad x(s) = x^s, \quad t \in [s, s+\theta],$$

where $s \geq 0$, $\theta \in (0, \infty)$, $x^s \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ are given. Denote by $\hat{v}[s, \theta, x^s; y](t)$ a solution of this problem (assuming that such exists). Then the step-wise approximation $(v^\sigma, x^\sigma, y^\sigma)$ of BaU for $\sigma = (h, \theta)$ can be written (recursively in k) as

$$v^\sigma(t) = \hat{v}[t_k, \theta, x^\sigma(t_k); y^\sigma(t_k)](t), \quad t \in [t_k, t_{k+1}], \tag{13}$$

$$(x^\sigma, y^\sigma) \text{ solves (7), (8) with } v = v^\sigma. \tag{14}$$

The initial conditions in (14) are given by (9) at $t_k = t_0 = 0$, while x^σ, y^σ are extended by continuity at each $t_k > 0$.

Clearly, in the context of the previous section the BaU approach has been practically the only one implemented by agents at all levels till the late $20^t h$ century, and is still dominating in many countries. It consists in being conservative in the understanding of nature's law. This BaU scenario, however, is rationally consistent.

2. Free riders (FR)

There exists a presumably large number of (identical or not) agents in the economy, which therefore can be viewed as represented (aggregated) by a single representative agent. The representative agent determines the collectively optimal policy $\hat{v}$ by solving the problem (6)–(10). In particular, the y-component of the solution, $\hat{y}$, gives the future development of the environment if the optimal policy were implemented by all agents. But some agents may free ride. A free rider is an agent that assumes that the other agents would implement the optimal policy, and since the influence of a single agent on the environment is negligible, she may take $\hat{y}(t)$ as predictor for the future environment and design her own optimal policy solving (6), (7), (9), (10) with $y = \hat{y}$ (which policy is, in general, different from the collectively optimal one, $\hat{v}$).

In this section we consider an "extreme" scenario in which all agents undertake free riding. In sections 4 and 5.3 we shall investigate the more realistic case of co-existing agents with different behavior, some following the optimal policy regarding the influence of the economy on the environment (OPT), some others practicing BaU or FR.

Formally, the predictor $\mathcal{E}$ of a FR is defined as follows. Denote by $(\hat{x}[s, x^s, y^s], \hat{y}[s, x^s, y^s])$ the optimal trajectory of problem (6)–(10) on the interval $[s, \infty)$ (instead of $[0, \infty)$) and with initial data $x(s) = x^s$, $y(s) = y^s$ at time s (replacing (9)). Then the predictor $\mathcal{E}$ is defined as

$$\mathcal{E}(y_{|[s-\kappa,s]}, x)(t) = \hat{y}[s, x, y(s)](t).$$

That is, the hypothetical collectively optimal evolution of the environment starting from the current data is used as a predictor.

3.3 Feedback definition of BaU-solution: existence and uniqueness

Here we present another point of view to the optimality concepts introduced above focusing for more transparency on the simplest BaU – that with $\kappa = 0$. The general case requires appropriate modifications and assumptions concerning the predictor $\mathcal{E}$.

Although in the assumptions below we do not seek generality (they are just reasonably general, so that several applications satisfy them) they are rather implicit and will be discussed in Appendix 2. Assume the following:

Assumption A1: (i) Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$ and $V \in \mathbf{R}^r$ be given sets and let E be a set containing all vectors $e(t, v, x)$ with $t \geq 0$, $v \in V$, $x \in X$. The functions $f(t, v, x, y)$, $g(t, e, y)$, $L(t, v, x, y)$, $e(t, v, x)$ are bounded, continuous in t and Lipschitz continuous in the rest of the variables, $v \in V$, $x \in X$, $y \in Y$ and $e \in E$, uniformly with respect to t. Moreover, $x^0 \in X$ and $y^0 \in Y$.
(ii) For every admissible control v, every $s \geq 0$, $x^s \in X$ and $y^s \in Y$ the solution of (17), (18), with initial conditions $x(s) = x^s$, $y(s) = y^s$, exists in $X \times Y$ on $[s, \infty)$.

Assumption (A2). (i) For every $s \geq 0$, $\theta \in (0, \infty)$, $x^s \in X$ and $y^s \in Y$ the optimal control $\hat{v}[s, \theta, x^s; y^s](\cdot)$ exists and is unique in the space $L_1([s, s + \theta])$.
(ii) There exist a number $\hat{l}$ and a continuous function $\delta : [0, \infty) \mapsto [0, \infty)$, with $\delta(0) = 0$, such that for every $t \geq s \geq 0$, $\tilde{\theta} \geq \theta \geq t - s$, x, $\tilde{x} \in X$, y, $\tilde{y} \in Y$ the following inequality holds:

$$\left| \hat{v}[s, \theta, x; y](t) - \hat{v}[t, \tilde{\theta}, \tilde{x}; \tilde{y}](t) \right| \leq \hat{l} \, (|x - \tilde{x}| + |y - \tilde{y}|) + \delta \left(|s - t| + \frac{1}{\theta} \right).$$

Let T be any positive number. Assumptions A2 together with the completeness of the space $C([s, T])$ imply that for every $s \in [0, T]$, $x^s \in X$ and $y^s \in Y$ the sequence $\hat{v}[s, \theta, x^s; y^s]$ converges in the space $C([s, T])$ whit $\theta \longrightarrow \infty$ to a continuous admissible control denoted by $\hat{v}[s, \infty, x^s; y^s]$.

Proposition 1 *Under A1, A2 a unique BaU-solution in the sense of Definition 1 exists and coincides with the unique solution of the feedback system*

$$\dot{x}(t) = f(t, \hat{v}[t, \infty, x(t); y(t)](t), x(t), y(t)), \quad x(0) = x^0, \tag{15}$$

$$\dot{y}(t) = g(t, e(t, \hat{v}[t, \infty, x(t); y(t)](t), x(t)), y(t)), \quad y(0) = y^0. \tag{16}$$

We skip the proof (which is a standard exercise in calculus), but in fact it is contained in the proof of the more complicated Proposition 2, which is given in Appendix 1.

The feedback system (15), (16) provides an alternative to Definition 1. This is an easy shortcut to the concept of BaU-solution. However, Definition 1 reflects in a better way the practical BaU-behavior, where the stepwise policy v_h^θ is implemented. Therefore the issue of convergence of the step-wise BaU-control v_h^θ is of key importance for the credibility of the idealization (15), (16) and, in addition, provides a numerical approach for finding the BaU-solution, implemented in our numerical analysis of the global warming problem in Section 5.

4 Co-existing agents with different behavior

In the previous section we considered the scenario in which all economic agents behave in the same way, therefore can be represented by a single agent which either optimizes by solving the full economic-environment problem (6)–(10) (we call such an agent OPT), or chooses the BaU or the FR behavior. In the present section we consider the case of co-existence of agents with different behavior. For reasons of transparency we focus on the simplest case of two agents (groups of countries): one OPT and one BaU.

Let v_1 and v_2 be the controls of the OPT and of the BaU, and x_1 and x_2 be their economic states. Assuming additivity of the environmental impact e of the economic activities of the two agents (which is the case if polluting emissions such as CO_2 are considered), we reformulate the dynamics of the economic-environment system as

$$\dot{x}_i(t) = f_i(t, v_i(t), x_i(t), y(t)), \quad x_i(0) = x_i^0, \quad i = 1, 2, \tag{17}$$

$$\dot{y}(t) = g(t, e_1(t, v_1(t), x_1(t)) + e_2(t, v_2(t), x_2(t)), y(t)), \quad y(0) = y^0, \tag{18}$$

with the control constraints

$$v_i(t) \in V, \quad i = 1, 2. \tag{19}$$

The objective function of the agent i is

$$\int_0^\infty L_i(v_i(t), x_i(t), y(t))\, \mathrm{d}t. \tag{20}$$

Here the finite dimensional vector $e_i(t, v_i, x_i)$ represents the instantaneous emission of the agent i resulting from control value v_i and economic state x_i at time t. Below we investigate the evolution of such an economic-environment system, adapting the concept of BaU developed in the previous section. Agent 2 optimizes her utility in a BaU manner, while agent 1 optimizes her utility regarding the environmental change. The complication of the coexisting BaU and OPT agents comes from the fact that they live in a common environment, $y(t)$, which they both influence, therefore their respective optimal policies are not independent of each other. Namely, in order to solve her problem OPT has to know the influence of BaU on the environment. On the other hand, BaU adapts her decision to the environmental change (in a similar manner as in the previous section), which on its turn depends on the policy of OPT.

The idea of the formal definition of co-existing OPT and BaU is as follows. The agent BaU (agent $i = 2$) first solves her optimal control problem (17), (19), (20) (with $i = 2$) on $[0, \theta]$ using the predictor $y(t) = y^0$. This defines a hypothetical emission $\eta_2(t) = e_2(t, v_2(t), x_2(t))$ of the BaU-agent for all $t \in [0, \theta]$. The BaU implements her control only on a short interval $[0, h]$, but nevertheless the agent OPT takes η_2 as a "prediction" for the emission of BaU[9] on $[0, \theta]$ and solves her problem

$$\max_{v_1(t) \in V} \int_0^\theta L_1(v_1(t), x_1(t), y(t))\, \mathrm{d}t \tag{21}$$

$$\dot{x}_1(t) = f_1(t, v_1(t), x_1(t), y(t)), \quad x_1(0) = x_1^0, \tag{22}$$

$$\dot{y}(t) = g(t, e_1(t, v_1(t), x_1(t)) + \eta_2(t), y(t)), \quad y(0) = y^0. \tag{23}$$

Agent OPT implements the so obtained control on $[0, h]$. The controls of both agents being chosen on $[0, h]$ determine the evolution of the overall system on this interval according to (17), (18). Then the same procedure is repeated on $[h, h + \theta]$ to determine the evolution on $[h, 2h]$, and so on.

To define formally the above step-wise procedure we introduce the following notations: similarly as in the previous section, $\hat{v}_2[s, \theta, x_2^s; y^s](t)$ will be an optimal control of the problem

$$\max_{v_2(t) \in V} \int_s^{s+\theta} L_2(v_2(t), x_2(t), y^s)\, \mathrm{d}t \tag{24}$$

$$\dot{x}_2(t) = f_2(t, v_2(t), x_2(t), y^s), \quad x_2(s) = x_2^s, \tag{25}$$

and $\hat{x}_2[s, \theta, x_2^s; y^s](t)$ – the corresponding optimal trajectory. Denote by $\hat{v}_1[s, \theta, x_1^s, y^s; \eta_2(\cdot)](t)$ a solution of the problem (21)–(23) on the interval $[s, s + \theta]$ (instead of $[0, \theta)$), and with initial data $x_1(s) = x_1^s$, $y(s) = y^s$, and with the function η_2 exogenously given.

Let $\sigma = (h, \theta)$ ba a fixed pair of a "small" $h > 0$ and a "large" $\theta > 0$ and let $t_k^\sigma = kh$. Fix a parameter $\rho > 1$. We shall define a step-wise OPT-BaU solution $(v_1^\sigma, v_2^\sigma, x_1^\sigma, x_2^\sigma, y^\sigma)$ recurrently,

[9] The assumption that OPT uses the emission of BaU resulting from the current BaU control for a long horizon ahead, although BaU implements the current control only on a short horizon, may look somewhat questionable in the context of the global warming. One supporting argument is that in reality OPT may not know in advance how frequently BaU would adjust her policy (that is, how large is h). An alternative definition of combined OPT-BaU behavior will be discussed in the end of the section. The numerical results obtained for the global warming problem using the two different definitions of OPT-BaU behavior are practically indistinguishable.

assuming that it is already defined till time t_k^σ (for $k = 0$ only the initial x_i^0 and y^0 are required for the recursion). Define
$\tilde{v}_2^\sigma(\tau) = \hat{v}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](\tau)$, $\tau \in [t_k^\sigma, t_k^\sigma + \theta]$,
$\tilde{x}_2^\sigma(\tau) = \hat{x}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](\tau)$, $\tau \in [t_k^\sigma, t_k^\sigma + \theta]$,
$\tilde{\eta}_2^\sigma(\tau) = e_2(\tau, \tilde{v}_2^\sigma(\tau), \tilde{x}_2^\sigma(\tau))$, $\tau \in [t_k^\sigma, t_k^\sigma + \theta]$,
$v_1^\sigma(t) = \hat{v}_1[t_k^\sigma, \theta, x_1^\sigma(t_k^\sigma), y^\sigma(t_k^\sigma); \tilde{\eta}_2^\sigma(\cdot)](t)$ and $v_2^\sigma(t) = \tilde{v}_2^\sigma(t)$ for $t \in [t_k^\sigma, t_{k+1}^\sigma)$,
$(x_1^\sigma(t), x_2^\sigma(t), y^\sigma(t))$ – the solution of (17), (18) on $[t_k^\sigma, t_{k+1}^\sigma]$ with the initial $(x_1^\sigma(t_k^\sigma), x_2^\sigma(t_k^\sigma), y^\sigma(t_k^\sigma))$ and the above defined controls v_1^σ and v_2^σ.

This completes the recurrent definition of step-wise OPT-BaU solution.

Definition 2 Every limit point of a sequence (v_1^σ, v_2^σ) defined as above whith $h \to 0$ and $\theta \to +\infty$ in $L_1^{\rm loc}(0, \infty)$ (if such exists) will be called *OPT-BaU solution* of the optimization system (17)–(20).

Below we formulate conditions for existence of an OPT-BaU solution. Moreover, it will be proved that the OPT-BaU solution is independent of the choice of the parameter $\rho > 1$ in the step-wise definition.

Assumption A1$'$: (i) Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$ and $V \in \mathbf{R}^r$ be given open sets and let E be a set containing all vectors $e_1(t, v_1, x_1) + e_2(t, v_2, x_2)$ with $t \ge 0$, $v_i \in V$, $x_i \in X$. The functions $f_i(t, v, x, y)$, $g(t, e, y)$, $L_i(t, v, x, y)$, $e_i(t, v, x)$ are bounded, continuous in t and Lipschitz continuous in the rest of the variables, $v \in V$, $x \in X$, $y \in Y$ and $e \in E$, uniformly with respect to t. Moreover, $x_i^0 \in X$ and $y^0 \in Y$.
(ii) For every admissible controls v_1 and v_2, every $s \ge 0$, $x_i^s \in X$ and $y^s \in Y$ the solution of (17), (18), with initial conditions $x_i(s) = x_i^s$, $y(s) = y^s$, exists in $X \times X \times Y$ on $[s, \infty)$.

Assumption A2$'$: (i) For every $s \ge 0$, $\theta \in (0, \infty)$, $x^s \in X$, $y^s \in Y$, and continuous $\eta_2 : [s, s+\theta] \to E$ the optimal controls $\hat{v}_1[s, \theta, x^s, y^s; \eta_2(\cdot)](\cdot)$ and $\hat{v}_2[s, \theta, x^s; y^s](\cdot)$ exist and are unique in $L_1(s, s+\theta)$.
(ii) There exist a number $\hat{l}$, and continuous functions δ, $\alpha : [0, \infty) \mapsto [0, \infty)$, with $\delta(0) = 0$, such that for every $t \ge s \ge 0$, $\tilde{\theta} \ge \theta \ge t - s$, x, $\tilde{x} \in X$, y, $\tilde{y} \in Y$ and continuous functions continuous $\eta_2, \tilde{\eta}_2 : [0, \infty) \mapsto E$ the following inequality holds:

$$\begin{aligned}&\left|\hat{v}_1[s, \theta, x, y; \eta_2](t) - \hat{v}_1[t, \tilde{\theta}, \tilde{x}, \tilde{y}; \tilde{\eta}_2](t)\right| \\ &\le \hat{l}\,(|x - \tilde{x}| + |y - \tilde{y}|) + \delta\left(|t - s| + \frac{1}{\theta}\right) + \int_t^{s+\theta} \alpha(\tau - s)\,|\eta_2(\tau) - \tilde{\eta}_2(\tau)|\,\mathrm{d}\tau.\end{aligned}$$

(iii) There exist continuous functions δ, $\beta : [0, \infty) \mapsto [0, \infty]$, with $\delta(0) = 0$ and $\int_0^\infty \alpha(\tau)\beta(\tau)\,\mathrm{d}\tau < \infty$, such that for every $t \ge s \ge 0$, $\tilde{\theta} \ge \theta \ge 0$, $\tau \in [t, s+\theta)$, x, $\tilde{x} \in X$ and y, $\tilde{y} \in Y$ the following inequality holds:

$$\begin{aligned}&\left|\hat{v}_2[s, \theta, x; y](\tau) - \hat{v}_2[t, \tilde{\theta}, \tilde{x}; \tilde{y}](\tau)\right| + \left|\hat{x}_2[s, \theta, x; y](\tau) - \hat{x}_2[t, \tilde{\theta}, \tilde{x}; \tilde{y}](\tau)\right| \\ &\le \beta(\tau - s)\,\left(|x - \tilde{x}| + |y - \tilde{y}| + \delta\left(|s - t| + \frac{1}{s + \theta - \tau}\right)\right).\end{aligned}$$

As mentioned in the introduction, assumptions A2$'$ look very cumbersome, although they just relevantly represent the requirement of Lipschitz dependence of the optimal solution on the initial data and on the length of the horizon. What "relevantly" does mean here? For example, the nastier looking term δ in A2$'$(iii) reflects the fact that the optimal control on a long horizon $[s, s+\theta]$ is strongly influenced near the end of the horizon by the fact that the optimization disregards what happens after $s+\theta$. Thus the solution on $[s, s+\theta]$ needs not be close to the one for a longer horizon $\tilde{\theta}$ for t near $s+\theta$. The next lemma claims that the dependence of the optimal solution on the length of the horizon is significant only near the end of the horizon, that is, it melts out if the horizon is infinite.

Lemma 1 *For every $s \geq 0$, $(x, y) \in X \times Y$ and continuous $\eta_2 : [0, \infty) \mapsto E$ the limits*

$$\hat{v}_1[s, \infty, x, y; \eta_2](s) := \lim_{\theta \to \infty} \hat{v}_1[s, \theta, x, y; \eta_2](s),$$
$$\hat{v}_2[s, \infty, x; y](\cdot) := \lim_{\theta \to \infty} \hat{v}_2[s, \theta, x; y](\cdot), \quad \hat{x}_2[s, \infty, x; y](\cdot) := \lim_{\theta \to \infty} \hat{x}_2[s, \theta, x; y](\cdot)$$

exist, the last two in $C(s, \infty)$.

The first claim follows from A2$'$(ii) applied for $t = s$, $\tilde{x} = x$, $\tilde{y} = y$, $\eta_2 = \tilde{\eta}_2$. The second and the third ones use similarly A2$'$(iii) and also the completeness of the space $C(0, T)$.

To shorten the notations we abbreviate

$$\eta_2[x_2, y](t, \cdot) = e_2(\cdot, \hat{v}_2[t, \infty, x_2; y](\cdot), \hat{x}_2[t, \infty, x_2; y](\cdot)),$$
$$v_1[x_1, x_2, y](t) = \hat{v}_1[t, \infty, x_1, y; \eta_2[x_2, y](t, \cdot)](t), \quad v_2[x_2, y](t) = \hat{v}_2[t, \infty, x_2; y](t),$$
$$\eta_1[x_1, x_2, y](t) = e_1(t, v_1[x_1, x_2, y](t), x_1).$$

Proposition 2 *Under A1$'$ and A2$'$, a combined OPT/BaU-optimal solution in the sense of Definition 2 exists and coincides with the unique solution of the feedback system*

$$\dot{x}_1 = f(t, v_1[x_1, x_2, y](t), x_1, y), \quad x_1(0) = x_1^0, \tag{26}$$
$$\dot{x}_2 = f(t, v_2[x_2, y](t), x_2, y), \quad x_2(0) = x_2^0, \tag{27}$$
$$\dot{y}(t) = g(t, \eta_1[x_1, x_2, y](t) + \eta_2[x_2, y](t, t), y), \quad y(0) = y^0, \tag{28}$$

where the argument t of $x_1(t)$, $x_2(t)$ and $y(t)$ is suppressed.

Notice that $\eta_2[x_2, y](t, t)$ in equation (28) equals just $e_2(t, \hat{v}_2[t, \infty, x_2; y](t), x_2)$.

The proof is given in Appendix 1.

Similarly as for the case of a pure BaU solution, the approximate calculation of OPT-BaU solutions makes use of the intuitive step-wise definition.

As we mentioned in footnote 9 a different reasonable definition of OPT-BaU solution may be given, which assumes more sophisticated behaviour of the OPT-agent. Namely, in the above step-wise definition OPT uses at every step the prediction of the future emission $\tilde{\eta}_2(t)$ of BaU, not taking into account that BaU may adapt in the next time-period to the change of the environmental variable y. A more involved OPT-agent may take into account the influence of her own decisions to the adjustment of the BaU-control.

Below we briefly formulate the resulting refined concept of OPT-BaU solution, considering directly the "idealized" limit-version with infinite horizon and instantaneous adjustment of BaU.

The definition will be of fixed-point type. Imagine that OPT bases her solution on a guess $\eta_2 \in L_1^{\text{loc}}(0,T)$ for the future emission of BaU. That is, OPT solves the problem

$$\max_{v_1(t)\in V} \int_0^\infty L_1(v_1(t), x_1(t), y(t))\,\mathrm{d}t$$

subject to

$$\begin{aligned} \dot{x}_1(t) &= f_1(t, v_1(t), x_1(t), y(t)), \quad x_1(0) = x_1^0, \\ \dot{y}(t) &= g(t, e_1(t, v_1(t), x_1(t)) + \eta_2(t), y(t)), \quad y(0) = y^0. \end{aligned}$$

Denote by $(v_1^*[\eta_2], x_1^*[\eta_2])$ the economic component of the solution, and by $\eta_1^*[\eta_2](t) = e_1(t, v_1^*[\eta_2](t), x_1^*[\eta_2](t))$ – the corresponding emission. Then BaU obtains the BaU-optimal solution (in the sense of Definition 1) of her problem (with the emission of OPT exogenously given by $\eta_1^*[\eta_2](t)$):

$$\max_{v_2(t)\in V} \int_0^\infty L_2(t, v_2(t), x_2(t), y(t))\,\mathrm{d}t$$

subject to

$$\begin{aligned} \dot{x}_2(t) &= f_2(t, v_2(t), x_2(t), y(t)), \quad x_2(0) = x_2^0, \\ \dot{y}(t) &= g(t, \eta_1[\eta_2](t) + e_2(t, v_2(t), x_2(t)), y(t)), \quad y(0) = y^0. \end{aligned}$$

Let $(v_2^*[\eta_2], x_2^*[\eta_2])$ be BaU-optimal solution, thus $\eta_2^*[\eta_2](t) = e_2(t, v_2^*[\eta_2](t), x_2^*[\eta_2](t))$ is the resulting emission of the BaU-agent. Now we complete the definition by requiring that the initial guess of OPT is consistent with the obtained real emission output:

$$\eta_2^*[\eta_2] = \eta_2. \tag{29}$$

This is a functional equation that determines the "right" initial guess η_2 of OPT.

We do not address the apparently complicated issue of existence of a solution to the above functional equation, since Definition 2 seems to be more relevant to the real behaviour of the OPT and BaU agents. In addition, iterating the fix-point equation (29), using as an initial guess the BaU emission $\eta_2(t)$ in the OPT-BaU solution in sense of Definition 2 for our global warming model, we establish that there is no considerable difference between the two OPT-BaU solution concepts for this particular model. Therefore, further we focus on the OPT-BaU solution concept given by Definition 2.

5 Numerical analysis of the world economy under global warming

In this section we provide numerical illustrations for the different decision patterns defined above, with an application to the global warming issue. Firstly, the computational model and its calibration are presented. Then we compute Business-as-usual (BaU) and Optimal (OPT) scenarios and, in the last subsection, we consider the case of co-existing agents.

5.1 The computational model and its calibration

The model is a Nordhaus-type integrated assessment model of the climate and the economy. The calibration is made in such a way that we replicate the central 2007 IPCC scenarios in terms of carbon emissions and temperature increase.[10] The abatement cost function and the damage function are calibrated to replicate Nordhaus' orders of magnitude. So it must be clear that there is nothing new in the model itself, because we want to focus on new behavioral rules. We compute numerically the economy described in section 2 over a period of $T = 300$ years. To get rid of boundary effects we display results only for the first 175 years. The time slice is the year, and for the discretization scheme we choose a time step $h = 0.05$ years. Table 1 displays the parameters value and initial conditions for the economic and climatic parts of the model. We have deliberately chosen a low value for the interest rate (with respect to the values chosen by Nordhaus in the DICE model) because of the arguments given by [11] and [17]. We consider that, given the length of our exercise, such a low value is necessary to fully take into account the welfare of far future generations. The initial values for the state variables are $k_0 = 733.2$, the initial value of physical capital (in billion dollars), $m_0 = 808.9$, the carbon concentration in 2005 (in GtC) and $\tau_0 = 0.7307$ the initial temperature increase (between 1900 and 2005, in °C).

Table 1: Parameters value

Economic parameters		
depreciation rate	δ	0.1
intertemporal elasticity of substitution	α	0.5
interest rate	r	0.005
capital elasticity	γ	0.75
Climate parameters		
temperature reabsorption	λ	0.11
climate sensitivity	η	0.0054
preindustrial carbon concentration	m_0^*	596.4
damage function parameter	θ_1	0.0057
damage function parameter	θ_2	2
CO_2 reabsorption rate	ν	0.0054

[10]The 2007 IPCC report is available on line at: *www.ipcc.ch*.

In what follows, we describe the functional forms for the functions p, e, φ, c and d. There exists an exogenous technological progress, $p(t)$. It is a linear function of time such that $p(0) = 1$ and $p(175) = 1.25$. The exogenous technology which reduces carbon emission intensity with time, $e(t)$, is an exponential function verifying that $e(0) = 0.0427$ and $e(75) = 0.75e(0)$; *i.e.* we assume an exogenous decrease in the emission output intensity by 25% in 75 years. All these assumptions are within common ranges.[11]

The damage function has a standard formulation $\varphi(\tau) = 1/(1+\theta_1\tau^{\theta_2})$. The calibration captures the impact of global warming on global productivity: a 2°C increase in the mean temperature reduces the global productivity by 3%. We assume $c(a) = 0.01a/(1-a)$; *i.e.* $c(0) = 0$ and $\lim_{a\to 1} c(a) = \infty$. The calibration implies that reducing the emission output intensity by 50% incurs a cost of 1% of GDP. The effect of CO_2 concentration on the average temperature increase is captured by the standard function $d(m) = \eta \ln(m/m_0^*)$: a doubling of CO_2 concentration increases the average temperature by 0.41°C per year, where $m_0^* = 596.4$ GtC is the preindustrial CO_2 concentration in the atmosphere.

5.2 A comparison between BaU and OPT scenarios

The first exercise we perform shows convergence of the BaU trajectory as revisions take place more frequently. Figure 1 shows the trajectory for the share of output to be consumed, u, under two BaU scenarios in which two revision schemes are considered, after 10 years (dashed line) and after 5 years (dotted line). One can see that the trajectory converges towards a smooth BaU as the revision scheme becomes shorter. When the revision scheme is slow, (every 10 years) the share of output that is consumed moves too slowly. In other words, too much capital is invested in order to have more consumption in the future. The trade-off between consumption and investment is biased because the agent under-estimates the climate damages that will occur in the far future. When the revision occurs, the agent realizes her mistake and she sharply decreases capital accumulation. But then again, she makes the same kind of mistake during a decade. This balance between the mistake (too much capital accumulation and too much emissions) and the revision becomes smoother as the economy approaches the steady state. It must be noticed that the correction is sharp because there are no adjustment costs in our model. The movements in consumption would be smoother if adjustment costs were considered on capital accumulation or on emission abatement.

Next we have computed the world economy in which the OPT policy maker can abate, i.e. she can devote a share of output to reduce CO_2 emissions. Results for the OPT and BaU scenarios are displayed in figure 2.

In the OPT scenario, the share of output that is consumed (panel a) is always larger than in the BaU scenario. It results that capital accumulation is always smaller (panel d). Interestingly, the gap between these two trajectories shrinks with time (panels d), and the capital stock is almost

[11] See the IPCC (2007), Nordhaus (2008), Stern (2008) and Yang (2009).

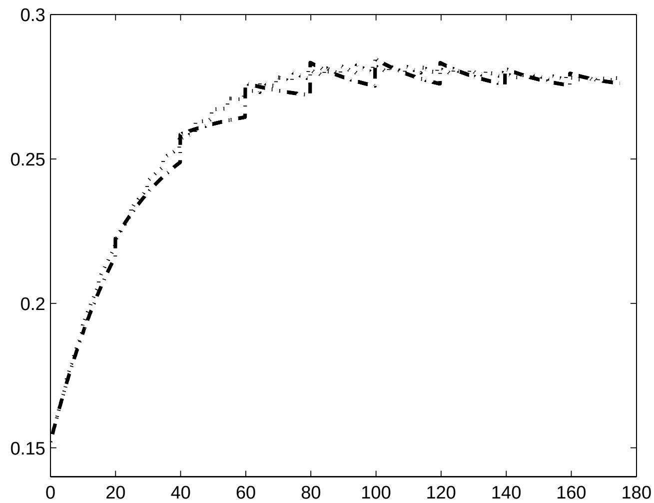

Figure 1: BaU convergence. Share of output consumed. Legend: BaU 10 years (dashes), BaU 5 years (dots).

the same in the two scenarios at the end of the simulation period. So, what makes the difference between the two scenarios is the transition path.

Actually, the consumption level is not always higher in OPT than in BaU (despite the fact that the share is always higher). Let us study the reasons: BaU's larger investment induces not only a larger level of physical capital and production in the future (all else equal) but also more emissions and consequently a larger temperature increase. Damages only come out in the long run. So at the beginning of the simulation period, consumption level is higher is OPT, but because the BaU accumulates more capital, it brings a larger consumption level between period 35 and 88. After period 85, the costs of global warming become large and capital accumulation is refrained in BaU. And consumption is again larger in OPT. The feed-back effect of climate on the economy is sharp: economic growth is almost zero in BaU after period 60.

Panel b) shows the abatement rate. Provided the abatement cost function we have chosen, an abatement rate of 45% is optimal since the very first time period. Abatement then increases and reaches its maximum after 40 years at roughly 52%. In the long run, emissions are thus cut by 33%. As a result, under this OPT scenario the temperature is reduced in the long-run by 2°C with respect to the BaU scenario.

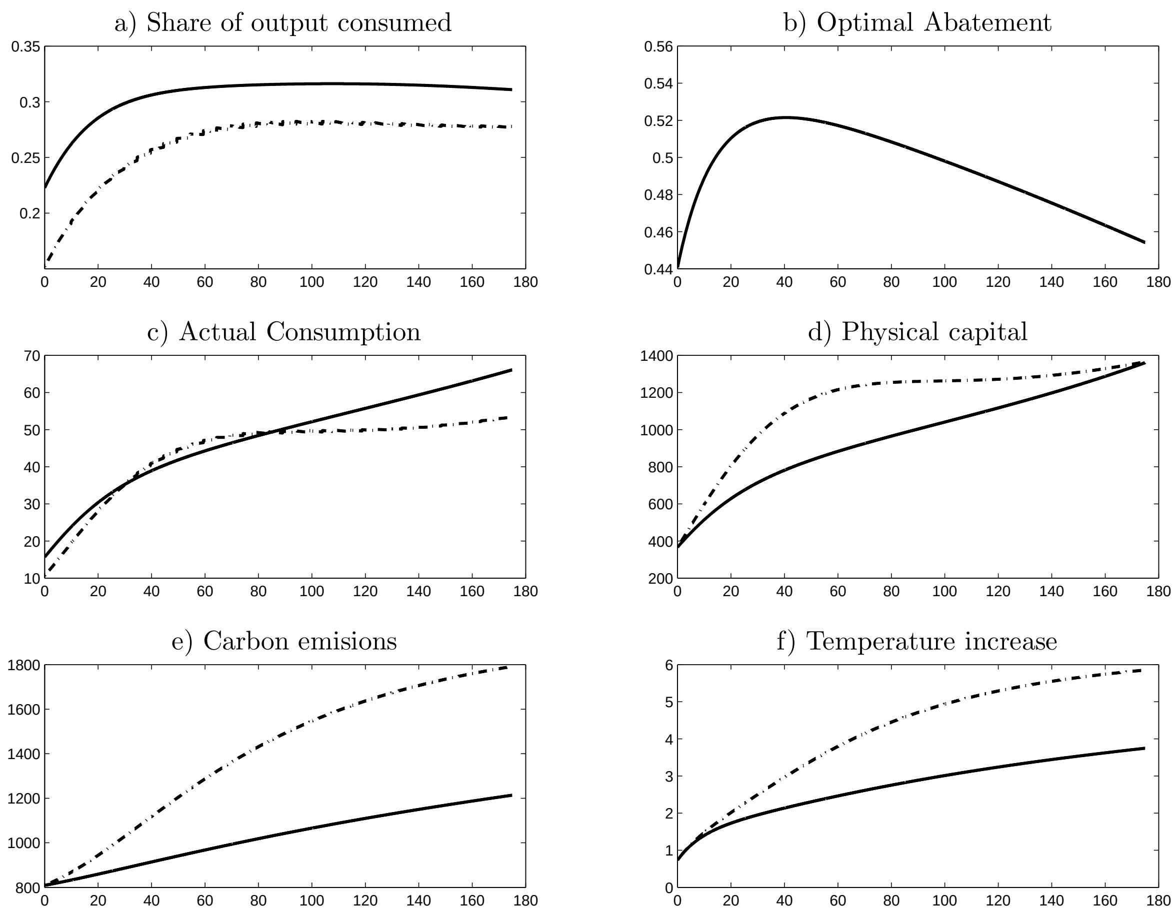

Figure 2: Optimal solutions under abatement. Legend: OPT (solid line), BaU (dash-dotted line)

5.3 Co-existing agents with different strategies

Let us now consider that there exist two agents in the economy. These two agents are of the same size and have similar initial capital endowment. One agent follows a BaU scenario, the other follows an OPT scenario. Naturally, they share the same global temperature increase.

The BaU-agent maximizes her utility by choosing her optimal trajectory $\{u(t)\}_{t\geq 0}$, assuming that the climate is not affected by her economic activity, *i.e.* she assumes $\tau(t) = \tau_0$, $\forall t \geq 0$. Actually, the climate is modified by her activity and her actual consumption will differ from the expected one. The OPT-agent chooses $\{u(t), a(t)\}_{t\geq 0}$ and takes into account the temperature increase due to her production plan. Besides, she knows the BaU-agent's decisions and incorporates them into the dynamics of carbon concentration in the atmosphere. It must be stressed that, because the BaU-agent's optimal decisions are not realized, the OPT-agent computes 'pseudo-optimal' trajectories. Provided BAU-agent and OPT-agent decisions, we compute in a final step actual CO_2 emissions and temperature change. Every five years both agents revise their decisions after noticing the divergence between their expected and realized consumption levels.

With these two agents we are able to compare three economies. In the two first, two similar agents co-exist, two BaU-agents or two OPT-agents. Actually, these two cases provide the same solution as the economy with a single BaU or a single OPT initially endowed with double amounts of physical capital and labor. The last economy that can be considered is the one where two different agents co-exist, one BaU and one OPT. In this latter case, even though the two agents are initially of the same size, they will become different as time goes on because they follow different decisions rules.

Figure 3 displays the main results. Let us stress that emissions, capital stock and consumption are expressed per capita.

In the economy with one BaU-agent and one OPT-agent, emissions and global warming are in between single OPT and single BaU, simply because one agent does not abate (see panel e and f). Nevertheless, global emissions and temperature are closer to the single BaU than to the single OPT, which reveals that the effectiveness of the OPT-agent abatement efforts is rather small. The striking result in this two-agent economy is that the BaU-agent can accumulate more capital than the OPT-agent, because she spends nothing in abatement. Thus she enjoys a higher consumption level than the OPT-agent during the whole simulation period. There exists a clear incentive to free-ride. In the meantime, the OPT-agent does her best to combat global warming. As a result, her consumption level turns out to be quite close to the single BaU scenario during the whole simulation period (see panel c). By playing BaU instead of OPT, the BaU-agent is better-off until period 115. Surprisingly, the OPT-agent is also better-off, but for a shorter time span (until period 80). Then, both are worst-off (*w.r.t.* two-OPT) because the climate is warming too much.

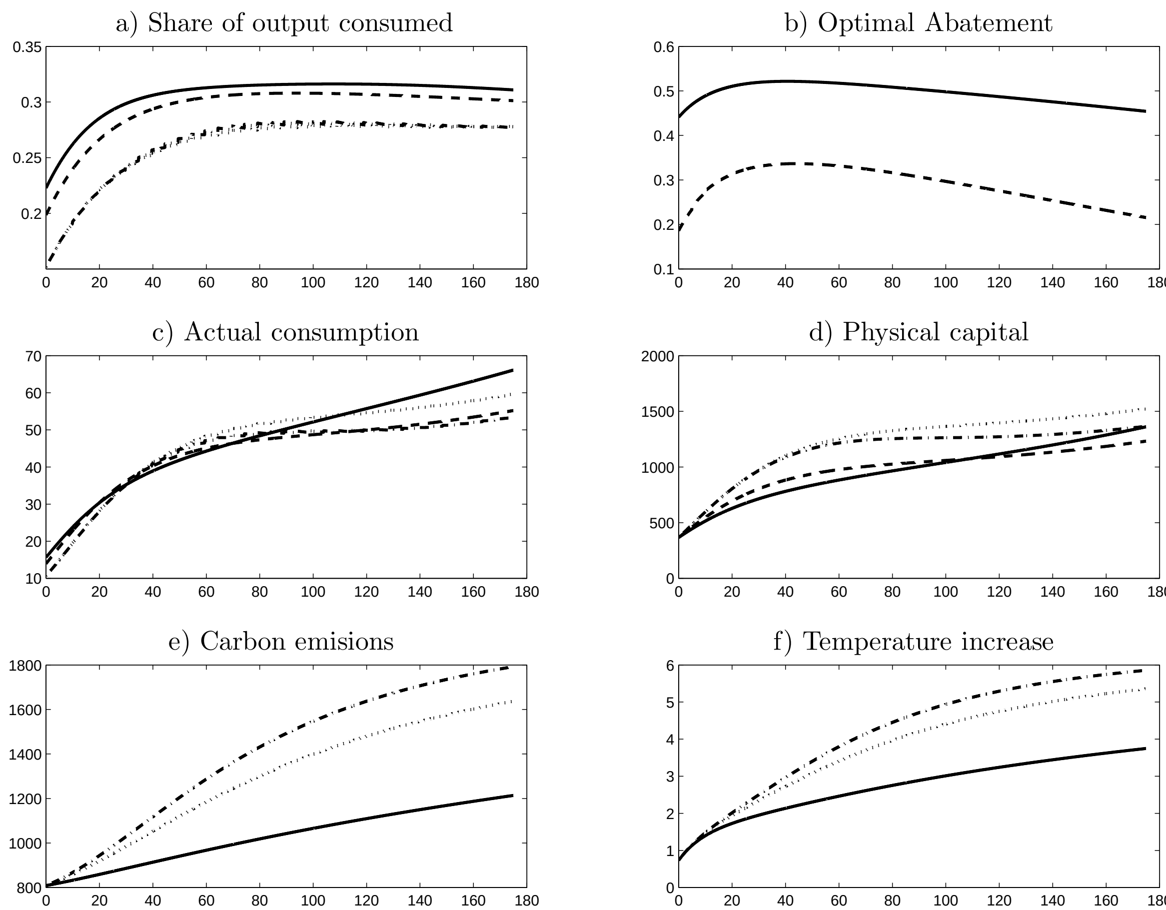

Figure 3: Two coexisting agents. Legend: OPT-OPT (solid line), OPT in OPT-BAU (dashed line), BAU in OPT-BAU (dotted line), BAU-BAU (dash-dotted line).

6 Conclusion

In this paper we reconsider agents decision rules in an integrated assessment model *à la* Nordhaus. To depart from the standard ill-defined Business-as-Usual scenario we propose alternative – and arguably more realistic – behavioral decision rules. The modeling technique we use adapts ideas from an area in the engendering-oriented control theory, known as *model predictive control.* This allows us to model two specific prediction mappings leading to Business-as-Usual (BaU) and Free-Riding decision rules, and to compare them with the Optimal scenario (OPT). In both scenarios, the agent revises her decisions on a regular basis by using exogenous predictions for the evolution of the environment. Existence and uniqueness of the BaU solution is established. We also consider the case of an economy with two co-existing agents having different decision rules, one following BaU and the other OPT. The existence is also establish. These concepts are then illustrated with a numerical application to the global warming issue. We first consider the BaU scenario and show how the revision process shapes the trajectory of the economy. By comparing BaU with OPT, it is shown that the over accumulation of capital in the former leads to large climate damages, thus stopping economic growth after a while. Finally, in a two-agent economy we show that there exists a strong incentive to play BaU instead of OPT.

Appendix 1: Proof of Proposition 2

First we mention that $\hat{v}_i$ and $\hat{x}_2$ satisfy A2'(ii) and A2'(iii) also with θ and $\tilde{\theta}$ (or only $\tilde{\theta}$) set formally to ∞. This is because $\hat{l}$, α and β are independent of θ. Then with x_i, $\tilde{x}_i \in X$, y, $\tilde{y} \in Y$ we have for $\tau \geq t$

$$\begin{aligned}
&|\eta_2[x_2, y](t,\tau) - \eta_2[\tilde{x}_2, \tilde{y}](t,\tau)| \\
&\leq l\,(|\hat{v}_2[t,\infty,x;y](\tau) - \hat{v}_2[t,\infty,\tilde{x};\tilde{y}](\tau)| + |\hat{x}_2[t,\infty,x;y](\tau) - \hat{x}_2[t,\infty,\tilde{x};\tilde{y}](\tau)|) \\
&\leq l\beta(\tau - t)(|x-\tilde{x}| + |y-\tilde{y}|)
\end{aligned}$$

where l is the Lipschitz constant of e_2. Hence,

$$\begin{aligned}
&|v_1[x_1,x_2,y](t) - v_1[\tilde{x}_1,\tilde{x}_2,\tilde{y}](t)| = |\hat{v}_1[t,\infty,x_1,y;\eta_2[x_2,y](t,\cdot)](t) - \hat{v}_1[t,\infty,\tilde{x}_1,\tilde{y};\eta_2[\tilde{x}_2,\tilde{y}](t,\cdot)](t)| \\
&\leq \hat{l}\left(|x-\tilde{x}| + |y-\tilde{y}| + \int_t^\infty \alpha(\tau - t)|\eta_2[x_2,y](t,\tau) - \eta_2[x_2,y](t,\tau)|\,\mathrm{d}\tau\right) \\
&\leq \hat{l}(|x-\tilde{x}| + |y-\tilde{y}|)\left(1 + \int_t^\infty \alpha(\tau-t)l\beta(\tau-t)\,\mathrm{d}\tau\right) \leq C(|x-\tilde{x}| + |y-\tilde{y}|),
\end{aligned}$$

where C is a finite number. Thus v_1 in (26)–(28) depends in a Lipschitz way on x_1, x_2 and y. The same applies obviously to v_2 and η_1. The function $\eta_2(t,t)$ that enters in (28) is also Lipschitz in x_2 and y according to the expression after Proposition 2, A1$'$ and A2$'$(iii).

Thus the right-hand side of system (26)–(28) is Lipschitz continuous with respect to x_1, x_2 and y. Then a unique solution (x_i^*, y^*) exists at least locally. Obviously this triple solves also (17), (18) with

$$v_1^*(t) = \hat{v}_1[t, \infty, x_1^*(t), y^*(t); \eta_2^*(t, \cdot)](t), \qquad v_2^*(t) = \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](t), \tag{30}$$

where

$$\eta_2^*(t, \cdot) = e_2(\cdot, \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\cdot), \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\cdot)) = \eta_2[x_2^*(t), y^*(t)](t, \cdot).$$

In particular, A1$'$(ii) implies that (x_i^*, y^*) is extendible to $[0, \infty)$.

Now we consider an arbitrary pair $\sigma = (h, \theta)$ with $h > 0$, $(\rho - 1)\theta > h$ (the last inequality is not restrictive, since later we shall let $h \to 0$, $\theta \to +\infty$). We shall compare (x_i^*, y^*) with the step-wise OPT-BaU solution $(v_i^\sigma, x_i^\sigma, y^\sigma)$, corresponding to σ. Denote $t_k^\sigma = kh$ and

$$\varepsilon_k^\sigma = \max_{t \in [t_{k-1}^\sigma, t_k^\sigma]} (|(x_1^\sigma(t) - x_1^*(t)| + |(x_2^\sigma(t) - x_2^*(t)| + |(y^\sigma(t) - y^*(t)|)$$

for $k = 0, 1, \ldots$, and with t_{-1}^σ redefined as 0.

Obviously we have $\varepsilon_0^\sigma = 0$. We shall recurrently estimate ε_{k+1}^σ by ε_k^σ. In doing this we use A2$'$ and the fact that

$$|\dot{x}_1^*(t)| + |\dot{x}_2^*(t)| + |\dot{y}^*(t)| \le M \qquad \text{for every } \ t \ge 0,$$

where M is an appropriate constant. Thus for $t \in [t_k^\sigma, t_{k+1}^\sigma]$ we have

$$|x_i^\sigma(t_k^\sigma) - x_i^*(t)| \le |x_i^\sigma(t_k^\sigma) - x_i^*(t_k^\sigma)| + |x_i^*(t_k^\sigma) - x_i^*(t)| \le \varepsilon_k^\sigma + Mh,$$

and similarly for y.

Using the A2$'$(iii) and the above inequality we obtain for $t \in [t_k^\sigma, t_{k+1}^\sigma]$

$$\begin{aligned} |v_2^\sigma(t) - v_2^*(t)| &= |\hat{v}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](t) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](t)| \\ &\le \beta(t - t_k^\sigma) \left[\varepsilon_k^\sigma + Mh + \delta\left(h + \frac{1}{t_k^\sigma + \rho\theta - t}\right)\right] \le \beta(t - t_k^\sigma) \left[\varepsilon_k^\sigma + Mh + \delta\left(h + \frac{1}{\theta}\right)\right] \\ &\le C_1 \left(\varepsilon_k^\sigma + Mh + \delta\left(h + \frac{1}{\theta}\right)\right), \end{aligned}$$

where C_1 is independent of σ. (Obviously one can assume without any restriction that δ from assumptions A2$'$(ii) and A2$'$(iii) is monotone increasing.)

Moreover, for $\tau \in [t_k^\sigma, t_k^\sigma + \theta]$ we estimate

$$\begin{aligned} &|\tilde{v}_2^\sigma(\tau) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\tau)| = |\hat{v}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](\tau) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\tau)| \\ &\le \beta(\tau - t_k^\sigma) \left[\varepsilon_k^\sigma + Mh + \delta\left(h + \frac{1}{t_k^\sigma + \rho\theta - \tau}\right)\right] \le \beta(\tau - t_k^\sigma) \left[\varepsilon_k^\sigma + Mh + \delta\left(h + \frac{1}{(\rho - 1)\theta}\right)\right]. \end{aligned}$$

In fact, according to A2$'$(iii) the same estimate holds for the sum of the above estimated difference and $|\tilde{x}_2^\sigma(\tau) - \hat{x}_2[t, \infty, x_2^*(t); y^*(t)](\tau)|$. Then

$$
\begin{aligned}
|\tilde{\eta}_2^\sigma(\tau) - \eta_2^*(t,\tau)| &= |e_2(\tau, \tilde{v}_2^\sigma(\tau), \tilde{x}_2^\sigma(\tau)) - e_2(\tau, \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\tau), \hat{x}_2[t, \infty, x_2^*(t); y^*(t)](\tau))| \\
&\le l\beta(\tau - t_k^\sigma)\left[\varepsilon_k^\sigma + Mh + \delta\left(h + \frac{1}{(\rho-1)\theta}\right)\right],
\end{aligned}
$$

where l is the Lipschitz constant of e_2. Hence for $t \in [t_k^\sigma, t_{k+1}^\sigma]$ we obtain successively

$$
\begin{aligned}
&|v_1^\sigma(t) - v_1^*(t)| = |\hat{v}_1[t_k^\sigma, \theta, x_1^\sigma(t_k^\sigma), y^\sigma(t_k^\sigma); \tilde{\eta}_2^\sigma(\cdot)](t) - \hat{v}_1[t, \infty, x_1^*(t), y^*(t); \eta_2^*(t, \cdot)](t)| \\
&\le \hat{l}(\varepsilon_k^\sigma + Mh) + \delta\left(h + \frac{1}{\theta}\right) + \int_t^{t_k^\sigma+\theta} \alpha(\tau - t_k^\sigma)|\tilde{\eta}_2^\sigma(\tau) - \eta_2^*(t,\tau)|\,\mathrm{d}\tau \\
&\le \hat{l}(\varepsilon_k^\sigma + Mh) + \delta\left(h + \frac{1}{\theta}\right) + l\left[\varepsilon_k^\sigma + Mh + \delta\left(h + \frac{1}{(\rho-1)\theta}\right)\int_0^\infty \alpha(z)\beta(z)\,\mathrm{d}z\right] \\
&\le C_2\left[\varepsilon_k^\sigma + h + \delta\left(h + \frac{1}{\gamma\theta}\right)\right],
\end{aligned}
$$

where $\gamma = \min\{1, \rho - 1\}$ and C_2 is independent of σ.

Denoting by λ the Lipschitz constant of the overall right-side in (17), (18) with respect to (v_i, x_i, y) and $C_3 = C_2 + C_3$ we obtain in a standard way (using the Grönwall inequality) that

$$
\begin{aligned}
\varepsilon_{k+1}^\sigma &\le e^{\lambda h}\left[\varepsilon_k^\sigma + \lambda C_3 h\left(\varepsilon_k^\sigma + h + \delta\left(h + \frac{1}{\gamma\theta}\right)\right)\right] \\
&= e^{\lambda h}(1 + \lambda C_3 h)\varepsilon_k^\sigma + e^{\lambda h}\left[\lambda C_3 h^2 + \lambda C_3 h\,\delta\left(h + \frac{1}{\gamma\theta}\right)\right].
\end{aligned}
$$

This easily implies (see e.g. Lemma 2.2 in [19]) existence of a number C_3 (independent of σ) such that

$$
\varepsilon_k^\sigma \le C_4 e^{\lambda k h}\left[h + \delta\left(h + \frac{1}{\gamma\theta}\right)\right] \quad \text{for every } k \ge 0.
$$

Then from the estimations for $|v_i^\sigma(t) - v_i^*(t)|$ obtained above we conclude that for every finite interval $[0, T]$ one can estimate

$$
|v_i^\sigma(t) - v_i^*(t)| \le \bar{C}_T\left[h + \delta\left(h + \frac{1}{\gamma\theta}\right)\right]
$$

with $\bar{C}_T$ independent of σ. Hence $v_i^\sigma \to v_i^*$ in $C(0, \infty)$ when $h \to 0$, $\theta \to +\infty$, which implies the claim of the proposition.

Appendix 2: Discussion of the assumptions

Below we provide a certain justification in the case of the global warming model of the assumptions made in sections 3 and 4, focusing on the more requiring assumptions A1$'$ and A2$'$.

The critical part of assumption A1$'$ is the (uniform in t) Lipschitz continuity in the set $X \times X \times Y$, combined with the invariance of this set with respect to the respective differential equations. Theoretically, due to purely physical arguments, the productivity function $p(t)$ must be bounded. Hence, according to (2) the capital stock is bounded, therefore the emission is also bounded (since the emission rate $e(t)$ is also bounded). From the stability of the atmospheric system (3), (4) the environmental states are also bounded. Hence, the sets X, Y, and E can be specified as bounded sets. Thus the only trouble for the Lipschitz continuity is caused by the unboundedness of our specification for $c(a)$ when $a \to 1$. However, this trouble is artificial – the specification for $c(a)$ is actually relevant only for values of a far below $a = 1$. Thus assumption A1$'$ is easily justifiable for the global warming model.

The justification of A2$'$ is much more complicated, since it involves the (unknown) optimal controls $\hat{v}_1$ and $\hat{v}_2$. In general, even the uniqueness assumption A2$'$(i) is problematic in the Nordhause type models, as it is shown in [10]. Namely, in the version of the DICE model considered in [10] the authors show existence of a threshold manifold in the economic-environment state space, starting at which the optimal policy is not unique: one leads to "lower warming", a second one – to a "higher warming". However, in that paper the abatement is not used as a control, rather just an exogenous parameter (there are also other small differences in the our model and the one in [10]). Multiple solutions appear only in the case of a low value of the abatement parameter. Most probably the non-uniqueness disappears when the abatement is used as a second control variable, since the numerical experiments clearly shown uniqueness for the calibration of the model described in Section 5.

Assumption A2$'$(iii) is not difficult to check, since $\hat{v}_2$ is the optimal investment control (the abatement control a obviously equals identically zero) in a version of the Ramsey model with a non-stationary but bounded parameter.

Assumption A2$'$(ii) is difficult for strict verification. Its claim concerning the dependence of $\hat{v}_1$ on θ is an well-know property, but general sufficient conditions for this property are, to our knowledge, not available in the literature. However, it is clear that this property relies on a certain dissipativity, and such is present in our model in a reasonable domain $X \times Y$.

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