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Abstract

The main result in this short note is that the integral form of the Leitmann-Stalford sufficiency conditions can be verified for a class of optimal control problems whose Hamiltonian is not concave with respect to the state variable. The main requirement for this class of problems is that the dynamics is sufficiently dissipative. An application to a Stackelberg differential game between a producer and a developer is exemplified. Using our result we show that the necessary conditions implied by Pontryagin's maximum principle are also sufficient. This allows a complete characterization of the solution.

Keywords: optimal control, sufficient optimality conditions, differential games, Stackelberg equilibrium

1   Introduction

Pontryagin's maximum principle provides necessary conditions for optimality in intertemporal optimization problems. It is well-known that these conditions are generally not sufficient for optimality. Mangasarian [1] has shown that, in addition to Pontryagin's necessary conditions, concavity of the Hamiltonian with respect to the state and control variables is sufficient for optimality in maximization problems.

Arrow [2], see also [3], extended Mangasarian's result to the concavity with respect to the state variables of the Hamiltonian maximized with respect to the control variable. Both concavity conditions are rather strong and not satisfied in many applications.

At the same time, Leitmann and Stalford ([4]) derived (what they called *direct*) sufficiency conditions; compare also [5, 6], [7, Sect. 15.6] and Theorem 1 in Section 2 below. Seierstad and Sydsaeter [8] gave a fine survey and new results on sufficient optimality conditions in control theory. Their theorems 3 and 4 can be viewed as extensions of the Leitmann-Stalford conditions to discontinuous infinite horizon problems, related to the subject of the present paper.

Although the Leitmann-Stalford theorem is more general than the Mangasarian-Arrow results, it is not easily applicable for problems that not satisfy the Arrow concavity condition. The reason for this is indicated in the end of Section 2. The main point of the present paper is that a slight modification of the Leitmann-Stalford sufficiency theorem (just reformulating it in an even more direct *integral* form) allows applications even in the case of non-concave Hamiltonian. Our main result shows that the integral form of the Leitmann-Stalford sufficiency conditions is fulfilled for a class of control problems for dissipative systems if certain relations involving the data of the problem are satisfied, however not implying concavity of the Hamiltonian. As an application we consider an economic differential game in the Stackelberg framework, where the leader's problem has a convex (with respect to the state variable) Hamiltonian. Using our main theorem, we prove that a Stackelberg equilibrium exists if certain quantitative conditions on the data are satisfied.

The paper is organized as follows. Section 2 presents the integral formulation of the Leitmann-Stalford sufficiency theorem. Section 3 contains our main result, which provides a tool for verification of the Leitmann-Stalford conditions. Section 4 presents the application to a Stackelberg game.

2 The Leitmann-Stalford sufficient conditions

In this section we remind the Leitmann-Stalford sufficient conditions [5] in a simplified situation – without state or terminal constraints. The formulation differs with the original only in that it has integral instead of pointwise form.

Consider the optimal control problem

$$\max \int_0^\infty e^{-rt} g_0(t, x(t), u(t)) \, \mathrm{d}t, \tag{1}$$

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subject to

$$\dot{x}(t) = g(t, x(t), u(t)), \quad x(0) = x^0, \qquad u(t) \in U, \tag{2}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$. Throughout the paper it is assumed that the discount rate r is positive, the set $U \subset \mathbf{R}^m$ is closed, and the functions g_0, $g : [0, \infty) \times \mathbf{R}^n \times U \mapsto \mathbf{R}$ (resp. $\mapsto \mathbf{R}^n$) are measurable in t, differentiable in x, and Lipschitz in u (uniformly with respect to t and x).

Let $\mathcal{U}$ be a set of measurable functions $u : [0, \infty) \mapsto U$ for which the solution $x[u]$ of (2) exists on $[0, \infty)$ and the integral in (1) is finite. The functions from the set $\mathcal{U}$ will be viewed as *admissible controls*.

Define the Hamiltonian

$$H(t, x, u, \lambda) = g_0(t, x, u) + \langle \lambda, g(t, x, u) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbf{R}^n$.

Theorem 1 *(Leitmann-Stalford condition in integral form.)*
Let $u^* \in \mathcal{U}$, let $x^* = x[u^*]$ be the corresponding trajectory, and let $\lambda : [0, \infty) \mapsto \mathbf{R}^n$ be absolutely continuous. Let following two conditions be fulfilled for every $u \in \mathcal{U}$ and $x = x[u]$:

(i) $\displaystyle \int_0^\infty e^{-rt} \Big[H(t, x^*(t), u^*(t), \lambda(t)) - H(t, x(t), u(t), \lambda(t))$

$$-\langle \dot{\lambda}(t) - r\lambda(t), x^*(t) - x(t) \rangle \Big] \, \mathrm{d}t \geq 0$$

and

(ii) $\lim_{t \to \infty} e^{-rt} \langle \lambda(t), x^*(t) - x(t) \rangle \leq 0.$

Then (u^*, x^*) is an optimal solution.

The proof is the same as in [5], with obvious modifications caused by the infinite horizon (cf. with Theorem 10 in [8]).

The reason we formulate condition (i) in integral form instead of the pointwise formulation in [5] is the following. Under the simplifying supposition that H_x is independent of u the pointwise version of condition (i) (as in [5]) is essentially equivalent to the Arrow condition if the solution of the corresponding adjoint equation is used as λ. In the next section we shall see that this is not the case if the integral formulation is used.

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3 Verifying the first Leitmann-Stalford condition without concavity of the Hamiltonian

In this section we consider a class of problems where the Arrow sufficiency conditions cannot be applied due to the non-concavity of the Hamiltonian with respect to the state variable, but the Leitmann-Stalford theorem is still applicable in its integral form. In the following we assume that $H_x(t, x, u, \lambda) = H_x(t, x, \lambda)$ is independent of u. This assumption is not required in principle, but it simplifies the estimations employed in the proof of our main result. Moreover, below we suppress the argument t of g_0, g and H.

In this section the set of admissible controls $\mathcal{U}$ will be assumed to be a subset of the weighted space $L_2(e^{-rt}; [0, \infty) \mapsto \mathbf{R}^m)$, consisting of those measurable functions $u : [0, \infty) \mapsto U$ for which

$$\|u\|_2 := \sqrt{\int_0^\infty e^{-rt} |u(t)|^2 \, dt}$$

is finite.

Denote by $X \subset \mathbf{R}^n$ a set that contains all states $x[u](t)$ for any $u \in \mathcal{U}$ and $t \geq 0$. Let us denote by L_g a Lipschitz constant of g with respect to $u \in U$, uniformly in $x \in X$ and $t \geq 0$.

Theorem 2 *Let $u^* \in \mathcal{U}$, let $x^* = x[u^*]$ be the corresponding trajectory, and let $\lambda : [0, \infty) \mapsto \mathbf{R}^n$ be absolutely continuous. Assume that λ satisfies the adjoint equation*

$$\dot{\lambda}(t) = r\lambda(t) - H_x(x^*(t), \lambda(t)). \tag{3}$$

Let there exist numbers $\alpha \geq 0$, β, and $\delta > 0$, such that inequalities (4)–(7) below are satisfied for almost every $t \geq 0$ and for every $x \in X$ and $u \in U$:

$$H(x^*(t), u^*(t), \lambda(t)) - H(x^*(t), u, \lambda(t)) \geq \alpha |u - u^*(t)|^2, \tag{4}$$

$$H(x^*(t), u, \lambda(t)) - H(x, u, \lambda(t)) - \langle H_x(x^*(t), \lambda(t)), \, x^*(t) - x \rangle \geq -\beta |x^*(t) - x|^2, \tag{5}$$

$$\langle g(x^*(t), u) - g(x, u), x^*(t) - x \rangle \leq -\delta |x - x^*(t)|^2, \tag{6}$$

$$\beta L_g^2 \leq 2r\alpha\delta. \tag{7}$$

Then the triple (u^*, x^*, λ) satisfies condition (i) in Theorem 1.

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Before proving the theorem we briefly comment conditions (4)–(7). If the set U is convex and H happens to be twice continuously differentiable in u, then condition (4) means that Pontryagin's condition

$$H(x^*(t), u^*(t), \lambda(t)) = \max_{u \in U} H(x^*(t), u, \lambda(t)) \tag{8}$$

holds for a.e. $t \geq 0$ and in addition, H is uniformly strongly concave with respect to u around $u^*(t)$ with coefficient α of strong concavity. Condition (5) is automatically fulfilled (with some β) if H is twice continuously differentiable in x and H_{xx} is bounded in X. Condition (6) is more restrictive since it requires dissipativity of the dynamic equation (2). The essential restriction is posed by condition (7). Its meaning is that the deviation from u^* directly damages the objective value more than the potential benefit that the corresponding deviation from x^* brings. This will be clearly seen in the proof.

Proof of Theorem 2. We have from (4) and (5)

$$\int_0^\infty e^{-rt} \left[H(x^*(t), u^*(t), \lambda(t)) - H(x(t), u(t), \lambda(t)) - \langle \dot{\lambda}(t) - r\lambda(t), x^*(t) - x(t) \rangle \right] \mathrm{d}t$$

$$= \int_0^\infty e^{-rt} \left[H(x^*(t), u^*(t), \lambda(t)) - H(x(t), u(t), \lambda(t)) + \langle H_x(x^*(t), \lambda(t)), x^*(t) - x(t) \rangle \right] \mathrm{d}t$$

$$= \int_0^\infty e^{-rt} \left[H(x^*(t), u^*(t), \lambda(t)) - H(x^*(t), u(t), \lambda(t)) \right] \mathrm{d}t$$

$$+ \int_0^\infty e^{-rt} \left[H(x^*(t), u(t), \lambda(t)) - H(x(t), u(t), \lambda(t)) + \langle H_x(x^*(t), \lambda(t)), x^*(t) - x(t) \rangle \right] \mathrm{d}t$$

$$\geq \alpha \int_0^\infty e^{-rt} |u^*(t) - u(t)|^2 \, \mathrm{d}t - \beta \int_0^\infty e^{-rt} |x^*(t) - x(t)|^2 \, \mathrm{d}t. \tag{9}$$

Below we shall estimate the last term. For shortness denote $\Delta x(t) = x^*(t) - x(t)$, $\Delta u(t) = u^*(t) - u(t)$. Using (6) we obtain that for those t for which $\Delta x(t) \neq 0$

$$|\Delta x(t)| \frac{\mathrm{d}}{\mathrm{d}t} |\Delta x(t)| = \frac{\mathrm{d}}{2\,\mathrm{d}t} |\Delta x(t)|^2 = \langle \Delta \dot{x}(t), \Delta x(t) \rangle$$

$$= \langle g(x^*(t), u^*(t)) - g(x(t), u(t)), \Delta x(t) \rangle$$

$$= \langle g(x^*(t), u^*(t)) - g(x^*(t), u(t)), \Delta x(t) \rangle + \langle g(x^*(t), u(t)) - g(x(t), u(t)), \Delta x(t) \rangle$$

$$\leq L_g |\Delta u(t)| |\Delta x(t)| - \delta |\Delta x(t)|^2.$$

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Thus for t for which $\Delta x(t) \neq 0$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta x(t)| \leq -\delta|\Delta x(t)| + L_g|\Delta u(t)|, \quad \Delta x(0) = 0.$$

This in a standard way implies the inequality[1]

$$|\Delta x(t)| \leq L_g \int_0^t e^{-\delta(t-s)}|\Delta u(s)|\,\mathrm{d}s.$$

Hence

$$|\Delta x(t)|^2 \leq L_g^2 \int_0^t e^{-2\delta(t-s)}\,\mathrm{d}s \int_0^t |\Delta u(s)|^2\,\mathrm{d}s \leq \frac{L_g^2}{2\delta} \int_0^t |\Delta u(s)|^2\,\mathrm{d}s.$$

Then we can estimate

$$
\begin{aligned}
\int_0^\infty e^{-rt}|x^*(t) - x(t)|^2\,\mathrm{d}t \quad &\leq \quad \frac{L_g^2}{2\delta}\int_0^\infty e^{-rt}\int_0^t |\Delta u(s)|^2\,\mathrm{d}s\,\mathrm{d}t \leq \frac{L_g^2}{2\delta}\int_0^\infty \int_s^\infty e^{-rt}\,\mathrm{d}t|\Delta u(s)|^2\,\mathrm{d}s \\
&\leq \quad \frac{L_g^2}{2r\delta}\int_0^\infty e^{-rt}|\Delta u(t)|^2\,\mathrm{d}t.
\end{aligned}
$$

Then (7) implies that the quantity in (9) is non-negative, hence (i) in Theorem 1.
  Q.E.D.

4   An application to a Stackelberg differential game

In this section we consider a differential game in which the first player—the leader (in the sense of Stackelberg games [9])—is a large firm or an industry, called further *Producer*. The second player—the follower—is the government or a non-profit institution, called further *Developer*. Producer aims at maximizing profits, Developer tries to help increasing the productivity of Producer in an efficient way by investments in R&D.

We consider the most simple model of the above type aimed to demonstrate an application of Theorem 2. The Producer's problem is

$$\max_{u(t) \geq 0} \int_0^\infty e^{-rt}\left[P(t)K(t) - cu(t)^2 \right]\,\mathrm{d}t$$

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[1] The Gronwall inequality can be applied in every subinterval in which $\Delta x(t) \neq 0$.

subject to
$$\dot{K}(t) = -\delta K(t) + u(t), \quad K(0) = K^0. \tag{10}$$

Here $K(t)$ is the capital stock at time t, $P(t)$ is the productivity of the capital (depending on the policy of Developer), K^0 is the initial capital stock, u is investment in physical capital, δ is depreciation rate, cu^2 is the cost of investments u (including adjustment costs).

The Developer (who is a follower, in the terminology of the Stackelberg games) has P as a state variable and solves the problem

$$\max_{v(t) \geq 0} \int_0^\infty e^{-rt} \left[\gamma \dot{P}(t) \dot{K}(t) - dv(t)^2 \right] \mathrm{d}t$$

subject to

$$\dot{P}(t) = v(t), \quad P(0) = P^0.$$

Here v is investment in R&D, dv^2 is the cost of investment v, P^0 is the initial productivity of capital. The objective function of Developer needs some more explanation. The new technologies that Developer has mastered in the time interval $[t, t + \Delta t]$ increase the productivity of Producer by $\dot{P}(t)\Delta t$. Developer assumes that these new technologies will be embodied in the increment of the capital of Producer, $\dot{K}(t)\Delta t$. Then the effect of the investment of the developer at time t is measured by $\dot{P}(t)\dot{K}(t)$. The coefficient $\gamma > 0$ represents the weight that Developer attributes to the particular industry. We stress that Developer is not profit-oriented, therefore her problem is only a meat to allocate (public) resources for R&D for the given industry, depending on the weight γ attributed to this industry, and on the information about the current investment intensity of the industry.

The solution of Developer's problem is obvious:

$$v(t) = \frac{\gamma}{2d} \dot{K}(t).$$

Hence, Producer's problem becomes

$$\max_{u \geq 0} \int_0^\infty e^{-rt} \left[\frac{\gamma}{2d} K(t)^2 + \left(P^0 - \frac{\gamma}{2d} K^0 \right) K(t) - cu(t)^2 \right] \mathrm{d}t \tag{11}$$

subject to (10). The set of admissible controls is as in Section 3: $\mathcal{U}$ is the set of measurable functions $u : [0, \infty) \mapsto [0, \infty)$ for which $\|u\|_2$ is finite. Notice that for $u \in \mathcal{U}$ the total discounted investment cost is finite.

Notice that the Producer's problem with an exogenously given productivity $P(t)$ is a standard linear-quadratic problem for which the Arrow sufficiency condition (among

others) is trivially fulfilled. In the Stackelberg scenario, however, this condition is not fulfilled, since the Hamiltonian resulting from (11) is not concave. However, Theorem 2 is still applied if the data represented by the coefficients r, δ, c, d, γ satisfy an inequality resembling (7). To see this we first check that conditions (4)–(7) are satisfied.

Let (x^*, u^*, λ) satisfy (3) and (8). We have

$$H(K, u, \lambda) = \frac{\gamma}{2d}K^2 + qK - cu^2 + \lambda(-\delta K + u),$$

where $q = P^0 - K^0\gamma/(2d)$.

Since (8) implies that $\lambda(t) = 2cu^*(t)$, condition (4) is fulfilled with $\alpha = c$. Condition (5) holds with $\beta = \gamma/(2d)$. Condition (6) obviously holds with the same δ. Finally, $L_g = 1$ and (7) reads now as

$$\gamma \leq 4cd\delta r. \tag{12}$$

To solve the Producer's problem first we shall determine an appropriate control $u^* \in \mathcal{U}$, which together with λ and the corresponding K, satisfies Pontryagin's maximum principle (3), (8). Thus the triple (u^*, K^*, λ) has to satisfy

$$
\begin{aligned}
\dot{K} &= u - \delta K, \quad K(0) = K^0, \\
\dot{\lambda} &= (r + \delta)\lambda - \frac{\gamma}{d}K - \left(P^0 - \frac{\gamma}{2d}K^0\right),
\end{aligned}
$$

where from (8)

$$u(t) = \frac{1}{2c}\lambda(t).$$

Thus we come up with the linear ("canonical") system

$$\dot{K} = -\delta K + \frac{1}{2c}\lambda, \quad K(0) = K^0, \tag{13}$$

$$\dot{\lambda} = -\frac{\gamma}{d}K + (r + \delta)\lambda - \left(P_0 - \frac{\gamma}{2d}K_0\right). \tag{14}$$

According to Theorem 2, any solution K^*, $\lambda \geq 0$ and corresponding $u^* = \lambda/(2c)$ satisfies condition (i) in the Leitmann-Stalford Theorem 1. Notice that no boundary

conditions for λ are specified so far. We shall determine $(K^*, \lambda, u^* = \lambda/(2c))$ in such a way that also condition (ii) in Theorem 1 is satisfied (then (u^*, K^*) will be optimal).

To do this, first we shall specify an initial state for λ so that the solution of the canonical system is bounded. Since the eigenvalues of the canonical system are

$$\frac{r \pm \sqrt{r^2 + 4\delta(r+\delta) - 4\gamma/(2cd)}}{2},$$

If

$$\gamma < 2cd\delta(r+\delta), \tag{15}$$

then the eigenvalues are real and one of them is negative. Due to the linearity of the system (13)-(14) the solution for the given initial K^0 and an initial $\lambda(0) = \lambda^0$ (the latter viewed as a free parameter) can be found explicitly. The initial λ^0 can be chosen in such a way that the solution converges monotonically to the unique steady state

$$K^\infty = \frac{2dP^0 - \gamma K^0}{2[2cd\delta(r+\delta) - \gamma]}, \quad \lambda^\infty = \frac{c\delta(2dP^0 - \gamma K^0)}{2cd\delta(r+\delta) - \gamma}.$$

As the denominator of above expressions is positive due to assumption (15) this steady state is feasible if and only if $P^0 > \gamma K^0/(2d)$, i.e. the initial productivity of the capital stock is sufficiently high. Due to the monotonicity of the trajectory this also guarantees that $K(t)$ as well as $\lambda(t)$ (and therefore also $u(t)$) remain positive along the solution.[2] Further on (u^*, K^*) will be the pair corresponding to the so defined λ.

To prove that (u^*, K^*) is optimal we shall verify condition (ii) in Theorem 1. For any $u \in \mathcal{U}$ we have (using the Cauchy-Bunjakowski-Schwarz inequality)

$$K[u](t) = e^{-\delta t} K^0 + \int_0^t e^{-\delta(t-s)} u(s) \, ds \le K^0 + \int_0^t e^{-\delta(t-s) + \frac{r}{2}s} e^{-\frac{r}{2}s} u(s) \, ds$$

$$= K^0 + \sqrt{\int_0^t e^{-2\delta(t-s) + rs} \, ds} \sqrt{\int_0^t e^{-rs} u(s)^2 \, ds} \le K^0 + \frac{1}{\sqrt{2\delta + r}} e^{\frac{r}{2}t} \|u\|_2.$$

Since λ is bounded, this obviously implies condition (ii) in Theorem 1, thus the specified control $u^* \in \mathcal{U}$ is optimal, provided that inequalities (12) and (15) are fulfilled.

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[2] A detailed proof is available from the authors upon request.

Although the above result aims only to demonstrate an application of Theorem 2, some interesting economic analysis can be carried out. If we substitute $e = d/\gamma$ (the cost-benefit ratio of Developer), we see that conditions (12) and (15) require, essentially, that e is large enough. That is, if the cost of R&D is large relative to the "importance" coefficient γ, then the industry would have a bounded optimal solution, while for a low cost-benefit ratio e of Developer the industry may grow infinitely. Moreover, the ratio of initial capital stock to initial productivity K^0/P^0 has to be less than twice the cost-benefit ratio e for the existence of bounded solutions converging to an interior feasible steady state.

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