

NATURAL REPARAMETERIZATION AND CONTINUITY OF THE SOLUTION MAP FOR POLYNOMIALS

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ABSTRACT. The optimal Sobolev regularity of the roots of a smooth curve of monic complex polynomials is $W^{1,q}$, for $q \in [1, \frac{d}{d-1})$, where d is the degree. This result is stable in the sense that the solution map from C^d coefficients to $W^{1,q}$ roots is continuous. We prove that, after a natural Lipschitz reparameterization, the roots are Lipschitz and the map from C^d coefficients to reparameterization and reparameterized roots is continuous with respect to the $W^{1,q}$ topology on the target spaces, for all $q \in [1, \infty)$. The result is based on a convergence and reparameterization theorem for Almgren's Q -valued Sobolev functions.

1. INTRODUCTION

The optimal Sobolev regularity of the roots of monic complex polynomials with smooth coefficients was established in [PR16, PR18, PR20]. If the coefficients are functions of class $C^{d-1,1}(\bar{I})$, where d is the degree of the polynomial and I a bounded open interval, then each continuous root is absolutely continuous, bounded, and $\lambda' \in L^q(I)$, for all $q \in [1, \frac{d}{d-1})$. In [PR24], stability of this result was proved, in the sense that the solution map which assigns to polynomials with coefficients in $C^d(\bar{I})$ their unordered root tuple of Sobolev class $W^{1,q}$, for $q \in [1, \frac{d}{d-1})$, is continuous. In this note, we show that, by means of a natural Lipschitz reparameterization, these results extend to all values $q \in [1, \infty)$.

Since there is no canonical ordering of the complex roots, we work with the space of unordered d -tuples and use Almgren's theory of multivalued Sobolev functions (see [Alm00], [DLS11]). We view the symmetric product $\text{Sym}^d(\mathbb{C}) = \mathbb{C}^d / S_d$ as the complete metric space $(\mathcal{A}_d(\mathbb{C}), \mathbf{d}_2)$ of unordered d -tuples $[z] = [z_1, \dots, z_d]$ of complex numbers, where

$$\mathbf{d}_2([z], [w]) := \min_{\sigma \in S_d} \|z - \sigma w\|_2 = \min_{\sigma \in S_d} \left(\sum_{j=1}^d |z_j - w_{\sigma(j)}|^2 \right)^{1/2}.$$

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There is a bi-Lipschitz embedding $\Delta : \mathcal{A}_d(\mathbb{C}) \rightarrow \mathbb{R}^N$, for some $N = N(d)$, which can be used to define the Sobolev space

$$W^{1,q}(I, \mathcal{A}_d(\mathbb{C})) := \{f : I \rightarrow \mathcal{A}_d(\mathbb{C}) : \Delta \circ f \in W^{1,q}(I, \mathbb{R}^N)\}, \quad q \in [1, \infty).$$

We call Δ an Almgren embedding. Pullback of the Sobolev norm on $W^{1,q}(I, \mathbb{R}^N)$ endows $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ with a complete metric; the induced topology does not depend on the choice of Δ . See Section 3.1 for more details.

Each element $[\lambda_1, \dots, \lambda_d] \in \mathcal{A}_d(\mathbb{C})$ corresponds in a one-to-one fashion to a complex polynomial

$$P_a(Z) = Z^d + \sum_{i=1}^d a_i Z^{d-i} = \prod_{j=1}^d (Z - \lambda_j). \quad (1.1)$$

Since the coefficient a_i is (up to sign) the i -th elementary symmetric function of the roots, this defines a bijective map $a = (a_1, \dots, a_d) : \mathcal{A}_d(\mathbb{C}) \rightarrow \mathbb{C}^d$. We denote by $\Lambda : \mathbb{C}^d \rightarrow \mathcal{A}_d(\mathbb{C})$ the inverse map which sends (the coefficient vector of) a polynomial to its unordered root tuple. The map a is locally Lipschitz and Λ is locally $\frac{1}{d}$ -Hölder (see [PR24, Lemma 6.4]).

Let us consider curves of polynomials P_a , that is curves $a : I \rightarrow \mathbb{C}^d$, where $I \subseteq \mathbb{R}$ is a bounded open interval. The unordered roots of P_a are given by the composite $\Lambda \circ a : I \rightarrow \mathcal{A}_d(\mathbb{C})$. We will call the nonlinear superposition operator $\mathcal{S} : a \mapsto \Lambda \circ a$ the *solution map* and write $\mathcal{S}(a) = \Lambda \circ a = \Lambda_a$ interchangeably.

Theorem 1.1 ([PR18, PR24]). *Let $d \in \mathbb{N}_{\geq 2}$ and $q \in [1, \frac{d}{d-1})$. Then:*

- (1) *The map $\mathcal{S} : C^{d-1,1}(\bar{I}, \mathbb{C}^d) \rightarrow W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ is well-defined and bounded.*
- (2) *The map $\mathcal{S} : C^d(\bar{I}, \mathbb{C}^d) \rightarrow W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ is continuous.*

The map $\mathcal{S}$ is not well-defined for $q \geq \frac{d}{d-1}$, even on polynomial curves in $\mathbb{C}^d$. In fact, the obstruction is already visible for $Z^2 = x$, $x \in (0, 1)$, where the first derivative of the roots fails to be in $L^2((0, 1))$. (In Theorem 1.1, we assume $d \geq 2$ so that $\frac{d}{d-1}$ is well-defined, but the assertions are trivially true for $d = 1$.)

Our main result is that the restriction on q can be removed after a natural Lipschitz reparameterization associated with the rate $\epsilon + \|(\Delta \circ \Lambda_a)'\|_2$, where Δ is an Almgren embedding and $\epsilon > 0$ is arbitrary. The parameter $\epsilon > 0$ guarantees that the rate is strictly positive and hence the reparameterization is invertible on I . The potentially unbounded root velocity near collisions is absorbed into the reparameterization of time in a stable manner.

Theorem 1.2. *Let $d \in \mathbb{N}_{\geq 1}$, $\epsilon > 0$, and $\Delta : \mathcal{A}_d(\mathbb{C}) \rightarrow \mathbb{R}^N$ an Almgren embedding. There is a reparameterization map $t = t(\epsilon, \Delta) : C^{d-1,1}(\bar{I}, \mathbb{C}^d) \rightarrow \text{Lip}(\mathbb{R}, \bar{I})$, $a \mapsto t_a$, such that the following properties hold for all $a \in C^{d-1,1}(\bar{I}, \mathbb{C}^d)$:*

- (1) *$t_a : \mathbb{R} \rightarrow \bar{I}$ is surjective, increasing, and strictly increasing on $t_a^{-1}(I)$;*
- (2) *t_a is globally Lipschitz with Lipschitz constant $\text{Lip}(t_a) \leq \frac{1}{\epsilon}$;*
- (3) *the composite $\Lambda_a \circ t_a : \mathbb{R} \rightarrow \mathcal{A}_d(\mathbb{C})$ is globally Lipschitz with $\text{Lip}(\Lambda_a \circ t_a) \leq 1$.*

The induced map

$$\begin{aligned} C^d(\bar{I}, \mathbb{C}^d) &\rightarrow W_{\text{loc}}^{1,q}(\mathbb{R}) \times W_{\text{loc}}^{1,q}(\mathbb{R}, \mathcal{A}_d(\mathbb{C})) \\ a &\mapsto (t_a, \Lambda_a \circ t_a) \end{aligned} \quad (1.2)$$

is continuous, for all $q \in [1, \infty)$.

While the construction of t depends on the specific choice of ϵ and Δ , the continuity of the map in (1.2) is independent.

This continuity is a consequence of the following convergence result, Theorem 1.3: C^d convergence of the coefficients implies uniform convergence as well as $W^{1,q}$ convergence, for all $q \in [1, \infty)$, of the reparameterization and the reparameterized roots.

Theorem 1.3. *Let $d \in \mathbb{N}_{\geq 1}$, $\epsilon > 0$, $\Delta : \mathcal{A}_d(\mathbb{C}) \rightarrow \mathbb{R}^N$ an Almgren embedding, and $t = t(\epsilon, \Delta)$ the reparameterization map from Theorem 1.2. If $a_n \rightarrow a$ in $C^d(\bar{I}, \mathbb{C}^d)$, then:*

- (1) $t_{a_n} \rightarrow t_a$ uniformly on $\mathbb{R}$ and in $W_{\text{loc}}^{1,q}(\mathbb{R})$ for all $q \in [1, \infty)$.
- (2) $\Lambda_{a_n} \circ t_{a_n} \rightarrow \Lambda_a \circ t_a$ uniformly on $\mathbb{R}$ and in $W_{\text{loc}}^{1,q}(\mathbb{R}, \mathcal{A}_d(\mathbb{C}))$ for all $q \in [1, \infty)$.
- (3) Assume that $\lambda_n : I \rightarrow \mathbb{C}^d$ is a continuous parameterization of the roots of P_{a_n} and λ_n converges in $C^0(\bar{I}, \mathbb{C}^d)$ to a continuous parameterization λ of the roots of P_a . Then, for each $\epsilon > 0$, there exist $t, t_n \in \text{Lip}(\mathbb{R}, \bar{I})$ with Lipschitz constants $\leq \frac{1}{\epsilon}$ such that $\lambda \circ t, \lambda_n \circ t_n \in \text{Lip}(\mathbb{R}, \mathbb{C}^d)$ with Lipschitz constants ≤ 1 and
 - $t_n \rightarrow t$ uniformly on $\mathbb{R}$ and in $W_{\text{loc}}^{1,q}(\mathbb{R})$ for all $q \in [1, \infty)$,
 - $\lambda_n \circ t_n \rightarrow \lambda \circ t$ uniformly on $\mathbb{R}$ and in $W_{\text{loc}}^{1,q}(\mathbb{R}, \mathbb{C}^d)$ for all $q \in [1, \infty)$.

Here t, t_n are independent of Δ .

Regarding (3), note that while there always exist continuous parameterizations of the roots on the interval I , not every continuous root parameterization of P_a is the limit of continuous root parameterizations of P_{a_n} as $a_n \rightarrow a$.

The proofs of Theorem 1.2 and Theorem 1.3 consist of two parts. First, we prove a general convergence and reparameterization result in the realm of Almgren's Q -valued Sobolev functions. We state it in Theorem 3.3 for Sobolev functions on open bounded intervals with values in $\mathcal{A}_Q(\mathbb{R}^n)$, the space of unordered Q -tuples of points in $\mathbb{R}^n$. In Theorem 3.7, we even obtain a weaker variant for absolutely continuous curves in arbitrary complete metric spaces. The natural Lipschitz reparameterization map t , which will be defined in this general setting in Section 2, results from a standard construction taken from [AGS08, Section 1.1].

The second step in the proof of Theorems 1.2 and 1.3 is essentially a simple combination of Theorem 3.3 with the continuity of the solution map provided by Theorem 1.1; this is carried out in Section 4.

Theorems 1.2 and 1.3 close the gap between Theorem 1.1 and the case of *hyperbolic polynomials*, where no reparameterization is necessary: the polynomial P_a in (1.1) is called hyperbolic if all roots λ_j are real. The coefficient vectors of the hyperbolic

polynomials of degree d form a proper semialgebraic subset $\text{Hyp}(d) \subseteq \mathbb{R}^d$. In contrast to the general case, there is canonical ordering of the roots by size which induces a locally Lipschitz map $\lambda_\uparrow : \text{Hyp}(d) \rightarrow \mathbb{R}^d$ and the superposition operator $\mathcal{S}_\uparrow : a \mapsto \lambda_\uparrow \circ a$. (Thus it is not necessary to work in the space of unordered tuples.)

Theorem 1.4 ([Bro79], [PR15], [PR24a]). *Let d be a positive integer. Then:*

- (1) *The map $\mathcal{S}_\uparrow : C^{d-1,1}(I, \text{Hyp}(d)) \rightarrow C^{0,1}(I, \mathbb{R}^d)$ is well-defined and bounded.*
- (2) *The map $\mathcal{S}_\uparrow : C^d(I, \text{Hyp}(d)) \rightarrow W_{\text{loc}}^{1,q}(I, \mathbb{R}^d)$ is continuous for all $q \in [1, \infty)$.*

The map $\mathcal{S}_\uparrow$ is *not* continuous with respect to the $C^{0,1}$ topology on the target space; see [PR24a, Example 1.12].

For characteristic polynomials of Hermitian and normal matrices, even stronger eigenvalue stability results hold true, see [PR26].

2. LIPSCHITZ REPARAMETERIZATION

This preliminary section is based on [AGS08, Section 1.1]. Throughout this section, (X, d) is a complete metric space. Let $(a, b) \subseteq \mathbb{R}$ be a bounded open interval.

2.1. Absolutely continuous curves and their metric slope. A map $f : (a, b) \rightarrow X$ is said to belong to $AC^q((a, b), X)$, for $q \in [1, \infty]$, if there exists $m \in L^q((a, b))$ such that

$$d(f(x), f(y)) \leq \int_x^y m(t) dt \quad \text{whenever } a < x \leq y < b. \quad (2.1)$$

Each $f \in AC^q((a, b), X)$ is uniformly continuous and admits a continuous extension to $[a, b]$. Furthermore, the limit

$$|f'| (x) := \lim_{y \rightarrow x} \frac{d(f(x), f(y))}{|x - y|}$$

exists for $\mathcal{L}^1$ -almost every $x \in (a, b)$ and $|f'| \in L^q((a, b))$. (The Lebesgue measure is the only measure we will use; therefore it will not be mentioned explicitly from now on.) The function $|f'|$ is called the *metric slope* or *metric derivative* of f . Note that (2.1) holds with $m = |f'|$ and, if m is such that (2.1) holds, then $|f'| \leq m$ almost everywhere in (a, b) .

2.2. Lipschitz reparameterization. Let $\epsilon > 0$ and $f \in AC^1((a, b), X)$. Following [AGS08, Lemma 1.1.4], we define reparameterization maps $s = s_{\epsilon, f}$ and $t = t_{\epsilon, f}$ as follows. Setting

$$s(x) := \int_a^x (\epsilon + |f'| (t)) dt$$

defines a strictly increasing absolutely continuous function $s : [a, b] \rightarrow [0, L]$, where

$$L = L_{\epsilon, f} := \epsilon(b - a) + \int_a^b |f'| (t) dt.$$

Let $t : [0, L] \rightarrow [a, b]$ be the inverse of s . Then t is Lipschitz with $\text{Lip}(t) \leq \frac{1}{\epsilon}$ and

$$t' \circ s = \frac{1}{\epsilon + |f'|} \quad \text{almost everywhere in } (a, b).$$

The composite $\hat{f} := f \circ t$ is Lipschitz with $\text{Lip}(\hat{f}) \leq 1$ and

$$|\hat{f}'| \circ s = \frac{|f'|}{\epsilon + |f'|} \quad \text{almost everywhere in } (a, b); \quad (2.2)$$

see [AGS08, Lemma 1.1.4].

The right endpoint L of the domain of t depends on f . It will be convenient to have the domain of t independent of f . Thus we extend t to $\mathbb{R}$ by setting $t(y) := a$ if $y < 0$ and $t(y) := b$ if $y > L$. Then $t : \mathbb{R} \rightarrow [a, b]$ is surjective, increasing, strictly increasing on $(0, L)$, and Lipschitz with $\text{Lip}(t) \leq \frac{1}{\epsilon}$. Clearly, $\hat{f} = f \circ t : \mathbb{R} \rightarrow X$ is still Lipschitz with $\text{Lip}(\hat{f}) \leq 1$.

Lemma 2.1. *Let $f, f_n \in AC^1((a, b), X)$, for $n \geq 1$, be such that $|f'_n| \rightarrow |f'|$ in $L^1((a, b))$. For fixed $\epsilon > 0$, let $s = s_{\epsilon, f}$, $s_n = s_{\epsilon, f_n}$, $t = t_{\epsilon, f}$, and $t_n = t_{\epsilon, f_n}$. Then:*

- (1) $s_n \rightarrow s$ uniformly on $[a, b]$;
- (2) $s'_n \rightarrow s'$ in $L^1((a, b))$;
- (3) $t_n \rightarrow t$ uniformly on $\mathbb{R}$.

We will see in Theorem 3.7 that $t'_n \rightarrow t'$ in $L^q_{\text{loc}}(\mathbb{R})$ for all $q \in [1, \infty)$.

Proof. (1) follows from

$$|s(x) - s_n(x)| = \left| \int_a^x |f'| (t) - |f'_n| (t) dt \right| \leq \int_a^b ||f'| (t) - |f'_n| (t)| dt.$$

(2) is evident, by definition.

(3) We will only use monotonicity and (1). By (1), $L_n = s_n(b) \rightarrow s(b) = L$. Define, for small $\delta > 0$,

$$g(\delta) := \min \left\{ \min_{x \in [a, b-\delta]} (s(x+\delta) - s(x)), \min_{x \in [a+\delta, b]} (s(x) - s(x-\delta)) \right\}.$$

Then $g(\delta) > 0$ because s is strictly increasing. Fix $\delta < \frac{b-a}{2}$. By (1), there is $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\|s_n - s\|_{C^0([a, b])} < \frac{g(\delta)}{2} \quad (2.3)$$

and $L_n \in (s(b-\delta), L+\delta)$.

We first show that, for all $y \in [0, s(b-\delta)]$ and $n \geq n_0$,

$$|t(y) - t_n(y)| < \delta. \quad (2.4)$$

The remaining y will be considered at the end of the proof.

Let $y \in [0, s(b-\delta)]$ and $x := t(y) \in [a, b-\delta]$.

If $x \in [a+\delta, b-\delta]$, then, by (2.3), for $n \geq n_0$,

$$\begin{aligned} s_n(x+\delta) &> s(x+\delta) - \frac{g(\delta)}{2} \geq \frac{s(x+\delta) + s(x)}{2} \geq s(x) = y, \\ s_n(x-\delta) &< s(x-\delta) + \frac{g(\delta)}{2} \leq \frac{s(x) + s(x-\delta)}{2} \leq s(x) = y. \end{aligned} \quad (2.5)$$

Therefore, there is a unique $x' \in (x - \delta, x + \delta)$ such that $s_n(x') = y$. As $y < L_n$, we conclude $t_n(y) = x'$ and

$$|t(y) - t_n(y)| = |x - x'| < \delta.$$

If $x \in [a, a + \delta)$, then (2.5) still holds. Thus $s_n(x + \delta) > y$ and $x + \delta > t_n(y)$. Since $x - t_n(y) < a + \delta - t_n(y) \leq \delta$, we conclude that

$$|t(y) - t_n(y)| = |x - t_n(y)| < \delta.$$

Thus (2.4) is proved.

If $y \in (s(b - \delta), L + \delta)$, then as t and t_n are $\frac{1}{\epsilon}$ -Lipschitz on $\mathbb{R}$ and $t(L) = b = t_n(L_n)$,

$$\begin{aligned} |t(y) - t_n(y)| &\leq |t(y) - t(L)| + |t_n(L_n) - t_n(y)| \leq \frac{|y - L| + |L_n - y|}{\epsilon} \\ &\leq \frac{2|y - L| + |L - L_n|}{\epsilon} \leq \frac{3}{\epsilon} \max\{\delta, s(b) - s(b - \delta)\}, \end{aligned}$$

for all $n \geq n_0$. Since s is uniformly continuous and $t(y) = t_n(y)$ for all $n \geq n_0$ if $y < 0$ or $y \geq L + \delta$, we conclude that $t_n \rightarrow t$ uniformly on $\mathbb{R}$. $\square$

3. CONVERGENCE AND REPARAMETERIZATION OF Q -VALUED SOBOLEV FUNCTIONS

The main objective of this section is Theorem 3.3.

3.1. Q -valued Sobolev functions. We follow [Alm00] and [DLS11]. Let Q be a positive integer. We consider the space of unordered sets of Q points in $\mathbb{R}^n$ admitting multiplicities,

$$\mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^Q \llbracket p_i \rrbracket : p_1, \dots, p_Q \in \mathbb{R}^n \right\},$$

where $\llbracket p_i \rrbracket$ denotes the Dirac mass of $p_i \in \mathbb{R}^n$, and endow it with the metric

$$\mathbf{d}_2 \left(\sum_{i=1}^Q \llbracket p_i \rrbracket, \sum_{i=1}^Q \llbracket q_i \rrbracket \right) := \min_{\sigma \in S_Q} \left(\sum_{i=1}^Q \|p_i - q_{\sigma(i)}\|_2^2 \right)^{1/2}.$$

The metric space $(\mathcal{A}_Q(\mathbb{R}^n), \mathbf{d}_2)$ is complete (see [DLS11, Remark 0.3]).

There exists an injective Lipschitz map $\Delta : \mathcal{A}_Q(\mathbb{R}^n) \rightarrow \mathbb{R}^N$ with Lipschitz constant $\text{Lip}(\Delta) \leq 1$, where $N = N(n, Q)$. The inverse $\Delta|_{\Delta(\mathcal{A}_Q(\mathbb{R}^n))}^{-1}$ is Lipschitz with Lipschitz constant bounded by $C(n, Q)$. Moreover, there is a Lipschitz retraction of $\mathbb{R}^N$ onto $\Delta(\mathcal{A}_Q(\mathbb{R}^n))$. See [DLS11, Theorem 2.1].

Almgren used this bi-Lipschitz embedding – we call Δ an *Almgren embedding* – to define Sobolev spaces of Q -valued functions: for open $U \subseteq \mathbb{R}^m$ and $q \in [1, \infty)$ we define

$$W^{1,q}(U, \mathcal{A}_Q(\mathbb{R}^n)) := \{f : U \rightarrow \mathcal{A}_Q(\mathbb{R}^n) : \Delta \circ f \in W^{1,q}(U, \mathbb{R}^N)\}.$$

For an equivalent intrinsic definition, see [DLS11, Definition 0.5 and Theorem 2.4]. Endowed with the metric

$$(f, g) \mapsto \|\Delta \circ f - \Delta \circ g\|_{W^{1,q}(U, \mathbb{R}^N)}, \quad (3.1)$$

$W^{1,q}(U, \mathcal{A}_Q(\mathbb{R}^n))$ is a complete metric space (where functions that coincide almost everywhere are identified); see [PR24, Lemma 3.1].

The topology induced by the metric (3.1) is independent of the choice of Δ . This was proved for $\mathcal{A}_Q(\mathbb{R}^2) \cong \mathcal{A}_Q(\mathbb{C})$ in [PR24, Theorem 3.11] in the case that U is an interval and in [PR26, Theorem 8.11] for general U . This proof generalizes to $\mathcal{A}_Q(\mathbb{R}^n)$.

Let $I \subseteq \mathbb{R}$ be a bounded open interval and $f \in W^{1,q}(I, \mathcal{A}_Q(\mathbb{R}^n))$. By [DLS11, Proposition 1.2], there exist $f_1, \dots, f_Q \in W^{1,q}(I, \mathbb{R}^n)$ such that

$$f = \sum_{i=1}^Q \llbracket f_i \rrbracket. \quad (3.2)$$

The following lemma generalizes [PR24, Lemma 11.1]; the proof is the same.

Lemma 3.1. *Let $f \in W^{1,q}(I, \mathcal{A}_Q(\mathbb{R}^n))$ and $f_1, \dots, f_Q \in W^{1,q}(I, \mathbb{R}^n)$ such that $f = \sum_{i=1}^Q \llbracket f_i \rrbracket$. Then, for almost every $x \in I$,*

$$|f'| (x) = \|(f'_1(x), \dots, f'_Q(x))\|_2.$$

In [PR24, Theorem 3.11], it was proved that $f_n \rightarrow f$ in $W^{1,q}(I, \mathcal{A}_Q(\mathbb{R}^2))$ if and only if $f_n \rightarrow f$ with respect to a semimetric $\mathbf{d}^{1,q}$ involving the selection (3.2); see [PR24, Definition 3.6]. As mentioned before, this generalizes to $\mathcal{A}_Q(\mathbb{R}^n)$. Together with Lemma 3.1, it implies the following (cf. [PR24, Corollary 1.4]):

Lemma 3.2. *If $f_n \rightarrow f$ in $W^{1,q}(I, \mathcal{A}_Q(\mathbb{R}^n))$ then $|f'_n| \rightarrow |f'|$ in $L^q(I)$.*

3.2. Strong convergence and reparameterization. Let $(a, b) \subseteq \mathbb{R}$ be a bounded open interval. We will consider functions $f \in W^{1,q}((a, b), \mathcal{A}_Q(\mathbb{R}^m))$. Then $f \in AC^q((a, b), \mathcal{A}_Q(\mathbb{R}^m))$, by Lemma 3.1 (see also [AGS08, Remark 1.1.3] and [DLS11, Proposition 1.2]). In particular, f is uniformly continuous and extends continuously to $[a, b]$.

Theorem 3.3. *Let $f_n \rightarrow f$ in $W^{1,q}((a, b), \mathcal{A}_Q(\mathbb{R}^m))$ for some $q > 1$ or, alternatively, $f_n \rightarrow f$ in $W^{1,1}((a, b), \mathcal{A}_Q(\mathbb{R}^m))$ and $f_n(x_0) \rightarrow f(x_0)$ for some $x_0 \in [a, b]$. Fix $\epsilon > 0$ and an Almgren embedding $\Delta : \mathcal{A}_Q(\mathbb{R}^m) \rightarrow \mathbb{R}^N$, and consider $t := t_{\epsilon, \Delta \circ f}$ and $t_n := t_{\epsilon, \Delta \circ f_n}$, provided by Section 2.2. Then $t_n \rightarrow t$ in $W^{1,q}_{\text{loc}}(\mathbb{R})$ and $f_n \circ t_n \rightarrow f \circ t$ in $W^{1,q}_{\text{loc}}(\mathbb{R}, \mathcal{A}_Q(\mathbb{R}^m))$, for all $q \in [1, \infty)$.*

Theorem 3.3 is a consequence of the following proposition, by composition with Δ .

Proposition 3.4. *Let $f_n \rightarrow f$ in $W^{1,q}((a, b), \mathbb{R}^N)$ for some $q > 1$ or, alternatively, $f'_n \rightarrow f'$ in $L^1((a, b), \mathbb{R}^N)$ and $f_n(x_0) \rightarrow f(x_0)$ for some $x_0 \in [a, b]$. Fix $\epsilon > 0$ and let $t := t_{\epsilon, f}$ and $t_n := t_{\epsilon, f_n}$. Then $t_n \rightarrow t$ in $W^{1,q}_{\text{loc}}(\mathbb{R})$ and $f_n \circ t_n \rightarrow f \circ t$ in $W^{1,q}_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$, for all $q \in [1, \infty)$.*

The rest of this section is dedicated to the proof of Proposition 3.4. For simplicity of notation, we write $\hat{f} := f \circ t$ and $\hat{f}_n := f_n \circ t_n$. Moreover, $s := s_{\epsilon, f}$, $s_n := s_{\epsilon, f_n}$, $L := s(b)$, and $L_n := s_n(b)$; see Section 2.2.

We will prove

- (1) $\hat{f}_n \rightarrow \hat{f}$ uniformly on $\mathbb{R}$,
- (2) $\hat{f}'_n \rightarrow \hat{f}'$ in $L^q_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$,
- (3) $t'_n \rightarrow t'$ in $L^q_{\text{loc}}(\mathbb{R})$,

for all $q \in [1, \infty)$. This will imply Proposition 3.4, because $t_n \rightarrow t$ uniformly on $\mathbb{R}$, by Lemma 2.1.

Let us check (1). If $f_n \rightarrow f$ in $W^{1,q}((a,b), \mathbb{R}^N)$ for some $q > 1$, then $f_n \rightarrow f$ uniformly on $[a,b]$, by Morrey's inequality, and (1) follows because $t_n \rightarrow t$ uniformly on $\mathbb{R}$ (and f is uniformly continuous on $[a,b]$). If on the other hand $f'_n \rightarrow f'$ in $L^1((a,b), \mathbb{R}^N)$ and $f_n(x_0) \rightarrow f(x_0)$ for some $x_0 \in [a,b]$, then

$$\|f(x) - f_n(x)\|_2 = \left\| f(x_0) - f_n(x_0) + \int_{x_0}^x f'(u) - f'_n(u) du \right\|_2 \rightarrow 0$$

uniformly for $x \in [a,b]$ and (1) follows as before.

Next we prove (2). Since t is increasing, $\hat{f} = f \circ t$ is differentiable almost everywhere and the chain rule holds (see e.g. [Leo09, Corollary 3.50]),

$$\hat{f}' = (f' \circ t) \cdot t' = (f' \circ t) \cdot \frac{1}{s' \circ t} = \frac{f'}{\epsilon + \|f'\|_2} \circ t \quad (3.3)$$

almost everywhere in $(0, L)$ and $\hat{f}' = 0$ in $\mathbb{R} \setminus [0, L]$. Similarly,

$$\hat{f}'_n = \frac{f'_n}{\epsilon + \|f'_n\|_2} \circ t_n \quad (3.4)$$

almost everywhere in $(0, L_n)$ and $\hat{f}' = 0$ in $\mathbb{R} \setminus [0, L_n]$.

In Lemma 3.5, we will prove that $\hat{f}'_n \rightarrow \hat{f}'$ in measure on $\mathbb{R}$. Since $\|\hat{f}'_n\|_2 \leq 1$ almost everywhere in $\mathbb{R}$ and hence the family $\{\hat{f}'_n\}$ is uniformly integrable on each bounded interval, this implies that $\hat{f}'_n \rightarrow \hat{f}'$ in $L^q_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ for all $q \in [1, \infty)$, by Vitali's convergence theorem. Thus (2) is proved.

Lemma 3.5. $\hat{f}'_n \rightarrow \hat{f}'$ in measure on $\mathbb{R}$.

Proof. Since $f_n \rightarrow f$ in $W^{1,1}((a,b), \mathbb{R}^N)$, after passing to a subsequence, $f'_n \rightarrow f'$ almost everywhere in (a,b) . It follows that

$$\frac{f'_n}{\epsilon + \|f'_n\|_2} \rightarrow \frac{f'}{\epsilon + \|f'\|_2} \quad \text{almost everywhere in } (a,b).$$

By the generalized dominated convergence theorem (see e.g. [Fol99, p. 59]), we conclude that

$$\frac{f'_n}{\epsilon + \|f'_n\|_2} \rightarrow \frac{f'}{\epsilon + \|f'\|_2} \quad \text{in } L^1((a,b), \mathbb{R}^N). \quad (3.5)$$

We proved (3.5) for a subsequence, but this implies its validity for the original sequence.

Let us put

$$\alpha_n := \frac{f'_n}{\epsilon + \|f'_n\|_2}, \quad \alpha := \frac{f'}{\epsilon + \|f'\|_2}.$$

Then $\alpha_n \rightarrow \alpha$ in $L^1((a, b), \mathbb{R}^N)$, by (3.5). We have $\hat{f}' = \alpha \circ t$ almost everywhere in $(0, L)$, by (3.3), and $\hat{f}'(y) = 0$ if $y < 0$ or $y > L$. By (3.4), $\hat{f}'_n = \alpha_n \circ t_n$ almost everywhere in $(0, L_n)$ and $\hat{f}'_n(y) = 0$ if $y < 0$ or $y > L_n$.

Let $\rho, \sigma > 0$ be fixed. We have to show that there exists $n_0 \geq 1$ such that

$$|\{y \in \mathbb{R} : \|\hat{f}'_n(y) - \hat{f}'(y)\|_2 \geq \rho\}| \leq \sigma, \quad n \geq n_0. \quad (3.6)$$

We claim that it suffices to show (3.6) for $y \in (0, L - \frac{\sigma}{4})$ (assuming of course that σ is small enough). Indeed, if $y < 0$ then $\hat{f}'_n(y) = \hat{f}'(y) = 0$ for all n . On the other hand, since $L_n \rightarrow L$, there is $n_1 \geq 1$ such that for all $n \geq n_1$ we have $L_n \in (L - \frac{\sigma}{4}, L + \frac{\sigma}{4})$ and hence $\hat{f}'_n(y) = \hat{f}'(y) = 0$ for all $y > L + \frac{\sigma}{4}$. Thus, for $n \geq n_1$, the left-hand side of (3.6) equals

$$|\{y \in (0, L + \frac{\sigma}{4}) : \|\hat{f}'_n(y) - \hat{f}'(y)\|_2 \geq \rho\}|$$

so that, in order to establish (3.6), it suffices to show

$$|\{y \in (0, L - \frac{\sigma}{4}) : \|\alpha_n(t_n(y)) - \alpha(t(y))\|_2 \geq \rho\}| \leq \frac{\sigma}{2}, \quad (3.7)$$

for all sufficiently large n . Note that, for all $n \geq n_1$, the restriction of t_n to $[0, L - \frac{\sigma}{4}]$ is invertible with inverse $s_n|_{[a, t_n(L - \frac{\sigma}{4})]}$.

Since $s'_n \rightarrow s'$ in $L^1((a, b))$ (see Lemma 2.1), the set $\{s'_n : n \geq 1\} \cup \{s'\}$ is uniformly integrable, i.e., there exists $\tau > 0$ such that, for all $n \geq 1$,

$$|s_n(E)| = \int_E s'_n(x) dx \leq \frac{\sigma}{6} \quad \text{and} \quad |s(E)| \leq \frac{\sigma}{6} \quad \text{if } |E| \leq \tau. \quad (3.8)$$

By Lusin's theorem, there exists a compact $K \subseteq [a, b]$ with $|[a, b] \setminus K| \leq \tau$ such that $\alpha|_K$ is continuous. By the Tietze–Dugundji theorem, there exists a continuous function $g : [a, b] \rightarrow \mathbb{C}^d$ such that $g|_K = \alpha|_K$.

For $y \in (0, L - \frac{\sigma}{4})$, consider

$$\alpha_n(t_n(y)) - \alpha(t(y)) = A_n(y) + B_n(y) + C_n(y) + D(y),$$

where

$$\begin{aligned} A_n(y) &:= \alpha_n(t_n(y)) - \alpha(t_n(y)), \\ B_n(y) &:= \alpha(t_n(y)) - g(t_n(y)), \\ C_n(y) &:= g(t_n(y)) - g(t(y)), \\ D(y) &:= g(t(y)) - \alpha(t(y)). \end{aligned}$$

It follows that, if

$$\|A_n(y)\|_2 < \frac{\rho}{2}, \quad B_n(y) = D(y) = 0, \quad \|C_n(y)\|_2 < \frac{\rho}{2},$$

then

$$\|\alpha_n(t_n(y)) - \alpha(t(y))\|_2 < \rho.$$

We have $\{y \in (0, L - \frac{\sigma}{4}) : \|A_n(y)\|_2 \geq \frac{\rho}{2}\} = s_n(F_n)$, where

$$F_n := \{x \in (a, t_n(L - \frac{\sigma}{4})) : \|\alpha_n(x) - \alpha(x)\|_2 \geq \frac{\rho}{2}\}.$$

Since $\alpha_n \rightarrow \alpha$ in $L^1((a, b), \mathbb{R}^N)$ and

$$\int_a^b \|\alpha_n(x) - \alpha(x)\|_2 dx \geq \int_{F_n} \|\alpha_n(x) - \alpha(x)\|_2 dx \geq \frac{\rho}{2} |F_n|,$$

we have $|F_n| \leq \tau$ for all sufficiently large n , so that by (3.8),

$$|\{y \in (0, L - \frac{\sigma}{4}) : \|A_n(y)\|_2 \geq \frac{\rho}{2}\}| = |s_n(F_n)| \leq \frac{\sigma}{6}.$$

Next we observe that $B_n(y) = 0$ if $y \in s_n(K)$ and that $|s_n([a, b] \setminus K)| \leq \frac{\sigma}{6}$, by (3.8). Similarly, $D(y) = 0$ if $y \in s(K)$ and $|s([a, b] \setminus K)| \leq \frac{\sigma}{6}$.

Finally, $\|C_n(y)\|_2 < \frac{\rho}{2}$ uniformly for all y , if n is sufficiently large, in view of Lemma 2.1.

We conclude that, there is $n_2 \geq n_1$ such that for all $n \geq n_2$,

$$|\{y \in (0, L - \frac{\sigma}{4}) : \|\alpha_n(t_n(y)) - \alpha(t(y))\|_2 \geq \rho\}| \leq \frac{\sigma}{6} + \frac{\sigma}{6} + \frac{\sigma}{6} = \frac{\sigma}{2}$$

which proves (3.7). $\square$

The following lemma follows by the same arguments.

Lemma 3.6. $t'_n \rightarrow t'$ in measure on $\mathbb{R}$.

Proof. We have

$$t' = \frac{1}{s' \circ t}$$

almost everywhere in $(0, L)$ and $t' = 0$ in $\mathbb{R} \setminus [0, L]$. Similarly,

$$t'_n = \frac{1}{s'_n \circ t_n}$$

almost everywhere in $(0, L_n)$ and $t' = 0$ in $\mathbb{R} \setminus [0, L_n]$.

Let $\rho, \sigma > 0$ be fixed. We have to show that there exists $n_0 \geq 1$ such that

$$|\{y \in \mathbb{R} : |t'_n(y) - t'(y)| \geq \rho\}| \leq \sigma, \quad n \geq n_0. \quad (3.9)$$

As in the proof of Lemma 3.5, we see that it suffices to show (3.9) for $y \in (0, L - \frac{\sigma}{4})$.

Since $s', s'_n \geq \epsilon$ almost everywhere in (a, b) ,

$$|t'_n - t'| = \left| \frac{1}{s'_n \circ t_n} - \frac{1}{s' \circ t} \right| \leq \epsilon^{-2} |s'_n \circ t_n - s' \circ t| \quad (3.10)$$

almost everywhere in $(0, L - \frac{\sigma}{4})$, for sufficiently large n .

In view of (3.10), setting $\alpha_n := s'_n$ and $\alpha := s'$, the assertion of the lemma follows from (3.7). $\square$

Lemma 3.6 implies $t'_n \rightarrow t'$ in $L^q_{\text{loc}}(\mathbb{R})$ for all $q \in [1, \infty)$, because $|t'_n| \leq \frac{1}{\epsilon}$ almost everywhere in $\mathbb{R}$. Thus (3) is proved.

This ends the proof of Proposition 3.4.

3.3. Metric slope convergence and reparameterization. The proof of Theorem 3.3 yields the following variant for absolutely continuous curves in an arbitrary complete metric space (X, d) .

Theorem 3.7. *Let $f, f_n \in AC^1((a, b), X)$ and $|f'_n| \rightarrow |f'|$ in $L^1((a, b))$. Fix $\epsilon > 0$ and let $\hat{f} := f \circ t$ and $\hat{f}_n := f_n \circ t_n$, where $t := t_{\epsilon, f}$ and $t_n := t_{\epsilon, f_n}$. Then $t_n \rightarrow t$ in $W_{\text{loc}}^{1,q}(\mathbb{R})$ and $|\hat{f}'_n| \rightarrow |\hat{f}'|$ in $L^q_{\text{loc}}(\mathbb{R})$, for all $q \in [1, \infty)$.*

Proof. We use the notation of the previous section. Lemmas 2.1 and 3.6 imply $t_n \rightarrow t$ in $W_{\text{loc}}^{1,q}(\mathbb{R})$, for all $q \in [1, \infty)$, as above. By (2.2),

$$|\hat{f}'| \circ s = \frac{|f'|}{\epsilon + |f'|} \quad \text{almost everywhere in } (a, b).$$

Consequently (note that s and t have the Lusin N -property),

$$|\hat{f}'| = \frac{|f'|}{\epsilon + |f'|} \circ t \quad \text{almost everywhere in } (0, L).$$

Since $\hat{f}$ is locally constant on $\mathbb{R} \setminus [0, L]$, we have $|\hat{f}'|(y) = 0$ for $y \in \mathbb{R} \setminus [0, L]$. Analogously,

$$|\hat{f}'_n| = \frac{|f'_n|}{\epsilon + |f'_n|} \circ t \quad \text{almost everywhere in } (0, L_n),$$

and $|\hat{f}'_n|(y) = 0$ for $y \in \mathbb{R} \setminus [0, L_n]$.

Similarly as (3.5), we find that

$$\frac{|f'_n|}{\epsilon + |f'_n|} \rightarrow \frac{|f'|}{\epsilon + |f'|} \quad \text{in } L^1((a, b)).$$

The proof of Lemma 3.5 (with $\alpha_n = \frac{|f'_n|}{\epsilon + |f'_n|}$ and $\alpha = \frac{|f'|}{\epsilon + |f'|}$) gives that $|\hat{f}'_n| \rightarrow |\hat{f}'|$ in measure on $\mathbb{R}$. Consequently, $|\hat{f}'_n| \rightarrow |\hat{f}'|$ in $L^q_{\text{loc}}(\mathbb{R})$, for all $q \in [1, \infty)$, because $|\hat{f}'_n| \leq 1$ almost everywhere in $\mathbb{R}$. $\square$

4. PROOF OF THEOREM 1.2 AND THEOREM 1.3

Let $d \in \mathbb{N}_{\geq 1}$, $\epsilon > 0$, and $I \subseteq \mathbb{R}$ a bounded open interval. For $a \in C^{d-1,1}(\bar{I}, \mathbb{C}^d)$ consider $\mathcal{S}(a) = \Lambda_a$ we belongs to $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$, for all $q \in [1, \frac{d}{d-1})$; see Theorem 1.1. Fix an Almgren embedding $\Delta : \mathcal{A}_d(\mathbb{C}) \rightarrow \mathbb{R}^N$. We define (cf. Section 2.2)

$$t_a := t_{\epsilon, \Delta \circ \Lambda_a} : \mathbb{R} \rightarrow \bar{I}$$

and thus obtain the reparameterization map $t : C^{d-1,1}(\bar{I}, \mathbb{C}^d) \rightarrow \text{Lip}(\mathbb{R}, \bar{I})$. As seen in Section 2.2, t has the properties (1)–(3) listed in Theorem 1.2. The continuity of the map (1.2), for all $q \in [1, \infty)$, is a consequence of Theorem 1.3.

Let us prove Theorem 1.3. Let $a_n \rightarrow a$ in $C^d(\bar{I}, \mathbb{C}^d)$. Then $\Lambda_{a_n} \rightarrow \Lambda_a$ uniformly on $\bar{I}$, by [PR24, Corollary 6.5], and in $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ for all $q \in [1, \frac{d}{d-1})$, by Theorem 1.1. By Lemma 2.1, $t_{a_n} \rightarrow t_a$ and hence also $\Lambda_{a_n} \circ t_{a_n} \rightarrow \Lambda_a \circ t_a$ uniformly on $\mathbb{R}$. Theorem 3.3 implies that $t_{a_n} \rightarrow t_a$ in $W_{\text{loc}}^{1,q}(\mathbb{R})$ and $\Lambda_{a_n} \circ t_{a_n} \rightarrow \Lambda_a \circ t_a$ in $W_{\text{loc}}^{1,q}(\mathbb{R}, \mathcal{A}_d(\mathbb{C}))$, for all $q \in [1, \infty)$.

To prove (3) of Theorem 1.3 note that, in the respective setup, $\lambda_n \rightarrow \lambda$ in $W^{1,q}(I, \mathbb{C}^d)$ for all $q \in [1, \frac{d}{d-1})$, by [PR24, Theorem 1.6]. Consequently, the statement follows from Proposition 3.4. Here $t_n := t_{\epsilon, \lambda_n}$ and $t := t_{\epsilon, \lambda}$.

The proofs of Theorem 1.2 and Theorem 1.3 are complete.

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