EFFECTIVE QUASIANALYTIC REMEZ INEQUALITIES ON TAME SETS

ARMIN RAINER

Abstract. We establish a Remez inequality for functions in quasianalytic Denjoy–Carleman classes $\mathcal{C}_M$ on a large family of fat compact sets $K \subseteq \mathbb{R}^n$ with tame geometry, including all fat compact sets definable in the o-minimal expansion of $\mathbb{R}$ by restricted $\mathcal{C}_M$ functions. The inequality generalizes the classical Remez inequality for polynomials and its quasianalytic versions on convex bodies, replacing the polynomial degree by the Bang degree, an integer associated with the weight M and the size of the function. The constants depend explicitly on the Bang degree and the geometry of K. We derive a range of quantitative consequences: estimates for the volume of sublevel sets, comparison of L^p-norms, effective inequalities of Łojasiewicz, Harnack, and Markov type with explicit constants, and decay estimates for oscillatory integrals.

Contents

Contents . . . 1
1. Introduction . . . 2
2. Weights, quasianalytic functions, and the Bang degree . . . 10
3. O-minimal structures . . . 13
4. Quasianalytic Remez inequalities . . . 15
5. Bounds along polynomial curves . . . 18
6. Proof of the Remez inequality on admissible sets . . . 20
7. Sublevel sets and applications . . . 23
8. Inequalities of Łojasiewicz type . . . 28
9. Inequalities of Harnack type . . . 33
10. Inequalities of Markov type . . . 36
11. Oscillatory integrals . . . 39
Appendix A. Examples of admissible weights . . . 42
Appendix B. Univariate Remez inequality . . . 45
References . . . 49

Date: June 8, 2026.

2020 *Mathematics Subject Classification.* Primary 41A17, 26E10; Secondary 03C64, 32B20, 26D10, 28A75, 42B20.

Key words and phrases. Quantitative Remez inequalities, quasianalytic functions, Denjoy–Carleman classes, tame sets, o-minimal structures, Łojasiewicz inequalities, Harnack inequalities, Markov inequalities, oscillatory integrals.

This research was funded in whole or in part by the Austrian Science Fund (FWF) DOI 10.55776/PAT1381823. For open access purposes, the authors have applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

0000-0003-3825-3313.

1. Introduction

The Remez inequality is a fundamental tool in approximation theory. The classical result of Remez [Rem36] states that for a Lebesgue measurable subset E of the unit interval $[0,1]$ with Lebesgue measure $|E|>0$ we have

$$\|p\|_{[0,1]} \le T_d\Big(\frac{2-|E|}{|E|}\Big)\,\|p\|_E,$$

for all polynomials $p\in\mathbb{R}[x]$ of degree at most d, where $T_d(x)=\cos(d\arccos x)$ is the d-th Chebyshev polynomial.

In higher dimensions, there is the generalization due to Yu. Brudnyi and Ganzburg [BG73] for convex bodies $K\subseteq\mathbb{R}^n$: for Lebesgue measurable subsets $E\subseteq K$ with $|E|>0$ we have the sharp inequality

$$\|p\|_K \le T_d\Big(\frac{|K|^{1/n}+(|K|-|E|)^{1/n}}{|K|^{1/n}-(|K|-|E|)^{1/n}}\Big)\,\|p\|_E, \tag{1.1}$$

for all polynomials $p\in\mathbb{R}[x_1,\dots,x_n]$ of degree at most d.

Beyond convex sets, (1.1) has been generalized by Pierzchała [Pie15] to a large family of compact sets $K\subseteq\mathbb{R}^n$ admitting cusps, including all compact fat subanalytic sets: there exist $\nu>0$ and $m\in\mathbb{N}_{\ge1}$, depending only on K, such that

$$\|p\|_K \le T_{md}\Big(\frac{|K|^{\nu}+(|K|-|E|)^{\nu}}{|K|^{\nu}-(|K|-|E|)^{\nu}}\Big)\,\|p\|_E, \tag{1.2}$$

for all Lebesgue measurable $E\subseteq K$ with $|E|>0$ and all $p\in\mathbb{R}[x_1,\dots,x_n]$ of degree at most d. (Recall that a compact set $K\subseteq\mathbb{R}^n$ is called *fat* if $K=\overline{\operatorname{int}(K)}$.)

The Remez inequality has been extended to various classes of functions (e.g. analytic [Bru99b], plurisubharmonic [Bru99a], and holomorphic [Bru08]), generally, in the univariate or convex multivariate case. Still in the convex setting, [Yom11] showed that, in the Remez inequality for polynomials, the Lebesgue measure can be replaced by a certain geometric invariant which may be nonzero for discrete or even finite sets E; see also [BY15].

In this paper, we focus on quasianalytic Denjoy–Carleman classes $\mathcal{C}_M$, that is, classes of smooth functions whose sequence of derivatives is bounded by a given positive sequence $M=(M_j)_{j\ge0}$ with suitable properties.

Nazarov, Sodin, and Volberg [NSV04] proved a univariate Remez inequality for functions f of class $\mathcal{C}_M$, where the so-called *Bang degree* replaces the polynomial degree. The Bang degree is a nonnegative integer which can be computed from the sequence M and which depends on the size of f as a parameter. This number was introduced by Bang [Ban53] and named Bang degree by [NSV04]. A multivariate Remez inequality for functions f of class $\mathcal{C}_M$ on convex bodies $K\subseteq\mathbb{R}^n$ was obtained in [Rai23], where also effective bounds on the zero sets were proved.

The main goal of this paper is to establish a Remez inequality for functions f of class $\mathcal{C}_M$ on a large family of compact sets $K\subseteq\mathbb{R}^n$ with "tame geometry". In addition, we develop several analytic and geometric consequences of this Remez inequality. The family of admissible sets K (the same as the one used by [Pie15]) includes all fat compact sets that are definable in $\mathbb{R}_{\mathcal{C}_M}$, the o-minimal expansion of the real ordered field by all restricted functions of class $\mathcal{C}_M$ (assuming that

M satisfies some standard regularity conditions). In particular, all fat compact subanalytic sets are admissible. The constants that appear in the Remez inequality depend only on the Bang degree associated with M and on the geometry of K and can be computed explicitly. Similarly, all the consequences, including sublevel set estimates, reverse Hölder inequalities, inequalities of Łojasiewicz, Harnack, and Markov type, and decay estimates for oscillatory integrals, have a quantitative character. The Bang degree plays the role of an effective complexity parameter, analogous to the polynomial degree in semialgebraic geometry.

In the following, we present our main results in more detail.

1.1. **Quasianalytic Remez inequalities on tame sets.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 0}$ be a positive infinite sequence satisfying $0 < \mu_1 \leq \mu_2 \leq \cdots$. Let us also assume here in the introduction (for simplicity of exposition) that $\mu_j \geq j$ for $j \geq 1$. Given $M_0 > 0$, we then set

$$M_j := M_0 \prod_{i=1}^{j} \mu_i, \quad j \geq 1,$$

and get a sequence $M = (M_j)_{j\geq 0}$, which conversely determines the pair $(\mu^{|\infty}, M_0)$ in a unique way.

For open $U \subseteq \mathbb{R}^n$, the *Denjoy–Carleman class* $\mathcal{C}_M(U)$ consists of all $f \in \mathcal{C}^\infty(U)$ such that for each compact $K \subseteq U$ there exist constants $C, \rho > 0$ such that

$$|\partial^\alpha f(x)| \leq C\rho^{|\alpha|} M_{|\alpha|}, \qquad \text{for all } x \in K,\ \alpha \in \mathbb{N}^n. \tag{1.3}$$

For instance, $\mathcal{C}_M(U)$ coincides with space of real analytic functions on U, if $\mu_j = j$. By the Denjoy–Carleman theorem, the class $\mathcal{C}_M$ is *quasianalytic*, i.e., the Taylor series homomorphism $\mathcal{C}_M(U) \to \mathbb{R}[\![x_1, \ldots, x_d]\!]$ at any point of U is injective, provided that U is connected, if and only if

$$\sum_{j\geq 1} \frac{1}{\mu_j} = \infty. \tag{1.4}$$

See [Rai22] and [Thi08] for background on Denjoy–Carleman classes. Throughout the introduction, we make the quasianalyticity assumption (1.4); this can be relaxed somewhat as shall be seen later.

Let $K \subseteq \mathbb{R}^n$ be nonempty and compact. It will be convenient to work with the following variant of condition (1.3): a function $f : K \to \mathbb{R}$ is called $(\mu^{|\infty}, M_0)$-*smooth* if $f \in \mathcal{C}^\infty(K)$ and

$$\|f\|_{j,K} := \sup_{x\in K} \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha f(x)| \leq M_j, \quad j \geq 0.$$

(This is no restriction relative to (1.3) because the constants C, ρ can be absorbed by M, by redefining $(\mu^{|\infty}, M_0)$.)

Given $\mu^{|\infty}$ and $b > 0$, we define the *Bang degree*

$$\mathbf{D}_{\mu^{|\infty}}(b) := \sup\Big\{n \in \mathbb{N} : \sum_{j=j_0(b)+1}^{n} \frac{1}{\mu_j} < e\Big\},$$

where $j_0(b) := \lceil \log(b^{-1}) \rceil_{\mathbb{N}}$ is the smallest nonnegative integer j with $j \geq \log(b^{-1})$. Thanks to (1.4), $\mathbf{D}_{\mu^{|\infty}}(b)$ is a nonnegative integer.

We will define a collection $\mathscr{K}(\mathbb{R}^n)$ of fat compact subsets of $\mathbb{R}^n$ admissible for the Remez inequality (see Section 4.4). Informally, the sets in $\mathscr{K}(\mathbb{R}^n)$ admit a uniformly controlled polynomial parametrization. For the purposes of this introduction, it suffices to note that $\mathscr{K}(\mathbb{R}^n)$ includes all compact UPC-sets, in particular, convex bodies, fat compact sets with Hölder boundary, and all fat compact sets that are subanalytic or definable in $\mathbb{R}_{\mathcal{C}_M}$, where $M = (M_j)_{j\geq 0}$ is any quasianalytic weight sequence fulfilling some standard regularity conditions (see Example 3.4).

With each $K \in \mathscr{K}(\mathbb{R}^n)$ we will associate constants

$$a_K > 0 \quad \text{and} \quad 0 < \nu_K \leq 1,$$

which depend only on the geometry of K (see Section 4.5). For instance, if $K \subseteq \mathbb{R}^n$ is a convex body then $a_K = \operatorname{diam}(K)$ and $\nu_K = 1/n$.

The following quasianalytic Remez inequality is our main result.

Theorem 1.1. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ and $M_0 > 0$ be as above. Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$. Let $b_0 \in (0, M_0)$. Then for each $(\mu^{|\infty}, M_0)$-smooth function $f : L \to \mathbb{R}$ with $\|f\|_K \geq b_0$ and each Lebesgue measurable set $E \subseteq K$ with $|E| > 0$ we have*

$$\|f\|_K \leq \mathbf{C_N} \left(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E|)^{\nu_K}}\right)^{\mathbf{N}} \|f\|_E,$$

where

$$\mathbf{N} = 2\,\mathbf{D}_{a_K\,\mu^{|\infty}}(\tfrac{b_0}{M_0})$$

and $\mathbf{C_N}$ is a constant depending only on $\mathbf{N}$ and $\mu^{|\infty}$.

Theorem 1.1 can be interpreted as a "quantitative propagation-of-smallness principle" for $(\mu^{|\infty}, M_0)$-smooth functions on general tame compact sets beyond the convex setting.

The constant $\mathbf{C_N}$ is explicitly given in Definition 2.13. For standard examples of $\mu^{|\infty}$, it is exponential in $\mathbf{N}$.

The main defining property of the sets K in $\mathscr{K}(\mathbb{R}^n)$ is that they can be covered by (possibly infinite) families of images of compact convex or star-shaped subsets of the unit cube, with positive measure uniformly bounded below, under polynomial maps $\mathbb{R}^n \to \mathbb{R}^n$ of uniformly bounded degree and uniformly controlled Jacobian. While ν_K can be computed (with some effort) from these data, the constant a_K is explicitly given as a function of the diameter $\operatorname{diam}(K)$ and the maximal degree d of the polynomial maps; see (4.2). It follows that a_K grows linearly in $\operatorname{diam}(K)$ and asymptotically quadratically in d.

The proof of Theorem 1.1 is based on a fundamental technical result of [Pie15] on sets in $\mathscr{K}(\mathbb{R}^n)$, combined with the univariate Remez inequality for $(\mu^{|\infty}, M_0)$-smooth functions as well as its multivariate version on convex bodies. In fact, a more general version of Theorem 1.1 is stated in Theorem 4.13 which will be proved in Sections 5 and 6. In particular, Theorem 4.13 provides a Remez inequality for vector-valued maps $f = (f_1, \dots, f_m) : L \to \mathbb{R}^m$.

Remark 1.2. Contrary to the Remez inequality for polynomials ((1.1) and (1.2)), the vector-valued Remez inequality of Theorem 4.13 does not simply follow from

its single-valued version applied component-wise. The reason is that we may have $\|f\|_K \geq b_0 > 0$ while $\|f_i\|_K$ is much smaller than b_0 for some i. A component-wise treatment would cause $\mathbf{N}$ to be much larger than actually needed.

Remark 1.3. The requirement that $\|f\|_K$ is bounded below by a positive number b_0, on which the degree $\mathbf{N}$ depends, is vital, as we will see in Example 1.4. Setting $b_0 = \|f\|_K$ we obtain a Remez inequality, where the Bang degree

$$\mathbf{N}_{f,K} = 2\,\mathbf{D}_{a_K\,\mu^{|\infty}}\big(\tfrac{\|f\|_K}{M_0}\big)$$

explicitly depends on $f|_K$, similarly to Remez's inequality for polynomials.

The Bang degree is however not a generalization of the polynomial degree and the analogy between the two is limited. For instance, the Bang degree $\mathbf{N}_{f,K}$ of a $(\mu^{|\infty}, M_0)$-smooth function $f : L \to \mathbb{R}$ tends to infinity as we shrink K to a zero of f, since then $\|f\|_K \to 0$. (The accompaning decreasing effect on a_K fails to prevent this; see Propositions A.1, A.3 and A.4.)

Example 1.4. Consider the sequence of polynomials $p_k(x) := \frac{x^k}{k!}$, for $k \geq 1$, on $\mathbb{R}$. Each $f_k := p_k|_{[0,1]}$ is $(\mu^{|\infty}, 1)$-smooth for $\mu_j \equiv 1$, or $\mu_j = j$ (if one insists on the condition $\mu_j \geq j$), or other appropriate choices of $\mu^{|\infty}$. On the other hand,

$$\|f_k\|_{[0,1]} = \frac{1}{k!} \quad \text{and} \quad \|f_k\|_{[0,\frac{1}{2}]} = \frac{1}{2^k k!}.$$

Thus a Remez inequality for f_k with a constant that is independent of the size of the functions f_k, i.e., independent of k, cannot hold.

The set of functions for which Theorem 1.1 applies (with specified μ^∞, M_0, and b_0) can be infinite dimensional.

Remark 1.5. Setting $K = B(x,r)$ and $b_0 = \|f\|_{B(x,r)}$ in Theorem 1.1 we get a bound for the *doubling exponent* of f on the ball $B(x,r)$:

$$\log \frac{\|f\|_{B(x,r)}}{\|f\|_{B(x,\frac{r}{2})}} \leq (n \log 2 + \log n)\, \mathbf{N}_f(x,r) + \log \mathbf{C}_{\mathbf{N}_f(x,r)},$$

where

$$\mathbf{N}_f(x,r) = 2\,\mathbf{D}_{2r\,\mu^{|\infty}}\big(\tfrac{\|f\|_{B(x,r)}}{M_0}\big).$$

1.2. **Finite determinacy.** So far we only considered $\mathcal{C}^\infty$-functions with derivatives of all orders controlled by the sequence $M = (M_j)$ and we assumed that the quasianalyticity assumption (1.4) is satisfied.

Actually, the result in Theorem 1.1 and many of its consequences depend only on the derivatives up to some finite order and remain true even for functions that are only differentiable up to said finite order. Also the quasianalyticity requirement can often be relaxed.

We will make this precise below, in particular, we will specify the finite order and use customized terminology; see Section 2.

Next we discuss various analytic and geometric consequences of Theorem 1.1 and retain its setting for simplicity.

1.3. **Volume of sublevel sets and applications.** In the setup of Theorem 1.1, we obtain the following corollaries; see Section 7, where more general versions are proved.

Corollary 1.6 (Volume of sublevel sets)**.** *The sublevel set* $K_t := \{x \in K : |f(x)| \leq t\}$ *of* $f|_K$ *satisfies*

$$|K_t| \leq \frac{|K|}{\nu_K} \Big(\frac{\mathbf{C_N}\, t}{\|f\|_K}\Big)^{1/\mathbf{N}}, \quad t > 0.$$

This simple consequence is very useful. For instance, it implies the following inequality of Nikolskii type which reverses Hölder's inequality.

Corollary 1.7 (Comparison of L^p-norms)**.** *For all* $0 < q < p \leq \infty$,

$$\|f\|_{L^p(K)} \leq \Big(\frac{\mathbf{C_N}}{\nu_K^{\mathbf{N}}}\Big)^{1-\frac{q}{p}} \Big(\frac{q\mathbf{N}+1}{|K|}\Big)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^q(K)}.$$

We conclude integrability of $|f|^{-\alpha}$ in the range $0 < \alpha < 1/\mathbf{N}$.

Corollary 1.8 (Integrability of $|f|^{-\alpha}$)**.** *The integral* $\int_K |f(x)|^{-\alpha}\, dx$ *converges, provided that* $0 < \alpha < 1/\mathbf{N}$.

We get a bound for the mean oscillation of $\log|f|$ over Lebesgue measurable sets.

Corollary 1.9 (Mean oscillation of $\log|f|$)**.** *For each Lebesgue measurable set* $E \subseteq K$ *with* $|E| > 0$, *the mean oscillation of* $\log|f|$ *over* E *satisfies*

$$\mathrm{mo}_E(\log|f|) \leq 2\log \mathbf{C_N} + 2\mathbf{N}\Big(1 + \log\Big(\frac{|K|}{\nu_K\,|E|}\Big)\Big).$$

This bound is not sufficient for concluding that $\log|f|$ is BMO, in contrast to polynomials or analytic functions; see [Bru99b]. What is missing is control near the zero set of f, since the Bang degree depends strongly on the size of the function; see Remark 1.3 and Remark 7.10.

1.4. **Inequalities of Łojasiewicz type.** Functions definable in polynomially bounded o-minimal structures satisfy inequalities of Łojasiewicz type (cf. [vdDM96, 4.14]), but the constants are typically unspecified. In Section 8, we establish several inequalities of Łojasiewicz type with *explicit* constants, depending on the Bang degree and the geometry of K.

Following [Pie22, Definition 1.16], we say that $K \in \mathscr{K}(\mathbb{R}^n)$ is (c_K, θ_K)-*thick* if there exist constants $c_K, \theta_K > 0$ such that

$$|K \cap B(x,r)| \geq c_K\, r^{\theta_K},$$

for all $x \in K$ and $0 < r \leq 1$. Every compact fat nonempty set $K \subseteq \mathbb{R}^n$ that is definable in a polynomially bounded o-minimal structure is (c_K, θ_K)-thick for some $c_K, \theta_K > 0$, as a consequence of the mentioned Łojasiewicz inequalities (see [Pie22, Theorem 1.18]). Each convex body $K \subseteq \mathbb{R}^n$ is (c_K, θ_K)-thick with $\theta_K = n$.

Corollary 1.10. *In the setup of Theorem 1.1, assume additionally that* K *is* (c_K, θ_K)-*thick. For any function* $g : K \to [-1, 1]$, *we have*

$$\|f\|_{B(x, \frac{|g(x)|}{2}) \cap K} \geq \frac{\|f\|_K}{\mathbf{C_N}} \Big(\frac{c_K\, \nu_K}{2^{\theta_K}\, |K|}\Big)^{\mathbf{N}} |g(x)|^{\theta_K \mathbf{N}},$$

for all $x \in K$.

Note that we do not require $f|_K^{-1}(0) \subseteq g^{-1}(0)$. See Theorem 8.3 for a more general version.

In particular, this applies to the distance to the zero set of f,

$$g(x) := \min\{\operatorname{dist}(x, f|_K^{-1}(0)), 1\}.$$

Taking $g \equiv r$ for $r \in (0,1)$, we get a bound on the order of vanishing.

Corollary 1.11. *The order of vanishing of* f *at each* $x \in K$ *is bounded by* $\theta_K \mathbf{N}$.

We deduce several inequalities for the gradient with completely explicit constants. For instance, the following corollary is a special case of Theorem 8.9.

Corollary 1.12. *Assume that* $(\operatorname{grad} f|_K)^{-1}(0) \subseteq f|_K^{-1}(0)$ *and* $M_0 \geq 1$. *Then,*

$$|\operatorname{grad} f(x)| \geq \frac{1}{2\sqrt{M_2}} \Big(\frac{\|f\|_K}{2\mathbf{C_N}}\Big)^{\frac{1}{\theta_K \mathbf{N}}} \Big(\frac{c_K\, \nu_K}{|K|}\Big)^{\frac{1}{\theta_K}} |f(x)|^{\frac{3}{2} - \frac{1}{\theta_K \mathbf{N}}}$$

in a neighborhood of $f|_K^{-1}(0)$ *in* K.

It is well-known (see Lemma 8.8) that there exist $c > 0$ and $\alpha \in [0,1)$ such that

$$|\operatorname{grad} f(x)| \geq c\, |f(x)|^\alpha,$$

but c and α are not explicit. We have the following variant with an explicit exponent < 1; see Corollary 8.10.

Corollary 1.13. *Assume that* $(\operatorname{grad} f|_K)^{-1}(0) \subseteq f|_K^{-1}(0)$ *and* $M_0 \geq 1$. *Then,*

$$\|\operatorname{grad} f\|_{B_x} \geq \frac{1}{2} \Big(\frac{2\|f\|_K}{3\mathbf{C_N}}\Big)^{\frac{1}{\theta_K \mathbf{N}}} \Big(\frac{c_K\, \nu_K}{|K|}\Big)^{\frac{1}{\theta_K}} |f(x)|^{1 - \frac{1}{\theta_K \mathbf{N}}}$$

for x *in a neighborhood of* $f|_K^{-1}(0)$ *in* K, *where* B_x *is a neighborhood of* x *in* L *that shrinks to* x *if* x *tends to* $f|_K^{-1}(0)$.

See Section 8 for further inequalities for the gradient.

1.5. **Inequalities of Harnack type.** If $f : L \to \mathbb{R}$ is nonvanishing on K and $\min_{x \in K} |f(x)| \geq m_0 > 0$, then Theorem 1.1 easily implies

$$\max_{x \in K} |f(x)| \leq 2\, \mathbf{C_N} \Big(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E_{m_0}|)^{\nu_K}}\Big)^{\mathbf{N}} \min_{x \in K} |f(x)|,$$

where $E_{m_0} = \{x \in K : |f(x)| \leq \min_{x \in K} |f(x)| + m_0\}$. But it is difficult to determine $|E_{m_0}|$.

In Section 9, we combine this inequality for f with the one for $1/f$ in order to obtain the following inequality of Harnack type with completely explicit constants. This requires an additional regularity assumption on $\mu^{|\infty}$ which allows to conclude that $1/f : L \to \mathbb{R}$ is $(\beta_0\, \mu^{|\infty}, \frac{2}{m_0})$-smooth, provided that $f : L \to \mathbb{R}$ is $(\mu^{|\infty}, M_0)$-smooth and $\min_{x \in L} |f(x)| \geq m_0 > 0$, where

$$\beta_0 = \beta_0(\mu^{|\infty}, n, M_0, m_0) > 1.$$

Corollary 1.14. *Then, for all $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$,*

$$\max_{x\in K} |f(x)| \leq \mathbf{C}_{\mathbf{N}}^2 \Big(\frac{2^{\nu_K}}{2^{\nu_K}-1}\Big)^{2\mathbf{N}} \min_{x\in K} |f(x)|,$$

where

$$\mathbf{N} = 2\,\mathbf{D}_{a_K \beta_0\, \mu^{|\infty}(\frac{m_0}{2M_0})}.$$

See Theorem 9.5, where β_0 is made fully explicit.

1.6. **Inequalities of Markov type.** The Bang degree gives an upper bound for the total number of zeros of an $(\mu^{|\infty}, M_0)$-smooth function on an interval, as proved in [Rai23]; see Proposition 10.1. Using this fact and the reverse Hölder inequality in Corollary 1.7, we obtain inequalities of Markov type in Section 10.

Here is an exemplary result for univariate functions. Since the zero bound is applied to the first derivative f', we work with the shifted weight $\mu_{+1}^{|\infty} = (\mu_{j+1})_{j\geq 1}$.

Corollary 1.15. *Let $I \subseteq \mathbb{R}$ be a compact interval. Let $b_1 \in (0, M_1)$. Every $(\mu^{|\infty}, M_0)$-smooth function $f : I \to \mathbb{R}$ such that $\|f'\|_I \geq b_1$ satisfies*

$$\|f'\|_I \leq \frac{2\,\mathbf{C}_{\mathbf{N}}}{|I|}(\mathbf{N}+1)^2\,\|f\|_I,$$

where

$$\mathbf{N} = 2\,\mathbf{D}_{|I|\,\mu_{+1}^{|\infty}(\frac{b_1}{M_1})}.$$

This has consequence for sampling the supremum norm; see Section 10.3.

In Section 10, we also provide multivariate versions of Corollary 1.15 on $K \in \mathscr{K}(\mathbb{R}^n)$.

1.7. **Oscillatory integrals.** In Section 11, we combine the estimate for the volume of sublevel sets in Corollary 1.6 and the bound for the number of zeros with the van der Corput lemma (which we recall in Lemma 11.1) in order to obtain decay estimates for oscillatory integrals whose phase functions are $(\mu^{|\infty}, M_0)$-smooth.

Corollary 1.16. *Let $I^n \subseteq \mathbb{R}^n$ be a compact cube. Let $b_2 \in (0, M_2)$. Let $f : I^n \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function. Assume, for some $1 \leq i_0 \leq n$, we have $\|\partial_{i_0}^2 f\|_\ell \geq b_2$ for each line segment ℓ in I^n with direction e_{i_0}. Let $g : I^n \to \mathbb{C}$ belong to $\mathcal{C}^1(I^n)$. Then,*

$$\Big|\int_{I^n} e^{i\lambda f(x)} g(x)\,dx\Big| \leq B\,\lambda^{-\frac{1}{\mathbf{N}+2}}, \qquad \text{if } \lambda \geq \Big(\frac{2}{b_2}\Big)^{1+\frac{2}{\mathbf{N}}},$$

where

$$\mathbf{N} = 2\,\mathbf{D}_{\sqrt{n}\,|I|\,\mu_{+2}^{|\infty}(\frac{\frac{b_2}{2}}{M_2+\frac{b_2}{2}})},$$

$$B = \frac{\mathbf{C}_{\mathbf{N}}^{1/\mathbf{N}}\, n\,|I|^n\,\|g\|_{I^n}}{\|\partial_{i_0}^2 f\|_{I^n}^{1/\mathbf{N}}} + 2C\,(\mathbf{N}+1)\,\big(|I|^{n-1}\,\|g\|_{I^n} + \|\partial_{i_0} g\|_{L^1(I^n)}\big),$$

and $C > 0$ is an absolute constant.

If $K \in \mathscr{K}(\mathbb{R}^d)$ is definable in $\mathbb{R}_{\mathcal{C}^M}$, we can use uniform finiteness in o-minimal structures and obtain the following variant for integrals over K.

Corollary 1.17. *In the setting of Corollary 1.16, assume additionally that $K \in \mathscr{K}(\mathbb{R}^n)$ is contained in the interior of I^n and definable in $\mathbb{R}_{\mathcal{C}_M}$. If $\|\partial_{i_0}^2 f\|_K \geq b_2' > 0$, then*

$$\Big|\int_K e^{i\lambda f(x)} g(x)\,dx\Big| \leq B\,\lambda^{-\frac{1}{\mathbf{N}+2}}, \qquad \text{for } \lambda > 0,$$

where

$$\mathbf{N} = 2\,\mathbf{D}_{\sqrt{n}\,|I|\,\mu_{+2}^{|\infty}\left(\frac{\frac{b_2}{2}}{M_2+\frac{b_2}{2}}\right)},$$

$$B = \frac{\mathbf{C}_{\mathbf{N}}^{1/\mathbf{N}}\,|K|\,\|g\|_K}{\nu_K\,\|\partial_{i_0}^2 f\|_K^{1/\mathbf{N}}} + 2C\,N_{f,K}\left(|I|^{n-1}\,\|g\|_{I^n} + \|\partial_{i_0} g\|_{L^1(I^n)}\right),$$

$C > 0$ is an absolute constant, and $N_{f,K} \in \mathbb{N}$ is an integer depending only on f and K.

1.8. **Metric entropy of sublevel sets.** Let $\epsilon > 0$ and $K \subseteq \mathbb{R}^n$ relatively compact. The *covering number* $N_\epsilon(K)$ of K is the minimal number of closed balls of radius ϵ covering K and $H_\epsilon(K) := \log_2 N_\epsilon(K)$ is the *ϵ-entropy* of K.

Corollary 1.18. *In the setup of Theorem 1.1, assume additionally that $K \in \mathscr{K}(\mathbb{R}^n)$ is definable in $\mathbb{R}_{\mathcal{C}_M}$ and $L = B(0,r)$. Then the covering number of the sublevel set K_t of $f|_K$ satisfies*

$$N_\epsilon(K_t) \leq C(n)\frac{|K|}{\nu_K}\left(\frac{\mathbf{C}_{\mathbf{N}}\,t}{\|f\|_K}\right)^{1/\mathbf{N}}\frac{1}{\epsilon^n} + C_{K_t}\left(1+\left(\frac{r}{\epsilon}\right)^{n-1}\right), \quad t > 0,$$

for a constant $C_{K_t} > 0$.

Proof. This follows from Corollary 1.6 and [YC04, Proposition 5.6]. □

The constant C_{K_t} is of the form

$$C_{K_t} = C(n)\max_{1\leq j\leq n-1}\omega_j \cdot B_{n-j}(K_t),$$

where $B_{n-j}(K_t)$ bounds the number of connected components of $P \cap K_t$, where P is any affine $(n-j)$-plane in $\mathbb{R}^n$. The bounds $B_{n-j}(K_t)$ exist because K_t is definable in $\mathbb{R}_{\mathcal{C}_M}$. It would be interesting to find explicit bounds via the Bang degree depending on the Denjoy–Carleman complexity of K_t.

1.9. **Organization of the paper.** In Section 2, we introduce the admissible weights that control the derivatives of quasianalytic functions and define the Bang degree. We also introduce terminology that accounts for the fact that often differentiability of finite order is sufficient.

In Section 3, we recall basics on o-minimal structures, in particular, the expansion of the real field by restricted function in a quasianalytic Denjoy–Carleman class.

In Section 4, we formulate the general version of the quasianalytic Remez inequality (Theorem 4.13) and discuss the collection $\mathscr{K}(\mathbb{R}^d)$ of sets that support this inequality. Its proof comprises Sections 5 and 6.

In Section 7, we deduce estimates for the volume of sublevel sets and obtain a number of consequences, including a reverse Hölder inequality and bounds on the mean oscillation of $\log|f|$.

Section 8 is devoted to several inequalities of Łojasiewicz type. In Section 9, inequalities of Harnack types are obtained. Combining the reverse Hölder inequality with a bound for the number of zeros in terms of the Bang degree, we obtain inequalities of Markov type in Section 10.

Finally, in Section 11, the sublevel set approach together with the van der Corput lemma are used to provide decay effective estimates for oscillatory integrals.

The paper ends with two appendices. In Section A, we establish precise upper and lower bounds for the Bang degree associated with specific admissible weights (the analytic weight and Denjoy weights). In Section B, we reexamine the proof of the univariate quasianalytic Remez inequality of [NSV04] and adjust it to the vector-valued case.

1.10. **Notation.** We use the Euclidean norm $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$ in $\mathbb{R}^n$. Sometimes we also use the 1-norm $|x|_1 = \sum_{j=1}^n |x_j|$. Recall that $|x| \le |x|_1 \le \sqrt{n}\,|x|$.

For differentiable functions $f : \mathbb{R}^n \supseteq K \to \mathbb{R}$ we typically use (semi)norms based on the 1-norm:

$$\|f\|_{j,K} := \sup_{x\in K} \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha f(x)|,$$

in particular,

$$\|\operatorname{grad} f\|_K := \sup_{x\in K} \sum_{i=1}^n |\partial_i f(x)| = \|f\|_{1,K}.$$

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and often we also write $|E| = \mathcal{L}^n(E)$ if the ambient dimension n is clear. Let ω_n denote the volume of the unit ball $B(0,1) \subseteq \mathbb{R}^n$. The standard basis vectors in $\mathbb{R}^n$ are denoted by $e_1, \dots, e_n$.

2. Weights, quasianalytic functions, and the Bang degree

This section introduces the central objects of the paper and sets the terminology used throughout.

2.1. Admissible weights and $(\mu^{|N}, M_0)$-smooth functions.

Definition 2.1 (Admissible weights)**.** An *admissible weight* is an infinite increasing sequence $\mu^{|\infty} = (\mu_j)_{j\ge1}$ of elements in $\mathbb{R}_{>0}$, i.e., $0 < \mu_1 \le \mu_2 \le \cdots$.

Definition 2.2 (Quasianalytic weights)**.** An admissible weight $\mu^{|\infty} = (\mu_j)_{j\ge1}$ is called *quasianalytic* if

$$\sum_{j\ge1} \frac{1}{\mu_j} = \infty.$$

Given an admissible weight $\mu^{|\infty} = (\mu_j)_{j\ge1}$ and any positive real number $M_0 > 0$, we define

$$M_j := M_0 \prod_{i=1}^j \mu_i, \quad j \ge 1. \tag{2.1}$$

Then the sequence $M = (M_j)_{j\ge0}$ satisfies the logarithmic convexity property

$$M_j^2 \le M_{j-1} M_{j+1}, \quad j \ge 1.$$

Conversely, $M = (M_j)_{j\geq 0}$ uniquely determines the pair $(\mu^{|\infty}, M_0)$ (setting $\mu_j = \frac{M_j}{M_{j-1}}$).

Definition 2.3 ($(\mu^{|N}, M_0)$-smooth functions and maps)**.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > 0$. Let $N \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. A function $f : K \to \mathbb{R}$, defined on a nonempty compact set $K \subseteq \mathbb{R}^n$, is called $(\mu^{|N}, M_0)$-*smooth* if f is of class $\mathcal{C}^N$ (in some open neighborhood of K) and

$$\|f\|_{j,K} \leq M_j, \quad 0 \leq j < N+1,$$

where

$$\|f\|_{j,K} := \sup_{x\in K} \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha f(x)|$$

and $\|f\|_K = \|f\|_{0,K} = \sup_{x\in K} |f(x)|$ denotes the sup-norm.

Similarly, a map $f = (f_1, \ldots, f_m) : K \to \mathbb{R}^m$ is called $(\mu^{|N}, M_0)$-*smooth* if f is of class $\mathcal{C}^N$ and

$$\|f\|_{j,K} \leq M_j, \quad 0 \leq j < N+1,$$

where

$$\|f\|_{j,K} := \sup_{x\in K} \sum_{i=1}^m \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha f_i(x)|.$$

We set $\|f\|_K := \|f\|_{0,K} = \sup_{x\in K} \sum_{i=1}^m |f_i(x)|$.

Remark 2.4. In the single-valued case, $\|f\|_{j,K}$ is the supremum over $x \in K$ of the 1-norm of the vector with entries $\partial^\alpha f(x)$, where $|\alpha| = j$ and the entry $\partial^\alpha f(x)$ appears $\frac{j!}{\alpha!}$ times. This is convenient because the directional derivatives satisfy

$$|d_v^j f(x)| = \big|\partial_t^j|_{t=0} f(x+tv)\big| = \Big|\sum_{|\alpha|=j} \frac{j!}{\alpha!} \partial^\alpha f(x) v^\alpha\Big| \leq |v|^j \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha f(x)|.$$

The vector-valued case behaves similarly.

2.2. **The Bang degree.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight. We consider the function $\Sigma_{\mu^{|\infty}} : \mathbb{N}_{\geq 1} \times \mathbb{N} \to [0, \infty)$ defined by

$$\Sigma_{\mu^{|\infty}}(m, n) := \sum_{j=m}^n \frac{1}{\mu_j}.$$

If $n < m$ then $\Sigma_{\mu^{|\infty}}(m, n) = 0$ (being an empty sum). The function $(m, n) \mapsto \Sigma_{\mu^{|\infty}}(m, n)$ is increasing in n and decreasing in m.

For $b > 0$, let $j_0(b)$ be the smallest integer $j \in \mathbb{N}$ with $j \geq \log(b^{-1})$, i.e.,

$$j_0(b) := \lceil \log(b^{-1}) \rceil_{\mathbb{N}}.$$

Definition 2.5 (Bang degree)**.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $b > 0$. We define the *Bang degree* $\mathbf{D}_{\mu^{|\infty}}(b) \in \mathbb{N} \cup \{\infty\}$ by setting

$$\mathbf{D}_{\mu^{|\infty}}(b) := \sup\big\{n \in \mathbb{N} : \Sigma_{\mu^{|\infty}}(j_0(b)+1, n) < e\big\}.$$

Lemma 2.6. *We have:*

(1) $\mathbf{D}_{\mu^{|\infty}}(b) \geq j_0(b)$ *and* $\mathbf{D}_{\mu^{|\infty}}(b) = j_0(b)$ *if and only if* $\mu_{j_0(b)+1} \leq 1/e$.

(2) $\mathbf{D}_{\mu^{|\infty}}(b) = \infty$ *if and only if* $\Sigma_{\mu^{|\infty}}(j_0(b)+1, n) < e$ *for all* n.

(3) *The map* $(a,b)\mapsto \mathbf{D}_{a\,\mu^{|\infty}}(b)$ *is increasing in* $a>0$ *and decreasing in* $b>0$.

(4) *If* $\mu^{|\infty}$ *is quasianalytic, then* $\mathbf{D}_{\mu^{|\infty}}(b)$ *is finite.*

Proof. All properties follow easily from the definition. □

Remark 2.7. Our notion of Bang degree differs slightly from the notion proposed in [NSV04], where it is attached to a given $(\mu^{|\infty},M_0)$-smooth function $f:[0,1]\to\mathbb{R}$ and equals $\mathbf{D}_{\mu^{|\infty}}(\|f\|_{[0,1]}/M_0)$ (in our notation). The unspecified argument b allows for more flexibility.

2.3. **Shifted weights.** Occasionally, we will deal with derivatives of $(\mu^{|\infty},M_0)$-smooth functions. This naturally requires a "shift" of the weight $\mu^{|\infty}$.

Given an admissible weight $\mu^{|\infty}=(\mu_j)_{j\geq 1}$, also the shifted sequence

$$\mu^{|\infty}_{+i}=(\mu_{j+i})_{j\geq 1}, \tag{2.2}$$

where $i\in\mathbb{N}_{\geq 1}$, is an admissible weight. Evidently, $\mu^{|\infty}$ is quasianalytic if and only if $\mu^{|\infty}_{+i}$ is quasianalytic.

The Bang degree for $\mu^{|\infty}_{+i}$ can always be reduced to the Bang degree for $\mu^{|\infty}$.

Lemma 2.8. *We have* $\mathbf{D}_{\mu^{|\infty}_{+i}}(b)=\mathbf{D}_{\mu^{|\infty}}(e^{-i}b)-i$.

Proof. This follows from $\Sigma_{\mu^{|\infty}_{+1}}(j_0(b)+1,n)=\Sigma_{\mu^{|\infty}}(j_0(b)+2,n+1)$. □

2.4. **Further definitions and conventions.** Let us adjust the definition of the Bang degree to its use in the Remez inequality. We will not automatically assume that $\mu^{|\infty}$ is quasianalytic. Instead we work with an admissibility condition that guarantees that the Bang degree is finite.

Definition 2.9 (Associated degree)**.** Let $\mu^{|\infty}=(\mu_j)_{j\geq 1}$ be an admissible weight and $M_0,a,b>0$. We say that $(\mu^{|\infty},M_0,a,b)$ is *admissible*, if

$$\sum_{j=j_0+1}^{\infty}\frac{1}{\mu_j}>a\,e,\qquad \text{where } j_0=\lceil\log(\tfrac{M_0}{b})\rceil_{\mathbb{N}}. \tag{2.3}$$

In that case, we define the integer

$$\mathbf{N}=\mathbf{N}(\mu^{|\infty},M_0,a,b):=2\,\mathbf{D}_{a\,\mu^{|\infty}}(\tfrac{b}{M_0}) \tag{2.4}$$

and say that $\mathbf{N}$ is the *degree associated with* $(\mu^{|\infty},M_0,a,b)$.

Remark 2.10. (1) By definition, for $r,s>0$, $(r\,\mu^{|\infty},s\,M_0,a,b)$ is admissible if and only if $(\mu^{|\infty},M_0,r\,a,\frac{b}{s})$ is admissible, and

$$\mathbf{N}(r\,\mu^{|\infty},s\,M_0,a,b)=\mathbf{N}(\mu^{|\infty},M_0,r\,a,\tfrac{b}{s}).$$

(2) Assuming admissibility, the integer $\mathbf{N}(\mu^{|\infty},M_0,a,b)$ is increasing in a and decreasing in b.

(3) If $\mu^{|\infty}$ is quasianalytic, then (2.3) is satisfied for all $M_0,a,b>0$.

Remark 2.11. In our results, the integer $\mathbf{N}$ will be useless if it is zero. We have $\mathbf{N}(\mu^{|\infty}, M_0, a, b) = 0$ if and only if $ae\mu_1 \leq 1$ and $M_0 \leq b$ (by Lemma 2.6(1)). Therefore, we will generally assume that $M_0 > b$ so that $\mathbf{N}(\mu^{|\infty}, M_0, a, b) \geq 2$.

In [Rai23], we occasionally used $M_0 = b = \|f\|_K$, for a $(\mu^{|\infty}, M_0)$-smooth function f, and took extra care for the case $\mathbf{N}(\mu^{|\infty}, M_0, a, b) = 0$. These occasions can simply be adjusted to the convention $M_0 > b$ by setting $b = \|f\|_K$ and $M_0 = 2\|f\|_K$. This possibly increases $\mathbf{N}$ a little, while f is still $(\mu^{|\infty}, M_0)$-smooth.

Definition 2.12 (The function $\gamma_{\tilde{\mu}}$)**.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight. Let $\tilde{\mu} : [1, \infty) \to (0, \infty)$ be an increasing continuous and (piecewise) $\mathcal{C}^1$ function such that

$$\mu_j = \tilde{\mu}(j), \quad j \geq 1.$$

For instance, we may take $\tilde{\mu}$ piecewise affine and work consistently with the left derivative at points, where $\tilde{\mu}$ is not differentiable.

Then we define (following [NSV04])

$$\gamma_{\tilde{\mu}}(n) := \sup_{1\leq s\leq n} \frac{s\tilde{\mu}'(s)}{\tilde{\mu}(s)}, \quad n \in \mathbb{N}_{\geq 1}.$$

Note that $\gamma_{\tilde{\mu}}$ depends on the choice of $\tilde{\mu}$ which is not unique.

Definition 2.13 (The constant $\mathbf{C_N}$)**.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0, a, b > 0$ with $M_0 > b$. Assume that $(\mu^{|\infty}, M_0, a, b)$ is admissible and let $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, a, b)$. Suppose that $\tilde{\mu}$ has been chosen. We define

$$\mathbf{C_N} := (4e^{4+\gamma_{\tilde{\mu}}(\mathbf{N})})^{\mathbf{N}}.$$

If $\sup_{1\leq s<\infty} \frac{s\tilde{\mu}'(s)}{\tilde{\mu}(s)}$ is finite (see e.g. Propositions A.2 and A.5), then $\mathbf{C_N}$ can be replaced by the potentially larger constant $C^{\mathbf{N}}$ for some suitable $C > 0$.

Remark 2.14. Our notation differs somewhat from the notation used in [Rai23]. For instance, the integer $\mathbf{N}$ defined in (2.4) is twice the integer denoted by the same symbol in [Rai23].

3. O-minimal structures

We recall the definition of an *o-minimal structure* over the real ordered field and some background; cf. [vdDM96] and [vdD98].

Definition 3.1 (O-minimal structure over the real ordered field)**.** A *structure* over the real ordered field $(\mathbb{R}, +, \cdot, <)$ is a sequence $\mathscr{S} = (\mathscr{S}_n)_{n\geq 1}$ of collections of subsets $\mathscr{S}_n$ of $\mathbb{R}^n$, such that for all $n, n' \geq 1$:

(1) $\mathscr{S}_n$ is a Boolean algebra with respect to the usual set-theoretic operations;

(2) $\mathscr{S}_n$ contains all semialgebraic subsets of $\mathbb{R}^n$;

(3) if $X \in \mathscr{S}_n$ and $X' \in \mathscr{S}_{n'}$, then $X \times X' \in \mathscr{S}_{n+n'}$;

(4) if $n \geq n'$ and $X \in \mathscr{S}_n$, then $\pi(X) \in \mathscr{S}_{n'}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n'}$ is the projection on the first n' coordinates.

A subset $X \subseteq \mathbb{R}^n$ is said to be *definable* in the structure $\mathscr{S}$ if $X \in \mathscr{S}_n$. A map $f : X \to \mathbb{R}^{n'}$ is called *definable* in $\mathscr{S}$ if its graph is definable. A structure $\mathscr{S}$ is called *o-minimal* if

(5) every set in $\mathscr{S}_1$ is a finite union of intervals and points.

A structure $\mathscr{S}$ is called *polynomially bounded* if for every function $f : \mathbb{R} \to \mathbb{R}$ that is definable in $\mathscr{S}$ there exists $N \in \mathbb{N}$ such that $f(t) = O(t^N)$ as $t \to \infty$. An o-minimal structure $\mathscr{S}$ either is polynomially bounded or the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is definable in $\mathscr{S}$ (see [Mil94]).

Example 3.2 (Semialgebraic sets)**.** The family of all semialgebraic sets forms a polynomially bounded o-minimal structure.

Example 3.3 (Globally subanalytic sets)**.** The family of globally subanalytic sets forms a polynomially bounded o-minimal structure denoted $\mathbb{R}_{\mathrm{an}}$. It is the smallest structure over the real ordered field containing all restricted analytic functions $f : \mathbb{R}^n \to \mathbb{R}$, i.e., $f|_{[-1,1]^n}$ is analytic and $f = 0$ outside $[-1,1]^n$.

Example 3.4 (The structure $\mathbb{R}_{\mathcal{C}_M}$)**.** Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > 0$. Let $M = (M_j)_{j\geq 0}$ be defined by (2.1). Consider the associated Denjoy–Carleman class $\mathcal{C}_M$ (see (1.3)).

Let us make the following additional assumptions on the admissible weight $\mu^{|\infty}$.

(1) The sequence
$$a_j := \Big(\frac{\mu_1 \cdots \mu_j}{j!}\Big)^{1/j}, \quad j \geq 1,$$
satisfies $\liminf_{j\to\infty} a_j > 0$.

(2) The sequence a_j is almost increasing, i.e., there exists a constant $H \geq 1$ such that $a_j \leq H a_k$ whenever $j \leq k$.

(3) There is a constant $C \geq 1$ such that $\mu_{j+1} \leq C^j$, equivalently, $M_{j+1} \leq C^j M_j$, for $j \geq 1$.

(4) $\mu^{|\infty}$ is quasianalytic.

Under the assumptions (1)–(4), the class $\mathcal{C}_M$ is quasianalytic, containes all real analytic functions, and is stable under composition, differentiation, and inverse and implicit functions (cf. [Rai22]).

Under these assumptions, the expansion $\mathbb{R}_{\mathcal{C}_M}$ of the real ordered field by all restricted $\mathcal{C}_M$-functions is a polynomially bounded o-minimal structure, by [RSW03]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *restricted $\mathcal{C}_M$-function* if there exists $\tilde{f} \in \mathcal{C}_M(U)$ for an open neighborhood U of $[-1,1]^n$ such that $f(x) = \tilde{f}(x)$ if $x \in [-1,1]^n$ and $f(x) = 0$ if $x \notin [-1,1]^n$.

Note that (2) is satisfied with $H = 1$ if the sequence μ_j/j is increasing (which means that $M_j/j!$ is logarithmically convex). Under this stronger condition, one can drop the assumption (3) (which guarantees that $\mathcal{C}_M$ is stable under differentiation): the expansion $\mathbb{R}_{\overline{\mathcal{C}}_M}$ of the real ordered field by all restricted $\overline{\mathcal{C}}_M$-functions is a polynomially bounded o-minimal structure, where
$$\overline{\mathcal{C}}_M := \bigcup_{k\geq 0} \mathcal{C}_{M_{+k}} \quad \text{and} \quad M_{+k} := (M_{j+k})_{j\geq 0}.$$

4. Quasianalytic Remez inequalities

After reviewing quasianalytic Remez inequalities on intervals and convex bodies, we discuss a large collection of sets in $\mathbb{R}^n$ to which the quasianalytic Remez inequality can be extended; see Theorem 4.13.

4.1. **Remez inequality for univariate functions.** The following theorem is a Remez inequality for univariate vector-valued functions. It generalizes [NSV04, Theorem B] for single-valued functions; see also [Rai23, Theorem 5.4]. It is proved in Section B for the reader's convenience.

Theorem 4.1. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > b_0 > 0$. Assume that $(\mu^{|\infty}, M_0, 1, b_0)$ is admissible and let $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, 1, b_0)$. Let $f : [0,1] \to \mathbb{R}^m$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth such that $\|f\|_{[0,1]} \geq b_0$. Then for each interval $I \subseteq [0,1]$ and each Lebesgue measurable set $E \subseteq I$ with $|E| > 0$ we have*

$$\|f\|_I \leq \mathbf{C}_{\mathbf{N}} \left(\frac{m\,|I|}{|E|}\right)^{\mathbf{N}} \|f\|_E.$$

4.2. **Remez inequality on convex bodies.** Theorem 4.1, for $m = 1$, was extended to convex bodies in $\mathbb{R}^n$ in [Rai23, Theorem 5.4]. Here we state a vector-valued version.

Theorem 4.2. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > b_0 > 0$. Let $K \subseteq L \subseteq \mathbb{R}^n$ be convex bodies. Assume that $(\mu^{|\infty}, M_0, \operatorname{diam}(K), b_0)$ is admissible and let $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, \operatorname{diam}(K), b_0)$. Let $f : L \to \mathbb{R}^m$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth such that $\|f\|_K \geq b_0$. If $E \subseteq K$ is a Lebesgue measurable subset with $|E| > 0$, then*

$$\|f\|_K \leq \mathbf{C}_{\mathbf{N}} \Big(\frac{m\,|K|^{1/n}}{|K|^{1/n} - (|K| - |E|)^{1/n}}\Big)^{\mathbf{N}} \|f\|_E.$$

Even for $m = 1$, the statement of Theorem 4.2 is slightly stronger than the one of [Rai23, Theorem 5.4]. In fact, in the latter, the potentially bigger integer $\mathbf{N}(\mu^{|\infty}, M_0, \operatorname{diam}(L), b_0)$ is used rather than $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, \operatorname{diam}(K), b_0)$. The stronger vector-valued version is obtained by the same proof (based on the technique of [BG73]). Alternatively, we shall see that Theorem 4.2 follows from Corollary 4.4 below.

4.3. **Star-shaped sets.** The following lemma, due to [Pie15, Lemma 4.1], will play an important role in the proof of Theorem 1.1.

Lemma 4.3. *Let $K \subseteq \mathbb{R}^n$ be a compact set that is star-shaped with respect to $x_0 \in K$ and satisfies $|K| > 0$. Let $E \subseteq K$ be Lebesgue measurable. Then there exists an affine curve $\gamma : [0,1] \to K$ with $\gamma(0) = x_0$ and such that $\gamma^{-1}(E)$ contains a Lebesgue measurable subset F satisfying*

$$\mathcal{L}^1(F) \geq 1 - \Big(1 - \frac{|E|}{|K|}\Big)^{1/n}. \tag{4.1}$$

We give a corollary which implies Theorem 4.2: for the convex body K, it suffices to choose $x_0 \in K$ such that $|f(x_0)|_1 = \|f\|_K$.

Corollary 4.4. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > b_0 > 0$. Let $K \subseteq \mathbb{R}^n$ be a compact set that is star-shaped with respect to $x_0 \in K$ and such that $|K| > 0$. Assume that $(\mu^{|\infty}, M_0, \operatorname{diam}(K), b_0)$ is admissible and let $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, \operatorname{diam}(K), b_0)$. Let $f : K \to \mathbb{R}^m$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth such that $|f(x_0)|_1 \geq b_0$. If $E \subseteq K$ is a Lebesgue measurable subset with $|E| > 0$, then*

$$|f(x_0)|_1 \leq \mathbf{C}_{\mathbf{N}} \Big(\frac{m\,|K|^{1/n}}{|K|^{1/n} - (|K| - |E|)^{1/n}} \Big)^{\mathbf{N}} \|f\|_E.$$

Proof. Let $\gamma : [0,1] \to K$, $\gamma(t) = x_0 + t(x_1 - x_0)$ with $x_1 = \gamma(1)$, be the affine curve from Lemma 4.3. We apply Theorem 4.1 to $g := f \circ \gamma$ which is $(\operatorname{diam}(K)\,\mu^{|\mathbf{N}}, M_0)$-smooth and to the set F from Lemma 4.3,

$$|f(x_0)|_1 = |g(0)|_1 \leq \mathbf{C}_{\mathbf{N}} \Big(\frac{m}{\mathcal{L}^1(F)} \Big)^{\mathbf{N}} \|g\|_F.$$

This implies the assertion, by (4.1). □

4.4. **Admissible sets.** We introduce a collection of sets in $\mathbb{R}^n$ admissible for the Remez inequality. The following definitions are based on [Pie15, Definition 9.1].

Definition 4.5 (Admissible collection)**.** Suppose that a family $(K_w, a_w, \Upsilon_w)_{w\in W}$ is given, where $K_w \subseteq [0,1]^n$ is compact, $a_w \in K_w$, $\Upsilon_w : [0,\infty)^n \to \mathbb{R}^n$ is a map, and W is an arbitrary set. This family is called an *admissible collection* in $\mathbb{R}^n$ if the following holds. There exist constants $\delta_1, \delta_2, \delta_3 > 0$, $d, q \in \mathbb{N}_{\geq 1}$, and polynomial maps $R_w : \mathbb{R}^n \to \mathbb{R}^n$ of degree at most d such that, for all $w \in W$:

(1) K_w is star-shaped with respect to a_w and $|K_w| \geq \delta_1$;

(2) $\Upsilon_w(x_1, \dots, x_n) = R_w(x_1^{1/q}, \dots, x_n^{1/q})$;

(3) $\delta_2 \leq \sum_\kappa |c_{w,\kappa}| \leq \delta_3$, where $c_{w,\kappa}$ are the coefficients of the polynomial $\operatorname{Jac}(R_w)$.

(4) $a_{w,i} = 0$ whenever Υ_w is not a polynomial map with respect to the variable x_i, where $a_w = (a_{w,1}, \dots, a_{w,n})$.

We also say that $(d, q, \delta_1, \delta_2, \delta_2)$ is the *characteristic* of $(K_w, a_w, \Upsilon_w)_{w\in W}$.

Definition 4.6 (Admissible set)**.** Let $K \subseteq \mathbb{R}^n$ be a nonempty compact set. We say that K is an *admissible set* if there exists an admissible collection $(K_w, a_w, \Upsilon_w)_{w\in K}$ in $\mathbb{R}^n$ such that

(1) $\Upsilon_w(a_w) = w$ for each $w \in K$;

(2) $K = \bigcup_{w\in K} \Upsilon_w(K_w)$.

We also say that K has the *characteristic* $(d, q, \delta_1, \delta_2, \delta_2)$ if $(K_w, a_w, \Upsilon_w)_{w\in K}$ has this characteristic.

Let us denote by $\mathscr{K}(\mathbb{R}^n)$ the family of all admissible sets in $\mathbb{R}^n$.

In particular, every compact set $K \subseteq \mathbb{R}^n$ with the *special parameterization property* as defined in [Pie15, Definition 1.3] is admissible.

Example 4.7. Each convex body $K \subseteq \mathbb{R}^n$ is an admissible set with characteristic $(1, 1, \delta_1, \delta_2, \delta_3)$. Indeed, we may use the admissible collection $(K_w, a_w, \Upsilon_w)_{w\in K}$, where $a_w := (\frac{1}{2}, \dots, \frac{1}{2})$, $T_w(x) := a_w + \frac{1}{2\operatorname{diam}(K)}(x - w)$, $K_w := T_w(K)$, and $\Upsilon_w := T_w^{-1}$.

Definition 4.8 (UPC set)**.** A subset $K \subseteq \mathbb{R}^n$ is *UPC* if there exists a map $P : \overline{K} \times \mathbb{R} \to \mathbb{R}^n$ and constants $A, B > 0$ and $d \in \mathbb{N}_{\geq 1}$ such that, for each $x \in \overline{K}$,

- $P(x, \cdot)$ is a polynomial curve of degree at most d;
- $P(x, 0) = x$;
- $\operatorname{dist}(P(x, t), \mathbb{R}^n \setminus K) \geq A\, t^B$, for $t \in [0, 1]$.

Proposition 4.9 ([Pie15, Lemma 3.8])**.** *Each compact UPC set is admissible with characteristic* $(\max\{d, \lceil B\rceil\}, 1, \delta_1, \delta_2, \delta_3)$*, where* $d \in \mathbb{N}_{\geq 1}$ *and* $B > 0$ *are the constants from Definition 4.8.*

Proof. The characteristic follows easily from the proof of Lemma 3.8 in [Pie15]. □

Example 4.10. Each compact fat subanalytic set is UPC and hence admissible, by [PP86]. More generally, each compact fat set that is definable in $\mathbb{R}_{\mathcal{C}_M}$ is UPC and hence admissible, where $\mathcal{C}_M$ is a suitable quasianalytic Denjoy–Carleman class; see Example 3.4. This is due to [Pie05].

Example 4.11. The closure of a bounded Lipschitz domain is UPC and thus admissible with characteristic $(1, 1, \delta_1, \delta_2, \delta_3)$. More generally, the closure of a bounded Hölder domain with Hölder exponent $\alpha \in (0, 1]$ is UPC and thus admissible with characteristic $(\lceil \alpha^{-1}\rceil, 1, \delta_1, \delta_2, \delta_3)$. Cf. [Rai24, Section 1.3].

4.5. **Remez inequality on admissible sets.**

Definition 4.12 (The constant a_K)**.** Let $K \in \mathscr{K}(\mathbb{R}^n)$. Assume that K has the characteristic $(d, q, \delta_1, \delta_2, \delta_3)$. Then we associate the positive number

$$a_K := \begin{cases} 4\,\frac{(d+1)^{d+1}}{(d-1)^{d-1}} \cdot \max\{\operatorname{diam}(K), 1\}, & \text{if } d \geq 2, \\ \operatorname{diam}(K), & \text{if } d = 1. \end{cases} \tag{4.2}$$

We are ready to state the general Remez inequality.

Theorem 4.13. *Let* $L \subseteq \mathbb{R}^n$ *be a convex body and* $K \in \mathscr{K}(\mathbb{R}^n)$ *such that* $K \subseteq L$*. There exists* $\nu_K \in (0, 1]$*, depending only on* K*, such that the following holds. Let* $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ *be an admissible weight satisfying* $\mu_j \geq j$ *for* $j \geq 1$*. Let* $M_0 > b_0 > 0$*. Let* $m \in \mathbb{N}_{\geq 1}$*. Assume that* $(\mu^{|\infty}, m\, M_0, a_K, b_0)$ *is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, m\, M_0, a_K, b_0).$$

Then, for each $(\mu^{|\mathbf{N}}, M_0)$*-smooth map* $f : L \to \mathbb{R}^m$ *with* $\|f\|_K \geq b_0$ *and each Lebesgue measurable set* $E \subseteq K$ *with* $|E| > 0$*, we have*

$$\|f\|_K \leq \mathbf{C}_{\mathbf{N}} \left(\frac{m\,|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E|)^{\nu_K}}\right)^{\mathbf{N}} \|f\|_E.$$

Clearly, Theorem 1.1 is a direct consequence of Theorem 4.13. The proof of Theorem 4.13 comprises Section 5 and Section 6.

Remark 4.14. It is possible to explicitly compute a number ν_K for which Theorem 4.13 holds; this requires the full characteristic $(d, q, \delta_1, \delta_2, \delta_3)$ of K as well as other specifics of K such as the ambient dimension n and $|K|$.

Remark 4.15. Theorem 4.13 extends Theorem 4.2; in fact, if K is a convex body we may take $d = 1$, $a_K = \operatorname{diam}(K)$, $\nu_K = 1/n$, and $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, a_K, b_0)$ (even if $m > 1$).

Remark 4.16. Noting that

$$\frac{(d+1)^{d+1}}{(d-1)^{d-1}} = \Big(\frac{d+1}{d-1}\Big)^{d-1}(d+1)^2 = \Big(1+\frac{2}{d-1}\Big)^{d-1}(d+1)^2$$

we see that the growth of a_K (as defined in (4.2)) is quadratic in d,

$$a_K \sim 4e^2(d+1)^2 \cdot \max\{\operatorname{diam}(K), 1\} \quad \text{as } d \to \infty.$$

Remark 4.17. The proof will show that Theorem 4.13 remains valid in the following more general setup: let $a > 0$, $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, m\, M_0, a, b_0)$, and assume that the maps $f : L \to \mathbb{R}^m$ with $\|f\|_K \geq b_0$ satisfy

- $f|_\ell$ is $(a\,\mu^{|\mathbf{N}}, M_0)$-smooth for each line segment $\ell \subseteq L$;
- $f \circ p$ is $(a\,\mu^{|\mathbf{N}}, M_0)$-smooth for each polynomial curve $p : [0,1] \to K$ of degree at most d.

5. Bounds along polynomial curves

In this section, we establish bounds for the composition of a $(\mu^{|N}, M_0)$-smooth function with a polynomial curve of degree at most d.

We will use the following corollary of the Bernstein–Walsh lemma; see e.g. [RS02, Corollary 12.1.6].

Lemma 5.1 (Bernstein's lemma)**.** *Let p be a polynomial of degree at most d. For $r > 1$, let $E_r \subseteq \mathbb{C}$ be the ellipse with foci ± 1 and semiaxes $\frac{1}{2}(r+r^{-1})$ and $\frac{1}{2}(r-r^{-1})$. Then*

$$|p(z)| \leq r^d\, \|p\|_{[-1,1]}, \quad z \in E_r. \tag{5.1}$$

The next proposition is the goal of this section.

Proposition 5.2. *Let $\mu^{|\infty}$ be an admissible weight satisfying $\mu_j \geq j$, for $j \geq 1$, and $M_0 > 0$. Let $K \subseteq \mathbb{R}^n$ be a compact nonempty set containing the origin. Let $f : K \to \mathbb{R}$ be a $(\mu^{|N}, M_0)$-smooth function and $p : [-1,1] \to K$ a polynomial curve of degree at most d. Then $f \circ p : [-1,1] \to \mathbb{R}$ is $(a\,\mu^{|N}, M_0)$-smooth, where*

$$a := \begin{cases} 2\,\frac{(d+1)^{d+1}}{(d-1)^{d-1}} \cdot \max\{\operatorname{diam}(K), 1\}, & \text{if } d \geq 2, \\ \frac{1}{2}\operatorname{diam}(K), & \text{if } d = 1. \end{cases} \tag{5.2}$$

Proof. If $d = 1$ then we may assume that $p(t) = \frac{1}{2}(x_0 + x_1) + \frac{t}{2}(x_1 - x_0)$, where $x_0, x_1 \in K$ and the assertion follows easily.

Let us assume $d \geq 2$. Write $p = (p_1, \dots, p_n)$. Since $0 \in K$, we have

$$\|p_i\|_{[-1,1]} \leq \max_{t\in[-1,1]} |p(t)| \leq \operatorname{diam}(K), \quad 1 \leq i \leq n. \tag{5.3}$$

Obviously, $\|f\circ p\|_{[-1,1]}\le M_0$. The derivative $(f\circ p)^{(k)}(t)$, for $1\le k\le N$ and $t\in[-1,1]$, coincides with the k-th derivative of

$$z\mapsto\sum_{|\alpha|\le k}(\partial^\alpha f)(p(t))\frac{(p(z)-p(t))^\alpha}{\alpha!} \tag{5.4}$$

at $z=t$. For fixed $t\in[-1,1]$, this is a polynomial in z and we may assume that $z\in\mathbb{C}$. By (5.1) and (5.3), we have

$$\begin{aligned}|(p(z)-p(t))^\alpha| &= \prod_{i=1}^n |p_i(z)-p_i(t)|^{\alpha_i}\\ &\le \prod_{i=1}^n((r^d+1)\|p_i\|_{[-1,1]})^{\alpha_i}\le (r^d+1)^{|\alpha|}\operatorname{diam}(K)^{|\alpha|}.\end{aligned}$$

Thus, the modulus of the polynomial (5.4) can be estimated on E_r by

$$\begin{aligned}&\sum_{|\alpha|\le k}\frac{1}{\alpha!}|\partial^\alpha f(p(t))|(r^d+1)^{|\alpha|}\operatorname{diam}(K)^{|\alpha|}\\ &\quad=\sum_{j=0}^k\sum_{|\alpha|=j}\frac{1}{\alpha!}|\partial^\alpha f(p(t))|(r^d+1)^j\operatorname{diam}(K)^j\\ &\quad\le\sum_{j=0}^k\frac{\|f\|_{j,K}}{j!}(r^d+1)^j\operatorname{diam}(K)^j\\ &\quad\le\sum_{j=0}^k\frac{M_j}{j!}((r^d+1)\operatorname{diam}(K))^j\\ &\quad\le\frac{M_k}{k!}\sum_{j=0}^k((r^d+1)\operatorname{diam}(K))^j,\end{aligned}$$

where we used $\mu_j\ge j$ in the last step.

For each $t\in[-1,1]$ the ball $B(t,\delta)$ with radius

$$\delta=\frac12(r+r^{-1})-1=\frac{(r-1)^2}{2r}$$

is contained in E_r. Thus, by Cauchy's estimates, we find

$$|(f\circ p)^{(k)}(t)|\le\frac{M_k}{\delta^k}\sum_{j=0}^k((r^d+1)\operatorname{diam}(K))^j, \tag{5.5}$$

for $1\le k\le N$ and $t\in[-1,1]$.

Let us estimate

$$\frac{1}{\delta^k}\sum_{j=0}^k((r^d+1)\operatorname{diam}(K))^j=\frac{2^kr^k}{(r-1)^{2k}}\sum_{j=0}^k((r^d+1)\operatorname{diam}(K))^j. \tag{5.6}$$

We distinguish two cases:

(1) $\operatorname{diam}(K) < 1$: Then (5.6) is bounded by

$$\frac{2^k r^k}{(r-1)^{2k}} \sum_{j=0}^{k} (2r^d)^j = \frac{2^k r^k}{(r-1)^{2k}} \frac{(2r^d)^{k+1} - 1}{2r^d - 1} \leq \frac{2^k r^k}{(r-1)^{2k}} \frac{(2r^d)^{k+1}}{2r^d - 1} \leq \Big(\frac{8r^{d+1}}{(r-1)^2} \Big)^k,$$

since $\frac{2r^d}{2r^d-1} \leq 2$. As $d \geq 2$, the minimum of $\frac{r^{d+1}}{(r-1)^2}$ for $r > 1$ is attained at $r_0 = \frac{d+1}{d-1}$ and amounts to $\frac{1}{4} \frac{(d+1)^{d+1}}{(d-1)^{d-1}}$. Thus (5.6) is bounded by

$$\Big(\frac{2(d+1)^{d+1}}{(d-1)^{d-1}} \Big)^k.$$

(2) $\operatorname{diam}(K) \geq 1$: In that case, $r^d \operatorname{diam}(K) \geq 1$ and hence (5.6) is bounded by

$$\begin{aligned} \frac{2^k r^k}{(r-1)^{2k}} \sum_{j=0}^{k} (2r^d \operatorname{diam}(K))^j &= \frac{2^k r^k}{(r-1)^{2k}} \frac{(2r^d \operatorname{diam}(K))^{k+1} - 1}{2r^d \operatorname{diam}(K) - 1} \\ &\leq \frac{2^k r^k}{(r-1)^{2k}} \frac{(2r^d \operatorname{diam}(K))^{k+1}}{2r^d \operatorname{diam}(K) - 1} \\ &\leq \Big(\frac{8r^{d+1} \operatorname{diam}(K)}{(r-1)^2} \Big)^k, \end{aligned}$$

since $\frac{2r^d \operatorname{diam}(K)}{2r^d \operatorname{diam}(K)-1} \leq 2$. As in case (1), we conclude that (5.6) is bounded by

$$\Big(\frac{2(d+1)^{d+1}}{(d-1)^{d-1}} \cdot \operatorname{diam}(K) \Big)^k.$$

In view of (5.5), we conclude that, for $1 \leq k \leq N$ and $t \in [-1, 1]$,

$$|(f \circ p)^{(k)}(t)| \leq a^k M_k,$$

where a is given by (5.2). This completes the proof. □

6. Proof of the Remez inequality on admissible sets

This section is dedicated to the proof of Theorem 4.13.

6.1. **Preparatory results.** We will make crucial use of the following result on admissible sets, due to [Pie15].

Proposition 6.1 ([Pie15, Corollary 9.11])**.** *Let $K \in \mathscr{K}(\mathbb{R}^n)$ be an admissible set with admissible collection $(K_w, a_w, \Upsilon_w)_{w \in K}$. There exist constants $\sigma, \eta > 0$, and $\varrho \in [0, 1)$ such that for each Lebesgue measurable $E \subseteq K$ and each $w \in K$,*

$$\frac{|\Upsilon_w^{-1}(E) \cap K_w|}{|K_w|} \geq \frac{1}{\eta(1-\lambda)^{1/\sigma} + 1} \quad \text{if } \lambda := \frac{|E|}{|K|} > \varrho.$$

The following lemma is a slight modification of [Pie15, Lemma 8.2].

Lemma 6.2. *Let $a, b, c > 0$. Consider the functions $\varphi : (-\infty, 1] \to (0, 1]$ and $\psi : (0, 1] \to [1, \infty)$ given by*

$$\varphi(t) = \frac{1}{a(1-t)^b + 1} \quad \text{and} \quad \psi(t) = \frac{1}{1-(1-t)^c}.$$

There exists $d = d(a, b, c) > 0$ such that

$$(\psi \circ \varphi)(t) \le \frac{1}{1-(1-t)^d}, \quad t \in (0, 1].$$

Proof. We claim that there exists $r = r(a) \in (0, 1]$ such that

$$as \le (1 + as)s^r, \quad \text{for all } s \ge 0. \tag{6.1}$$

Therefore, setting $s = (1-t)^b$,

$$\varphi(t) \ge 1 - (1-t)^{br}, \quad t \in (-\infty, 1],$$

and, consequently, as ψ is decreasing,

$$(\psi \circ \varphi)(t) \le \psi\big(1-(1-t)^{br}\big) = \frac{1}{1-(1-t)^{bcr}}$$

for $t \in (0, 1]$.

It remains to prove (6.1). If $a \in (0, 1]$, it suffices to set $r := 1$. Let us assume that $a > 1$. The inequality is always true for $s = 0$ and $s \ge 1$. So let $s \in (0, 1)$. By substituting $u = s^r$, (6.1) is equivalent to

$$u^{\frac{1-r}{r}}(1-u) \le \frac{1}{a}.$$

The maximum of the left-hand side for $u \in (0, 1)$ as attained at $u = 1 - r$. Hence, for $a > 1$, letting r be the unique solution of

$$r(1-r)^{\frac{1-r}{r}} = \frac{1}{a}$$

in the interval $(0, 1)$, implies (6.1). (Such r exists because the left-hand side defines a strictly increasing surjective function $(0, 1) \to (0, 1)$.) □

6.2. **A preliminary Remez inequality.** The following Remez inequality holds if the ratio $|E|/|K|$ is not too small.

Theorem 6.3. *Let $K \in \mathscr{K}(\mathbb{R}^n)$. Then there exist $\nu \in (0, 1]$ and $\varrho \in [0, 1)$ such that the following holds. Let $\mu^{|\infty} = (\mu_j)_{j \ge 1}$ be an admissible weight satisfying $\mu_j \ge j$ for $j \ge 1$. Let $M_0 > b_0 > 0$. Let $m \in \mathbb{N}_{\ge 1}$. Assume that $(\mu^{|\infty}, m\, M_0, a_K, b_0)$ is admissible and let $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, m\, M_0, a_K, b_0)$. Let $f : K \to \mathbb{R}^m$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth such that $\|f\|_K \ge b_0$. Then for each Lebesgue measurable set $E \subseteq K$ we have*

$$\|f\|_K \le \mathbf{C}_{\mathbf{N}} \left(\frac{m}{1-(1-\lambda)^\nu}\right)^{\mathbf{N}} \|f\|_E \quad \text{if } \lambda := \frac{|E|}{|K|} > \varrho.$$

Proof. We may assume that K contains the origin; indeed, the statement of Theorem 6.3 is invariant under translations. Let $(K_w, a_w, \Upsilon_w)_{w \in K}$ be an admissible collection for K in $\mathbb{R}^n$ with characteristic $(d, q, \delta_1, \delta_2, \delta_3)$. Let $\sigma, \eta > 0$ and $\varrho \in [0, 1)$ be the constants provided by Proposition 6.1. Assume that $E \subseteq K$ is Lebesgue measurable and $\lambda := \frac{|E|}{|K|} > \varrho$.

Fix $w \in K$ such that $|f(w)|_1 = \|f\|_K$. The preimage $\Upsilon_w^{-1}(E) \subseteq [0,\infty)^n$ is Lebesgue measurable (indeed, E is the union of a F_σ-set and a set of measure zero and the preimage under Υ_w of the latter is again of measure zero).

By Lemma 4.3, there exists an affine curve $\gamma_w : [0,1] \to K_w$ such that $\gamma_w(0) = a_w$ and $\gamma_w^{-1}(\Upsilon_w^{-1}(E) \cap K_w)$ contains a Lebesgue measurable subset F_0 satisfying

$$\mathcal{L}^1(F_0) \geq 1 - \Big(1 - \frac{|\Upsilon_w^{-1}(E) \cap K_w|}{|K_w|}\Big)^{1/n}.$$

By Proposition 6.1, we conclude that

$$\mathcal{L}^1(F_0) \geq 1 - \Big(1 - \frac{1}{\eta(1-\lambda)^{1/\sigma} + 1}\Big)^{1/n}. \tag{6.2}$$

Consider the function $G_q : [0,1] \to [0,1]$, $G_q(t) = t^q$, and $F_1 := G_q^{-1}(F_0)$. Then (by [LL01, Theorem 1.13] and [Pie15, Lemma 7.1])

$$\mathcal{L}^1(F_0) \leq 1 - (1 - \mathcal{L}^1(F_1))^q$$

and, hence, using (6.2),

$$\mathcal{L}^1(F_1) \geq 1 - (1 - \mathcal{L}^1(F_0))^{1/q} \geq 1 - \Big(1 - \frac{1}{\eta(1-\lambda)^{1/\sigma} + 1}\Big)^{1/(nq)}.$$

Thus, by Lemma 6.2, there exists $\nu > 0$ such that

$$\frac{1}{\mathcal{L}^1(F_1)} \leq \frac{1}{1 - (1-\lambda)^\nu}. \tag{6.3}$$

Replacing ν by $\min\{\nu, 1\}$, we may assume that $\nu \in (0,1]$.

The composite $p := \Upsilon_w \circ \gamma_w \circ G_q : [0,1] \to K$ is a polynomial curve of degree at most d, by Definition 4.5(4) (cf. [Pie15, Lemma 9.6]).

We claim that $f \circ p : [0,1] \to \mathbb{R}^m$ is $(a_K\, \mu|^{\mathbf{N}}, m\, M_0)$-smooth. Let $\ell : [-1,1] \to [0,1]$ be given by $\ell(t) = \frac{1}{2}(t+1)$. Then $p \circ \ell : [-1,1] \to K$ is a polynomial curve of degree at most d. Since $\|f\|_K = |f(w)|_1 = |f(p(0))|_1$, we have

$$b_0 \leq \|f \circ p\|_{[0,1]} = \|f\|_K \leq M_0.$$

By Proposition 5.2, for $1 \leq k \leq \mathbf{N}$,

$$\begin{aligned}\|(f \circ p)^{(k)}\|_{[0,1]} &= \sup_{x \in [0,1]} \sum_{i=1}^m |(f_i \circ p)^{(k)}(x)| = \sup_{x \in [0,1]} \sum_{i=1}^m |(f_i \circ p \circ \ell \circ \ell^{-1})^{(k)}(x)| \\ &= 2^k \sup_{x \in [-1,1]} \sum_{i=1}^m |(f_i \circ p \circ \ell)^{(k)}(x)| \leq m\, M_0\, a_K^k\, \mu_1 \mu_2 \cdots \mu_k,\end{aligned}$$

where a_K is defined in (4.2). This implies the claim.

In view of Remark 2.10(1), we may apply Theorem 4.1 to $f \circ p : [0,1] \to \mathbb{R}^m$ and $F_1 \subseteq [0,1]$. Therefore,

$$\|f\|_K = \|f \circ p\|_{[0,1]} \leq \mathbf{C}_{\mathbf{N}} \Big(\frac{m}{\mathcal{L}^1(F_1)}\Big)^{\mathbf{N}} \|f \circ p\|_{F_1}.$$

By (6.3) and $f(p(F_1)) = f(\Upsilon_w(\gamma_w(F_0))) \subseteq f(\Upsilon_w(\Upsilon_w^{-1}(E) \cap K_w)) \subseteq f(E)$, the proof is complete. □

6.3. **Proof of Theorem 4.13.** Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$. Let $M_0 > b_0 > 0$. Let $m \in \mathbb{N}_{\geq 1}$. Assume that $(\mu^{|\infty}, m\, M_0, a_K, b_0)$ is admissible and let $\mathbf{N} = \mathbf{N}(\mu^{|\infty}, m\, M_0, a_K, b_0)$. Let $f : L \to \mathbb{R}^m$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth such that $\|f\|_K \geq b_0$. Let $E \subseteq K$ be Lebesgue measurable with $|E| > 0$ and set $\lambda := \frac{|E|}{|K|}$.

We may assume without loss of generality that L is the convex hull of K and hence $\operatorname{diam}(L) = \operatorname{diam}(K)$. Indeed the convex hull $\operatorname{co}(K)$ of K is contained in L and the restriction $f|_{\operatorname{co}(K)}$ is $(\mu^{|\mathbf{N}}, M_0)$-smooth.

Let $\nu \in (0, 1]$ and $\varrho \in [0, 1)$ be the constants associated with K in Theorem 6.3. Assume that $\varrho > 0$ (otherwise we are done). If $\lambda \in (\varrho, 1]$, then the statement follows from Theorem 6.3. Suppose that $\lambda \in (0, \varrho]$.

Set $\mu := \frac{|K|}{|L|} \in (0, 1]$. There is a unique $\epsilon_0 \in (0, 1]$ such that $1 - \mu\varrho = (1 - \varrho)^{\epsilon_0}$. Thus $u(t) := (1 - t)^{\epsilon_0} - 1 + \mu t$ vanishes at $t = 0$ and $t = \varrho$ and is concave in $[0, \varrho]$ since $u''(t) = \epsilon_0(\epsilon_0 - 1)(1 - t)^{\epsilon_0 - 2} \leq 0$. It follows that

$$(1 - \mu t) \leq (1 - t)^{\epsilon_0} \quad \text{for } t \in [0, \varrho]. \tag{6.4}$$

Since $\operatorname{diam}(L) = \operatorname{diam}(K) \leq a_K$ (see (4.2)), $(\mu^{|\infty}, m\, M_0, \operatorname{diam}(L), b)$ is admissible and

$$\mathbf{N}' = \mathbf{N}(\mu^{|\infty}, m\, M_0, \operatorname{diam}(L), b) \leq \mathbf{N}(\mu^{|\infty}, m\, M_0, a_K, b) = \mathbf{N}$$

as well as $\mathbf{C}_{\mathbf{N}'} \leq \mathbf{C}_{\mathbf{N}}$ (see Remark 2.10(2)). Thus, by Theorem 4.2 and (6.4),

$$\|f\|_K \leq \|f\|_L \leq \mathbf{C}_{\mathbf{N}'} \left(\frac{m}{1 - (1 - \mu\lambda)^{1/n}}\right)^{\mathbf{N}'} \|f\|_E \leq \mathbf{C}_{\mathbf{N}} \left(\frac{m}{1 - (1 - \lambda)^{\epsilon_0/n}}\right)^{\mathbf{N}} \|f\|_E.$$

It follows that Theorem 4.13 holds with $\nu_K := \min\{\nu, \epsilon_0/n\}$, which depends only on K, because we assumed that $L = \operatorname{co}(K)$.

7. Sublevel sets and applications

In this section, we deduce from Theorem 4.13 a precise bound for the volume growth of sublevel sets and several applications.

For the rest of the paper, let a_K be defined by (4.2) and let $\nu_K \in (0, 1)$ be the constant provided by Theorem 4.13 with the convention that $a_K = \operatorname{diam}(K)$ and $\nu_K = 1/n$, in the case that $K \subseteq \mathbb{R}^n$ is a convex body.

The following setup applies throughout this section.

7.1. **Setup.** Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$.

Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$. Let $M_0 > b_0 > 0$. Assume that $(\mu^{|\infty}, M_0, a_K, b_0)$ is admissible and let

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, a_K, b_0).$$

Let $f : L \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}}, M_0)$-smooth function satisfying $\|f\|_K \geq b_0$. We only consider single-valued functions in this section (i.e. $m = 1$).

Remark 7.1. Most of the results in this section were obtained in [Rai23] in the special case that $K \subseteq \mathbb{R}^n$ is a convex body. In that case, the condition $\mu_j \geq j$ for $j \geq 1$ is not needed.

7.2. **Sublevel sets.** Let $K_t := \{x \in K : |f(x)| \leq t\}$ denote the sublevel set of $f|_K$.

Theorem 7.2. *We have*

$$|K_t| \leq \frac{|K|}{\nu_K} \Big(\frac{\mathbf{C_N}\, t}{\|f\|_K}\Big)^{1/\mathbf{N}}, \quad t > 0.$$

Proof. We may assume that $\lambda_t := \frac{|K_t|}{|K|} > 0$. By Theorem 4.13,

$$\|f\|_K \leq \mathbf{C_N} \Big(\frac{1}{1-(1-\lambda_t)^{\nu_K}}\Big)^{\mathbf{N}} t$$

and hence

$$1-(1-\lambda_t)^{\nu_K} \leq \Big(\frac{\mathbf{C_N}\, t}{\|f\|_K}\Big)^{1/\mathbf{N}}.$$

So the assertion follows from the fact that $\nu\, s \leq 1-(1-s)^\nu$, for $s \in [0,1]$ and $\nu \in (0,1]$. □

7.3. **Comparison of L^p-norms.** Recall that the *distribution function* d_f of a Borel measurable function $f : \mathbb{R}^n \supseteq X \to \mathbb{R}$ is defined by

$$d_f(t) := |\{x \in X : |f(x)| > t\}|$$

and the *decreasing rearrangement* f^* of f by

$$f^*(y) := \inf\{t > 0 : d_f(t) \leq y\}.$$

Lemma 7.3. *Let $E \subseteq K$ be a Lebesgue measurable set with $|E| > 0$. Then*

$$(f|_E)^*(|E|\,\lambda) \geq \frac{\|f\|_K}{\mathbf{C_N}} \Big(\frac{\nu_K\, |E|\,(1-\lambda)}{|K|}\Big)^{\mathbf{N}}, \quad \lambda \in (0,1). \tag{7.1}$$

Proof. For $E_t := \{x \in E : |f(x)| \leq t\}$, we have

$$d_{f|_E}(t) = |E|\,\lambda_t, \qquad \text{where } \lambda_t := 1 - \frac{|E_t|}{|E|}.$$

Let s_λ denote the right-hand side of (7.1). If $t \in (0, s_\lambda)$ then, by Theorem 7.2,

$$|E|(1-\lambda_t) = |E_t| \leq \frac{|K|}{\nu_K} \Big(\frac{\mathbf{C_N}\, t}{\|f\|_K}\Big)^{1/\mathbf{N}} < \frac{|K|}{\nu_K} \Big(\frac{\mathbf{C_N}\, s_\lambda}{\|f\|_K}\Big)^{1/\mathbf{N}} = |E|(1-\lambda),$$

implying that $d_{f|_E}(t) = |E|\,\lambda_t > |E|\,\lambda$, for $t \in (0, s_\lambda)$. Thus, $(f|_E)^*(|E|\,\lambda) \geq s_\lambda$ and (7.1) is proved. □

Let us write

$$\|f\|^\sharp_{L^p(E)} := \Big(\frac{1}{|E|}\int_E |f(x)|^p\, dx\Big)^{1/p}, \quad 0 < p < \infty,$$
$$\|f\|^\sharp_{L^\infty(E)} := \operatorname*{ess\,sup}_E |f|,$$

for the normalized L^p-norms (or quasinorms if $0 < p < 1$) of f on a Lebesgue measurable set E with $0 < |E| < \infty$. Then, as a consequence of Hölder's inequality,

$$\|f\|^\sharp_{L^q(E)} \le \|f\|^\sharp_{L^p(E)}, \qquad \text{if } 0 < q \le p \le \infty.$$

In the setup of Section 7.1, also opposite inequalities hold.

Theorem 7.4. *Let $E \subseteq K$ be a Lebesgue measurable subset with $|E| > 0$. Then, for all $0 < q < p \le \infty$,*

$$\|f\|^\sharp_{L^p(K)} \le \Big(\mathbf{C_N}\Big(\frac{|K|}{\nu_K\,|E|}\Big)^{\mathbf{N}}\Big)^{1-\frac{q}{p}} (q\,\mathbf{N}+1)^{\frac1q-\frac1p} (\|f\|^\sharp_{L^q(K)})^{\frac{q}{p}} (\|f\|^\sharp_{L^q(E)})^{1-\frac{q}{p}}. \quad (7.2)$$

In particular,

$$\|f\|^\sharp_{L^p(K)} \le \Big(\frac{\mathbf{C_N}}{\nu_K^{\mathbf{N}}}\Big)^{1-\frac{q}{p}} (q\,\mathbf{N}+1)^{\frac1q-\frac1p} \|f\|^\sharp_{L^q(K)} \quad (7.3)$$

and

$$\|f\|^\sharp_{L^p(K)} \le \mathbf{C_N}\Big(\frac{|K|}{\nu_K\,|E|}\Big)^{\mathbf{N}} (q\,\mathbf{N}+1)^{\frac1q} \|f\|^\sharp_{L^q(E)}. \quad (7.4)$$

Proof. Both (7.3) and (7.4) are simple consequences of (7.2): it suffices to specialize to $E = K$ and to use $\|f\|^\sharp_{L^q(K)} \le \|f\|^\sharp_{L^p(K)}$, respectively.

Let us prove (7.2). By Lemma 7.3,

$$\begin{aligned}\frac{1}{|E|}\int_E |f(x)|^q\,dx &= \frac{1}{|E|}\int_0^{|E|} \big((f|_E)^*(y)\big)^q\,dy = \int_0^1 \big((f|_E)^*(|E|\,\lambda)\big)^q\,d\lambda \\ &\ge \frac{\|f\|^q_K}{\mathbf{C}^q_{\mathbf{N}}}\Big(\frac{\nu_K\,|E|}{|K|}\Big)^{q\mathbf{N}} \int_0^1 (1-\lambda)^{q\mathbf{N}}\,d\lambda \\ &= \frac{\|f\|^q_K}{\mathbf{C}^q_{\mathbf{N}}}\Big(\frac{\nu_K\,|E|}{|K|}\Big)^{q\mathbf{N}} \frac{1}{q\mathbf{N}+1} \end{aligned} \quad (7.5)$$

and thus

$$\|f\|_K \le \mathbf{C_N}\Big(\frac{|K|}{\nu_K\,|E|}\Big)^{\mathbf{N}} (q\,\mathbf{N}+1)^{\frac1q} \|f\|^\sharp_{L^q(E)} \quad (7.6)$$

which is (7.2) for $p = \infty$.

If $0 < q < p < \infty$, then (7.6) implies

$$\begin{aligned}\frac{1}{|K|}\int_K |f(x)|^p\,dx &\le \|f\|^{p-q}_K \frac{1}{|K|}\int_K |f(x)|^q\,dx = \|f\|^{p-q}_K\,(\|f\|^\sharp_{L^q(K)})^q \\ &\le \Big(\mathbf{C_N}\Big(\frac{|K|}{\nu_K\,|E|}\Big)^{\mathbf{N}} (q\,\mathbf{N}+1)^{\frac1q}\Big)^{p-q} (\|f\|^\sharp_{L^q(K)})^q (\|f\|^\sharp_{L^q(E)})^{p-q}\end{aligned}$$

from which (7.2) follows easily. □

7.4. **Integrability of $|f|^{-\alpha}$.**

Theorem 7.5. *The integral*

$$\int_K |f(x)|^{-\alpha}\,dx$$

converges, provided that $0 < \alpha < 1/\mathbf{N}$.

Proof. By [Rai23, Theorem 4.6], the zero set Z_f of $f : L \to \mathbb{R}$ has Lebesgue measure zero. Thus $K \ni x \mapsto |f(x)|^{-\alpha}$ is defined almost everywhere and measurable. A computation similar to (7.5) (using Lemma 7.3 for $E = K$) shows

$$\begin{aligned}\int_K |f(x)|^{-\alpha}\,dx &= |K| \int_0^1 \big((f|_K)^*(|K|\,\lambda)\big)^{-\alpha}\,d\lambda \\ &\le |K| \Big(\frac{\|f\|_K\,\nu_K^{\mathbf{N}}}{\mathbf{C_N}}\Big)^{-\alpha} \int_0^1 (1-\lambda)^{-\alpha\mathbf{N}}\,d\lambda.\end{aligned}$$

The integral on the right-hand side converges precisely if $\alpha\mathbf{N} < 1$. □

7.5. **On the mean oscillation of** $\log|f|$**.** Recall that the *mean oscillation* of a locally integrable function $g : \mathbb{R}^n \supseteq X \to \mathbb{R}$ over a Lebesgue measurable set $E \subseteq X$ with $0 < |E| < \infty$ is given by

$$\mathrm{mo}_E(g) := \frac{1}{|E|}\int_E |g(x) - g_E|\,dx,$$

where g_E is the average of g over E,

$$g_E := \frac{1}{|E|}\int_E g(x)\,dx.$$

In the Setup 7.1, we have the following bound for the mean oscillation of $\log|f|$.

Corollary 7.6. *For each Lebesgue measurable set $E \subseteq K$ with $|E| > 0$,*

$$\mathrm{mo}_E(\log|f|) \le 2\log\mathbf{C_N} + 2\mathbf{N}\Big(1 + \log\Big(\frac{|K|}{\nu_K\,|E|}\Big)\Big). \tag{7.7}$$

Proof. Observe that

$$\mathrm{mo}_E(\log|f|) \le \frac{2}{|E|}\int_E \Big|\log\frac{|f(x)|}{\|f\|_K}\Big|\,dx.$$

Indeed,

$$\begin{aligned}\Big|\log|f| - \frac{1}{|E|}\int_E \log|f|\,dx\Big| &\le \Big|\log|f| - \log\|f\|_K\Big| + \Big|\log\|f\|_K - \frac{1}{|E|}\int_E \log|f|\,dx\Big| \\ &= \Big|\log\frac{|f|}{\|f\|_K}\Big| + \Big|\frac{1}{|E|}\int_E \log\|f\|_K - \log|f|\,dx\Big| \\ &\le \Big|\log\frac{|f|}{\|f\|_K}\Big| + \frac{1}{|E|}\int_E \Big|\log\frac{|f|}{\|f\|_K}\Big|\,dx.\end{aligned}$$

By Lemma 7.3,

$$\begin{aligned}\frac{1}{|E|}\int_E \Big|\log\frac{|f(x)|}{\|f\|_K}\Big|\,dx &= \frac{1}{|E|}\int_0^{|E|} \Big|\log\frac{(f|_E)^*(y)}{\|f\|_K}\Big|\,dy = \int_0^1 \Big|\log\frac{(f|_E)^*(|E|\,\lambda)}{\|f\|_K}\Big|\,d\lambda \\ &\le \int_0^1 -\log\Big(\frac{1}{\mathbf{C_N}}\Big) - \mathbf{N}\log\Big(\frac{\nu_K\,|E|\,(1-\lambda)}{|K|}\Big)\,d\lambda \\ &= \log\mathbf{C_N} + \mathbf{N}\Big(1 + \log\Big(\frac{|K|}{\nu_K\,|E|}\Big)\Big)\end{aligned} \tag{7.8}$$

and (7.7) follows. □

Corollary 7.6 provides an explicit bound on how fast $\mathrm{mo}_E(\log|f|)$ can grow as $|E|$ tends to zero. In the following corollary, we derive a different bound, where the dependence is integrated in the integer $\mathbf{N}$. We only consider complex bodies and balls therein, for simplicity.

Corollary 7.7. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible quasianalytic weight. Let $f : K \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function which is not identically zero and where $M_0 > 0$ is such that $M_0 > \|f\|_K$. Then, for each ball $B \subseteq K$,*

$$\mathrm{mo}_B(\log|f|) \leq 2\log \mathbf{C}_{\mathbf{N}_B} + 2(1+\log n)\,\mathbf{N}_B,$$

where $\mathbf{N}_B := \mathbf{N}(\mu^{|\infty}, M_0, \mathrm{diam}(K), \|f\|_B)$.

Proof. Since $M_0 > \|f\|_K$, we have $\mathbf{N}_B \in \mathbb{N}_{\geq 2}$, by Remark 2.11, and thus the constant $\mathbf{C}_{\mathbf{N}_B} > 0$ is well-defined. By Lemma 7.3 (applied to $E = K = L = B$),

$$(f|_B)^*(|B|\,\lambda) \geq \frac{\|f\|_B}{\mathbf{C}_{\mathbf{N}_B}} \Big(\frac{1-\lambda}{n}\Big)^{\mathbf{N}_B}.$$

Then, the computation (7.8) implies the assertion. □

Remark 7.8. We refer to Proposition A.1, Proposition A.3, and Proposition A.4 for precise growth estimates of $\mathbf{N}(\mu^{|\infty}, 1, a, b)$ as a function of a and b, for particular admissible weights $\mu^{|\infty}$. For instance, in the case $\mu^{|\infty}_{\mathrm{an}} = (j)_{j\geq 1}$, $\mathbf{N}_B$ is of the same order as

$$\Big\lceil \log \frac{M_0}{\|f\|_B} \Big\rceil_{\mathbb{N}}.$$

We cannot infer that $\log|f|$ is BMO, but we have the following.

Corollary 7.9. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible quasianalytic weight. Let $M_0 > b_0 > 0$. Let $f : K \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function such that $M_0 > \|f\|_K$. Then, for each ball $B \subseteq K$ satisfying $\|f\|_B \geq b_0$,*

$$\mathrm{mo}_B(\log|f|) \leq 2\log \mathbf{C}_{\mathbf{N}} + 2(1+\log n)\,\mathbf{N},$$

where $\mathbf{N} := \mathbf{N}(\mu^{|\infty}, M_0, \mathrm{diam}(K), b_0)$. In particular, for each $b_0 > 0$, the restriction of $\log|f|$ to the set $\{x \in K : |f(x)| \geq b_0\}$ is BMO.

Proof. Since $\mathbf{N}(\mu^{|\infty}, M_0, \mathrm{diam}(K), \|f\|_B) \leq \mathbf{N}(\mu^{|\infty}, M_0, \mathrm{diam}(K), b_0)$, this is a consequence of Corollary 7.7. □

Remark 7.10. A locally integrable function $g : \mathbb{R}^n \to \mathbb{R}$ is BMO if and only if $g = C_1 \log\omega$ for some constant C_1 and some nonnegative $\omega : \mathbb{R}^n \to \mathbb{R}$ that satisfies a reverse Hölder inequality in the sense that there exist $p = p(\omega) > 1$ and $C_2 = C_2(\omega) > 0$ such that

$$\|\omega\|^{\sharp}_{L^p(B)} \leq C_2\,\|\omega\|^{\sharp}_{L^1(B)},$$

for all balls B. See [Ste93, Chapter V]. In our reverse Hölder inequality (7.3) for $K = B$, the constant depends on B (through $\mathbf{N}$).

8. Inequalities of Łojasiewicz type

The goal of this section is to establish several inequalities of Łojasiewicz type with explicit constants, which depend on $\mathbf{N}$ and the geometry of the underlying admissible set K.

8.1. **Thickness of a set.** Let us recall the definition of thickness, following [Pie22, Definition 1.16].

Definition 8.1 (Thick sets)**.** Let $K \subseteq \mathbb{R}^n$ be a Lebesgue measurable nonempty set. We say that K is (c_K, θ_K)-*thick* if there exist constants $c_K, \theta_K > 0$ such that

$$|K \cap B(x,r)| \geq c_K\, r^{\theta_K}$$

for all $x \in K$ and $0 < r \leq 1$.

Every compact fat nonempty $K \subset \mathbb{R}^n$ that is definable in a polynomially bounded o-minimal expansion of the real field is (c_K, θ_K)-*thick* for some $c_K, \theta_K > 0$. This follows easily from the Łojasiewicz inequality that is available for polynomially bounded o-minimal structures; see e.g. the proof of [Pie22, Theorem 1.18].

Remark 8.2. Each convex body $K \subseteq \mathbb{R}^n$ is (c_K, θ_K)-thick with

$$\theta_K = n \quad \text{and} \quad c_k = \omega_n \Big(\frac{\rho}{H}\Big)^n,$$

where $H := \max\{\operatorname{diam}(K), 1\}$ and $\rho > 0$ is the radius of the largest ball contained in K. Indeed, there exists a ball $B(x_0, \rho) \subseteq K$. For given $x \in K$ and $0 < r \leq 1$, consider the ball $B := \{x + \frac{r}{H}(y - x) : y \in B(x_0, \rho)\}$. Then $B \subseteq K$, by convexity, and $B \subseteq B(x, r)$ since $\frac{r}{H}|y - x| \leq \frac{r}{H} \operatorname{diam}(K) \leq r$. It follows that

$$|K \cap B(x,r)| \geq |B| = \omega_n (\tfrac{\rho}{H})^n\, r^n.$$

8.2. **Inequalities of Łojasiewicz type with explicit constants.** The following theorem is formulated for vector-valued maps, since among other things we will apply it to the gradient.

Theorem 8.3. *Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ with $K \subseteq L$. Assume that K is (c_K, θ_K)-thick. Let $\mu^{|\infty} = (\mu_j)_{j \geq 1}$ be an admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$. Let $M_0 > b_0 > 0$. Let $m \in \mathbb{N}_{\geq 1}$. Assume that $(\mu^{|\infty}, m\, M_0, a_K, b_0)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, m\, M_0, a_K, b_0).$$

Let $f : L \to \mathbb{R}^m$ be a $(\mu^{|\mathbf{N}}, M_0)$-smooth map satisfying $\|f\|_K \geq b_0$. For any function $g : K \to [-1, 1]$, we have

$$\|f\|_{B(x, \frac{|g(x)|}{2}) \cap K} \geq \frac{\|f\|_K}{\mathbf{C}_\mathbf{N}} \Big(\frac{c_K\, \nu_K}{m\, 2^{\theta_K}\, |K|}\Big)^{\mathbf{N}} |g(x)|^{\theta_K \mathbf{N}}, \tag{8.1}$$

for all $x \in K$.

Note that we do not require that $f|_K^{-1}(0) \subseteq g^{-1}(0)$.

Proof. If $x \in g^{-1}(0)$, then (8.1) holds trivially; in that case, the left-hand side of (8.1) equals $|f(x)|_1$. Fix $x \in K \setminus g^{-1}(0)$ and set $B := B(x, \frac{r}{2}) \cap K$ for $r := |g(x)|$. Note that $r \in (0, 1]$.

We formulated Theorem 7.2 for single-valued function, but it remains true in the vector-valued setting with the same proof. Thus, for $t > 0$,

$$|K_t| \le \frac{m\,|K|}{\nu_K} \Big(\frac{\mathbf{C_N}\, t}{\|f\|_K}\Big)^{1/\mathbf{N}}, \tag{8.2}$$

where $K_t := \{x \in K : |f(x)|_1 \le t\}$.

Let T be the set of all $t > 0$ such that

$$\frac{m\,|K|}{\nu_K} \Big(\frac{\mathbf{C_N}\, t}{\|f\|_K}\Big)^{1/\mathbf{N}} < c_K \Big(\frac{r}{2}\Big)^{\theta_K}. \tag{8.3}$$

That means $t \in T$ if and only if

$$0 < t < \frac{\|f\|_K}{\mathbf{C_N}} \Big(\frac{c_K\,\nu_K}{m\,2^{\theta_K}\,|K|}\Big)^{\mathbf{N}} r^{\theta_K \mathbf{N}}. \tag{8.4}$$

Since $|B| \ge c_K(\frac{r}{2})^{\theta_K}$ because K is (c_K, θ_K)-thick, we have $B \not\subseteq K_t$ and hence $\|f\|_B > t$ for all $t \in T$, by (8.2) and (8.3). In view of (8.4), this implies (8.1). □

Let us apply this to the distance function.

Corollary 8.4. *In the setup of Theorem 8.3 with $m = 1$, we have, for all $x \in K$,*

$$\|f\|_{B(x,\frac{r(x)}{2})\cap K} \ge \frac{\|f\|_K}{\mathbf{C_N}} \Big(\frac{c_K\,\nu_K}{2^{\theta_K}\,|K|}\Big)^{\mathbf{N}} r(x)^{\theta_K \mathbf{N}},$$

where

$$r(x) := \min\{\operatorname{dist}(x, f|_K^{-1}(0)), 1\}, \quad x \in K,$$

or, alternatively,

$$r(x) := \frac{\operatorname{dist}(x, f|_K^{-1}(0))}{\max\{\operatorname{diam}(K), 1\}}, \quad x \in K.s$$

8.3. **Order of vanishing.** The order of vanishing of a continuous function $f : K \to \mathbb{R}$ at $x \in K$ is defined by

$$\operatorname{ord}_x f = \limsup_{r\to 0} \frac{\log \|f\|_{B(x,r)\cap K}}{\log r}.$$

If f is smooth and K is open, then $\operatorname{ord}_x f$ is the order of the smallest nonzero partial derivative of f at x.

Corollary 8.5. *In the setup of Theorem 8.3 with $m = 1$, we have $\operatorname{ord}_x f \le \theta_K \mathbf{N}$ for all $x \in K$.*

Proof. Applying Theorem 8.3 to $g \equiv r$ for $r \in (0, 1)$, gives

$$\|f\|_{B(x,\frac{r}{2})\cap K} \ge \frac{\|f\|_K}{\mathbf{C_N}} \Big(\frac{c_K\,\nu_K}{|K|}\Big)^{\mathbf{N}} \Big(\frac{r}{2}\Big)^{\theta_K \mathbf{N}},$$

for all $x \in K$. This implies the assertion. □

8.4. **Inequalities for the gradient.** We will deduce several inequalities for the gradient.

8.4.1. *First gradient inequality.* The first is a direct application of Theorem 8.3 for $\operatorname{grad} f$ and f in place of f and g. It requires a "shift" of the weight $\mu^{|\infty}$, see (2.2).

Corollary 8.6. *Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$. Assume that K is (c_K, θ_K)-thick. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$. Let $M_1 = M_0\mu_1 > b_1 > 0$. Assume that $(\mu^{|\infty}_{+1}, n\, M_1, a_K, b_1)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}_{+1}, n\, M_1, a_K, b_1).$$

Let $f : L \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}+1}, M_0)$-smooth function satisfying $\|f\|_K \leq 1$ and $\|\operatorname{grad} f\|_K \geq b_1$. Then, for all $x \in K$,

$$\|\operatorname{grad} f\|_{B(x, \frac{|f(x)|}{2})\cap K} \geq \frac{\|\operatorname{grad} f\|_K}{\mathbf{C_N}} \Big(\frac{c_K\, \nu_K}{n\, 2^{\theta_K}\, |K|}\Big)^{\mathbf{N}} |f(x)|^{\theta_K \mathbf{N}}. \tag{8.5}$$

We do not require $(\operatorname{grad} f|_K)^{-1}(0) \subseteq f|_K^{-1}(0)$.

Proof. It is easy to see that $\operatorname{grad} f : L \to \mathbb{R}^n$ is $(\mu^{|\infty}_{+1}, M_1)$-smooth. Thus the statement follows from Theorem 8.3. □

Remark 8.7. Alternatively, using that $\|\operatorname{grad} f\|_K \geq b_1$ implies $\|\partial_{j_0} f\|_K \geq b_1/n$ for some j_0 and Theorem 8.3 for $\partial_{j_0} f$ implies the following version of (8.5):

$$\|\operatorname{grad} f\|_{B(x, \frac{|f(x)|}{2})\cap K} \geq \frac{b_1}{\mathbf{C_N}\, n} \Big(\frac{c_K\, \nu_K}{2^{\theta_K}\, |K|}\Big)^{\mathbf{N}} |f(x)|^{\theta_K \mathbf{N}}.$$

with $\mathbf{N} = \mathbf{N}(\mu^{|\infty}_{+1}, n\, M_1, a_K, b_1)$.

8.4.2. *Second gradient inequality.* The constants in (8.5) are completely explicit, but the exponent $\theta_K \mathbf{N}$ of $|f(x)|$ is typically large. In the analytic or quasianalytic setting (see Lemma 8.8), one has an (unspecified) exponent $\alpha \in (0, 1)$. In the following, we discuss gradient inequalities with explicit constants and *small* exponents of $|f(x)|$.

Lemma 8.8 ([BM04, Theorem 6.3])**.** *Let $\mathcal{C}_M$ be a quasianalytic Denjoy–Carleman class satisfying the properties listed in Example 3.4. Let $U \subseteq \mathbb{R}^n$ be open and $f \in \mathcal{C}_M(U)$. Let $K \subseteq U$ be compact and assume that $(\operatorname{grad} f|_K)^{-1}(0) \subseteq f|_K^{-1}(0)$. Then there exist $c > 0$ and $\alpha \in [0, 1)$ such that*

$$|\operatorname{grad} f(x)| \geq c\, |f(x)|^{\alpha}$$

in a neighborhood of K.

Theorem 8.9. *Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq \operatorname{int}(L)$. Assume that K is (c_K, θ_K)-thick. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be a quasianalytic admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$ and all the properties listed in Example 3.4. Let $M_0 \geq 1$ and $M_0 > b_0 > 0$ and*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, a_K, b_0).$$

Let $f : L \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}}, M_0)$-smooth function satisfying $\|f\|_K \geq b_0$. Assume that $(\operatorname{grad} f|_K)^{-1}(0) \subseteq f|_K^{-1}(0)$. Then,

$$|\operatorname{grad} f(x)| \geq \frac{1}{2\sqrt{M_2}} \Big(\frac{\|f\|_K}{2\mathbf{C_N}}\Big)^{\frac{1}{\theta_K \mathbf{N}}} \Big(\frac{c_K\, \nu_K}{|K|}\Big)^{\frac{1}{\theta_K}} |f(x)|^{\frac{3}{2} - \frac{1}{\theta_K \mathbf{N}}} \tag{8.6}$$

in a neighborhood of $f|_K^{-1}(0)$ in K.

Proof. By Lemma 8.8, there exist $c > 0$ and $\alpha \in [0, 1)$ such that, for $x \in K$,

$$\frac{|f(x)|}{|\operatorname{grad} f(x)|} = \frac{|f(x)|^{\alpha}|f(x)|^{1-\alpha}}{|\operatorname{grad} f(x)|} \leq \frac{|f(x)|^{1-\alpha}}{c}.$$

Thus there is a neighborhood U in K of $f|_K^{-1}(0)$ such that

$$|f(x)| \leq 1 \quad \text{and} \quad \frac{|f(x)|}{|\operatorname{grad} f(x)|} \leq 1 \quad \text{for } x \in U.$$

In particular, the function $g : K \to \mathbb{R}$ which is 0 on $(\operatorname{grad} f|_K)^{-1}(0)$ and

$$g(x) := \frac{|f(x)|^{3/2}}{\sqrt{M_2}\,|\operatorname{grad} f(x)|} \quad \text{for } x \notin (\operatorname{grad} f|_K)^{-1}(0), \tag{8.7}$$

is well-defined on K and satisfies $0 \leq g \leq 1$ on U.

Let $x \in U$ and $y \in B(x, \frac{g(x)}{2}) \cap L$. Then

$$\begin{aligned} |f(y) - f(x)| &\leq |\langle \operatorname{grad} f(x), y - x\rangle| + \frac{M_2}{2}|x - y|^2 \\ &\leq \frac{1}{2\sqrt{M_2}}|f(x)|^{3/2} + \frac{1}{8}|f(x)| \end{aligned}$$

and consequently (noting that $M_2 \geq 1$),

$$|f(y)| \leq 2\,|f(x)|. \tag{8.8}$$

By Theorem 8.3 applied to the choice (8.7) of g and (8.8), for $x \in U$,

$$2\,|f(x)| \geq \|f\|_{B(x, \frac{g(x)}{2})\cap K} \geq \frac{\|f\|_K}{\mathbf{C_N}} \left(\frac{c_K\,\nu_K}{2^{\theta_K}\,|K|}\right)^{\mathbf{N}} \left(\frac{|f(x)|^{3/2}}{\sqrt{M_2}\,|\operatorname{grad} f(x)|}\right)^{\theta_K \mathbf{N}},$$

implying

$$|\operatorname{grad} f(x)|^{\theta_K \mathbf{N}} \geq \frac{\|f\|_K}{2\mathbf{C_N}\,M_2^{\theta_K \mathbf{N}/2}} \left(\frac{c_K\,\nu_K}{2^{\theta_K}\,|K|}\right)^{\mathbf{N}} |f(x)|^{\frac{3}{2}\theta_K \mathbf{N} - 1}.$$

This gives (8.6). □

Corollary 8.10. *In the setup of Theorem 8.9,*

$$\|\operatorname{grad} f\|_{B_x} \geq \frac{1}{2}\left(\frac{2\|f\|_K}{3\mathbf{C_N}}\right)^{\frac{1}{\theta_K \mathbf{N}}} \left(\frac{c_K\,\nu_K}{|K|}\right)^{\frac{1}{\theta_K}} |f(x)|^{1 - \frac{1}{\theta_K \mathbf{N}}}$$

for x in a neighborhood of $f|_K^{-1}(0)$ in K, where B_x is a neighborhood of x in L that shrinks to x if x tends to $f|_K^{-1}(0)$.

Proof. We follow the proof of Theorem 8.9 and replace (8.7) by

$$g(x) := \frac{|f(x)|}{|\operatorname{grad} f(x)|}.$$

Then the function $g : K \to \mathbb{R}$ is well-defined and satisfies $0 \leq g \leq 1$ on U. Moreover, $g(x) \to 0$ as x tends to $f|_K^{-1}(0)$.

Set $B_x := B(x, \frac{g(x)}{2}) \cap L$. For $x \in U$ and $y \in B_x'$, we have

$$|f(y) - f(x)| \leq \sup_{z \in B_x} |\operatorname{grad} f(z)| \frac{|f(x)|}{2\,|\operatorname{grad} f(x)|} \leq \frac{\|\operatorname{grad} f\|_{B_x}}{2\,|\operatorname{grad} f(x)|}\,|f(x)|,$$

(using $|\operatorname{grad} f(z)| \leq |\operatorname{grad} f(z)|_1$) and thus

$$|f(y)| \leq \Big(1 + \frac{\|\operatorname{grad} f\|_{B_x}}{2\,|\operatorname{grad} f(x)|}\Big)\,|f(x)| \leq \frac{3}{2}\,\|\operatorname{grad} f\|_{B_x}\,\frac{|f(x)|}{|\operatorname{grad} f(x)|}.$$

Then Theorem 8.3 applied to this choice of g gives

$$\frac{3}{2}\,\|\operatorname{grad} f\|_{B_x}\,\frac{|f(x)|}{|\operatorname{grad} f(x)|} \geq \frac{\|f\|_K}{\mathbf{C_N}}\Big(\frac{c_K\,\nu_K}{2^{\theta_K}\,|K|}\Big)^{\mathbf{N}}\Big(\frac{|f(x)|}{|\operatorname{grad} f(x)|}\Big)^{\theta_K\mathbf{N}},$$

and hence

$$\|\operatorname{grad} f\|_{B_x}^{\theta_K\mathbf{N}} \geq \frac{2\|f\|_K}{3\mathbf{C_N}}\Big(\frac{c_K\,\nu_K}{2^{\theta_K}\,|K|}\Big)^{\mathbf{N}}\,|f(x)|^{\theta_K\mathbf{N}-1}.$$

which implies the statement. □

8.4.3. *Third gradient inequality.* We close this section with another inequality of Łojasiewicz type based on the coarea formula, which we recall in the following lemma in the form we will use (cf. [Fed69]).

Lemma 8.11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then, for all Borel measurable functions $g : \mathbb{R}^n \to [0,\infty]$,*

$$\int_{\mathbb{R}^n} g(x)\,|\operatorname{grad} f(x)|\,dx = \int_{\mathbb{R}}\int_{f^{-1}(y)} g\,d\mathcal{H}^{n-1}\,dy.$$

Theorem 8.12. *Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$. Let $M_0 > b_0 > 0$. Assume that $(\mu^{|\infty}, M_0, a_K, b_0)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, a_K, b_0).$$

Let $f : L \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}}, M_0)$-smooth function satisfying $\|f\|_K \geq b_0$. Assume that there exist $\alpha \geq 0$ and $l, t_0 > 0$ such that

$$\mathcal{H}^{n-1}(\{|f| = y\} \cap K) \geq l\,y^\alpha \quad \text{for almost every } y \in [0, t_0].$$

Then, for $K_t := \{x \in K : |f(x)| \leq t\}$,

$$\|\operatorname{grad} f\|_{K_t} \geq \frac{l\,\nu_K}{(\alpha+1)\,|K|}\Big(\frac{\|f\|_K}{\mathbf{C_N}}\Big)^{1/\mathbf{N}}\,\|f\|_{K_t}^{\alpha+1-\frac{1}{\mathbf{N}}}, \quad 0 \leq t \leq t_0.$$

Proof. First we observe that $|\operatorname{grad}|f(x)|| = |\operatorname{grad} f(x)|$ if $f(x) \neq 0$. The zero set of f has Lebesgue measure zero, by [Rai23, Theorem 4.6]. Set $K_t' := \{x \in K : 0 < |f(x)| \leq t\}$. By Lemma 8.11,

$$|K_t'| = \int_0^t \int_{\{|f|=y\}\cap K} \frac{\mathbf{1}_{K_t'}}{|\operatorname{grad} f|}\,d\mathcal{H}^{n-1}\,dy.$$

We conclude that, for $0 \leq t \leq t_0$,

$$\begin{aligned}|K_t'| &\geq \frac{1}{\|\operatorname{grad} f\|_{K_t'}}\int_0^t \int_{\{|f|=y\}\cap K} \mathbf{1}_{K_t'}\,d\mathcal{H}^{n-1}\,dy \\ &= \frac{1}{\|\operatorname{grad} f\|_{K_t'}}\int_0^t \mathcal{H}^{n-1}(\{|f| = y\} \cap K)\,dy \geq \frac{l\,t^{\alpha+1}}{(\alpha+1)\,\|\operatorname{grad} f\|_{K_t'}}.\end{aligned}$$

Combined with Theorem 7.2, this implies the assertion. □

9. Inequalities of Harnack type

In this section, we prove two inequalities of Harnack type for $(\mu^{|N}, M_0)$-smooth functions. The first is a simple consequence of the Remez inequality, but the involved constant is somewhat implicit and not easy to compute. The second inequality has a completely explicit constant but requires more work.

9.1. First inequality of Harnack type.

Proposition 9.1. *Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$. Let $M_0 > b_0 > 0$. Assume that $(\mu^{|\infty}, M_0, a_K, b_0)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, a_K, b_0).$$

Let $f : L \to \mathbb{R}$ be $(\mu^{|\mathbf{N}}, M_0)$-smooth with $\|f\|_K \geq b_0$ and $\min_{x\in K} |f(x)| \geq m_0 > 0$. Then

$$\max_{x\in K} |f(x)| \leq 2\,\mathbf{C}_{\mathbf{N}} \Big(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E_{m_0}|)^{\nu_K}}\Big)^{\mathbf{N}} \min_{x\in K} |f(x)|,$$

where $E_{m_0} := \{x \in K : |f(x)| \leq \min_{x\in K} |f(x)| + m_0\}$.

Proof. The set E_{m_0} is closed and $|E_{m_0}| > 0$, by the continuity of f. The statement follows from Theorem 4.13 and the fact that $\|f\|_{E_{m_0}} \leq 2 \min_{x\in K} |f(x)|$. □

9.2. Bounds for $1/f$.

Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > 0$. Let us assume that the sequence

$$a_j := \Big(\frac{\mu_1 \cdots \mu_j}{j!}\Big)^{1/j}, \quad j \geq 1, \tag{9.1}$$

is almost increasing (cf. Example 3.4), i.e., there exists a constant $H \geq 1$ such that

$$a_j \leq H\, a_k \quad \text{whenever } j \leq k. \tag{9.2}$$

Lemma 9.2. *Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight satisfying (9.2) with constant $H \geq 1$ and let $M_0 \geq m_0 > 0$. Let $I \subseteq \mathbb{R}$ be a compact interval and $f : I \to \mathbb{R}$ a $(\mu^{|N}, M_0)$-smooth function such that $\min_{x\in I} |f(x)| \geq m_0$. Then the function $1/f : I \to \mathbb{R}$ is well-defined and $\big(H(1 + \frac{2M_0}{m_0})\,\mu^{|N}, \frac{2}{m_0}\big)$-smooth.*

Proof. We follow the proof of [Rud62, Theorem 2.1]. Recall that $N \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. For $1 \leq n < N + 1$, define r_n by

$$H\, a_n r_n = \frac{m_0}{m_0 + 2M_0}. \tag{9.3}$$

Fix $x_0 \in I$ and consider the polynomial $P(z) := \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} z^j$. Using (9.2), we find that, if $|z| \leq r_n$,

$$\begin{aligned} |P(z)| &\geq |f(x_0)| - \sum_{j=1}^n \frac{|f^{(j)}(x_0)|}{j!} |z|^j \\ &\geq m_0 - \sum_{j=1}^n \frac{M_j}{j!} r_n^j = m_0 - M_0 \sum_{j=1}^n (a_j r_n)^j \geq m_0 - M_0 \sum_{j=1}^n (H a_n r_n)^j \end{aligned}$$

$$= m_0 - M_0 \sum_{j=1}^{n} \Big(\frac{m_0}{m_0+2M_0}\Big)^j > m_0 - \frac{m_0 M_0}{m_0+2M_0}\frac{m_0+2M_0}{2M_0} = \frac{m_0}{2}.$$

Since $f^{(j)}(x_0) = P^{(j)}(0)$, for $0 \le j \le n$, we have $(1/f)^{(n)}(x_0) = (1/P)^{(n)}(0)$ and hence, by (9.1) and (9.3),

$$\begin{aligned}|(1/f)^{(n)}(x_0)| &= \Big|\frac{n!}{2\pi i}\int_{|z|=r_n} \frac{dz}{z^{n+1}P(z)}\Big| \\ &\le \frac{2}{m_0}\frac{n!}{r_n^n} = \frac{2}{m_0}\Big(H\Big(1+\frac{2M_0}{m_0}\Big)\Big)^n \mu_1\cdots\mu_n.\end{aligned}$$

This implies the assertion. □

Lemma 9.3. *Let $T : (\mathbb{R}^n)^j \to \mathbb{R}$,*

$$T(v_1,\dots,v_j) = \sum_{i_1,\dots,i_j=1}^{n} T_{i_1,\dots,i_j} v_1^{i_1}\cdots v_j^{i_j}$$

be a j-linear symmetric map, where $v_k = (v_k^1,\dots,v_k^n)$. Then

$$\|T\|_2 = \Big(\sum_{i_1,\dots,i_j=1}^{n} T_{i_1,\dots,i_j}^2\Big)^{1/2} \le n^{\frac{j-1}{2}} \sup_{|v|\le 1} |T(v,\dots,v)|.$$

Proof. We proceed by induction on j. For $j=1$, we have equality. Assume $j \ge 2$. By induction hypothesis, for the symmetric $(j-1)$-linear maps $T_i := T(e_i,\cdots)$,

$$\|T_i\|_2 \le n^{\frac{j-2}{2}} \sup_{|v|\le 1} |T_i(v,\dots,v)|.$$

Therefore,

$$\|T\|_2^2 = \sum_{i_1=1}^{n} \|T_{i_1}\|_2^2 \le \sum_{i_1=1}^{n} n^{j-2} \sup_{|v|\le 1} |T_{i_1}(v,\dots,v)|^2 \le n^{j-1} \sup_{|v|\le 1} |T(v,\dots,v)|^2,$$

where the last inequality is a consequence of

$$\sup_{|v_1|\le 1,\dots,|v_j|\le 1} |T(v_1,\dots,v_j)| = \sup_{|v|\le 1} |T(v,\dots,v)|,$$

see [BS71, Proposition 1]. □

Lemma 9.4. *Let $\mu^{|\infty} = (\mu_j)_{j\ge 1}$ be an admissible weight satisfying (9.2) with constant $H \ge 1$ and let $M_0 \ge m_0 > 0$. Let $K \subseteq \mathbb{R}^n$ be a nonempty fat compact set and $f : K \to \mathbb{R}$ a $(\mu^{|N}, M_0)$-smooth function such that $\min_{x\in K} |f(x)| \ge m_0$. Then the function $1/f : K \to \mathbb{R}$ is well-defined and $\big(nH(1+\frac{2M_0}{m_0})\,\mu^{|N}, \frac{2}{m_0}\big)$-smooth.*

Proof. Fix $x \in \operatorname{int}(K)$ and $v \in \mathbb{S}^{n-1}$. Let $g(t) := f(x+tv)$ for $t \in I$, where $I \subseteq \mathbb{R}$ is a compact interval such that $x \in x + Iv \subseteq K$. Then $g : I \to \mathbb{R}$ is $(\mu^{|N}, M_0)$-smooth. By Lemma 9.2,

$$|(1/g)^{(j)}(0)| = |d_v^j(1/f)(x)| \le \frac{2}{m_0} H^j \Big(1+\frac{2M_0}{m_0}\Big)^j \mu_1\cdots\mu_j.$$

Since $\sum_{|\alpha|=j} \frac{j!}{\alpha!} = n^j$, we have

$$\sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha (1/f)(x)| \le n^{j/2} \Big(\sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha (1/f)(x)|^2 \Big)^{1/2}$$

and, by Lemma 9.3, the right-hand side is bounded by

$$n^{j/2} \cdot n^{(j-1)/2} \sup_{|v|\le 1} \Big| \sum_{|\alpha|=j} \frac{j!}{\alpha!} \partial^\alpha (1/f)(x) v^\alpha \Big| = \frac{n^j}{\sqrt{n}} \sup_{|v|\le 1} |d_v^j (1/f)(x)|.$$

It follows that, for $j \ge 1$,

$$\sum_{|\alpha|=j} \frac{j!}{\alpha!} |\partial^\alpha (1/f)(x)| \le \frac{2}{\sqrt{n}\, m_0} (nH)^j \Big(1 + \frac{2M_0}{m_0}\Big)^j \mu_1 \cdots \mu_j,$$

for all $x \in \operatorname{int}(K)$. By continuity and since K is fat, the estimate holds for all $x \in K$. Clearly, $\|1/f\|_K \le 1/m_0$. □

9.3. **Second inequality of Harnack type.**

Theorem 9.5. *Let $L \subseteq \mathbb{R}^n$ be a convex body and $K \in \mathscr{K}(\mathbb{R}^n)$ such that $K \subseteq L$. Let $\mu^{|\infty} = (\mu_j)_{j\ge 1}$ be an admissible weight satisfying $\mu_j \ge j$ for $j \ge 1$ as well as (9.2) with constant $H \ge 1$. Let $M_0 \ge m_0 > 0$. Set*

$$\beta_0 := nH(1 + \tfrac{2M_0}{m_0}).$$

Assume that $(\mu^{|\infty}, 2M_0, a_K\beta_0, m_0)$ is admissible and let

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, 2M_0, a_K\beta_0, m_0).$$

If $f : L \to \mathbb{R}$ is $(\mu^{|\mathbf{N}}, M_0)$-smooth and satisfies $\min_{x\in L} |f(x)| \ge m_0$, then

$$\max_{x\in K} |f(x)| \le \mathbf{C}_{\mathbf{N}}^2 \Big(\frac{2^{\nu_K}}{2^{\nu_K} - 1}\Big)^{2\mathbf{N}} \min_{x\in K} |f(x)|.$$

Proof. Let us set $m_f := \min_{x\in K} |f(x)|$ and $M_f := \max_{x\in K} |f(x)|$. Clearly, $m_0 \le m_f \le M_f \le M_0$. The set $E_- := \{x \in K : |f(x)| \le \sqrt{M_f m_f}\}$ has positive measure. By Theorem 4.13,

$$M_f \le \mathbf{C}_{\mathbf{N}'} \Big(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E_-|)^{\nu_K}}\Big)^{\mathbf{N}'} \sqrt{M_f m_f},$$

where $\mathbf{N}' := \mathbf{N}(\mu^{|\infty}, M_0, a_K, m_0)$. Since $\mathbf{N}' \le \mathbf{N}$ (see Remark 2.10), we conclude

$$M_f \le \mathbf{C}_{\mathbf{N}}^2 \Big(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E_-|)^{\nu_K}}\Big)^{2\mathbf{N}} m_f.$$

By Lemma 9.4, the function $1/f : L \to \mathbb{R}$ is $\big(\beta_0\, \mu^{|\mathbf{N}}, \frac{2}{m_0}\big)$-smooth and satisfies $\min_{x\in L} |1/f(x)| \ge 1/M_0$. Note that $m_{1/f} := \min_{x\in K} |1/f(x)| = 1/M_f$ and $M_{1/f} := \max_{x\in K} |1/f(x)| = 1/m_f$. Observe that

$$\mathbf{N}(\mu^{|\infty}, \tfrac{2}{m_0}, a_K\beta_0, \tfrac{1}{M_0}) = \mathbf{N}(\mu^{|\infty}, 2M_0, a_K\beta_0, m_0).$$

Thus, Theorem 4.13 applied to $1/f$, gives

$$M_{1/f} \le \mathbf{C}_{\mathbf{N}} \Big(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E_+|)^{\nu_K}}\Big)^{\mathbf{N}} \sqrt{M_{1/f} m_{1/f}},$$

where $E_+ := \{x \in K : |1/f(x)| < \sqrt{M_{1/f} m_{1/f}}\}$, which implies

$$M_f \leq \mathbf{C}_{\mathbf{N}}^2 \Big(\frac{|K|^{\nu_K}}{|K|^{\nu_K} - (|K| - |E_+|)^{\nu_K}} \Big)^{2\mathbf{N}} m_f.$$

Now $E_+ = \{x \in K : \sqrt{M_f m_f} < |f(x)|\}$ so that E_- and E_+ form a partition of K. Therefore, $|E_-| \geq |K|/2$ or $|E_+| \geq |K|/2$. Consequently,

$$M_f \leq \mathbf{C}_{\mathbf{N}}^2 \Big(\frac{2^{\nu_K}}{2^{\nu_K} - 1} \Big)^{2\mathbf{N}} m_f.$$

The proof is complete. □

10. Inequalities of Markov type

Here we discuss a few inequalities of Markov type which are consequences of an upper bound for the number of zeros in terms of $\mathbf{N}$ which we recall in Proposition 10.1 and the comparison of the L^1- and L^∞-norm, see Theorem 7.4.

10.1. **Number of zeros.**

Proposition 10.1 ([Rai23, Corollary 3.6 and Remark 3.7])**.** *Let $I \subseteq \mathbb{R}$ be a compact interval. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_0 > b_0 > 0$. Assume that $(\mu^{|\infty}, M_0, |I|, b_0)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}, M_0, |I|, b_0).$$

Let $f : I \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}}, M_0)$-smooth function satisfying $\|f\|_I \geq b_0$. Then the total number of zeros of f in I (counted with multiplicities) is finite and bounded by $\mathbf{N}$.

10.2. **Inequalities of Markov type for univariate functions.**

The following lemma is a variant of [Pie22, Lemma 8.1].

Lemma 10.2. *Let $I \subseteq \mathbb{R}$ be a compact interval. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_1 = M_0\mu_1 > b_1 > 0$. Assume that $(\mu_{+1}^{|\infty}, M_1, |I|, b_1)$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu_{+1}^{|\infty}, M_1, |I|, b_1).$$

Let $f : I \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}+1}, M_0)$-smooth function satisfying $\|f'\|_I \geq b_1$. Then, for each nonempty Lebesgue measurable subset $E \subseteq I$,

$$\|f'\|_{L^1(E)} \leq 2(\mathbf{N}+1)\, \|f\|_E.$$

Proof. By Proposition 10.1 f' has at most $\mathbf{N}$ zeros in I (counted with multiplicities). Thus there is sequence $t_1 < \cdots < t_n$, with $n \leq \mathbf{N}$, of points in I such that the sign of f' is constant on each interval $I_j := (t_j, t_{j+1})$, for $j = 1, \ldots, n-1$, $I_0 := I \cap (-\infty, t_1)$ and $I_n := I \cap (t_n, \infty)$. Then

$$\begin{aligned} \int_E |f'(x)|\, dx &= \sum_{j=0}^{n} \int_{E\cap I_j} |f'(x)|\, dx \\ &\leq \sum_{j=0}^{n} |f(\sup E \cap I_j) - f(\inf E \cap I_j)| \leq 2(n+1)\, \|f\|_E \end{aligned}$$

which implies the assertion. □

Combining Lemma 10.2 with Theorem 7.4, we obtain the following inequality of Markov type.

Proposition 10.3. *In the setting of Lemma 10.2, for each Lebesgue measurable subset $E \subseteq I$ with $|E| > 0$,*

$$\|f'\|_I \le 2\,\mathbf{C_N}\,\frac{|I|^{\mathbf{N}}}{|E|^{\mathbf{N}+1}}(\mathbf{N}+1)^2\,\|f\|_E.$$

Proof. Combine Lemma 10.2 with (7.4) for f'. The condition $\mu_j \ge j$ for $j \ge 1$ is not needed; see Remark 7.1. □

Corollary 10.4. *Let $I \subseteq \mathbb{R}$ be a compact interval. Let $\mu^{|\infty} = (\mu_j)_{j\ge1}$ be a quasianalytic admissible weight and $M_0 > 0$. Let $f : I \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function. Then, for all $j \ge 1$,*

$$\|f^{(j)}\|_I \le \Big(\frac{2}{|I|}\Big)^j \prod_{i=1}^{j} \mathbf{C}_{\mathbf{N}_i}(\mathbf{N}_i+1)^2\,\|f\|_I,$$

where $\mathbf{N}_i = \mathbf{N}(\mu^{|\infty}_{+i}, M_i, |I|, \frac{\|f^{(i)}\|_I}{2})$.

Proof. Iterate Proposition 10.3. In the definition of $\mathbf{N}_i$ we put $\frac{\|f^{(i)}\|_I}{2}$ to guarantee that $M_i > \frac{\|f^{(i)}\|_I}{2}$ and hence $\mathbf{N}_i \in \mathbb{N}_{\ge2}$. □

10.3. **Sampling the supremum norm.** We need an easy consequence of Proposition 10.3.

Corollary 10.5. *Let $\mu^{|\infty} = (\mu_j)_{j\ge1}$ be a quasianalytic admissible weight, $M_0 > 0$, and $\mu_1 > 1$. Let $f : [0,1] \to \mathbb{R}$ be a nonzero $(\mu^{|\infty}, M_0)$-smooth function. Then*

$$\|f'\|_{[0,1]} \le 2\,\mathbf{C_N}\,(\mathbf{N}+1)^2\,\|f\|_{[0,1]},$$

where $\mathbf{N} = \mathbf{N}(\mu^{|\infty}_{+1}, M_1, 1, \|f\|_{[0,1]})$.

Proof. The assertion is trivially true if $\|f'\|_{[0,1]} \le \|f\|_{[0,1]}$. If $\|f'\|_{[0,1]} > \|f\|_{[0,1]}$, then the statement follows from Proposition 10.3. The assumption $\mu_1 > 1$ guarantees that $\|f\|_{[0,1]} < M_1$ so that $\mathbf{N} \in \mathbb{N}_{\ge2}$. □

Recall that an *ϵ-net* for $[0,1]$ is a subset $X \subseteq [0,1]$ such that for each $y \in [0,1]$ there exists $x \in X$ with $|x - y| \le \epsilon$.

Corollary 10.6. *In the setting of Corollary 10.5, let X be any ϵ-net for $[0,1]$, where*

$$\epsilon := \frac{1}{4\,\mathbf{C_N}\,(\mathbf{N}+1)^2}. \tag{10.1}$$

Then

$$\|f\|_X \le \|f\|_{[0,1]} \le 2\,\|f\|_X. \tag{10.2}$$

Proof. Let $y \in [0,1]$. There exists $x \in X$ such that $|x - y| \le \epsilon$. By Corollary 10.5,

$$|f(y)| \le |f(x)| + |x-y|\,\|f'\|_{[0,1]} \le \|f\|_X + 2\epsilon\,\mathbf{C_N}\,(\mathbf{N}+1)^2\,\|f\|_{[0,1]}.$$

With the choice (10.1) for ϵ, the statement follows easily. □

So given a nonzero $(\mu^{|\infty}, M_0)$-smooth function $f : [0,1] \to \mathbb{R}$, we may compute $|f(x_0)|$ at a random point x_0. If $|f(x_0)| \neq 0$, then Corollary 10.6 implies that, if X consists of $4\,\mathbf{C_N}\,(\mathbf{N}+1)^2+1$ equidistant points in $[0,1]$, where $\mathbf{N} = \mathbf{N}(\mu^{|\infty}_{+1}, M_1, 1, |f(x_0)|)$, then X satisfies (10.2).

10.4. **Inequalities of Markov type for multivariate functions.** Let $f : I^n \to \mathbb{R}$ be a function defined on a compact cube $I^n \subseteq \mathbb{R}^n$. If $n = 1$ we set $V(f) := \sup_{x,y\in I} |f(x) - f(y)|$. In general, for $1 \leq i \leq n$,

$$V_i(f) := \inf_{\ell} V(f|_\ell),$$

where ℓ ranges over all line segments in I^n with direction e_i.

Corollary 10.7. *Let $I^n \subseteq \mathbb{R}^n$ be a compact cube. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be a quasi-analytic admissible weight and $M_0 > 0$. Let $f : I^n \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function. If $V_i(f) > 0$, then*

$$\|\partial_i f\|_{I^n} \leq \frac{2\,\mathbf{C_N}}{|I|}\,(\mathbf{N}+1)^2\,\|f\|_{I^n}$$

where $\mathbf{N} = \mathbf{N}(\mu^{|\infty}_{+1}, M_1, |I|, \frac{V_i(f)}{2|I|})$.

Proof. If $\ell \subseteq I$ is a line segment with direction e_i, then $\|\partial_i f\|_\ell \geq V_i(f)/|I|$ for all $x \in \ell$. The statement follows from Proposition 10.3. (We have $\mathbf{N} \in \mathbb{N}_{\geq 2}$ since $M_1 > \frac{V_i(f)}{2|I|}$.) □

The next lemma is a variant of [Pie22, Lemma 8.1]. If $K \subseteq \mathbb{R}^n$ is a nonempty compact set, let $K_i \subseteq \mathbb{R}^{n-1}$ denote the image of K under the projection in direction e_i. In the following, the measure $\mathcal{L}^{n-1}(K_i)$ will appear; for $n = 1$, we set $\mathcal{L}^0(K_1) := 1$, by convention.

Lemma 10.8. *Let $I^n \subseteq \mathbb{R}^n$ be a compact cube. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be a quasi-analytic admissible weight and $M_0 > 0$. Let $f : I^n \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function. If $V_i(f) > 0$, then for each nonempty compact subset $K \subseteq I^n$,*

$$\|\partial_i f\|_{L^1(K)} \leq 2\mathcal{L}^{n-1}(K_i)\,(\mathbf{N}+1)\,\|f\|_K,$$

where $\mathbf{N} = \mathbf{N}(\mu^{|\infty}_{+1}, M_1, |I|, \frac{V_i(f)}{2|I|})$.

Proof. The case $n = 1$ follows from Lemma 10.2. Let $n \geq 2$ and $i = 1$. Then, by Fubini's theorem and the case $n = 1$,

$$\int_K |\partial_1 f(x_1, x')|\,dx \leq \int_{K_1} 2\,(\mathbf{N}+1)\,\|f\|_K\,dx' = 2\mathcal{L}^{n-1}(K_1)\,(\mathbf{N}+1)\,\|f\|_K.$$

For $i \geq 2$, we argue analogously. □

Let us combine Lemma 10.8 with Theorem 7.4.

Corollary 10.9. *Let $I^n \subseteq \mathbb{R}^n$ be a compact cube. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be a quasi-analytic admissible weight satisfying $\mu_j \geq j$ for $j \geq 1$. Let $M_1 = M_0\mu_1 > b_1 > 0$.*

Let $K \in \mathscr{K}(\mathbb{R}^n)$ be a subset of I^n. Let $f : I^n \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function such that $\|\partial_i f\|_K \geq b_1$. If $V_i(f) > 0$, then

$$\|\partial_i f\|_K \leq \frac{2\mathbf{C_N}\,\mathcal{L}^{n-1}(K_i)}{\nu_K^{\mathbf{N}}\,|K|}\,(\mathbf{N}+1)^2\,\|f\|_K,$$

where $\mathbf{N} = \mathbf{N}(\mu_{+1}^{|\infty}, M_1, \max\{a_K, |I|\}, \min\{b_1, \frac{V_i(f)}{|I|}\})$.

Proof. The assertion is immediate from Lemma 10.8 and (7.3) applied to $\partial_i f$. (Note that $\mathbf{N} \in \mathbb{N}_{\geq 2}$, because $M_1 > b_1$.) □

11. Oscillatory integrals

The sublevel set estimates of Theorem 7.2 and the bound for the numbers of zeros of Proposition 10.1 can be combined with the van der Corput lemma to provide estimates for oscillatory integrals whose phase functions are $(\mu^{|\infty}, M_0)$-smooth. A similar approach was pursued by [Loi21] for phase functions that are definable in an o-minimal structure.

11.1. **Van der Corput lemma.**

Lemma 11.1 ([CCW99, Proposition 2.2], [Ste93])**.** *There exists an absolute constant $C > 0$ such that for any compact interval $I \subseteq \mathbb{R}$, any $k \geq 2$, and any integrable function $f : I \to \mathbb{R}$ satisfying $|f^{(k)}| \geq 1$ in the sense of distributions,*

$$\Big|\int_I e^{i\lambda f(x)}\,dx\Big| \leq C\,k\,\lambda^{-1/k}, \quad \lambda > 0.$$

11.2. **Univariate phase and amplitude.**

Proposition 11.2. *Let $I \subseteq \mathbb{R}$ be a compact interval. Let $\mu^{|\infty} = (\mu_j)_{j\geq 1}$ be an admissible weight and $M_2 = M_0\mu_1\mu_2 \geq b_2 > 0$. Assume that $(\mu_{+2}^{|\infty}, M_2 + \frac{b_2}{2}, |I|, \frac{b_2}{2})$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu_{+2}^{|\infty}, M_2 + \tfrac{b_2}{2}, |I|, \tfrac{b_2}{2}).$$

Let $f : I \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}+2}, M_0)$-smooth function satisfying $\|f''\|_I \geq b_2$. Let $g : I \to \mathbb{C}$ belong to $\mathcal{C}^1(I)$. Then,

$$\Big|\int_I e^{i\lambda f(x)} g(x)\,dx\Big| \leq B\,\lambda^{-\frac{1}{\mathbf{N}+2}}, \quad \text{if } \lambda \geq \Big(\frac{2}{b_2}\Big)^{1+\frac{2}{\mathbf{N}}}, \tag{11.1}$$

where

$$B = \frac{\mathbf{C}_{\mathbf{N}}^{1/\mathbf{N}}\,|I|\,\|g\|_I}{\|f''\|_I^{1/\mathbf{N}}} + 2C\,(\mathbf{N}+1)\big(\|g\|_I + \|g'\|_{L^1(I)}\big)$$

and $C > 0$ is an absolute constant.

Proof. Consider the sublevel set $I_t := \{x \in I : |f''(x)| \leq t\}$ of f''. By Theorem 7.2 and Remark 7.1, we have

$$|I_t| \leq |I|\Big(\frac{\mathbf{C_N}\,t}{\|f''\|_I}\Big)^{1/\mathbf{N}}, \quad t > 0.$$

Consequently,

$$\Big|\int_{I_t} e^{i\lambda f(x)} g(x)\,dx\Big| \leq \|g\|_I\,|I|\,\Big(\frac{\mathbf{C_N}\,t}{\|f''\|_I}\Big)^{1/\mathbf{N}}, \quad t > 0.$$

For $0 < t \le \frac{b_2}{2}$, we have

$$\|f'' \pm t\|_I \le \|f''\|_I + t \le M_2 + \tfrac{b_2}{2}$$

and

$$\|f'' \pm t\|_I \ge \|f''\|_I - t \ge b_2 - \tfrac{b_2}{2} = \tfrac{b_2}{2}$$

so that, by Proposition 10.1, $|f''| - t$ has at most $\mathbf{N}$ zeros in I. We conclude that, for $0 < t \le \frac{b_2}{2}$, the set $I \setminus I_t = \{x \in I : |f''(x)| > t\}$ consists of at most $\mathbf{N}+1$ intervals (a_i, b_i) (the intervals containing the endpoints of I are half-open). By partial integration,

$$\begin{aligned}\int_{a_i}^{b_i} e^{i\lambda f(x)} g(x)\,dx &= \int_{a_i}^{b_i} \partial_x\Big(\int_{a_i}^{x} e^{i\lambda f(u)}\,du\Big) g(x)\,dx \\ &= \Big(\int_{a_i}^{b_i} e^{i\lambda f(u)}\,du\Big) g(b_i) - \int_{a_i}^{b_i} \Big(\int_{a_i}^{x} e^{i\lambda f(u)}\,du\Big) g'(x)\,dx.\end{aligned}$$

By Lemma 11.1, for $\lambda > 0$ and $x \in [a_i, b_i]$,

$$\Big|\int_{a_i}^{x} e^{i\lambda f(u)}\,du\Big| = \Big|\int_{a_i\sqrt{t}}^{x\sqrt{t}} e^{i\lambda f(y/\sqrt{t})}\,\frac{dy}{\sqrt{t}}\Big| \le 2C\,(\lambda t)^{-1/2}.$$

Thus,

$$\Big|\int_{a_i}^{b_i} e^{i\lambda f(x)} g(x)\,dx\Big| \le 2C\,(\lambda t)^{-1/2}\big(\|g\|_I + \|g'\|_{L^1(I)}\big).$$

It follows that, for $0 < t \le \frac{b_2}{2}$ and $\lambda > 0$,

$$\begin{aligned}&\Big|\int_I e^{i\lambda f(x)} g(x)\,dx\Big| \\ &\qquad \le \|g\|_I\,|I|\,\Big(\frac{\mathbf{C_N}\,t}{\|f''\|_I}\Big)^{1/\mathbf{N}} + 2C\,(\mathbf{N}+1)\,(\lambda t)^{-1/2}\big(\|g\|_I + \|g'\|_{L^1(I)}\big)\end{aligned}$$

and setting $t = \lambda^{-\frac{\mathbf{N}}{\mathbf{N}+2}}$, we conclude (11.1). □

11.3. **Multivariate phase and amplitude.** The next theorem generalizes Proposition 11.2 to several variables.

Theorem 11.3. *Let $I^n \subseteq \mathbb{R}^n$ be a compact cube. Let $\mu^{|\infty} = (\mu_j)_{j\ge 1}$ be an admissible weight and $M_2 = M_0\mu_1\mu_2 \ge b_2 > 0$. Assume that $(\mu^{|\infty}_{+2}, M_2 + \frac{b_2}{2}, \sqrt{n}\,|I|, \frac{b_2}{2})$ is admissible and let*

$$\mathbf{N} = \mathbf{N}(\mu^{|\infty}_{+2}, M_2 + \tfrac{b_2}{2}, \sqrt{n}\,|I|, \tfrac{b_2}{2}).$$

Let $f : I^n \to \mathbb{R}$ be a $(\mu^{|\mathbf{N}+2}, M_0)$-smooth function. Assume, for some $1 \le i_0 \le n$, we have $\|\partial_{i_0}^2 f\|_\ell \ge b_2$ for each line segment ℓ in I^n with direction e_{i_0}. Let $g : I^n \to \mathbb{C}$ belong to $\mathcal{C}^1(I^n)$. Then,

$$\Big|\int_{I^n} e^{i\lambda f(x)} g(x)\,dx\Big| \le B\,\lambda^{-\frac{1}{\mathbf{N}+2}}, \qquad \text{if } \lambda \ge \Big(\frac{2}{b_2}\Big)^{1+\frac{2}{\mathbf{N}}}, \tag{11.2}$$

where

$$B = \frac{\mathbf{C}_\mathbf{N}^{1/\mathbf{N}}\, n\,|I|^n\,\|g\|_{I^n}}{\|\partial_{i_0}^2 f\|_{I^n}^{1/\mathbf{N}}} + 2C\,(\mathbf{N}+1)\,\big(|I|^{n-1}\,\|g\|_{I^n} + \|\partial_{i_0} g\|_{L^1(I^n)}\big),$$

and $C > 0$ is an absolute constant.

Proof. Let us modify the proof of Proposition 11.2. For simplicity of notation assume $i_0 = 1$. For $I_t^n := \{x \in I^n : |\partial_1^2 f(x)| \le t\}$, Theorem 7.2 and Remark 7.1 give

$$|I_t^n| \le n\,|I|^n \Big(\frac{\mathbf{C_N}\, t}{\|\partial_1^2 f\|_{I^n}}\Big)^{1/\mathbf{N}}, \quad t > 0.$$

Consequently,

$$\Big|\int_{I_t^n} e^{i\lambda f(x)} g(x)\,dx\Big| \le \|g\|_{I^n}\, n\,|I|^n \Big(\frac{\mathbf{C_N}\, t}{\|\partial_1^2 f\|_{I^n}}\Big)^{1/\mathbf{N}}, \quad t > 0.$$

For $0 < t \le \frac{b_2}{2}$, the function $I \ni s \mapsto |\partial_1^2 f(s, x')| - t$ has at most $\mathbf{N}$ zeros, uniformly in $x' \in I^{n-1}$, by Proposition 10.1. So each line with direction e_1 intersects $I^n \setminus I_t^n$ in a union of at most $\mathbf{N}+1$ segments. For each such segment, say with endpoints (a_i, x') and (b_i, x') (where a_i, b_i depend on x'), Lemma 11.1 implies, similarly as in the proof of Proposition 11.2, that

$$\Big|\int_{a_i}^{b_i} e^{i\lambda f(s,x')} g(s,x')\,ds\Big| \le 2C\,(\lambda t)^{-1/2}\Big(\|g\|_{I^n} + \int_{a_i}^{b_i} |\partial_1 g(s,x')|\,ds\Big). \quad (11.3)$$

Thus, for $0 < t \le \frac{b_2}{2}$ and $\lambda > 0$,

$$\begin{aligned}\Big|\int_{I^n\setminus I_t^n} e^{i\lambda f(x)} g(x)\,ds\Big| &\le \int_{I^{n-1}} \sum_i \Big|\int_{a_i}^{b_i} e^{i\lambda f(s,x')} g(s,x')\,ds\Big|\,dx' \\ &\le 2C\,(\mathbf{N}+1)\,(\lambda t)^{-1/2}\Big(|I|^{n-1}\,\|g\|_{I^n} + \|\partial_1 g\|_{L^1(I^n)}\Big).\end{aligned}$$

Setting $t = \lambda^{-\frac{\mathbf{N}}{\mathbf{N}+2}}$, we conclude (11.2). □

In the following theorem, we consider oscillatory integrals over sets $K \in \mathscr{K}(\mathbb{R}^n)$ that are definable in the polynomially bounded o-minimal structure $\mathbb{R}_{\mathcal{C}_M}$; see Example 3.4. We assume that $\mu^{|\infty} = (\mu_j)_{j\ge 1}$ is a quasianalytic admissible weight satisfying $\mu_j \ge j$ as well as all properties listed in Example 3.4.

Theorem 11.4. *Let $I^n \subseteq \mathbb{R}^n$ be a compact cube. Let $K \in \mathscr{K}(\mathbb{R}^n)$ be definable in $\mathbb{R}_{\mathcal{C}_M}$ such that $K \subseteq \operatorname{int}(I^n)$. Let $M_2 = M_0\mu_1\mu_2, b_2, b_2' > 0$. Let*

$$\mathbf{N} = \mathbf{N}(\mu_{+2}^{|\infty}, M_2 + \tfrac{b_2}{2}, \max\{a_K, |I|\}, \min\{\tfrac{b_2}{2}, b_2'\}).$$

Let $f : I^n \to \mathbb{R}$ be a $(\mu^{|\infty}, M_0)$-smooth function. Assume, for some $1 \le i_0 \le n$, we have $\|\partial_{i_0}^2 f\|_\ell \ge b_2$ for each line segment ℓ in I^n with direction e_{i_0} as well as $\|\partial_{i_0}^2 f\|_K \ge b_2'$. Let $g : I^n \to \mathbb{C}$ belong to $\mathcal{C}^1(I^n)$. Then,

$$\Big|\int_K e^{i\lambda f(x)} g(x)\,dx\Big| \le B\,\lambda^{-\frac{1}{\mathbf{N}+2}}, \quad \text{if } \lambda > 0, \quad (11.4)$$

where

$$B = \frac{\mathbf{C_N}^{1/\mathbf{N}}\,|K|\,\|g\|_K}{\nu_K\,\|\partial_{i_0}^2 f\|_K^{1/\mathbf{N}}} + 2CN_{f,K}\,\big(|I|^{n-1}\,\|g\|_{I^n} + \|\partial_{i_0} g\|_{L^1(I^n)}\big),$$

$C > 0$ is an absolute constant, and $N_{f,K} \in \mathbb{N}$ is an integer depending only on f and K.

Proof. We follow the proof of Theorem 11.3. Let $i_0 = 1$ and $K_t := \{x \in K : |\partial_1^2 f(x)| \leq t\}$. Then Theorem 7.2 gives

$$\Big| \int_{K_t} e^{i\lambda f(x)} g(x)\, dx \Big| \leq \|g\|_K \, \frac{|K|}{\nu_K} \Big(\frac{\mathbf{C}_{\mathbf{N}}\, t}{\|\partial_1^2 f\|_K} \Big)^{1/\mathbf{N}}, \quad t > 0.$$

Since K and $f|_K$ are definable in $\mathbb{R}_{\mathcal{C}_M}$, so is the set

$$\mathcal{K} := \{(x_1, x', t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : (x_1, x') \in K,\, |\partial_1^2 f(x_1, x')| > t\}.$$

Hence there exists $N_{f,K} \in \mathbb{N}$ such that

$$\mathcal{K}_{(x',t)} = \{x_1 \in \mathbb{R} : (x_1, x', t) \in \mathcal{K}\} = \{x_1 \in \mathbb{R} : (x_1, x') \in K,\, |\partial_1^2 f(x_1, x')| > t\}$$

has at most $N_{f,K}$ connected components, for all $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ (see e.g. [vdDM96, 4.4]).

Let $t > 0$ and $x' \in I^{n-1}$. If a connected component of $\mathcal{K}_{(x',t)}$ has the endpoints (a_i, x') and (b_i, x'), then Lemma 11.1 implies as in the proof of Theorem 11.3 that (11.3) holds. Consequently, for $t, \lambda > 0$,

$$\Big| \int_{K \setminus K_t} e^{i\lambda f(x)} g(x)\, ds \Big| \leq 2C N_{f,K}\, (\lambda t)^{-1/2} \big(|I|^{n-1}\, \|g\|_{I^n} + \|\partial_1 g\|_{L^1(I^n)} \big).$$

Setting $t = \lambda^{-\frac{\mathbf{N}}{\mathbf{N}+2}}$, we conclude (11.4). □

Appendix A. Examples of admissible weights

In this section, we establish upper and lower bounds for the Bang degree $\mathbf{D}_{a\,\mu^{|\infty}}(b)$, as a function in a and b, for particular weights $\mu^{|\infty}$, namely the analytic weight and the Denjoy weights. Moreover, we estimate the associated function $\gamma_{\tilde{\mu}}$.

A.1. **The analytic weight.** Let us define $\mu_{\mathrm{an}}^{|\infty} := (j)_{j \geq 1}$. Clearly, $\mu_{\mathrm{an}}^{|\infty}$ is an admissible quasianalytic weight which satisfies all properties listed in Example 3.4. In this case, $\Sigma_{\mu_{\mathrm{an}}^{|\infty}}(1, n)$ is the partial harmonic sum $H_n := \sum_{j=1}^n \frac{1}{j}$, and, more generally,

$$\Sigma_{\mu_{\mathrm{an}}^{|\infty}}(m, n) = H_n - H_{m-1}$$

if $1 \leq m \leq n$, where $H_0 := 0$.

Proposition A.1. *Let $a, b > 0$ and set $j_0(b) = \lceil \log(b^{-1}) \rceil_{\mathbb{N}}$. If $0 < ae + H_{j_0(b)} \leq 1$ (consequently, $j_0(b) = 0$), then*

$$\mathbf{D}_{a\,\mu_{\mathrm{an}}^{|\infty}}(b) = 0. \tag{A.1}$$

If $ae + H_{j_0(b)} > 1$, then

$$(j_0(b) + 1)\, e^{ae-1} - 1 \leq \mathbf{D}_{a\,\mu_{\mathrm{an}}^{|\infty}}(b) \leq (j_0(b) + 1)\, e^{ae+1} - 1. \tag{A.2}$$

Proof. Lemma 2.6 implies (A.1). The function $g(t) = \log t$, for $t > 0$, is concave so that

$$g(j+1) - g(j) \leq g'(j) \leq g(j) - g(j-1),$$

where the left inequality holds for all integers $j \geq 1$, the right for all integers $j \geq 2$. Summing over j, we find

$$\log(n+1) \leq H_n \leq \log n + 1, \quad n \geq 1. \tag{A.3}$$

Let $x > 1$ and define the positive integer $n(x)$ by

$$H_{n(x)} < x \le H_{n(x)+1}. \tag{A.4}$$

By (A.3), we conclude

$$e^{x-1} - 1 \le n(x) < e^x - 1. \tag{A.5}$$

By (A.4), we have

$$\Sigma_{\mu_{\rm an}^{|\infty}}(m, n(x)) < x - H_{m-1} \le \Sigma_{\mu_{\rm an}^{|\infty}}(m, n(x)+1).$$

Setting $x = ae + H_{m-1}$, we get

$$\Sigma_{\mu_{\rm an}^{|\infty}}(m, n) < ae \le \Sigma_{\mu_{\rm an}^{|\infty}}(m, n+1)$$

with $n = n(ae + H_{m-1})$, provided $ae + H_{m-1} > 1$. Thus, by (A.3) and (A.5),

$$m\, e^{ae-1} - 1 \le n \le m\, e^{ae+1} - 1,$$

which also holds if $m - 1 = 0$ (by direct verification). This implies (A.2). □

It is easy to compute $\gamma_{\tilde{\mu}_{\rm an}}$; see Definition 2.12.

Proposition A.2. *For $\tilde{\mu}_{\rm an}(t) = t$, we get $\gamma_{\tilde{\mu}_{\rm an}} \equiv 1$.* □

A.2. **Denjoy weights.** Let $s \in (0, 1]$. Define $\mu_j := j(\log(ej))^s$ for $j \ge 1$. Then, for each $s \in (0, 1]$, $\mu_s^{|\infty} := (\mu_j)_{j\ge1}$ is a quasianalytic admissible weight satisfying all properties listed in Example 3.4. Set $A_n^s := \sum_{j=1}^n \frac{1}{\mu_j}$, for $n \ge 1$, and $A_0^s := 0$.

Proposition A.3 (The case $s = 1$)**.** *Let $a, b > 0$ and set $j_0(b) = \lceil \log(b^{-1}) \rceil_{\mathbb{N}}$. If $0 < ae + A^1_{j_0(b)} \le 1$ (consequently, $j_0(b) = 0$), then*

$$\mathbf{D}_{a\,\mu_1^{|\infty}}(b) = 0. \tag{A.6}$$

If $ae + A^1_{j_0(b)} > 1$, then

$$(j_0(b)+1)^{e^{ae-1}} e^{e^{ae-1}-1} - 1 \le \mathbf{D}_{a\,\mu_1^{|\infty}}(b) \le (j_0(b)+1)^{e^{ae+1}} e^{e^{ae+1}-1} - 1. \tag{A.7}$$

Proof. Lemma 2.6 implies (A.6). Consider $g(t) = \log\log(et)$, for $t \ge 1$. Then

$$g'(t) = \frac{1}{t\log(et)} \quad \text{and} \quad g''(t) = -\frac{1+\log(et)}{(t\log(et))^2}$$

so that g is concave. Thus

$$g(j+1) - g(j) \le g'(j) \le g(j) - g(j-1),$$

where the left inequality holds for $j \ge 1$ and the right for $j \ge 2$. Summing over j, we find

$$g(n+1) - g(1) \le A_n^1 \quad \text{and} \quad A_n^1 - A_1^1 \le g(n) - g(1)$$

which implies

$$\log\log(e(n+1)) \le A_n^1 \le \log\log(en) + 1, \quad n \ge 1. \tag{A.8}$$

For $x > 1$ define the positive integer $n(x)$ by

$$A^1_{n(x)} < x \le A^1_{n(x)+1}. \tag{A.9}$$

By (A.8) and (A.9),

$$e^{e^{x-1}-1} - 1 \le n(x) < e^{e^x-1} - 1. \tag{A.10}$$

By (A.9), we have

$$\Sigma_{\mu_1^{|\infty}}(m,n(x)) < x - A^1_{m-1} \leq \Sigma_{\mu_1^{|\infty}}(m,n(x)+1)$$

so that, setting $x = ae + A^1_{m-1}$, we get

$$\Sigma_{\mu_1^{|\infty}}(m,n) < ae \leq \Sigma_{\mu_1^{|\infty}}(m,n+1)$$

with $n = n(ae + A^1_{m-1})$, provided $ae + A^1_{m-1} > 1$. Thus, by (A.8) and (A.10),

$$m^{e^{ae-1}} e^{e^{ae-1}-1} - 1 \leq n \leq m^{e^{ae+1}} e^{e^{ae+1}-1} - 1$$

which also holds if $m - 1 = 0$ (by direct verification). This implies (A.7) □

Proposition A.4 (The case $s \in (0,1)$)**.** *Let $a, b > 0$ and set $j_0(b) = \lceil \log(b^{-1}) \rceil_{\mathbb{N}}$. If $ae + A^s_{j_0(b)} \leq 1$ (consequently, $j_0(b) = 0$), then*

$$\mathbf{D}_{a\,\mu_s^{|\infty}}(b) = 0. \tag{A.11}$$

If $ae + A^s_{j_0(b)} > 1$, then

$$e^{(f_s(a,b)-(1-s))^{\frac{1}{1-s}}-1} - 1 \leq \mathbf{D}_{a\,\mu_s^{|\infty}}(b) \leq e^{(f_s(a,b)+(1-s))^{\frac{1}{1-s}}-1} - 1 \tag{A.12}$$

where

$$f_s(a,b) := (1-s)ae + (1 + \log(j_0(b)+1))^{1-s}.$$

Proof. Lemma 2.6 implies (A.11). Fix $s \in (0,1)$ and consider $g_s(t) = (\log(et))^{1-s}$ for $t \geq 1$. Then

$$g_s'(t) = \frac{1-s}{t(\log(et))^s} \quad \text{and} \quad g_s''(t) = -\frac{(1-s)(\log(et))^s + s(1-s)(\log(et))^{s-1}}{(t(\log(et))^s)^2},$$

so that g_s is concave. Thus

$$g_s(j+1) - g_s(j) \leq g_s'(j) \leq g_s(j) - g_s(j-1),$$

where the left inequality holds for $j \geq 1$ and the right for $j \geq 2$. Summing over j, we find

$$g_s(n+1) - g_s(1) \leq (1-s)A^s_n \quad \text{and} \quad (1-s)(A^s_n - A^s_1) \leq g_s(n) - g_s(1)$$

which implies

$$(\log(e(n+1)))^{1-s} - 1 \leq (1-s)A^s_n \leq (\log(en))^{1-s} - s, \quad n \geq 1. \tag{A.13}$$

For $x > 1$ define the positive integer $n(x)$ by

$$A^s_{n(x)} < x \leq A^s_{n(x)+1}. \tag{A.14}$$

By (A.13) and (A.14),

$$e^{((1-s)x+s)^{\frac{1}{1-s}}-1} - 1 \leq n(x) < e^{((1-s)x+1)^{\frac{1}{1-s}}-1} - 1. \tag{A.15}$$

By (A.14),

$$\Sigma_{\mu_s^{|\infty}}(m,n(x)) < x - A^s_{m-1} \leq \Sigma_{\mu_s^{|\infty}}(m,n(x)+1).$$

Setting $x = ae + A^s_{m-1}$, we get

$$\Sigma_{\mu_s^{|\infty}}(m,n) < ae \leq \Sigma_{\mu_s^{|\infty}}(m,n+1)$$

with $n = n(ae + A^s_{m-1})$, provided $ae + A^s_{m-1} > 1$. Thus, by (A.13) and (A.15),

$$e^{((1-s)(ae-1)+(\log(em))^{1-s})^{\frac{1}{1-s}}-1} - 1 \leq n \leq e^{((1-s)(ae+1)+(\log(em))^{1-s})^{\frac{1}{1-s}}-1} - 1$$

which also holds if $m-1=0$ (by direct verification). This implies (A.12) □

Let us determine $\gamma_{\tilde{\mu}_s}$.

Proposition A.5. *Let $s\in(0,1]$. For $\tilde{\mu}_s(t)=t(\log(et))^s$, we have $\gamma_{\tilde{\mu}_s}\equiv 1+s$.*

Proof. We have

$$\begin{aligned}\gamma_{\tilde{\mu}_s}(n) &= \sup_{1\le t\le n}\frac{t\tilde{\mu}_s'(t)}{\tilde{\mu}_s(t)} = \sup_{1\le t\le n}\frac{t((\log(et))^s+s(\log(et))^{s-1})}{t(\log(et))^s}\\ &= \sup_{1\le t\le n}\Big(1+\frac{s}{\log(et)}\Big)=1+s\end{aligned}$$

as claimed. □

Appendix B. Univariate Remez inequality

The goal of this section is to prove the following vector-valued version of [NSV04, Theorem B], by adjusting the original proof.

Theorem B.1. *Let $\mu^{|\infty}=(\mu_j)_{j\ge1}$ be an admissible weight and $M_0>b_0>0$. Assume that $(\mu^{|\infty},M_0,1,b_0)$ is admissible and let $\mathbf{N}=\mathbf{N}(\mu^{|\infty},M_0,1,b_0)$. Let $f:[0,1]\to\mathbb{R}^m$ be $(\mu^{|\mathbf{N}},M_0)$-smooth such that $\|f\|_{[0,1]}\ge b_0$. Then for each interval $I\subseteq[0,1]$ and each Lebesgue measurable set $E\subseteq I$ with $|E|>0$ we have*

$$\|f\|_I\le \mathbf{C}_{\mathbf{N}}\Big(\frac{m\,|I|}{|E|}\Big)^{\mathbf{N}}\|f\|_E. \tag{B.1}$$

Let us set $\mathbf{D}:=\mathbf{D}_{\mu^{|\infty}}(\frac{b_0}{M_0})$ so that $\mathbf{N}=2\mathbf{D}$ and note that $\mathbf{D}\in\mathbb{N}_{\ge1}$. See Definition 2.13 for the definition of $\mathbf{C}_{\mathbf{N}}$.

Recall that a function $f=(f_1,\dots,f_m):[0,1]\to\mathbb{R}^m$ is $(\mu^{|\mathbf{N}},M_0)$ smooth if $f\in\mathcal{C}^{\mathbf{N}}([0,1])$ and

$$\|f\|_{j,[0,1]}=\sup_{x\in[0,1]}\sum_{i=1}^m|f_i^{(j)}(x)|\le M_j,\quad 0\le j\le\mathbf{N}.$$

We also use $\|f\|_{[0,1]}=\|f\|_{0,[0,1]}=\sup_{x\in[0,1]}\sum_{i=1}^m|f_i(x)|$.

In the following, we write $m_j:=M_j/j!$, for $j\ge1$, and assume $M_0=m_0=1$.

Proof. We may assume that $M_0=1$, by replacing f by f/M_0 and b_0 by b_0/M_0. Choose points $x_1<x_2<\dots<x_{\mathbf{N}}$ in E such that $\min_j(x_{j+1}-x_j)\ge\frac{|E|}{\mathbf{N}-1}$. Put $Q(x):=(x-x_1)(x-x_2)\cdots(x-x_{\mathbf{N}})$. Then, by the Lagrange interpolation formula, for $x\in I$,

$$f_i(x)=\sum_{j=1}^{\mathbf{N}}\frac{Q(x)f_i(x_j)}{Q'(x_j)(x-x_j)}+\frac{Q(x)f_i^{(\mathbf{N})}(\xi)}{\mathbf{N}!},\quad \xi=\xi_{i,x}\in I.$$

It follows that

$$\|f\|_I\le\Big(\sum_{j=1}^{\mathbf{N}}\frac{1}{|Q'(x_j)|}\Big)\|f\|_E|I|^{\mathbf{N}-1}+m_{\mathbf{N}}|I|^{\mathbf{N}}.$$

Now

$$\begin{aligned}|Q'(x_j)| &= (x_j - x_{j-1})\cdots(x_j - x_1)(x_{j+1} - x_j)\cdots(x_{\mathbf{N}} - x_j)\\ &\geq \frac{(j-1)!(\mathbf{N}-j)!}{(\mathbf{N}-1)^{\mathbf{N}-1}}|E|^{\mathbf{N}-1} > \frac{(j-1)!(\mathbf{N}-j)!}{(\mathbf{N}-1)!}\Big(\frac{|E|}{e}\Big)^{\mathbf{N}-1}\end{aligned}$$

so that

$$\sum_{j=1}^{\mathbf{N}} \frac{1}{|Q'(x_j)|} < \Big(\frac{2e}{|E|}\Big)^{\mathbf{N}-1}.$$

Thus

$$\|f\|_I \leq \Big(\frac{2e|I|}{|E|}\Big)^{\mathbf{N}-1}\|f\|_E + m_{\mathbf{N}}|I|^{\mathbf{N}}.$$

We claim that

$$m_{\mathbf{N}}|I|^{\mathbf{N}} \leq e^{-\mathbf{N}(\log m + 3 + \gamma_{\tilde{\mu}}(\mathbf{N}))} \implies m_{\mathbf{N}}|I|^{\mathbf{N}} \leq \frac{2}{3}\|f\|_I. \tag{B.2}$$

If (B.2) is satisfied, then we conclude

$$\|f\|_I \leq \Big(\frac{2e|I|}{|E|}\Big)^{\mathbf{N}}\|f\|_E. \tag{B.3}$$

Let us suppose that (B.2) is not satisfied. Choose a subinterval $I_1 \subseteq I$ such that (B.2) is satisfied for I_1, as well as

$$|E \cap I_1| \geq |E|\frac{|I_1|}{|I|},$$

and

$$m_{\mathbf{N}}|I_1|^{\mathbf{N}} \geq 2^{-\mathbf{N}}e^{-\mathbf{N}(\log m + 3 + \gamma_{\tilde{\mu}}(\mathbf{N}))}.$$

Then, using (B.2) and (B.3) for I_1,

$$\begin{aligned}1 &\leq \Big(2e^{\log m + 3 + \gamma_{\tilde{\mu}}(\mathbf{N})}\Big)^{\mathbf{N}} m_{\mathbf{N}}|I_1|^{\mathbf{N}} \leq \frac{2}{3}\Big(2e^{\log m + 3 + \gamma_{\tilde{\mu}}(\mathbf{N})}\Big)^{\mathbf{N}}\|f\|_{I_1}\\ &\leq \frac{2}{3}\Big(2e^{\log m + 3 + \gamma_{\tilde{\mu}}(\mathbf{N})}\Big)^{\mathbf{N}}\Big(\frac{2e|I_1|}{|E\cap I_1|}\Big)^{\mathbf{N}}\|f\|_{E\cap I_1} \leq \Big(4e^{\log m + 4 + \gamma_{\tilde{\mu}}(\mathbf{N})}\frac{|I|}{|E|}\Big)^{\mathbf{N}}\|f\|_E.\end{aligned}$$

This implies (B.1) because $\|f\|_I \leq \|f\|_{[0,1]} \leq M_0 = 1$. □

The proof of the claim (B.2) comprises the Lemmas B.2—B.7.

Lemma B.2 ([Rai23, Lemma 3.1])**.** *The function*

$$\beta(x) := \sup_{0\leq j\leq \mathbf{N}} \frac{\sum_{i=1}^m |f_i^{(j)}(x)|}{e^j M_j}, \quad x \in [0,1],$$

is continuous. More precisely, for all $1 \leq j \leq \mathbf{N}$ and all distinct $x, y \in [0,1]$,

$$\beta(x) < \max\{\beta(y), e^{-j}\}e^{e|x-y|\mu_j}. \tag{B.4}$$

Proof. [Rai23, Lemma 3.1] is formulated and proved in the case $m = 1$, but the argument works for general m. □

Lemma B.3. $\min_{x\in[0,1]} \beta(x) > e^{-\mathbf{D}-1}$.

Proof. Suppose $\min_{x\in[0,1]}\beta(x) \le e^{-\mathbf{D}-1}$ and let $x_0 \in [0,1]$ be such that $\beta(x_0) = e^{-\mathbf{D}-1}$. Since $\|f\|_{[0,1]} \ge b_0$, we have $\max_{x\in[0,1]}\beta(x) \ge \frac{b_0}{M_0} \ge e^{-j_0}$, where $j_0 := \lceil \log \frac{M_0}{b_0} \rceil_{\mathbb{N}}$. Since β is continuous, there is a monotonic sequence $x_j \in [0,1]$ such that $\beta(x_j) = e^{-j}$ for all $j_0 \le j \le \mathbf{D}+1$. By (B.4), we have

$$|x_j - x_{j-1}| > \frac{1}{e\,\mu_j} \qquad \text{for } j_0+1 \le j \le \mathbf{D}+1,$$

so that, summing over j, we find

$$e > \sum_{j=j_0+1}^{\mathbf{D}+1} \frac{1}{\mu_j},$$

contradicting the definition of $\mathbf{D}$. □

Lemma B.4. *For $0 \le k \le \mathbf{D}$,*

$$\Big(\frac{1}{\mathbf{N}-k} - \frac{1}{\mathbf{N}}\Big)\log m_{\mathbf{N}} - \frac{1}{\mathbf{N}-k}\log m_k < \gamma_{\tilde\mu}(\mathbf{N}) - 1.$$

Proof. The left-hand side equals

$$\begin{aligned}
&\Big(\frac{1}{\mathbf{N}-k} - \frac{1}{\mathbf{N}}\Big)\sum_{j=1}^{\mathbf{N}} \log(\tfrac{\mu_j}{j}) - \frac{1}{\mathbf{N}-k}\sum_{j=1}^{k}\log(\tfrac{\mu_j}{j}) \\
&= \frac{1}{\mathbf{N}-k}\sum_{j=k+1}^{\mathbf{N}} \big(\log(\tfrac{\mu_j}{j}) - \log\mu_1\big) - \frac{1}{\mathbf{N}}\sum_{j=1}^{\mathbf{N}}\big(\log(\tfrac{\mu_j}{j}) - \log\mu_1\big) \\
&= \int_1^{\mathbf{N}} \frac{\hat\mu'(t)}{\hat\mu(t)} S(t)\,dt,
\end{aligned}$$

where $\hat\mu(t) := \tilde\mu(t)/t$ and

$$S(t) := \frac{1}{\mathbf{N}-k}\sum_{j=k+1}^{\mathbf{N}} \mathbf{1}_{[1,j]}(t) - \frac{1}{\mathbf{N}}\sum_{j=1}^{\mathbf{N}}\mathbf{1}_{[1,j]}(t).$$

Since $0 \le S(t) < t/\mathbf{N}$, we find

$$\int_1^{\mathbf{N}} \frac{\hat\mu'(t)}{\hat\mu(t)} S(t)\,dt \le \sup_{1\le t\le \mathbf{N}} \frac{t\hat\mu'(t)}{\hat\mu(t)} = \sup_{1\le t\le \mathbf{N}} \frac{t\tilde\mu'(t)}{\tilde\mu(t)} - 1 = \gamma_{\tilde\mu}(\mathbf{N}) - 1.$$

□

Lemma B.5. *Let I be a subinterval of $[0,1]$. If*

$$m_{\mathbf{N}}|I|^{\mathbf{N}} \le e^{-\mathbf{N}(\log m + 3 + \gamma_{\tilde\mu}(\mathbf{N}))} \tag{B.5}$$

then, for $0 \le k \le \mathbf{D}$,

$$m_{\mathbf{N}}|I|^{\mathbf{N}} \le \frac{e^{-2\mathbf{N}}}{m} m_k |I|^k.$$

Proof. By (B.5),

$$\frac{m_{\mathbf{N}}|I|^{\mathbf{N}}}{m_k|I|^k} \le e^{-(\mathbf{N}-k)(\log m + 3 + \gamma_{\tilde\mu}(\mathbf{N}))} \frac{m_{\mathbf{N}}^{k/\mathbf{N}}}{m_k}.$$

By Lemma B.4,

$$\Big(\frac{m_{\mathbf{N}}^{k/\mathbf{N}}}{m_k}\Big)^{1/(\mathbf{N}-k)} = \frac{m_{\mathbf{N}}^{\frac{1}{\mathbf{N}-k}-\frac{1}{\mathbf{N}}}}{m_k^{\frac{1}{\mathbf{N}-k}}} < e^{\gamma_{\bar\mu}(\mathbf{N})-1}.$$

It follows that

$$\begin{aligned}\frac{m_{\mathbf{N}}|I|^{\mathbf{N}}}{m_k|I|^k} &\le e^{-(\mathbf{N}-k)(\log m+3+\gamma_{\bar\mu}(\mathbf{N}))+(\mathbf{N}-k)(\gamma_{\bar\mu}(\mathbf{N})-1)}\\ &= e^{-(\mathbf{N}-k)(\log m+4)} \le e^{-2\mathbf{N}-\log m} = \frac{e^{-2\mathbf{N}}}{m},\end{aligned}$$

if $k \le \mathbf{D} = \mathbf{N}/2$. □

Lemma B.6. *For some* $0 \le k \le \mathbf{D}$,

$$m_k\Big(\frac{|I|}{2}\Big)^k \le m e^{(2+\frac{1}{e})\mathbf{D}}\Big(\|f\|_I + m_{\mathbf{N}}\Big(\frac{|I|}{2}\Big)^{\mathbf{N}}\Big).$$

Proof. Let c be the center of I. Then, for $x \in I$,

$$f_i(x) = T_c^{\mathbf{N}-1} f_i(x) + \frac{f_i^{(\mathbf{N})}(\xi)}{\mathbf{N}!}(x-c)^{\mathbf{N}},$$

where $T_c^{\mathbf{N}-1} f_i(x)$ is the Taylor polynomial of degree $\mathbf{N}-1$ of f_i at c and $\xi_{i,x} \in I$. By [Man52, 6.4.III],

$$\begin{aligned}|f_i^{(k)}(c)| = |(T_c^{\mathbf{N}-1} f_i)^{(k)}(c)| &\le \Big(\frac{2}{|I|}\Big)^k (\mathbf{N}-1)^k \, \|T_c^{\mathbf{N}-1} f_i\|_I\\ &\le \Big(\frac{2}{|I|}\Big)^k (\mathbf{N}-1)^k \Big(\|f_i\|_I + m_{\mathbf{N}}\Big(\frac{|I|}{2}\Big)^{\mathbf{N}}\Big).\end{aligned}$$

Thus

$$\begin{aligned}\frac{|f_i^{(k)}(c)|}{e^k M_k} &\le \frac{1}{m_k}\Big(\frac{2}{|I|}\Big)^k \Big(\frac{\mathbf{N}-1}{e}\Big)^k \frac{1}{k!}\Big(\|f_i\|_I + m_{\mathbf{N}}\Big(\frac{|I|}{2}\Big)^{\mathbf{N}}\Big)\\ &\le \frac{1}{m_k}\Big(\frac{2}{|I|}\Big)^k e^{\frac{\mathbf{N}-1}{e}}\Big(\|f_i\|_I + m_{\mathbf{N}}\Big(\frac{|I|}{2}\Big)^{\mathbf{N}}\Big).\end{aligned}$$

By Lemma B.3, $e^{-\mathbf{D}-1} < \beta(c)$. Therefore, there is some $0 \le k \le \mathbf{D}$ such that

$$e^{-\mathbf{D}-1} < \sum_{i=1}^m \frac{|f_i^{(k)}(c)|}{e^k M_k}.$$

This implies the assertion. □

Lemma B.7. *If* (B.5)*, then* $m_{\mathbf{N}}|I|^{\mathbf{N}} < \frac{2}{3}\|f\|_I$.

Proof. By Lemmas B.5 and B.6, there exists $0 \le k \le \mathbf{D}$ such that

$$\begin{aligned}m_{\mathbf{N}}|I|^{\mathbf{N}} \le \frac{e^{-2\mathbf{N}}}{m} m_k |I|^k &\le e^{k-2\mathbf{N}} e^{(2+\frac{1}{e})\mathbf{D}}\Big(\|f\|_I + m_{\mathbf{N}}\Big(\frac{|I|}{2}\Big)^{\mathbf{N}}\Big)\\ &\le e^{-(1-\frac{1}{e})\mathbf{D}}\Big(\|f\|_I + \frac{1}{4} m_{\mathbf{N}}|I|^{\mathbf{N}}\Big).\end{aligned}$$

This implies the assertion, since $\frac{3}{2} < e^{1-\frac{1}{e}} - \frac{1}{4}$. □

Acknowledgement. This research was funded in whole or in part by the Austrian Science Fund (FWF) DOI 10.55776/PAT1381823. For open access purposes, the authors have applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

The author acknowledges the use of ChatGPT, Claude, and Perplexity for assistance with literature review.

References

[Ban53] T. Bang, *The theory of metric spaces applied to infinitely differentiable functions*, Math. Scand. **1** (1953), 137–152.

[BM04] E. Bierstone and P. D. Milman, *Resolution of singularities in Denjoy-Carleman classes*, Selecta Math. (N.S.) **10** (2004), no. 1, 1–28.

[BS71] J. Bochnak and J. Siciak, *Polynomials and multilinear mappings in topological vector-spaces*, Studia Math. **39** (1971), no. 1, 59–76.

[BG73] Yu. A. Brudnyi and M. I. Ganzburg, *A certain extremal problem for polynomials in n variables*, Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 344–355.

[Bru99a] A. Brudnyi, *Local inequalities for plurisubharmonic functions*, Ann. Math. (2) **149** (1999), no. 2, 511–533.

[Bru99b] ———, *On local behavior of analytic functions*, J. Funct. Anal. **169** (1999), no. 2, 481–493.

[Bru08] ———, *On local behavior of holomorphic functions along complex submanifolds of* $\mathbb{C}^N$, Invent. Math. **173** (2008), no. 2, 315–363§.

[BY15] A. Brudnyi and Y. Yomdin, *Norming sets and related Remez-type inequalities*, J. Aust. Math. Soc. **100** (2016), no. 2, 163–181.

[CCW99] A. Carbery, M. Christ, and J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Am. Math. Soc. **12** (1999), no. 4, 981–1015.

[vdDM96] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), no. 2, 497–540.

[vdD98] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.

[Fed69] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[LL01] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[Loi21] T. L. Loi, *Volumes of sub-level sets and the decay of oscillatory integrals*, J. Math. Inequal. **15** (2021), no. 2, 767–779.

[Man52] S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952.

[Mil94] C. Miller, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 257–259.

[NSV04] F. Nazarov, M. Sodin, and A. Volberg, *Lower bounds for quasianalytic functions. I. How to control smooth functions*, Math. Scand. **95** (2004), no. 1, 59–79.

[PP86] W. Pawłucki and W. Pleśniak, *Markov's inequality and* C^∞ *functions on sets with polynomial cusps*, Math. Ann. **275** (1986), no. 3, 467–480.

[Pie05] R. Pierzchała, *UPC condition in polynomially bounded o-minimal structures*, J. Approx. Theory **132** (2005), no. 1, 25–33.

[Pie15] ———, *Remez-type inequality on sets with cusps*, Adv. Math. **281** (2015), 508–552.

[Pie22] ———, *Polynomial inequalities, o-minimality and Denjoy-Carleman classes*, Adv. Math. **407** (2022), Paper No. 108565, 61.

[RS02] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press Oxford University Press, Oxford, 2002.

[Rai22] A. Rainer, *Ultradifferentiable extension theorems: a survey*, Expositiones Mathematicae **40** (2022), no. 3, 679–757, https://doi.org/10.1016/j.exmath.2021.12.001.

[Rai23] ———, *Quantitative tame properties of differentiable functions with controlled derivatives*, Nonlinear Anal. **237** (2023), 113372, https://doi.org/10.1016/j.na.2023.113372.

[Rai24] ———, *On real analytic functions on closed subanalytic domains*, Arch. Math. (Basel) **122** (2024), no. 6, 639–650, https://doi.org/10.1007/s00013-024-01983-1.

[Rem36] E. Remez, *Sur une propriété extremale des polynômes de Tchebychef*, Commun. Inst. Sci. Math. Mecan. Univ. Kharkow Soc. Math. Kharkow (4) **13** (1936), 93–95 (French).

[RSW03] J.-P. Rolin, P. Speissegger, and A. J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), no. 4, 751–777 (electronic).

[Rud62] W. Rudin, *Division in algebras of infinitely differentiable functions*, J. Math. Mech. **11** (1962), 797–809.

[Ste93] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[Thi08] V. Thilliez, *On quasianalytic local rings*, Expo. Math. **26** (2008), no. 1, 1–23.

[YC04] Y. Yomdin and G. Comte, *Tame geometry with application in smooth analysis*, Springer Berlin Heidelberg, 2004.

[Yom11] Y. Yomdin, *Remez-type inequality for discrete sets*, Israel Journal of Mathematics **186** (2011), no. 1, 45–60.

Armin Rainer: Faculty of Mathematics and Geoinformation, Institute for Statistics and Mathematical Methods in Economics, E105-04, TU Wien, Wiedner Hauptstrasse 8, 1040 Vienna, Austria

Email address: armin.rainer@tuwien.ac.at