EIGENVALUE STABILITY OF HERMITIAN AND NORMAL MATRICES

ADAM PARUSIŃSKI AND ARMIN RAINER

Abstract. The ordered eigenvalues define a Lipschitz map on the real vector space of Hermitian $d \times d$ matrices. We prove that this map acts continuously, but not uniformly continuously, by superposition on the Sobolev spaces $W^{1,q}$, for all $1 \le q < \infty$, on bounded open domains. For $q = \infty$, the action is still well-defined and bounded but not continuous. We show that this stability result extends to normal matrices, where the eigenvalues are naturally interpreted as multivalued Sobolev functions in the sense of Almgren. Several applications are given, including the stability of singular values, condition numbers of matrices, surface area of eigenvalue graphs, and compact self-adjoint operators in Hilbert space.

Contents

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1. Introduction

The perturbation theory of Hermitian and normal operators has a long history. The analytic theory started with Rellich's work [Rel37], culminated in Kato's monograph [Kat76], and extended to smooth perturbation theory in more recent papers, e.g., [AKLM98], [KM03, KMR11, KMR12], [Rai09, Rai13, Rai22], and [PR20]. For a more comprehensive account of the history, we refer to the recent survey article [PR25].

For the present paper, Weyl's perturbation theorem [Wey12] plays a fundamental role. It asserts that the ordered eigenvalues form a Lipschitz map on the space of Hermitian matrices with Lipschitz constant 1, when the space is equipped with the operator norm and the eigenvalues with the ∞-norm. Together with Löwner's theorem [Löw34], which provides the corresponding result for the 2-norm, and its extension to normal matrices by Hoffman and Wielandt [HW53], this theorem is foundational for nearly all of the results in this paper. Of course, in the case of normal matrices, the eigenvalues must be equipped with a metric that accounts for the fact that they are general complex numbers not admitting a coherent ordering. See also [BDM83] for a variant with the operator norm (where, however, the Lipschitz constant C satisfies $1 < C < 3$). For locally Lipschitz curves of normal matrices, each continuous choice of the eigenvalues is actually locally Lipschitz, as shown by [Rai13]. If the (real) parameter space is at least 2-dimensional, continuous choices of the eigenvalues of normal matrices may not exist.

The Lipschitz continuity of the eigenvalues as functions of the matrices immediately implies that Lipschitz families of Hermitian or normal matrices are assigned to the Lipschitz families of their eigenvalues, in a bounded way, that is, with controlled Lipschitz constants. This defines a map which we call the *characteristic map*: it is given by superposition with the ordered eigenvalue map, in the Hermitian case, and with the unordered eigenvalue map, in the normal case. In this paper, we are interested in the continuity properties of the characteristic map.

The characteristic map is not continuous with respect to the Lipschitz topology (on the eigenvalues), but it is continuous with respect to the Sobolev $W^{1,q}$ topology, for every $1 \le q < \infty$. Specifically, we prove that for Hermitian matrices the characteristic map is continuous, but not uniformly continuous, from $W^{1,q}$ matrices to ordered $W^{1,q}$ eigenvalues, while for normal matrices it is continuous from $W^{1,q}$ matrices to unordered $W^{1,q}$ eigenvalues — in both cases, for every $1 \le q < \infty$. In the latter setting, the unordered eigenvalues are naturally treated as multivalued Sobolev functions in the sense of Almgren [Alm00]; see also [DLS11].

These results are considerably stronger than the related continuity properties of the solution map for hyperbolic and general polynomials established, respectively, in the recent papers [PR24a] and [PR24b]. The solution map is defined in analogy to the characteristic map. Recall that Bronshtein's theorem [Bro79] implies that the ordered roots of a $C^{d-1,1}$ family of hyperbolic (i.e. monic real-rooted) polynomials of degree d are locally Lipschitz; explicit bounds for the Lipschitz constants of the roots in terms of the $C^{d-1,1}$ norms of the coefficients were obtained by [PR15]. By [PR24a], the solution map for hyperbolic polynomials of degree d is continuous from C^d coefficients to $C^{0,1}$ roots with respect to the $W^{1,q}_{\text{loc}}$ topology on the roots, for each $1 \le q < \infty$ — but not with respect to the $C^{0,1}$ topology. For general complex polynomials, the optimal Sobolev regularity $W^{1,q}$, for $1 \le q < \frac{d}{d-1}$, of the roots was established in [PR16, PR18, PR20a], and the existence and boundedness of the solution map rests on these results. As shown in [PR24b], the solution map is then continuous from C^d coefficients to unordered $W^{1,q}$ roots, for each $1 \le q < \frac{d}{d-1}$.

Two further points underline the superiority of the matrix setting. First, unlike the polynomial case, the size of the matrices plays no role. Second, the Sobolev spaces $W^{1,q}$ of Hermitian matrices and their eigenvalues are defined globally, whereas for hyperbolic polynomials the spaces of coefficients and roots are necessarily local.

These results also have concrete applications. We discuss several, including singular values, the condition number for matrices, the surface area of eigenvalue graphs, and compact self-adjoint operators in Hilbert space (made possible by independence of dimension).

It should be mentioned that the orthonormal eigenvectors do not share the regularity of the eigenvalues and, in general, fail to be even continuous in this setting.

In addition to Weyl's perturbation theorem and its relatives, a key ingredient in the proofs is a splitting lemma for complex triangular matrices due to [PR20], where it appears in a slightly different context. It provides a local uniform unitary block-diagonalization of Hermitian and normal matrices whenever not all eigenvalues coincide. This suggests an inductive argument on the size of the matrices and has the advantage of working directly with the matrices themselves rather than through their characteristic polynomials.

In the following, we present our results in more detail.

1.1. Hermitian matrices.

Let $M_d(\mathbb{C})$ be the complex vector space of complex $d \times d$ matrices. We endow $M_d(\mathbb{C})$ with the Frobenius norm $\|A\|_2 = (\sum_{i,j=1}^d |a_{ij}|^2)^{1/2}$.

Let $\mathrm{Herm}(d) = \{A \in M_d(\mathbb{C}) : A^* = A\}$ denote the real vector space of complex Hermitian $d \times d$ matrices. We associate with $A \in \mathrm{Herm}(d)$ its increasingly ordered eigenvalues

$$\lambda_1^\uparrow(A) \le \lambda_2^\uparrow(A) \le \cdots \le \lambda_d^\uparrow(A)$$

and thus obtain a continuous map

$$\lambda^\uparrow = (\lambda_1^\uparrow, \ldots, \lambda_d^\uparrow) : \mathrm{Herm}(d) \to \mathbb{R}^d. \tag{1.1}$$

In fact, by a result of Löwner [Löw34] (see Proposition 3.2), this map is even Lipschitz continuous: for $A, B \in \mathrm{Herm}(d)$,

$$\|\lambda^\uparrow(A) - \lambda^\uparrow(B)\|_2 \le \|A - B\|_2. \tag{1.2}$$

As a consequence (see Proposition 3.5), the map (1.1) induces a bounded map

$$\mathcal{E} := (\lambda^\uparrow)_* : W^{1,q}(U, \mathrm{Herm}(d)) \to W^{1,q}(U, \mathbb{R}^d), \quad A \mapsto \lambda^\uparrow \circ A, \tag{1.3}$$

for each $1 \le q < \infty$, which takes $W^{1,q}$ families of Hermitian matrices to their increasingly ordered $W^{1,q}$ eigenvalues, where $U \subseteq \mathbb{R}^m$ is open and bounded. (This also true for $q = \infty$ but this case will be discussed separately below.)

We will call $\mathcal{E} = (\lambda^\uparrow)_*$ the *characteristic map*, independently of the domain and codomain which may vary in different contexts.

We will show that the characteristic map (1.3) is continuous.

Theorem 1.1. *Let $1 \le q < \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. Then the characteristic map*

$$\mathcal{E} : W^{1,q}(U, \mathrm{Herm}(d)) \to W^{1,q}(U, \mathbb{R}^d), \quad A \mapsto \lambda^\uparrow \circ A,$$

is continuous.

Theorem 1.1 will be proved in Section 5 for $m = 1$ and the general case will be concluded from this special case in Section 6.

Remark 1.2. Note that superposition by any Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ induces a continuous map $f_* : W^{1,q}(U) \to W^{1,q}(U)$ for every $1 \le q < \infty$, where $U \subseteq \mathbb{R}^m$ is open and bounded, by [MM79]. However, there are Lipschitz maps $f : \mathbb{R}^k \to \mathbb{R}^\ell$, with $k \ge 2$, such that the induced map $f_* : W^{1,q}(U, \mathbb{R}^k) \to W^{1,q}(U, \mathbb{R}^\ell)$ is not continuous; see [Mus91], where also sufficient conditions for continuity are given. But our proof of Theorem 1.1 will follow a different strategy.

Remark 1.3. The image of the characteristic map is

$$\mathcal{E}(W^{1,q}(U, \mathrm{Herm}(d))) = W^{1,q}(U, \mathbb{R}^d_\uparrow),$$

the space of $f \in W^{1,q}(U, \mathbb{R}^d)$ with $f(U) \subseteq \mathbb{R}^d_\uparrow := \{x \in \mathbb{R}^d : x_1 \le \cdots \le x_d\}$. The continuous surjection $\mathcal{E} : W^{1,q}(U, \mathrm{Herm}(d)) \to W^{1,q}(U, \mathbb{R}^d_\uparrow)$ admits a continuous right-inverse which takes $\lambda = (\lambda_1, \ldots, \lambda_d) \in W^{1,q}(U, \mathbb{R}^d_\uparrow)$ to the diagonal matrix with the diagonal entries $\lambda_1, \ldots, \lambda_d$.

We state a simple consequence of Theorem 1.1. Here $\|f\|_{L^q(U,\mathbb{R}^d)} := \big\| \|f\|_2 \big\|_{L^q(U)}$, see Section 1.7 for notation.

Corollary 1.4. *Let $1 \le q < \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. If $A_n \to A$ in $W^{1,q}(U, \mathrm{Herm}(d))$ as $n \to \infty$, then, for all $1 \le j \le m$,*

$$\big\| \|\partial_j(\mathcal{E}(A))\|_2 - \|\partial_j(\mathcal{E}(A_n))\|_2 \big\|_{L^q(U)} \to 0$$

and

$$\|\partial_j(\mathcal{E}(A_n))\|_{L^q(U,\mathbb{R}^d)} \to \|\partial_j(\mathcal{E}(A))\|_{L^q(U,\mathbb{R}^d)}$$

as $n \to \infty$.

Proof. Let us set $\lambda := \mathcal{E}(A)$ and $\lambda_n := \mathcal{E}(A_n)$. Then

$$\begin{aligned} \big| \|\partial_j\lambda\|_{L^q(U,\mathbb{R}^d)} - \|\partial_j\lambda_n\|_{L^q(U,\mathbb{R}^d)} \big| &= \Big| \big\| \|\partial_j\lambda\|_2 \big\|_{L^q(U)} - \big\| \|\partial_j\lambda_n\|_2 \big\|_{L^q(U)} \Big| \\ &\le \big\| \|\partial_j\lambda\|_2 - \|\partial_j\lambda_n\|_2 \big\|_{L^q(U)} \le \big\| \|\partial_j\lambda - \partial_j\lambda_n\|_2 \big\|_{L^q(U)} = \|\partial_j\lambda - \partial_j\lambda_n\|_{L^q(U,\mathbb{R}^d)} \end{aligned}$$

so that the assertions follow from Theorem 1.1. □

Let us now turn to the Lipschitz case (i.e., $q = \infty$). Clearly, (1.2) also induces a bounded map

$$\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,1}(\overline{U}, \mathbb{R}^d), \quad A \mapsto \lambda^\uparrow \circ A,$$

taking Lipschitz families of Hermitian matrices to their increasingly ordered Lipschitz eigenvalues, with Lipschitz constants satisfying

$$|\mathcal{E}(A)|_{C^{0,1}(\overline{U},\mathbb{R}^d)} \le |A|_{C^{0,1}(\overline{U},M_d(\mathbb{C}))}, \tag{1.4}$$

with respect to the 2-norms on $\mathrm{Herm}(d)$ and $\mathbb{R}^d$.

As an immediate consequence of Theorem 1.1, we get the following theorem which answers Question 1.11 in [PR24a].

Theorem 1.5. *Let $U \subseteq \mathbb{R}^m$ be open and bounded. Then the characteristic map*

$$\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,1}_q(\overline{U}, \mathbb{R}^d), \quad A \mapsto \lambda^{\uparrow} \circ A,$$

is continuous, for all $1 \le q < \infty$, where $C^{0,1}_q(\overline{U}, \mathbb{R}^d)$ denotes the set $C^{0,1}(\overline{U}, \mathbb{R}^d)$ equipped with the trace topology of the inclusion $C^{0,1}(\overline{U}, \mathbb{R}^d) \to W^{1,q}(U, \mathbb{R}^d)$.

But we will see in Example 7.1 that the map $\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,1}(\overline{U}, \mathbb{R}^d)$ is *not* continuous: the natural topology on the target $C^{0,1}(\overline{U}, \mathbb{R}^d)$ is too strong.

As a corollary of Theorem 1.5, we find that $\mathcal{E}$ is continuous into the Hölder space $C^{0,\alpha}(\overline{U}, \mathbb{R}^d)$, carrying its natural topology, for all $0 < \alpha < 1$.

Corollary 1.6. *Let $U \subseteq \mathbb{R}^m$ be a bounded open Lipschitz domain. Then the characteristic map*

$$\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,\alpha}(\overline{U}, \mathbb{R}^d), \quad A \mapsto \lambda^{\uparrow} \circ A,$$

is continuous, for all $0 < \alpha < 1$, but not for $\alpha = 1$.

In Corollary 1.6, which will be proved in Section 6, U is a Lipschitz domain, since we use Morrey's inequality.

In the setup of Theorem 1.5, we will actually prove a stronger result, in Section 5 (for $m = 1$) and Section 6:

Theorem 1.7. *Let $U \subseteq \mathbb{R}^m$ be an open set. Let $A_n \to A$ in $C^{0,1}(U, \mathrm{Herm}(d))$ as $n \to \infty$. Then, for each $1 \le j \le m$ and almost every $x \in U$,*

$$\partial_j(\mathcal{E}(A_n))(x) \to \partial_j(\mathcal{E}(A))(x) \quad \text{as } n \to \infty. \tag{1.5}$$

As a consequence, we obtain a second proof of Theorem 1.5 based on the dominated convergence theorem; see Theorem 6.2. Clearly, (1.5) generally fails, if we only assume $A_n \to A$ in $W^{1,q}(U, \mathrm{Herm}(d))$ for some $1 \le q < \infty$ as $n \to \infty$.

By Egorov's theorem [Ego11], we may conclude that $\partial_j(\mathcal{E}(A_n)) \to \partial_j(\mathcal{E}(A))$ almost uniformly on U as $n \to \infty$. In general, the convergence is not uniform on U; see Example 7.1.

1.2. **Normal matrices.** The complex normal $d \times d$ matrices form a real algebraic subset

$$\mathrm{Norm}(d) := \{A \in M_d(\mathbb{C}) : A^*A = AA^*\}$$

of the vectorspace $M_d(\mathbb{C})$ of all complex $d \times d$ matrices. The set $\mathrm{Norm}(d)$ is a real singular cone; see [Huh01] for background on its geometry.

With $A \in \mathrm{Norm}(d)$ we associate its d eigenvalues $\lambda_1(A), \ldots, \lambda_d(A)$ (with multiplicities) and the unordered eigenvalue vector

$$\Lambda(A) = [\lambda_1(A), \ldots, \lambda_d(A)].$$

In this way, we obtain a map

$$\Lambda : \mathrm{Norm}(d) \to \mathcal{A}_d(\mathbb{C}), \tag{1.6}$$

where $\mathcal{A}_d(\mathbb{C})$ is the space of unordered complex d-tuples (see Section 8.1), a complete metric space with respect to the metric

$$\mathbf{d}_2([z],[w]) := \min_{\sigma \in \mathrm{S}_d} \Big(\sum_{i=1}^d |z_i - w_{\sigma(i)}|^2 \Big)^{1/2},$$

where S_d is the symmetric group. The map (1.6) is Lipschitz continuous, by a result of Hoffman and Wielandt [HW53] (see Proposition 3.3): for $A, B \in \mathrm{Norm}(d)$,

$$\mathbf{d}_2(\Lambda(A), \Lambda(B)) \le \|A - B\|_2. \tag{1.7}$$

Due to Almgren [Alm00], there exists a bi-Lipschitz embedding $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$, where $N = N(d)$, which can be used to define Sobolev spaces of $\mathcal{A}_d(\mathbb{C})$-valued functions: for open $U \subseteq \mathbb{R}^m$ and $1 \le q \le \infty$, we set

$$W^{1,q}(U, \mathcal{A}_d(\mathbb{C})) := \{ f : U \to \mathcal{A}_d(\mathbb{C}) : \Delta \circ f \in W^{1,q}(U, \mathbb{R}^N) \}.$$

An equivalent intrinsic definition of $W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$ is due to De Lellis and Spadaro [DLS11] (see also Definition 8.4). The space $W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$ carries the metric

$$\rho^{1,q}_\Delta(f,g) := \|\Delta \circ f - \Delta \circ g\|_{W^{1,q}(U,\mathbb{R}^N)}$$

which turns it into a complete metric space; see [PR24b, Lemma 3.1]. The topology induced by this metric does not depend on the choice of the Almgren embedding Δ, see Theorem 8.11.

The map (1.6) induces a bounded map

$$\mathcal{E}_u := \Lambda_* : W^{1,q}(U, \mathrm{Norm}(d)) \to W^{1,q}(U, \mathcal{A}_d(\mathbb{C})), \quad A \mapsto \Lambda \circ A,$$

for each $1 \le q < \infty$, where $U \subseteq \mathbb{R}^m$ is a bounded open set, see Proposition 9.1. (The Lipschitz case $q = \infty$ is discussed below.) By $W^{1,q}(U, \mathrm{Norm}(d))$ we mean the (nonlinear) space of all $A \in W^{1,q}(U, M_d(\mathbb{C}))$ such that $A(U) \subseteq \mathrm{Norm}(d)$, topologized by its inclusion in $W^{1,q}(U, M_d(\mathbb{C}))$.

In this setup, we call $\mathcal{E}_u = \Lambda_*$ the *characteristic map.*

Theorem 1.8. *Let $1 \le q < \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. Then the characteristic map*

$$\mathcal{E}_u : W^{1,q}(U, \mathrm{Norm}(d)) \to W^{1,q}(U, \mathcal{A}_d(\mathbb{C})), \quad A \mapsto \Lambda \circ A,$$

is continuous.

It turns out that Theorem 1.8 is a generalization of Theorem 1.1; see Section 8.4.

In the following corollary of Theorem 1.8, $|\dot\Lambda|$ denotes the metric speed and $\mathscr{E}_q(\Lambda)$ the q-energy of the curve $\Lambda \in W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$; see Section 2.4 for definitions.

Corollary 1.9. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Norm}(d))$ as $n \to \infty$ and set $\Lambda := \mathcal{E}_u(A), \Lambda_n := \mathcal{E}_u(A_n) : I \to \mathcal{A}_d(\mathbb{C})$. Then*

$$\|\mathbf{d}_2(\Lambda, \Lambda_n)\|_{L^\infty(I)} \to 0,$$
$$\big\| |\dot\Lambda| - |\dot\Lambda_n| \big\|_{L^q(I)} \to 0,$$
$$\big| \mathscr{E}_q(\Lambda) - \mathscr{E}_q(\Lambda_n) \big| \to 0,$$

as $n \to \infty$.

In the setup of Corollary 1.9, there always exist $W^{1,q}$ parameterizations $\lambda, \lambda_n : I \to \mathbb{C}^d$ of the eigenvalues of A, A_n, respectively; see Proposition 8.5. We will see that Corollary 1.9 follows from the following corollary; for their proofs see Section 10.4.

Corollary 1.10. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Norm}(d))$ as $n \to \infty$. Let $\lambda, \lambda_n \in W^{1,q}(I, \mathbb{C}^d)$ be parameterizations of the eigenvalues of A, A_n, respectively. Then*

$$\big\| \|\lambda'\|_2 - \|\lambda_n'\|_2 \big\|_{L^q(I)} \to 0,$$
$$\|\lambda_n'\|_{L^q(I,\mathbb{C}^d)} \to \|\lambda'\|_{L^q(I,\mathbb{C}^d)},$$

as $n \to \infty$.

See also Theorem 1.13, Theorem 1.14, and (1.8) below.

Let us now consider the Lipschitz case ($q = \infty$). We immediately get from (1.7) that, for all $A, B \in \mathrm{Norm}(d)$,

$$\|\mathbf{d}_2(\mathcal{E}_u(A), \mathcal{E}_u(B))\|_{L^\infty(U)} \le \|A - B\|_{L^\infty(U, M_d(\mathbb{C}))},$$

But, as evidenced by the Hermitian case, the (by (1.7) induced) bounded map

$$\mathcal{E}_u : C^{0,1}(\overline{U}, \mathrm{Norm}(d)) \to C^{0,1}(\overline{U}, \mathcal{A}_d(\mathbb{C}))$$

is not continuous. Nevertheless, as an immediate consequence of Theorem 1.8, we have the following result.

Theorem 1.11. *Let $U \subseteq \mathbb{R}^m$ be open and bounded. The characteristic map*

$$\mathcal{E}_u : C^{0,1}(\overline{U}, \mathrm{Norm}(d)) \to W^{1,q}(U, \mathcal{A}_d(\mathbb{C})), \quad A \mapsto \Lambda \circ A,$$

is continuous, for all $1 \le q < \infty$.

Clearly, $f \in C^{0,\alpha}(\overline{U}, \mathcal{A}_d(\mathbb{C}))$ if and only if $\Delta \circ f \in C^{0,\alpha}(\overline{U}, \mathbb{R}^N)$ which makes the following corollary meaningful; see Section 11 for its proof.

Corollary 1.12. *Let $U \subseteq \mathbb{R}^m$ be a bounded open Lipschitz domain. If $A_n \to A$ in $C^{0,1}(\overline{U}, \mathrm{Norm}(d))$, then $\mathcal{E}_u(A_n) \to \mathcal{E}_u(A)$ in $C^{0,\alpha}(\overline{U}, \mathcal{A}_d(\mathbb{C}))$ as $n \to \infty$, for all $0 < \alpha < 1$, but not for $\alpha = 1$, in the sense that*

$$\|\Delta \circ \mathcal{E}_u(A) - \Delta \circ \mathcal{E}_u(A_n)\|_{C^{0,\alpha}(\overline{U}, \mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty,$$

for each Almgren embedding $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$.

We have the following variants of Theorem 1.8 and Theorem 1.11.

Theorem 1.13. *Let $1 \le q < \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. Let $A_n \to A$ in $W^{1,q}(U, \mathrm{Norm}(d))$ as $n \to \infty$. Assume that $\lambda, \lambda_n \in W^{1,q}(U, \mathbb{C}^d)$ are parameterizations of the eigenvalues of A, A_n, respectively. Assume that $\lambda_n \to \lambda$ in L^∞ on $\mathcal{L}^{m-1}$-almost all line segments in U parallel to the coordinate axes. Then $\partial_j \lambda_n \to \partial_j \lambda$ in $L^q(U, \mathbb{C}^d)$ as $n \to \infty$, for all $1 \le j \le m$.*

Theorem 1.14. *Let $U \subseteq \mathbb{R}^m$ be open and bounded. Let $A_n \to A$ in $C^{0,1}(\overline{U}, \mathrm{Norm}(d))$ as $n \to \infty$. Assume that $\lambda, \lambda_n \in C^{0,1}(\overline{U}, \mathbb{C}^d)$ are parameterizations of the eigenvalues of A, A_n, respectively. Assume that $\lambda_n \to \lambda$ in L^∞ on $\mathcal{L}^{m-1}$-almost all line segments in U parallel to the coordinate axes. Then $\partial_j \lambda_n \to \partial_j \lambda$ almost everywhere in U as $n \to \infty$, for all $1 \le j \le m$.*

Theorem 1.13 and Theorem 1.14 will be proved in Section 10.3 (for $m = 1$) and Section 11. It is evident (see the proof of Corollary 1.4), that in their setup, for all $1 \le j \le m$,

$$\begin{aligned} \big\| \|\partial_j \lambda\|_2 - \|\partial_j \lambda_n\|_2 \big\|_{L^q(U)} &\to 0, \\ \|\partial_j \lambda_n\|_{L^q(U,\mathbb{C}^d)} &\to \|\partial_j \lambda\|_{L^q(U,\mathbb{C}^d)}, \end{aligned} \tag{1.8}$$

as $n \to \infty$. However, if $m \ge 2$ the desired parameterizations λ, λ_n may not always exist. (For $q > m$, the elements of $W^{1,q}(U)$, where $U \subseteq \mathbb{R}^m$, admit continuous representatives, but the eigenvalues of multiparameter families of normal matrices cannot always be represented by continuous functions, e.g.,

$$\begin{pmatrix} 0 & x \\ |x| & 0 \end{pmatrix},$$

for $x \in \mathbb{C}$.)

1.3. **Optimality of the results.** The results on eigenvalue stability obtained in this paper are essentially optimal. Here we briefly mention several examples that demonstrate this fact; these examples will then be discussed in detail in Section 7. We also formulate an open problem.

We will focus on the limitations in the Hermitian case, which clearly entail corresponding negative results in the more general normal case.

(1) *The map $\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,1}(\overline{U}, \mathbb{R}^d)$ is not continuous.*

For instance,

$$A_n(x) = \begin{pmatrix} \frac{1}{n} & x \\ x & -\frac{1}{n} \end{pmatrix} \to \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} = A(x) \quad \text{as } n \to \infty \tag{1.9}$$

uniformly in all derivatives on every compact interval in $\mathbb{R}$. Then $\mathcal{E}(A_n) = (-a_n, a_n)$ and $\mathcal{E}(A) = (-a, a)$, where $a_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$ and $a(x) = |x|$. We will see in Example 7.1 that $|a - a_n|_{C^{0,1}(\overline{I})} \ge 2 - \sqrt{2}$, for all bounded open intervals I containing 0 and all sufficiently large n. Also note that off 0 the derivatives a_n' tend pointwise to a' but not uniformly in any neighborhood of 0.

(2) *For no $1 \le q < \infty$, the map $\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,1}_q(\overline{U}, \mathbb{R}^d)$ is uniformly continuous.*

Indeed, the matrices

$$A_n(x) = \begin{pmatrix} \frac{1}{n} & \varphi_n(x) \\ \varphi_n(x) & -\frac{1}{n} \end{pmatrix}, \quad B_n(x) = \begin{pmatrix} \frac{1}{2n} & \varphi_n(x) \\ \varphi_n(x) & -\frac{1}{2n} \end{pmatrix},$$

where $\varphi_n : [0,1] \to [0, \frac{1}{n}]$ is a sawtooth function with Lipschitz constant 1, satisfy $\|A_n - B_n\|_{C^{0,1}([0,1], M_2(\mathbb{C}))} \to 0$ as $n \to \infty$, but for their nonnegative eigenvalues a_n and b_n we find $\|a_n' - b_n'\|_{L^q([0,1])} \ge \frac{1}{12 \cdot 2^{1/q}}$, in Example 7.2.

(3) *For no $\alpha \in (0,1)$, the map $\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,\alpha}(\overline{U}, \mathbb{R}^d)$ is uniformly continuous.*

This is demonstrated in Example 7.3, by the matrices

$$A_n(x) = \begin{pmatrix} \frac{1}{n^r} & nx \\ nx & -\frac{1}{n^r} \end{pmatrix}, \quad B_n(x) = \begin{pmatrix} \frac{1}{2n^r} & nx \\ nx & -\frac{1}{2n^r} \end{pmatrix},$$

where $r = \frac{\alpha}{1-\alpha}$.

Even tough the maps in (2) and (3) are not uniformly continuous, it is very desirable to find effective moduli of continuity for their restrictions to interesting (for applications) compact subspaces of $C^{0,1}(\overline{U}, \mathrm{Herm}(d))$. For instance, for given $C > 0$ and $\beta \in (0,1]$, the set

$$K := \big\{A \in C^{1,\beta}(\overline{U}, \mathrm{Herm}(d)) : \|A\|_{C^{1,\beta}(\overline{U}, M_d(\mathbb{C}))} \le C\big\}, \tag{1.10}$$

is a relatively compact subset of $C^{0,1}(\overline{U}, \mathrm{Herm}(d))$, by the Arzelà–Ascoli theorem. Thus, $\mathcal{E}|_K : K \to C^{0,1}_q(\overline{U}, \mathbb{R}^d)$, for $1 \le q < \infty$, and $\mathcal{E}|_K : K \to C^{0,\alpha}(\overline{U}, \mathbb{R}^d)$, for $0 < \alpha < 1$, are uniformly continuous, by Theorem 1.5 and Corollary 1.6.

Problem 1.15. *Find effective moduli of continuity for the restrictions of the characteristic map $\mathcal{E}$ to interesting compact subspaces.*

For instance, using the sequence A_n in (1.9) we will see in Example 7.4 that, for $1 \le q < \infty$ and $q^{-1} < \alpha \le 1$, the map $\mathcal{E}|_K : K \to C^{0,1}_q([0,1], \mathbb{R}^d)$ is not α-Hölder continuous.

Remark 1.16. Due to Rellich [Rel37], a real analytic curve of Hermitian matrices admits a real analytic system of eigenvalues and a real analytic orthonormal frame of eigenvectors. There are various extensions of this result, e.g., to normal matrices or to other categories (smooth quasianalytic classes, formal power series), see [AKLM98], [Rai13], and [PR20]. But in general there is no continuous orthonormal frame of the eigenvectors, as seen by the following C^∞ example from [Rel37]:

$$A(x) := e^{-1/x^2} \begin{pmatrix} \cos x^{-1} & \sin x^{-1} \\ \sin x^{-1} & -\cos x^{-1} \end{pmatrix} \qquad \text{for } x \in \mathbb{R} \setminus \{0\}, \quad A(0) := 0.$$

Let us suppose that $I \ni x \mapsto A(x)$ is a smooth curve of Hermitian matrices with smooth distinct eigenvalues λ_j and corresponding smooth orthonormal eigenvectors v_j. Differentiating the equation $A(x)v_j(x) = \lambda_j(x)v_j(x)$ and taking the result in the Hermitian inner product with $v_k \ne v_j$ gives

$$\langle v_k(x), v_j'(x)\rangle = \frac{\langle v_k(x), A'(x)v_j(x)\rangle}{\lambda_j(x) - \lambda_k(x)}$$

showing that the projection of the velocity v_j' to v_k is unbounded as λ_j and λ_k approach each other faster than the numerator tends to zero. This is connected to adiabatic theorems in physics and the quantum Hall effect.

1.4. **Singular values.** Let us consider the vector space $M_{D,d}(\mathbb{C})$ of complex $D \times d$ matrices, where $d \le D$ (without loss of generality). The singular values of $A \in M_{D,d}(\mathbb{C})$ are the nonnegative square roots of the eigenvalues of the Hermitian matrix A^*A, usually ordered decreasingly

$$\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_d(A) \ge 0.$$

This defines a map $\sigma = (\sigma_1, \ldots, \sigma_d) : M_{D,d}(\mathbb{C}) \to \mathbb{R}^d$.

Observing that the Hermitian matrix

$$\mathbf{A} := \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^* & 0 \end{pmatrix},$$

where $\tilde{A}$ is the $D \times D$ matrix resulting from A by adding $D-d$ columns consisting of zeros, has the eigenvalues

$$\sigma_1(A), \ldots, \sigma_d(A), 0, \ldots, 0, -\sigma_d(A), \ldots, -\sigma_1(A),$$

we conclude from (3.3) that, for $A, B \in M_{D,d}(\mathbb{C})$ and $1 \le i \le d$,

$$\begin{aligned}\sqrt{2}\,\|\sigma(A) - \sigma(B)\|_2 \le \|\mathbf{A} - \mathbf{B}\|_2 &= |\operatorname{Tr}\big((\mathbf{A}-\mathbf{B})^*(\mathbf{A}-\mathbf{B})\big)|^{1/2} \\ &= |2 \operatorname{Tr}\big((\tilde{A}-\tilde{B})^*(\tilde{A}-\tilde{B})\big)|^{1/2} = \sqrt{2}\,\|A-B\|_2,\end{aligned}$$

that is,

$$\|\sigma(A) - \sigma(B)\|_2 \le \|A - B\|_2. \tag{1.11}$$

Consequently, for each $1 \le q < \infty$, the map

$$\sigma_* : W^{1,q}(U, M_{D,d}(\mathbb{C})) \to W^{1,q}(U, \mathbb{R}^d), \quad A \mapsto \sigma \circ A,$$

is well-defined and bounded, as well as

$$\sigma_* : C^{0,1}(\overline{U}, M_{D,d}(\mathbb{C})) \to C^{0,1}(\overline{U}, \mathbb{R}^d), \quad A \mapsto \sigma \circ A.$$

Furthermore, Theorem 1.1, Theorem 1.5, Corollary 1.6, and Theorem 1.7 immediately give the following result.

Theorem 1.17. *Let $U \subseteq \mathbb{R}^m$ be open and bounded. Then:*

1. *For each $1 \le q < \infty$, the map $\sigma_* : W^{1,q}(U, M_{D,d}(\mathbb{C})) \to W^{1,q}(U, \mathbb{R}^d)$ is continuous.*
2. *The map $\sigma_* : C^{0,1}(\overline{U}, M_{D,d}(\mathbb{C})) \to C^{0,1}_q(\overline{U}, \mathbb{R}^d)$ is continuous, for all $1 \le q < \infty$.*
3. *Assume additionally that U is a Lipschitz domain. Then the map $\sigma_* : C^{0,1}(\overline{U}, M_{D,d}(\mathbb{C})) \to C^{0,\alpha}(\overline{U}, \mathbb{R}^d)$ is continuous, for all $0 < \alpha < 1$, but not for $\alpha = 1$.*
4. *If $A_n \to A$ in $C^{0,1}(\overline{U}, M_{D,d}(\mathbb{C}))$, then $\partial_j(\sigma_*(A_n))(x) \to \partial_j(\sigma_*(A))(x)$ as $n \to \infty$, for each $1 \le j \le m$ and almost every $x \in U$.*

This theorem answers a question raised in [PR24a, Section 7.5].

1.5. **Applications.** Let us present a few applications of our results.

1.5.1. *Functions defined on the spectrum or the singular values.* Clearly, the stability results for the eigenvalues and singular values have immediate applications, e.g., for the spectral gaps $\lambda_{i+1}(A) - \lambda_i(A)$ or the Ky Fan norms $\sum_{i=1}^k \sigma_i(A)$, for $1 \le k \le d$. This can be extended using a result of [MM79].

Corollary 1.18. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Let $1 \le q < \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. Then the map*

$$W^{1,q}(U, M_{D,d}(\mathbb{C})) \to W^{1,q}(U, \mathbb{R}), \quad A \mapsto f \circ \sigma_i \circ A, \tag{1.12}$$

is well-defined and continuous, where $\sigma_1(A) \ge \cdots \ge \sigma_d(A)$ are the singular values of A. The statement remains true if $M_{D,d}(\mathbb{C})$ is replaced by $\operatorname{Herm}(d)$ *and the ordered eigenvalues are used instead of the singular values.*

Proof. The map (1.12) is the composite

$$W^{1,q}(U, M_{D,d}(\mathbb{C})) \xrightarrow{(\sigma_i)_*} W^{1,q}(U, \mathbb{R}) \xrightarrow{f_*} W^{1,q}(U, \mathbb{R}).$$

The first map is continuous by Theorem 1.17, the second by [MM79]. For Hermitian matrices use Theorem 1.1 instead of Theorem 1.17. □

For instance:

(1) The p-th power of the Schatten p-norm $\sum_{i=1}^{d} \sigma_i(A)^p$, for $1 \le p < \infty$, induces a continuous map $W^{1,q}(U, M_{D,d}(\mathbb{C})) \to W^{1,q}(U, \mathbb{R})$.

(2) If $A \in W^{1,q}(U, \mathrm{Herm}(d))$ is a family of density matrices (i.e. $\mathrm{Tr}(A) = 1$) such that $\inf_{x \in U} \min_{1 \le i \le d} \lambda_i(A(x)) > 0$, then the von Neumann entropy $-\sum_{i=1}^{d} \lambda_i(A) \log \lambda_i(A)$ belongs to $W^{1,q}(U)$ and varies continuously in A.

(3) In the setup of (2), the Rényi entropy $\frac{1}{1-\alpha} \log \left(\sum_{i=1}^{d} \lambda_i(A)^\alpha \right)$, for $\alpha > 1$, belongs to $W^{1,q}(U)$ and varies continuously in A.

1.5.2. *Condition numbers.* Let $A \in M_d(\mathbb{C})$ and let $\sigma_1(A) \ge \cdots \ge \sigma_d(A)$ be the singular values of A. In numerical analysis, the *condition number* of A is defined by

$$\kappa(A) := \frac{\sigma_1(A)}{\sigma_d(A)}.$$

Considering the linear equation $Ax = b$ and assuming that A is nonsingular (i.e., $\sigma_d(A) > 0$), $\kappa(A)$ describes the maximum ratio of the relative error in x to the relative error in b. If $\kappa(A)$ is much larger than 1, then the problem is considered to be ill-conditioned.

Theorem 1.19. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Suppose that $A_0 \in W^{1,q}(I, M_d(\mathbb{C}))$ and $\inf_{x \in I} \sigma_d(A_0(x)) > 0$. Then $\kappa(A) \in W^{1,q}(I, \mathbb{R})$ is well-defined in a neighborhood $\mathcal{U}$ of A_0 in $W^{1,q}(I, M_d(\mathbb{C}))$ and $\kappa : \mathcal{U} \to W^{1,q}(I, \mathbb{R})$ is continuous.*

Theorem 1.19 will be proved in Section 5.4.

1.5.3. *Surface area.* Let $U \subseteq \mathbb{R}^m$ be a bounded open set and $f \in W^{1,1}(U, \mathbb{R})$. Then, by the area formula, the surface area of the graph of f is given by

$$\mathrm{Area}(f) := \mathcal{H}^m(\overline{f}(U)),$$

where $\overline{f} : U \to U \times \mathbb{R}$, $x \mapsto (x, f(x))$ is the graph mapping and $\mathcal{H}^m$ the m-dimensional Hausdorff measure; see [MSZ03].

Theorem 1.20. *Let $U \subseteq \mathbb{R}^m$ be open and bounded. Suppose that $A_n \to A$ in $C^{0,1}(\overline{U}, \mathrm{Herm}(d))$ as $n \to \infty$. Set $\lambda_j := \mathcal{E}(A)_j$ and $\lambda_{n,j} := \mathcal{E}(A_n)_j$, for $1 \le j \le d$. Then:*

(1) *For $1 \le j \le d$,*

$$\mathrm{Area}(\overline{\lambda}_{n,j}) \to \mathrm{Area}(\overline{\lambda}_j) \quad \text{as } n \to \infty.$$

(2) *Let* $Z := \bigcup_{j=1}^d \overline{\lambda}_j(U) \subseteq U \times \mathbb{R}$*, resp.* $Z_n := \bigcup_{j=1}^d \overline{\lambda}_{n,j}(U)$*, be the total zero set of the characteristic polynomial of* A*, resp.* A_n*. Then*

$$\liminf_{n\to\infty} \mathcal{H}^m(Z_n) \ge \mathcal{H}^m(Z).$$

Proof. This follows from Theorem 1.5 and Theorem 1.7 combined with the proof of Corollary 7.7 and Corollary 7.8 in [PR24a]. (Use (1.2) instead of Bronshtein's theorem.) □

1.5.4. *Compact self-adjoint operators.* The spectrum of a compact self-adjoint nonnegative operator A in Hilbert space is a countable set consisting of eigenvalues of finite multiplicities which we order decreasingly

$$\lambda_1(A) \ge \lambda_2(A) \ge \cdots$$

and possibly zero. The eigenvalues $\lambda_i(A)$ accumulate (only) at zero.

Theorem 1.21. *Let* $U \subseteq \mathbb{R}^m$ *be a bounded open set and* H *a Hilbert space. Suppose that* $A, A_n \in C^{0,1}(\overline{U}, K(H))$*, for* $n \ge 1$*, are Lipschitz families of compact self-adjoint nonnegative operators (w.r.t. the operator norm on the space* $K(H)$ *of compact operators on* H*). Assume that* $A(x)$ *is positive definite for all* $x \in U$ *and* $A_n \to A$ *in* $C^{0,1}(\overline{U}, K(H))$ *as* $n \to \infty$*. Then:*

(1) *The decreasingly ordered eigenvalues* $\lambda_i(A)$ *and* $\lambda_i(A_n)$*, for all* i *and large enough* n*, belong to* $C^{0,1}(\overline{U})$*.*

(2) *For all* $1 \le j \le m$ *and almost every* $x \in U$*,*

$$\lim_{n\to\infty} \partial_j(\lambda_i(A_n))(x) = \partial_j(\lambda_i(A))(x), \quad i = 1, 2, \ldots$$

(3) *For every* $1 \le q < \infty$*,*

$$\lim_{n\to\infty} \|\lambda_i(A) - \lambda_i(A_n)\|_{W^{1,q}(U)} = 0, \quad i = 1, 2, \ldots$$

This theorem will be proved in Section 12.

1.6. **Structure of the paper.** We separate the Hermitian and the normal case in a large part of the proofs, because the normal case requires more machinery for unordered eigenvalue maps and multivalued Sobolev functions. The reader only interested in the Hermitian case can safely skip Sections 8–11.

In Section 2, we introduce the relevant function spaces and discuss some tools that are used frequently in the paper. In Section 3, we recall matrix norms and results on the spectral variation of Hermitian and normal matrices. As a consequence, we introduce the characteristic map for Hermitian matrices. Section 4 is devoted to the local uniform unitary block-diagonalization of Hermitian and normal matrices and to related pointwise bounds for the derivatives of absolutely continuous curves of such matrices. In this section, we treat the Hermitian and normal case simultaneously.

The proofs of the main results in the Hermitian case are completed in Section 5 and Section 6. We treat the one-parameter cases in Section 5 and then deduce the multiparameter cases by sectioning arguments in Section 6. In Section 7, we discuss four examples that show optimality of our results.

For the treatment of the normal case, we provide in Section 8 the necessary background on the metric space of unordered d-tuples of complex numbers and on Sobolev maps with values in this space. This allows us to introduce the characteristic map for normal matrices in Section 9. Then we complete the proofs of the main results for normal matrices in Section 10 (one-parameter case) and Section 11 (multiparameter case).

The final Section 12 is dedicated to an application of some of our results to compact self-adjoint operators in Hilbert space.

1.7. **Notation.** The m-dimensional Lebesgue measure in $\mathbb{R}^m$ is denoted by $\mathcal{L}^m$. If not stated otherwise, 'measurable' means 'Lebesgue measurable' and 'almost everywhere' means 'almost everywhere with respect to Lebesgue measure'. For measurable $E \subseteq \mathbb{R}^m$, we usually write $|E| = \mathcal{L}^m(E)$. We will also use the k-dimensional Hausdorff measure $\mathcal{H}^k$.

For $1 \le p \le \infty$, $\|x\|_p$ denotes the p-norm of $x \in \mathbb{R}^d$. If $f : E \to \mathbb{R}^d$, for measurable $E \subseteq \mathbb{R}^m$, is a measurable map, then we set

$$\|f\|_{L^p(E,\mathbb{R}^d)} := \big\|\|f\|_2\big\|_{L^p(E)}.$$

In the following, a set is called *countable* if it is either finite or has the cardinality of $\mathbb{N}$.

We consider the standard action of the symmetric group S_d on $\mathbb{C}^d$ by permuting the coordinates: for $\sigma \in \mathrm{S}_d$ and $z \in \mathbb{C}^d$,

$$\sigma z = \sigma(z_1, \ldots, z_d) = (z_{\sigma(1)}, \ldots, z_{\sigma(d)}).$$

We write $X^{<2>} = \{(x, y) \in X \times X : x \neq y\}$ for the cartesian product with the diagonal removed.

We use the notation $C(d, \ldots)$ to denote a constant that depends only on $d, \ldots$; its value may change from line to line.

2. Function spaces

Let us fix notation and recall background on the function spaces used in this paper.

2.1. **Lebesgue spaces.** Let $U \subseteq \mathbb{R}^m$ be open and $1 \le q \le \infty$. We denote by $L^q(U)$ the Lebesgue space with respect to the m-dimensional Lebesgue measure $\mathcal{L}^m$, and $\|\cdot\|_{L^q(U)}$ is the corresponding L^q-norm. We will also use the space $L^q_{\mathrm{loc}}(U)$ of measurable functions $f : U \to \mathbb{R}$ satisfying $\|f\|_{L^q(K)} < \infty$ for all compact subsets $K \subseteq U$. For Lebesgue measurable sets $E \subseteq \mathbb{R}^m$ we also write $|E| = \mathcal{L}^m(E)$. We remark that for continuous functions $f : U \to \mathbb{R}$ we have (and use interchangeably) $\|f\|_{L^\infty(U)} = \|f\|_{C^0(\overline{U})}$.

A map $f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ is measurable if and only if each f_i is measurable. Now f belongs to $L^q(U, \mathbb{R}^n)$ if $f_i \in L^q(U)$ for all $1 \le i \le n$, or equivalently

$$\|f\|_{L^q(U,\mathbb{R}^n)} := \big\|\|f\|_2\big\|_{L^q(U)} < \infty.$$

2.2. **Sobolev spaces.** For $1 \le q \le \infty$, we consider the Sobolev space

$$W^{1,q}(U) := \{f \in L^q(U) : \partial_i f \in L^q(U) \text{ for } 1 \le i \le m\},$$

where $\partial_i f$ are distributional derivatives, endowed with the norm

$$\|f\|_{W^{1,q}(U)} := \|f\|_{L^q(U)} + \sum_{i=1}^m \|\partial_i f\|_{L^q(U)}.$$

A map $f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ belongs to $W^{1,q}(U, \mathbb{R}^n)$ if $f_j \in W^{1,q}(U)$ for all $1 \le j \le m$, or equivalently,

$$\|f\|_{W^{1,q}(U,\mathbb{R}^n)} := \|f\|_{L^q(U,\mathbb{R}^n)} + \sum_{i=1}^m \|\partial_i f\|_{L^q(U,\mathbb{R}^n)} < \infty.$$

Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $1 \le q < \infty$. A function $f : I \to \mathbb{R}$ belongs to $W^{1,q}(I)$ if and only if f has an absolutely continuous representative whose classical derivative belongs to $L^q(I)$. Moreover, $f \in W^{1,\infty}(I)$ if and only if f has a Lipschitz continuous representative.

Let $U \subseteq \mathbb{R}^m$ be bounded and open. Let $1 \le q < \infty$. A function $u \in L^q(U)$ belongs to $W^{1,q}(U)$ if and only if it has a representative that is absolutely continuous on $\mathcal{L}^{m-1}$-almost all line segments in U that are parallel to the coordinate axes and whose classical first-order partial derivatives belong to $L^q(U)$. They coincide $\mathcal{L}^m$-almost everywhere with the weak derivatives of f. If U is a Lipschitz domain, then $f \in W^{1,\infty}(U)$ if and only if f has a Lipschitz continuous representative.

2.3. **Lipschitz and Hölder spaces.** Let (M_i, d_i), $i = 1, 2$, be metric spaces. A map $f : M_1 \to M_2$ is *α-Hölder* continuous, for $\alpha \in (0, 1]$, if

$$|f|_{C^{0,\alpha}(M_1,M_2)} := \sup_{x \ne y \in M_1} \frac{\mathsf{d}_2(f(x), f(y))}{\mathsf{d}_1(x, y)^\alpha} < \infty.$$

In the case $\alpha = 1$, f is called *Lipschitz* continuous and we sometimes also write $\operatorname{Lip}(f)$ for the Lipschitz constant $|f|_{C^{0,1}(M_1,M_2)}$.

Most often, M_1 will be a bounded open set $U \subseteq \mathbb{R}^m$ and $M_2 = \mathbb{R}^n$, both equipped with the metrics induced by the 2-norm. Then, for $\alpha \in (0, 1]$, we consider the Banach space $C^{0,\alpha}(\overline{U}, \mathbb{R}^n)$ consisting of continuous maps $f : U \to \mathbb{R}^n$ that extend continuously to $\overline{U}$ with the norm

$$\|f\|_{C^{0,\alpha}(\overline{U},\mathbb{R}^n)} := \sup_{x \in U} \|f(x)\|_2 + |f|_{C^{0,\alpha}(\overline{U},\mathbb{R}^n)} < \infty.$$

We write $C^{0,\alpha}(U, \mathbb{R}^n)$ for the space of continuous functions on U that belong to $C^{0,\alpha}(\overline{V}, \mathbb{R}^n)$ for each relatively compact open $V \Subset U$, and endow $C^{0,\alpha}(U, \mathbb{R}^n)$ with its natural Fréchet topology.

2.4. **Absolutely continuous curves in a metric space.** Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $1 \le q \le \infty$. A curve $\gamma : I \to X$ in a complete metric space (X, d) belongs to $AC^q(I, X)$ if there exists $m \in L^q(I)$ such that

$$\mathsf{d}(\gamma(x), \gamma(y)) \le \int_x^y m(t)\, dt, \qquad \text{for all } x, y \in I,\ x \le y. \tag{2.1}$$

In that case, the limit

$$\lim_{h\to 0}\frac{\mathsf{d}(\gamma(x+h),\gamma(x))}{|h|}=:|\dot\gamma|(x)$$

exists for almost every $x\in I$ and is called the *metric speed* of γ at x. Furthermore, $|\dot\gamma|\in L^q(I)$ and (2.1) holds with m replaced by $|\dot\gamma|$; one has $|\dot\gamma|\le m$ almost everywhere in I for every m that satisfies (2.1). See [AGS08, Definition 1.1.1].

For $1\le q<\infty$, the q-*energy* $\mathscr{E}_q(\gamma)$ of $\gamma\in AC^q(I,X)$ is defined by

$$\mathscr{E}_q(\gamma):=\int_I(|\dot\gamma|(t))^q\,dt.$$

We also write AC for AC^1.

2.5. **Vitali's convergence theorem.** Let $(X,\mathcal{A},\mu)$ be a measure space with nonnegative measure μ (finite or with values in $[0,\infty]$). A set of functions $\mathcal{F}\subseteq L^1(\mu)$ is called *uniformly integrable* if

$$\lim_{C\to+\infty}\sup_{f\in\mathcal{F}}\int_{|f|>C}|f|\,d\mu=0.$$

Theorem 2.1 (De la Vallée Poussin's criterion [Bog07, Theorem 4.5.9])**.** *Let μ be a finite nonnegative measure. A family $\mathcal{F}$ of μ-integrable functions is uniformly integrable if and only if there exists a nonnegative increasing function G on $[0,\infty)$ such that*

$$\lim_{t\to+\infty}\frac{G(t)}{t}=\infty\qquad\text{and}\qquad\sup_{f\in\mathcal{F}}\int G(|f(x)|)\,\mu(dx)<\infty.$$

In such a case, one can choose a convex increasing function G.

Recall that a sequence of complex valued measurable functions f_n on X is said to *converge in measure* to f if, for all $\epsilon>0$,

$$\mu(\{x\in X:|f(x)-f_n(x)|\ge\epsilon\})\to 0\qquad\text{as } n\to\infty.$$

Theorem 2.2 (Vitali's convergence theorem [Bog07, Theorem 4.5.4])**.** *Let μ be a finite nonnegative measure. Suppose that f is a μ-measurable function and $\{f_n\}$ is a sequence of μ-integrable functions. Then the following assertions are equivalent:*

(1) *$f_n\to f$ in measure and $\{f_n\}$ is uniformly integrable.*
(2) *f is integrable and $f_n\to f$ in $L^1(\mu)$.*

The next lemma is a simple application of Vitali's convergence theorem which we state here for later reference.

Lemma 2.3. *Let $(X,\mathcal{A},\mu)$ be a finite measure space. Suppose $\{f_n\}$ is a sequence of nonnegative μ-integrable functions and $\{X_i\}$ is a countable cover of X by μ-measurable sets. If $\{f_n\}$ is uniformly integrable and*

$$\int_{X_i}f_n\,d\mu\to 0\qquad\text{as } n\to\infty,$$

for each i, then

$$\int_X f_n\,d\mu\to 0\qquad\text{as } n\to\infty.$$

Proof. For each i, there is a subsequence (n_k^i) of (n) such that $f_{n_k^i} \to 0$ μ-almost everywhere in X_i as $k \to \infty$. By choosing the subsequences successively, we may assume that (n_k^{i+1}) is a subsequence of (n_k^i) for all i. Then, for $n_k := n_k^k$, we find that $f_{n_k} \to 0$ μ-almost everywhere in X as $k \to \infty$.

Because $\{f_n\}$ is uniformly integrable, Vitali's convergence theorem 2.2 implies that

$$\int_X f_{n_k}\, d\mu \to 0 \qquad \text{as } k \to \infty;$$

noting that on a finite measure space almost everywhere convergence implies convergence in measure, by Egorov's theorem. This implies the assertion. □

2.6. **Slope-convergence.** The results presented here are used in the proof of Theorem 1.7 in Section 5.3.

Definition 2.4 (Slope convergence)**.** Let $U \subseteq \mathbb{R}^m$ be a bounded open set and $U^{<2>} := \{(x,y) \in U \times U : x \neq y\}$. For $f \in C^{0,1}(\overline{U}, \mathbb{R}^\ell)$ consider the *slope function* $s_f : U^{<2>} \to \mathbb{R}$ defined by

$$s_f(x,y) := \frac{\|f(x) - f(y)\|_2}{\|x-y\|_2}.$$

Let $f, f_n \in C^{0,1}(\overline{U}, \mathbb{R}^\ell)$, for $n \geq 1$. We say that f_n *slope-converges* to f if

$$\|s_f - s_{f_n}\|_{L^\infty(U^{<2>})} \to 0 \qquad \text{as } n \to \infty, \tag{2.2}$$

i.e., the slope functions converge uniformly.

We have the following characterization of slope convergence.

Lemma 2.5. *Let $f, f_n \in C^{0,1}(\overline{U}, \mathbb{R}^\ell)$, for $n \geq 1$. Then f_n slope-converges to f if and only if, for each subset $F \subseteq U^{<2>}$,*

$$\sup_F s_{f_n} \to \sup_F s_f \qquad \textit{as } n \to \infty. \tag{2.3}$$

In particular, for each subset $E \subseteq U$, we have convergence of the Lipschitz constants,

$$|f_n|_{C^{0,1}(E,\mathbb{R}^\ell)} \to |f|_{C^{0,1}(E,\mathbb{R}^\ell)} \qquad \textit{as } n \to \infty.$$

Proof. For arbitrary $(x,y) \in U^{<2>}$, we have

$$s_f(x,y) \leq s_{f_n}(x,y) + \|s_f - s_{f_n}\|_{L^\infty(U^{<2>})}$$

so that, for each subset $F \subseteq U^{<2>}$,

$$\sup_{(x,y)\in F} s_f(x,y) \leq \sup_{(x,y)\in F} s_{f_n}(x,y) + \|s_f - s_{f_n}\|_{L^\infty(U^{<2>})}.$$

Hence, by symmetry,

$$\Big| \sup_{(x,y)\in F} s_f(x,y) - \sup_{(x,y)\in F} s_{f_n}(x,y) \Big| \leq \|s_f - s_{f_n}\|_{L^\infty(U^{<2>})}$$

which shows that (2.2) implies (2.3).

Conversely, assume that (2.2) fails. Then, there exist $\epsilon > 0$ and sequences $n_k \to \infty$ and $(x_k, y_k) \in U^{<2>}$ such that

$$|s_f(x_k,y_k) - s_{f_{n_k}}(x_k,y_k)| \geq \epsilon.$$

After passing to a subsequence of k, we may assume that

$$s_f(x_k, y_k) - s_{f_{n_k}}(x_k, y_k) \geq \epsilon \quad \text{or} \quad s_{f_{n_k}}(x_k, y_k) - s_f(x_k, y_k) \geq \epsilon.$$

In the former case, setting $F := \{(x_k, y_k) : k \geq 1\}$,

$$\sup_{(x,y)\in F} s_f(x,y) \geq \sup_{(x,y)\in F} s_{f_{n_k}}(x,y) + \epsilon$$

which contradicts (2.3). The latter case is analogous. □

The next lemma shows that $C^{0,1}$ convergence implies slope convergence.

Lemma 2.6. *Let $f, f_n \in C^{0,1}(\overline{U}, \mathbb{R}^\ell)$, for $n \geq 1$. If*

$$|f - f_n|_{C^{0,1}(\overline{U},\mathbb{R}^\ell)} \to 0 \quad \text{as } n \to \infty,$$

then f_n slope-converges to f.

Proof. We have

$$\begin{aligned}\big|s_f(x,y) - s_{f_n}(x,y)\big| &= \Big|\frac{\|f(x)-f(y)\|_2}{\|x-y\|_2} - \frac{\|f_n(x)-f_n(y)\|_2}{\|x-y\|_2}\Big| \\ &\leq \frac{\|f(x)-f(y)-(f_n(x)-f_n(y))\|_2}{\|x-y\|_2} = s_{f-f_n}(x,y)\end{aligned}$$

and hence

$$\sup_{(x,y)\in U^{<2>}} \big|s_f(x,y) - s_{f_n}(x,y)\big| \leq \sup_{(x,y)\in U^{<2>}} s_{f-f_n}(x,y) = |f-f_n|_{C^{0,1}(\overline{U},\mathbb{R}^\ell)}$$

which shows the assertion. □

3. Hermitian and normal matrices

In this section, we recall some preliminaries on Hermitian and normal matrices.

3.1. **Matrix norms.** Let $M_d(\mathbb{C})$ denote the vector space of complex $d \times d$ matrices. The *operator norm* of $A \in M_d(\mathbb{C})$ is given by

$$\|A\|_{\mathrm{op}} := \sup\{\|Ax\|_2 : x \in \mathbb{C}^d,\ \|x\|_2 = 1\}.$$

We will also use the *Frobenius norm* or 2-*norm* of $A = (a_{ij}) \in M_d(\mathbb{C})$,

$$\|A\|_2 := \big(\operatorname{Tr}(A^*A)\big)^{1/2} = \Big(\sum_{i,j=1}^d |a_{ij}|^2\Big)^{1/2}.$$

If $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_d(A) \geq 0$ are the decreasingly ordered singular values of A, then

$$\|A\|_{\mathrm{op}} = \sigma_1(A) \quad \text{and} \quad \|A\|_2 = \Big(\sum_{j=1}^d \sigma_j(A)^2\Big)^{1/2}.$$

Both operator norm and Frobenius norm are unitarily invariant (i.e., $\|A\|_{\mathrm{op}} = \|UAV\|_{\mathrm{op}}$ and $\|A\|_2 = \|UAV\|_2$ for unitary U, V). It is easy to see that

$$\|A\|_{\mathrm{op}} \leq \|A\|_2 \leq \sqrt{d}\, \|A\|_{\mathrm{op}}.$$

If A is normal and $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of A, then

$$\|A\|_{\mathrm{op}} = \max_{1 \le j \le d} |\lambda_j| \quad \text{and} \quad \|A\|_2 = \Big(\sum_{j=1}^d |\lambda_j|^2 \Big)^{1/2}.$$

If not stated otherwise, we use the Frobenius norm on $M_d(\mathbb{C})$ (and on its subsets).

3.2. **Spectral variation of Hermitian and normal matrices.** Recall that $\lambda^\uparrow : \mathrm{Herm}(d) \to \mathbb{R}^d$ is the map assigning to a Hermitian $d \times d$ its d-tuple of increasingly ordered eigenvalues; see (1.1).

Proposition 3.1 (Weyl's perturbation theorem [Wey12]; see e.g. [Bha97, III.2.6])**.** *Let* $A, B \in \mathrm{Herm}(d)$*. Then*

$$\|\lambda^\uparrow(A) - \lambda^\uparrow(B)\|_\infty \le \|A - B\|_{\mathrm{op}}. \tag{3.1}$$

In [Bha97], the result is stated for eigenvalue vectors with decreasing eigenvalues, but reversing the order evidently leaves the left-hand side of (3.1) unchanged.

Most of the time we will work with 2-norms for which the following variant due to Löwner [Löw34] holds.

Proposition 3.2 ([Löw34], see e.g. [Bha97, III.6.15])**.** *Let* $A, B \in \mathrm{Herm}(d)$*. Then*

$$\|\lambda^\uparrow(A) - \lambda^\uparrow(B)\|_2 \le \|A - B\|_2. \tag{3.2}$$

Löwner's result was generalized to normal matrices by Hoffmann and Wielandt [HW53]. There is no canonical order on the spectrum of a normal matrix. Instead we consider the map $\Lambda : \mathrm{Norm}(d) \to \mathcal{A}_d(\mathbb{C})$ which takes a normal $d \times d$ matrix to its unordered d-tuple of eigenvalues; see (1.6). Recall that $\mathcal{A}_d(\mathbb{C})$ carries the metric $\mathbf{d}_2([z],[w]) = \min_{\sigma \in \mathrm{S}_d} \|z - \sigma w\|_2$, where $\sigma w = \sigma(w_1, \ldots, w_d) = (w_{\sigma(1)}, \ldots, w_{\sigma(d)})$.

Proposition 3.3 ([HW53], see e.g. [Bha97, VI.4.1])**.** *Let* $A, B \in \mathrm{Norm}(d)$*. Then*

$$\mathbf{d}_2(\Lambda(A), \Lambda(B)) \le \|A - B\|_2. \tag{3.3}$$

If A and B are Hermitian, then (3.3) reduces to (3.2); see Lemma 8.3.

With respect to the operator norm, there is the following result due to Bhatia, Davis, and McIntosh [BDM83], where $\mathbf{d}_\infty([z],[w]) = \min_{\sigma \in \mathrm{S}_d} \|z - \sigma w\|_\infty$.

Proposition 3.4 ([BDM83], see e.g. [Bha97, VII.4.1])**.** *Let* $A, B \in \mathrm{Norm}(d)$*. Then*

$$\mathbf{d}_\infty(\Lambda(A), \Lambda(B)) \le C\, \|A - B\|_{\mathrm{op}},$$

where C *is a universal constant satisfying* $1 < C < 3$*.*

3.3. **The characteristic map for Hermitian matrices.**

Proposition 3.5. *Let* $1 \le q \le \infty$*. Let* $U \subseteq \mathbb{R}^m$ *be open and bounded. Then the characteristic map*

$$\mathcal{E} : W^{1,q}(U, \mathrm{Herm}(d)) \to W^{1,q}(U, \mathbb{R}^d), \quad A \mapsto \lambda^\uparrow \circ A$$

is well-defined and bounded, satisfying

$$\|\mathcal{E}(A)(x)\|_2 \le \|A(x)\|_2 \quad \text{and} \quad \|\nabla(\mathcal{E}(A))(x)\|_2 \le \|\nabla A(x)\|_2 \tag{3.4}$$

for almost every $x \in U$*.*

Proof. It is well-known (see [ADM90]) that superposition with the 1-Lipschitz map $\lambda^\uparrow$ (see Proposition 3.2) defines a bounded map from $W^{1,q}(U,\mathrm{Herm}(d))$ to $W^{1,q}(U,\mathbb{R}^d)$, where

$$\|\lambda^\uparrow \circ A\|_2 \le \|A\|_2,$$

because $\lambda^\uparrow(0)=0$, and

$$\|\nabla(\lambda^\uparrow \circ A)\|_2 \le \|\nabla A\|_2$$

almost everywhere in U. This implies the assertion. □

In Proposition 9.1, we will encounter a variant of Proposition 3.5 for normal matrices; its formulation requires some background on multivalued Sobolev functions, which will be recalled in Section 8.

4. Uniform block-diagonalization of Hermitian and normal matrices

4.1. **Splitting lemma for complex triangular matrices.** The following splitting lemma for complex triangular matrices follows from the proof of Hensel's lemma of [PR20, Lemma 2.1] which is stated there in a different setup, namely for normal matrices over the ring of formal power series $\mathbb{C}[\![X]\!]$, where $X=(X_1,\dots,X_m)$.

We shall state this lemma in the Nash real algebraic setup because the unitary group $U_d(\mathbb{C})$ is not a complex manifold but only a nonsingular real algebraic variety. Recall that a map is a real Nash map if it is real analytic and its graph is semialgebraic. Nash maps between open subsets of nonsingular real algebraic varieties form the smallest family of maps that contains the polynomial morphisms and admits the Implicit Function Theorem, see e.g. [BCR98] for more on Nash maps.

Lemma 4.1 (Local unitary block-triangularization of matrices)**.** *Let $A_0 \in M_d(\mathbb{C})$ be a complex block-triangular matrix,*

$$A_0 = \begin{pmatrix} B_0 & 0 \\ D_0 & C_0 \end{pmatrix},$$

with $B_0 \in M_{d_1}(\mathbb{C})$, $C_0 \in M_{d_2}(\mathbb{C})$, $D_0 \in M_{d_2,d_1}(\mathbb{C})$, for $d=d_1+d_2$, and such that the characteristic polynomials of B_0 and C_0 are coprime. Then there is a neighborhood $\mathcal{V}$ of A_0 in $M_d(\mathbb{C})$ and a Nash map to the unitary matrices,

$$U:\mathcal{V}\to U_d(\mathbb{C}), \quad U(A_0)=\mathbb{I}_d,$$

such that, for all $A\in\mathcal{V}$,

$$U^*(A)A\,U(A) = \begin{pmatrix} B(A) & 0 \\ D(A) & C(A) \end{pmatrix}.$$

Proof. Consider

$$\Psi=(\Psi_1,\Psi_2,\Psi_3,\Psi_4): U_d(\mathbb{C})\times M_{d_1}(\mathbb{C})\times M_{d_2}(\mathbb{C})\times M_{d_2,d_1}(\mathbb{C}) \to M_d(\mathbb{C}),$$

defined by

$$(U,Y_1,Y_2,Y_3)\to U^*\begin{pmatrix} B_0+Y_1 & 0 \\ D_0+Y_3 & C_0+Y_2 \end{pmatrix} U = \begin{pmatrix} \Psi_1 & \Psi_4 \\ \Psi_3 & \Psi_2 \end{pmatrix}.$$

Recall that a tangent vector at $\mathbb{I}_d$ to $U_d(\mathbb{C})$ is a matrix $\mathbf{u}$ that is skew-Hermitian, $\mathbf{u}=-\mathbf{u}^*$. We shall write it as

$$\mathbf{u}=\begin{pmatrix} \mathbf{z}_1 & \mathbf{x} \\ -\mathbf{x}^* & \mathbf{z}_2 \end{pmatrix},$$

where $\mathbf{z}_1$ and $\mathbf{z}_2$ are skew-Hermitian. The differential of Ψ at $(\mathbb{I}_d, 0, 0, 0)$ on the vector $(\mathbf{u}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ is given by

$$\begin{aligned}
d\Psi_1(\mathbf{u}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= \mathbf{y}_1 - \mathbf{z}_1 B_0 + B_0 \mathbf{z}_1 - \mathbf{x} D_0,\\
d\Psi_2(\mathbf{u}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= \mathbf{y}_2 - \mathbf{z}_2 C_0 + C_0 \mathbf{z}_2 + D_0 \mathbf{x},\\
d\Psi_3(\mathbf{u}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= \mathbf{y}_3 + D_0 \mathbf{z}_1 - \mathbf{z}_2 D_0 + \mathbf{x}^* B_0 - C_0 \mathbf{x}^*,\\
d\Psi_4(\mathbf{u}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= B_0 \mathbf{x} - \mathbf{x} C_0.
\end{aligned}$$

This differential is a linear epimorphism if and only if so is $d\Psi_4$ and the latter holds true thanks to the following lemma of [Coh84, Lemma 2.3], see also [Zur93] or [PR20, Lemma 2.4].

Lemma 4.2. *Let R be an unitary commutative ring, $A \in M_p(R)$, $B \in M_q(R)$, $C \in M_{p,q}(R)$, such that the characteristic polynomials P_A and P_B are coprime, i.e., there exist polynomials U and V such that $UP_A + VP_B = 1$. Then there is a matrix $M \in M_{p,q}(R)$ such that $AM - MB = C$.*

By the Nash IFT, see e.g. [BCR98, Corollary 2.9.8], there exist a neighborhood $\mathcal{V}$ of A_0 in $M_d(\mathbb{C})$ and a Nash map

$$\Phi = (U, Y_1, Y_2, Y_3) : \mathcal{V} \to U_d(\mathbb{C}) \times M_{d_1}(\mathbb{C}) \times M_{d_2}(\mathbb{C}) \times M_{d_2,d_1}(\mathbb{C}),$$

$\Phi(A_0) = (\mathbb{I}_d, 0, 0, 0)$, that satisfies $\Psi \circ \Phi(A) = A$ for $A \in \mathcal{V}$. Thus it suffices to take $U(A) := \Phi_1^*(A)$. □

Corollary 4.3 (Local unitary block-diagonalization of normal matrices)**.** *With the assumptions of Lemma 4.1 if, moreover, A_0 is normal (resp. Hermitian) then A_0 is block-diagonal, i.e., $D_0 = 0$, and thus for all normal (resp. Hermitian) $A \in \mathcal{V}$ the matrix*

$$U^*(A) A\, U(A) = \begin{pmatrix} B(A) & 0 \\ 0 & C(A) \end{pmatrix}$$

is normal (resp. Hermitian).

Proof. It immediately follows from the facts that conjugation by unitary matrices preserves normal (resp. Hermitian) matrices, and that every block-triangular normal matrix is block-diagonal. □

4.2. **Uniform unitary block-diagonalization.**

Definition 4.4 (Spaces of Hermitian and normal matrices)**.** Let $\mathrm{Herm}_T(d)$ (resp. $\mathrm{Norm}_T(d)$) denote the space of Hermitian (resp. normal) $d \times d$ matrices with trace zero and $\mathrm{Herm}_T^0(d)$ (resp. $\mathrm{Norm}_T^0(d)$) its compact subspace of matrices of norm 1, i.e.,

$$\begin{aligned}
\mathrm{Herm}_T(d) &= \{A \in \mathrm{Herm}(d) : \mathrm{Tr}(A) = 0\},\\
\mathrm{Herm}_T^0(d) &= \{A \in \mathrm{Herm}_T(d) : \|A\|_2 = 1\},\\
\mathrm{Norm}_T(d) &= \{A \in \mathrm{Norm}(d) : \mathrm{Tr}(A) = 0\},\\
\mathrm{Norm}_T^0(d) &= \{A \in \mathrm{Norm}_T(d) : \|A\|_2 = 1\}.
\end{aligned}$$

Let $A_0 \in \mathrm{Norm}_T(d) \setminus \{0\}$. Then A_0 has at least two distinct eigenvalues. Therefore, after a unitary change of coordinates, we may assume that A_0 is block-diagonal,

$$A_0 = \begin{pmatrix} B_0 & 0 \\ 0 & C_0 \end{pmatrix},$$

where B_0, C_0 are normal and the resultant of the characteristic polynomials of B_0 and C_0 is nonzero.

If $A_0 \in \mathrm{Herm}_T(d) \setminus \{0\}$, then B_0, C_0 are Hermitian. In this case, we may assume furthermore that the largest eigenvalue of B_0 is strictly smaller than the smallest eigenvalue of C_0.

By Corollary 4.3, there is an open neighborhood $\mathcal{V} = \mathcal{V}_{A_0}$ of A_0 in $M_d(\mathbb{C})$ and a Nash map $U : \mathcal{V} \to U_d(\mathbb{C})$, with $U(A_0) = \mathbb{I}_d$, such that, for all $A \in \mathcal{V} \cap \mathrm{Norm}(d)$,

$$U^*(A)A\,U(A) = \begin{pmatrix} B(A) & 0 \\ 0 & C(A) \end{pmatrix}, \tag{4.1}$$

where $B(A)$ and $C(A)$ are normal and satisfy $B(A_0) = B_0$ and $C(A_0) = C_0$. Shrinking $\mathcal{V}$ slightly, we may assume that the derivatives of all orders of the map $U : \mathcal{V} \to U_d(\mathbb{C})$ are bounded. In addition, we may assume that $\mathcal{V}$ is convex.

Moreover, by shrinking $\mathcal{V}$ further if necessary, we may assume that there exists $\chi = \chi_{A_0} > 0$ such that, for all $A_1, A_2 \in \mathcal{V} \cap \mathrm{Norm}(d)$ and $B_j := B(A_j)$, $C_j := C(A_j)$, for $j = 1, 2$, given by (4.1), we have

$$|\mu_1 - \nu_2| > \chi \tag{4.2}$$

for all eigenvalues μ_1 of B_1 and all eigenvalues ν_2 of C_2.

If A_0 is Hermitian, then, for $A \in \mathcal{V} \cap \mathrm{Herm}(d)$, we have (4.1) with Hermitian $B(A)$ and $C(A)$. Shrinking $\mathcal{V}$ if necessary, we may assume that, for all $A \in \mathcal{V} \cap \mathrm{Herm}(d)$, the largest eigenvalue of $B(A)$ is strictly smaller than the smallest eigenvalue of $C(A)$.

The open neighborhoods $\mathcal{V}_{A_0}$, for $A_0 \in \mathrm{Norm}_T^0(d)$, form an open cover of the compact space $\mathrm{Norm}_T^0(d)$. We fix a finite subcover $\{\mathcal{V}_1, \ldots, \mathcal{V}_s\}$. Then, for each $i = 1, \ldots, s$, we have a Nash map $U_i : \mathcal{V}_i \to U_d(\mathbb{C})$ with bounded derivatives on $\mathcal{V}_i$ such that

$$U_i^*(A)A\,U_i(A) = \begin{pmatrix} B_i(A) & 0 \\ 0 & C_i(A) \end{pmatrix}, \quad A \in \mathcal{V}_i \cap \mathrm{Norm}(d), \tag{4.3}$$

where $B(A)$ and $C(A)$ are normal. By the Lebesgue covering lemma, there exists $\delta > 0$ such that each subset of $\mathrm{Norm}_T^0(d)$ of diameter less than δ is contained in some $\mathcal{V}_i$. Choose $r \in (0, \min\{\delta/2, 1\})$. Then for each $A_0 \in \mathrm{Norm}_T^0(d)$ there exists $1 \le i \le s$ such that

$$B(A_0, r) \cap \mathrm{Norm}_T^0(d) \subseteq \mathcal{V}_i \cap \mathrm{Norm}_T^0(d). \tag{4.4}$$

Thanks to the property (4.2), there exists $\chi > 0$ such that, for all $i \in \{1, \ldots, s\}$ and $A_1, A_2 \in \mathcal{V}_i \cap \mathrm{Norm}(d)$, we have

$$|\mu_1 - \nu_2| > \chi \tag{4.5}$$

for all eigenvalues μ_1 of $B_i(A_1)$ and all eigenvalues ν_2 of $C_i(A_2)$, where $B_i(A_1)$ and $C_i(A_2)$ are defined by (4.3).

Since $\mathrm{Herm}^0_T(d) \subseteq \mathrm{Norm}^0_T(d)$, also $\mathrm{Herm}^0_T(d)$ is covered by $\{\mathcal{V}_1, \ldots, \mathcal{V}_s\}$ and (4.3) holds with Hermitian $B_i(A), C_i(A)$ and such that the largest eigenvalue of $B_i(A)$ is strictly smaller than the smallest eigenvalue of $C_i(A)$, for all $A \in \mathcal{V}_i \cap \mathrm{Herm}(d)$.

Definition 4.5 (Uniform unitary block-diagonalization)**.** We say that the data

$$(\{U_i : \mathcal{V}_i \to U_d(\mathbb{C})\}_{i=1}^s, r, \chi)$$

that we just fixed is a *uniform unitary block-diagonalization* for $\mathrm{Norm}^0_T(d)$, respectively, for $\mathrm{Herm}^0_T(d)$.

4.3. **Pointwise bounds for absolutely continuous curves of Hermitian and normal matrices.** Let us fix a uniform unitary block-diagonalization $(\{U_i : \mathcal{V}_i \to U_d(\mathbb{C})\}_{i=1}^s, r, \chi)$ for $\mathrm{Norm}^0_T(d)$ for the rest of the section. Fix $i \in \{1, \ldots, s\}$ and let us simply write $U : \mathcal{V} \to U_d(\mathbb{C})$ for $U_i : \mathcal{V}_i \to U_d(\mathbb{C})$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Then U induces a map

$$U_* : AC(I, \mathcal{V}) \to AC(I, U_d(\mathbb{C})), \quad A \mapsto U \circ A$$

and the chain rule holds; see e.g. [MM72].

In the following, we use the norm

$$\|f\|_{C^k} := \max_{0 \le j \le k} \sup_{x \in \mathcal{V}} \|d^j f(x)\|_{L_j(\mathbb{R}^m, \mathbb{R}^\ell)}$$

on the space $C^k(\overline{\mathcal{V}}, \mathbb{R}^\ell)$, where $\mathcal{V} \subseteq \mathbb{R}^m$ is open and bounded and $L_j(\mathbb{R}^m, \mathbb{R}^\ell)$ is the space of j-linear maps with j arguments in $\mathbb{R}^m$ and values in $\mathbb{R}^\ell$.

Lemma 4.6. *Let $A_1, A_2 \in AC(I, \mathcal{V})$. Then, for every $x \in I$,*

$$\|U_*(A_1)(x) - U_*(A_2)(x)\|_2 \le \|U\|_{C^1} \|A_1(x) - A_2(x)\|_2 \tag{4.6}$$

and, for almost every $x \in I$ and $j = 1, 2$,

$$\begin{aligned} &\|(U_*(A_1))'(x) - (U_*(A_2))'(x)\|_2 \\ &\quad \le \|U\|_{C^2} \big(\|A_j'(x)\|_2 \|A_1(x) - A_2(x)\|_2 + \|A_1'(x) - A_2'(x)\|_2 \big). \end{aligned} \tag{4.7}$$

Proof. The mean value theorem implies (4.6). We have

$$\begin{aligned} \|(U_*(A_1))'(x) - (U_*(A_2))'(x)\|_2 &= \|dU(A_1(x))A_1'(x) - dU(A_2(x))A_2'(x)\|_2 \\ &\le \|(dU(A_1(x)) - dU(A_2(x)))A_1'(x)\|_2 + \|dU(A_2(x))(A_1'(x) - A_2'(x))\|_2 \\ &\le \|U\|_{C^2} \|A_1(x) - A_2(x)\|_2 \|A_1'(x)\|_2 + \|U\|_{C^1} \|A_1'(x) - A_2'(x)\|_2 \end{aligned}$$

which implies (4.7) for $j = 1$. The assertion for $j = 2$ follows by symmetry. □

Let $A \in AC(I, \mathrm{Norm}_T(d))$ and assume that $0 \notin A(I)$. Then

$$\underline{A}(x) := \frac{A(x)}{\|A(x)\|_2} \in \mathrm{Norm}^0_T(d)$$

is well-defined for $x \in I$. If $\underline{A}(I) \subseteq \mathcal{V}_i$ for some $i \in \{1, \ldots, s\}$, then

$$U_i^*(\underline{A})\underline{A}\, U_i(\underline{A}) = \begin{pmatrix} \underline{B}_i(\underline{A}) & 0 \\ 0 & \underline{C}_i(\underline{A}) \end{pmatrix}$$

and consequently

$$U_i^*(\underline{A}) A\, U_i(\underline{A}) = \begin{pmatrix} B_i(A) & 0 \\ 0 & C_i(A) \end{pmatrix}, \tag{4.8}$$

where $B_i(A) = \|A\|_2 \cdot \underline{B}_i(\underline{A})$ and $C_i(A) = \|A\|_2 \cdot \underline{C}_i(\underline{A})$ are normal.

By the property (4.5), there exists $\chi > 0$ such that the following holds. Assume that $A_1, A_2 \in AC(I, \mathrm{Norm}_T(d)\setminus\{0\})$ and $\underline{A}_1(I), \underline{A}_2(I) \subseteq \mathcal{V}_i$ for some $i \in \{1, \dots, s\}$. Then, for all $x \in I$,

$$\big|\|A_2(x)\|_2\, \mu_1(x) - \|A_1(x)\|_2\, \nu_2(x)\big| > \chi\, \|A_1(x)\|_2 \|A_2(x)\|_2 \tag{4.9}$$

for all eigenvalues μ_1 of $B_i(A_1)$ and all eigenvalues ν_2 of $C_i(A_2)$, where $B_i(A_1)$ and $C_i(A_2)$ are defined by (4.8).

If $A \in AC(I, \mathrm{Herm}_T(d))$, then $\underline{A}$, $B_i(A)$, and $C_i(A)$ are Hermitian and the largest eigenvalue of $B_i(A)$ is strictly smaller than the smallest eigenvalue of $C_i(A)$, on I, i.e.,

$$\mathcal{E}(A) = (\mathcal{E}(B_i(A)), \mathcal{E}(C_i(A))). \tag{4.10}$$

In Lemma 4.8, we derive pointwise bounds for the difference of the expression (4.8) for two absolutely continuous curves A_1, A_2 of normal matrices. In particular, they apply to Hermitian matrices. We need the following preliminary lemma.

Lemma 4.7. *Let $A \in AC(I, M_d(\mathbb{C}))$ and assume that $0 \notin A(I)$. Then, almost everywhere in I,*

$$\|A\|_2' = \mathrm{Re}\,\mathrm{Tr}(\underline{A}^* A'), \tag{4.11}$$

$$\big|\|A\|_2'\big| \le \|A'\|_2, \tag{4.12}$$

$$\|\underline{A}'\|_2 \le 2\,\frac{\|A'\|_2}{\|A\|_2}. \tag{4.13}$$

Proof. We have

$$\begin{aligned}\|A\|_2' = (\mathrm{Tr}(A^*A)^{1/2})' &= \frac{\mathrm{Tr}(A^*A') + \mathrm{Tr}((A^*)'A)}{2\|A\|_2} \\ &= \frac{\mathrm{Tr}(A^*A') + \mathrm{Tr}((A')^*A)}{2\|A\|_2} = \frac{\mathrm{Tr}(A^*A') + \overline{\mathrm{Tr}(A^*A')}}{2\|A\|_2} = \mathrm{Re}\,\mathrm{Tr}(\underline{A}^*A').\end{aligned}$$

Consequently,

$$\big|\|A\|_2'\big| \le \|\underline{A}^*\|_2 \|A'\|_2 = \|A'\|_2.$$

Finally,

$$\|\underline{A}'\|_2 = \Big\|\frac{A'\|A\|_2 - A\|A\|_2'}{\|A\|_2^2}\Big\|_2 \le \frac{\|A'\|_2}{\|A\|_2} + \frac{\big|\|A\|_2'\big|}{\|A\|_2} \le 2\,\frac{\|A'\|_2}{\|A\|_2}$$

and the lemma is proved. □

Lemma 4.8. *For $j = 1, 2$, let $A_j \in AC(I, \mathrm{Norm}_T(d))$ be such that $0 \notin A_j(I)$. Suppose that $\underline{A}_1(I), \underline{A}_2(I) \subseteq \mathcal{V}_i$ for some $i \in \{1, \dots, s\}$. For simplicity, we write $U : \mathcal{V} \to U_d(\mathbb{C})$ for $U_i : \mathcal{V}_i \to U_d(\mathbb{C})$. Then, pointwise on I,*

$$\|U^*(\underline{A}_1)A_1U(\underline{A}_1) - U^*(\underline{A}_2)A_2U(\underline{A}_2)\|_2 \le C\,\|A_1 - A_2\|_2 \tag{4.14}$$

and, almost everywhere on I and for $j = 1, 2$,

$$\begin{aligned}&\|(U^*(\underline{A}_1)A_1U(\underline{A}_1))' - (U^*(\underline{A}_2)A_2U(\underline{A}_2))'\|_2 \\ &\quad\le C\Big(\|A_1' - A_2'\|_2 + \big(\|A_1'\|_2 + \|A_2'\|_2\big)\|A_j\|_2^{-1}\|A_1 - A_2\|_2\Big),\end{aligned} \tag{4.15}$$

where C is a universal constant depending only on d.

Proof. We have, for $j = 1, 2$,

$$\|\underline{A}_1 - \underline{A}_2\|_2 \le 2\,\|A_j\|_2^{-1}\|A_1 - A_2\|_2, \tag{4.16}$$

since

$$\begin{aligned}\|\underline{A}_1 - \underline{A}_2\|_2 &= \Big\|\frac{A_1}{\|A_1\|_2} - \frac{A_2}{\|A_2\|_2}\Big\|_2 \\ &\le \frac{\|A_1 - A_2\|_2\|A_2\|_2 + \|A_2\|_2\,\big|\|A_1\|_2 - \|A_2\|_2\big|}{\|A_1\|_2\|A_2\|_2} \le 2\,\frac{\|A_1 - A_2\|_2}{\|A_1\|_2}\end{aligned}$$

and the case $j = 2$ follows by symmetry. Thus, by (4.6) and (4.16),

$$\begin{aligned}&\|U^*(\underline{A}_1)A_1U(\underline{A}_1) - U^*(\underline{A}_2)A_2U(\underline{A}_2)\|_2 \\ &\quad\le \|(U^*(\underline{A}_1) - U^*(\underline{A}_2))A_1\,U(\underline{A}_1)\|_2 + \|U^*(\underline{A}_2)A_1(U(\underline{A}_1) - U(\underline{A}_2)\|_2 \\ &\qquad + \|U^*(\underline{A}_2)(A_1 - A_2)U(\underline{A}_2)\|_2 \\ &\quad\lesssim \|A_1\|_2\|\underline{A}_1 - \underline{A}_2\|_2 + \|A_1 - A_2\|_2 \lesssim \|A_1 - A_2\|_2\end{aligned}$$

showing (4.14).

Let us prove (4.15). By (4.11),

$$\begin{aligned}\big|\|A_1\|_2' - \|A_2\|_2'\big| &= \big|\operatorname{Re}\operatorname{Tr}(\underline{A}_1^*A_1') - \operatorname{Re}\operatorname{Tr}(\underline{A}_2^*A_2')\big| \\ &= \Big|\operatorname{Re}\operatorname{Tr}((\underline{A}_1^* - \underline{A}_2^*)A_1') + \operatorname{Re}\operatorname{Tr}(\underline{A}_2^*(A_1' - A_2'))\Big| \\ &\le \|\underline{A}_1^* - \underline{A}_2^*\|_2\|A_1'\|_2 + \|\underline{A}_2^*\|_2\|A_1' - A_2'\|_2 \\ &= \|\underline{A}_1 - \underline{A}_2\|_2\|A_1'\|_2 + \|A_1' - A_2'\|_2,\end{aligned}$$

so that by symmetry, for $j = 1, 2$,

$$\big|\|A_1\|_2' - \|A_2\|_2'\big| \le \|\underline{A}_1 - \underline{A}_2\|_2\|A_j'\|_2 + \|A_1' - A_2'\|_2.$$

Using (4.16), we conclude, for $j = 1, 2$,

$$\big|\|A_1\|_2' - \|A_2\|_2'\big| \lesssim \frac{\|A_j'\|_2}{\|A_j\|_2}\|A_1 - A_2\|_2 + \|A_1' - A_2'\|_2. \tag{4.17}$$

We claim that

$$\|\underline{A}_1' - \underline{A}_2'\|_2 \lesssim \frac{\|A_2'\|_2\|A_1 - A_2\|_2}{\|A_1\|_2\|A_2\|_2} + \frac{\|A_1' - A_2'\|_2}{\|A_1\|_2}. \tag{4.18}$$

Indeed

$$\begin{aligned}\|\underline{A}_1' - \underline{A}_2'\|_2 &= \Big\|\frac{A_1'\|A_1\|_2 - A_1\|A_1\|_2'}{\|A_1\|_2^2} - \frac{A_2'\|A_2\|_2 - A_2\|A_2\|_2'}{\|A_2\|_2^2}\Big\|_2 \\ &\le \Big\|\frac{A_1'\|A_1\|_2\|A_2\|_2^2 - A_2'\|A_1\|_2^2\|A_2\|_2}{\|A_1\|_2^2\|A_2\|_2^2}\Big\|_2 \\ &\quad + \Big\|\frac{A_2\|A_1\|_2^2\|A_2\|_2' - A_1\|A_2\|_2^2\|A_1\|_2'}{\|A_1\|_2^2\|A_2\|_2^2}\Big\|_2.\end{aligned}$$

The first summand equals

$$\Big\|\frac{A_1'\|A_2\|_2 - A_2'\|A_1\|_2}{\|A_1\|_2\|A_2\|_2}\Big\|_2 = \Big\|\frac{(A_1' - A_2')\|A_2\|_2 - A_2'(\|A_1\|_2 - \|A_2\|_2)}{\|A_1\|_2\|A_2\|_2}\Big\|_2$$

$$\lesssim \frac{\|A_1' - A_2'\|_2}{\|A_1\|_2} + \frac{\|A_2'\|_2\|A_1 - A_2\|_2}{\|A_1\|\|A_2\|_2}$$

and, by (4.12) and (4.17), the second summand is bounded by

$$\Big\|\frac{\|A_2\|_2'(A_2\|A_1\|_2^2 - A_1\|A_2\|_2^2)}{\|A_1\|_2^2\|A_2\|_2^2}\Big\|_2 + \Big\|\frac{A_1\|A_2\|_2^2(\|A_2\|_2' - \|A_1\|_2')}{\|A_1\|_2^2\|A_2\|_2^2}\Big\|_2$$

$$\begin{aligned}
&= \frac{\big|\|A_2\|_2'\big|\,\big\|A_2\|A_1\|_2^2 - A_2\|A_1\|_2\|A_2\|_2 + A_2\|A_1\|_2\|A_2\|_2 - A_1\|A_2\|_2^2\big\|_2}{\|A_1\|_2^2\|A_2\|_2^2} \\
&\quad + \frac{\big|\|A_1\|_2' - \|A_2\|_2'\big|}{\|A_1\|_2} \\
&\lesssim \frac{\|A_2'\|_2\,\big|\|A_1\|_2 - \|A_2\|_2\big|}{\|A_1\|_2\|A_2\|_2} + \frac{\|A_2'\|_2\,\big\|A_2\|A_1\|_2 - A_1\|A_2\|_2\big\|_2}{\|A_1\|_2^2\|A_2\|_2} \\
&\quad + \frac{\|A_2'\|_2\|A_1 - A_2\|_2}{\|A_1\|_2\|A_2\|_2} + \frac{\|A_1' - A_2'\|_2}{\|A_1\|_2} \\
&\lesssim \frac{\|A_2'\|_2\|A_1 - A_2\|_2}{\|A_1\|_2\|A_2\|_2} + \frac{\|A_1' - A_2'\|_2}{\|A_1\|_2}
\end{aligned}$$

and (4.18) follows.

By (4.7), (4.13), (4.16), and (4.18),

$$\begin{aligned}
\|(U(\underline{A}_1) - U(\underline{A}_2))'\|_2 &\lesssim \|\underline{A}_2'\|_2\|\underline{A}_1 - \underline{A}_2\|_2 + \|\underline{A}_1' - \underline{A}_2'\|_2 \\
&\lesssim \frac{\|A_2'\|_2}{\|A_2\|_2}\frac{\|A_1 - A_2\|_2}{\|A_1\|_2} + \frac{\|A_1' - A_2'\|_2}{\|A_1\|_2}, \qquad (4.19)
\end{aligned}$$

similarly for U^*.

Finally we want to bound

$$\begin{aligned}
&\|(U^*(\underline{A}_1)A_1U(\underline{A}_1))' - (U^*(\underline{A}_2)A_2U(\underline{A}_2))'\|_2 \\
&\quad \le \|((U^*(\underline{A}_1) - U^*(\underline{A}_2))A_1U(\underline{A}_1))'\|_2 + \|(U^*(\underline{A}_2)A_1(U(\underline{A}_1) - U(\underline{A}_2)))'\|_2 \\
&\qquad + \|(U^*(\underline{A}_2)(A_1 - A_2)U(\underline{A}_2))'\|_2.
\end{aligned}$$

The first summand can be bounded by

$$\begin{aligned}
&\|(U^*(\underline{A}_1) - U^*(\underline{A}_2))'\|_2\|A_1\|_2 + \|U^*(\underline{A}_1) - U^*(\underline{A}_2)\|_2\|A_1'\|_2 \\
&\quad + \|U^*(\underline{A}_1) - U^*(\underline{A}_2)\|_2\|A_1\|_2\|(U(\underline{A}_1))'\|_2 \\
&\lesssim \frac{(\|A_1'\|_2 + \|A_2'\|_2)}{\|A_2\|_2}\|A_1 - A_2\|_2 + \|A_1' - A_2'\|_2,
\end{aligned}$$

using (4.6), (4.13), (4.16), and (4.19). The bound for the second summand is similar and the third summand can be bounded by

$$\begin{aligned}
&\|(U^*(\underline{A}_2))'\|_2\|A_1 - A_2\|_2 + \|A_1' - A_2'\|_2 + \|A_1 - A_2\|_2\|U(\underline{A}_2))'\|_2 \\
&\lesssim \frac{\|A_2'\|_2}{\|A_2\|_2}\|A_1 - A_2\|_2 + \|A_1' - A_2'\|_2.
\end{aligned}$$

Thus (4.15) follows. □

Remark 4.9. In the proof we did not use the fact that the matrices are normal.

Remark 4.10. Lemma 4.8 remains valid if $\underline{A}_1(I) \subseteq \mathcal{V}_{i_1}$, $\underline{A}_2(I) \subseteq \mathcal{V}_{i_2}$, for $i_1 \neq i_2 \in \{1, \ldots, s\}$, and $\|\underline{A}_1(x) - \underline{A}_2(x)\|_2 \geq \epsilon$, for some universal constant $\epsilon > 0$, in the following sense: pointwise on I,

$$\|U^*_{i_1}(\underline{A}_1) A_1 U_{i_1}(\underline{A}_1) - U^*_{i_2}(\underline{A}_2) A_2 U_{i_2}(\underline{A}_2)\|_2 \leq C \, \|A_1 - A_2\|_2$$

and, almost everywhere on I and for $j = 1, 2$,

$$\begin{aligned} &\|(U^*_{i_1}(\underline{A}_1) A_1 U_{i_1}(\underline{A}_1))' - (U^*_{i_2}(\underline{A}_2) A_2 U_{i_2}(\underline{A}_2))'\|_2 \\ &\qquad \leq C \Big(\|A_2' - A_1'\|_2 + \big(\|A_1'\|_2 + \|A_2'\|_2\big) \|A_j\|_2^{-1} \|A_1 - A_2\|_2 \Big), \end{aligned}$$

where C is a universal constant depending only on d.

Indeed,

$$\|U_{i_1}(\underline{A}_1) - U_{i_2}(\underline{A}_2)\|_2 \leq \|U_{i_1}(\underline{A}_1)\|_2 + \|U_{i_2}(\underline{A}_2)\|_2 \lesssim \epsilon \leq \|\underline{A}_1 - \underline{A}_2\|_2$$

and

$$\begin{aligned} \|(U_{i_1}(\underline{A}_1) - U_{i_2}(\underline{A}_2))'\|_2 &\leq \|(U_{i_1}(\underline{A}_1))'\|_2 + \|(U_{i_2}(\underline{A}_2))'\|_2 \\ &\lesssim (\|\underline{A}_1'\|_2 + \|\underline{A}_2'\|_2)\epsilon \leq (\|\underline{A}_1'\|_2 + \|\underline{A}_2'\|_2) \|\underline{A}_1 - \underline{A}_2\|_2. \end{aligned}$$

Note that the sizes of the blocks in (4.8) corresponding to i_1 and i_2 may be different.

5. Eigenvalue stability for Hermitian matrices: one-parameter case

This section is devoted to the proofs of Theorem 1.1 and Theorem 1.7 in the one-parameter case. In addition, we prove Theorem 1.19 in Section 5.4.

The following theorem is a version of Theorem 1.1 for $m = 1$.

Theorem 5.1. *Let $1 \leq q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}(d))$, i.e.,*

$$\|A - A_n\|_{W^{1,q}(I, M_d(\mathbb{C}))} \to 0 \qquad \text{as } n \to \infty.$$

Then

$$\|\mathcal{E}(A) - \mathcal{E}(A_n)\|_{W^{1,q}(I, \mathbb{R}^d)} \to 0 \qquad \text{as } n \to \infty.$$

The proof of Theorem 5.1 comprises Section 5.1 and Section 5.2.

5.1. Preliminary observations and reductions.

Lemma 5.2. *Let $A \in W^{1,q}(I, M_d(\mathbb{C}))$, where $1 \leq q < \infty$ and $I \subseteq \mathbb{R}$ is a bounded open interval. Then, for all $x \in I$,*

$$\|A(x)\|_2 \leq |I|^{-1/q} \, \|A\|_{L^q(I, M_d(\mathbb{C}))} + |I|^{1-1/q} \, \|A'\|_{L^q(I, M_d(\mathbb{C}))}. \tag{5.1}$$

Proof. For $x, y \in I$, we have

$$A(x) - A(y) = \int_y^x A'(t) \, dt$$

and integrating over y, we get

$$A(x) - A_I = \frac{1}{|I|} \int_I \int_y^x A'(t) \, dt \, dy,$$

where $A_I := |I|^{-1} \int_I A(y)\, dy$. Therefore,

$$\|A(x) - A_I\|_2 \leq \frac{1}{|I|} \int_I \int_I \|A'(t)\|_2 \, dt \, dy = \|A'\|_{L^1(I, M_d(\mathbb{C}))}.$$

Consequently,

$$\begin{aligned} \|A(x)\|_2 &\leq \|A_I\|_2 + \|A(x) - A_I\|_2 \\ &\leq |I|^{-1} \|A\|_{L^1(I, M_d(\mathbb{C}))} + \|A'\|_{L^1(I, M_d(\mathbb{C}))} \\ &\leq |I|^{-1/q} \|A\|_{L^q(I, M_d(\mathbb{C}))} + |I|^{1-1/q} \|A'\|_{L^q(I, M_d(\mathbb{C}))}, \end{aligned}$$

by Hölder's inequality. □

Corollary 5.3. *Let $1 \leq q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, M_d(\mathbb{C}))$ as $n \to \infty$. Then*

$$\|A - A_n\|_{L^\infty(I, M_d(\mathbb{C}))} \to 0 \qquad \text{as } n \to \infty. \tag{5.2}$$

Proof. Apply (5.1) to $A - A_n$ instead of A. □

Corollary 5.4. *Let $1 \leq q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}(d))$ as $n \to \infty$. Then*

$$\|\mathcal{E}(A) - \mathcal{E}(A_n)\|_{L^\infty(I, \mathbb{R}^d)} \to 0 \qquad \text{as } n \to \infty.$$

Proof. This follows from (3.2) and Corollary 5.3. □

To complete the proof of Theorem 5.1 it suffices, by Corollary 5.4, to show that

$$\|\mathcal{E}(A)' - \mathcal{E}(A_n)'\|_{L^q(I, \mathbb{R}^d)} \to 0 \qquad \text{as } n \to \infty.$$

With $A \in W^{1,q}(I, \mathrm{Herm}(d))$ we associate $\tilde{A} := A - \frac{1}{d} \mathrm{Tr}(A) \mathbb{I}_d \in W^{1,q}(I, \mathrm{Herm}_T(d))$. Note that

$$\mathcal{E}(A) - \mathcal{E}(\tilde{A}) = \frac{1}{d} \mathrm{Tr}(A)(1, 1, \ldots, 1). \tag{5.3}$$

Lemma 5.5. *Let $1 \leq q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Assume that $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}(d))$ as $n \to \infty$. Then:*

(1) *$\tilde{A}_n \to \tilde{A}$ in $W^{1,q}(I, \mathrm{Herm}_T(d))$ as $n \to \infty$.*

(2) *Moreover,*

$$\|\mathcal{E}(A) - \mathcal{E}(A_n)\|_{W^{1,q}(I, \mathbb{R}^d)} \to 0$$

if and only if

$$\|\mathcal{E}(\tilde{A}) - \mathcal{E}(\tilde{A}_n)\|_{W^{1,q}(I, \mathbb{R}^d)} \to 0$$

as $n \to \infty$.

Proof. Clearly, $\mathrm{Tr}(A_n) \to \mathrm{Tr}(A)$ in $W^{1,q}(I)$ as $n \to \infty$. The second assertion follows easily from (5.3). □

By Lemma 5.5, we may assume that $\mathrm{Tr}(A) = \mathrm{Tr}(A_n) = 0$ for all $n \geq 1$.

5.2. **Proof of Theorem 5.1.** By the preliminary reductions in Section 5.1, it suffices to show the following proposition:

Proposition 5.6. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}_T(d))$ as $n \to \infty$. Then*

$$\|\mathcal{E}(A)' - \mathcal{E}(A_n)'\|_{L^q(I,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{5.4}$$

We will proceed by induction on d. If $d = 1$ the assertion is trivially true. So assume that $d \ge 2$.

Let us first consider the zero set

$$Z_A := \{x \in I : A(x) = 0\} = \{x \in I : \mathcal{E}(A)(x) = 0\} =: Z_{\mathcal{E}(A)}$$

of A. Since $\mathcal{E}(A) \in W^{1,q}(I, \mathbb{R}^d)$, the derivative $\mathcal{E}(A)'$ exists almost everywhere in I. If x_0 belongs to the set of accumulation points of Z_A, denoted by $\mathrm{acc}(Z_A)$, and $\mathcal{E}(A)'(x_0)$ exists, then necessarily $\mathcal{E}(A)'(x_0) = 0$.

Lemma 5.7. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Assume that $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}(d))$ as $n \to \infty$. Then*

$$\|\mathcal{E}(A_n)' - \mathcal{E}(A)'\|_{L^q(Z_A,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Proof. Let E be the set of $x \in \mathrm{acc}(Z_A)$, where the derivatives $A'(x)$, $\mathcal{E}(A)'(x)$ and $A_n'(x)$, $\mathcal{E}(A_n)'(x)$ for all $n \ge 1$ exist. For each $x \in E$, $A'(x) = 0$, $\mathcal{E}(A)'(x) = 0$, and

$$\big\|\mathcal{E}(A_n)'(x)\big\|_2 \le \|A_n'(x)\|_2,$$

by (3.2) or (3.4). Since E has full measure in Z_A, we thus get

$$\begin{aligned}\|\mathcal{E}(A_n)' - \mathcal{E}(A)'\|_{L^q(Z_A,\mathbb{R}^d)} &= \|\mathcal{E}(A_n)'\|_{L^q(Z_A,\mathbb{R}^d)} \\ &\le \|A_n'\|_{L^q(Z_A,M_d(\mathbb{C}))} = \|A_n' - A'\|_{L^q(Z_A,M_d(\mathbb{C}))},\end{aligned}$$

which implies the assertion. □

The next lemma takes care of the complement of Z_A, where around each point we can find an interval on which we have a simultaneous unitary block-diagonalization for sufficiently large n.

Let us fix a uniform unitary block-diagonalization $(\{U_i : \mathcal{V}_i \to U_d(\mathbb{C})\}_{i=1}^s, r, \chi)$ for $\mathrm{Herm}_T^0(d)$; see Definition 4.5.

Lemma 5.8. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Assume that $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}_T(d))$ as $n \to \infty$. Let $x_0 \in I \setminus Z_A$. Then there exist an open interval J with $x_0 \in J \subseteq I \setminus Z_A$, $n_0 \ge 1$, and $i \in \{1, \ldots, s\}$ such that*

1. *for all $n \ge n_0$, the curves $\underline{A} := A/\|A\|_2$ and $\underline{A}_n := A_n/\|A_n\|_2$ belong to $W^{1,q}(J, \mathrm{Herm}_T^0(d))$ and satisfy $\underline{A}(J), \underline{A}_n(J) \subseteq \mathcal{V}_i$;*
2. *on J and for all $n \ge n_0$, we have*

$$U_i^*(\underline{A}) A\, U_i(\underline{A}) = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad U_i^*(\underline{A}_n) A_n\, U_i(\underline{A}_n) = \begin{pmatrix} B_n & 0 \\ 0 & C_n \end{pmatrix}, \tag{5.5}$$

where $B, B_n \in W^{1,q}(J, \mathrm{Herm}(d_1))$ and $C, C_n \in W^{1,q}(J, \mathrm{Herm}(d_2))$ with $d_1 + d_2 = d$;

(3) *on J and for all $n \geq n_0$, we have*
$$\mathcal{E}(A) = (\mathcal{E}(B), \mathcal{E}(C)), \quad \mathcal{E}(A_n) = (\mathcal{E}(B_n), \mathcal{E}(C_n)); \tag{5.6}$$

(4) *we have*
$$\|B - B_n\|_{W^{1,q}(J,M_{d_1}(\mathbb{C}))} \to 0, \quad \|C - C_n\|_{W^{1,q}(J,M_{d_2}(\mathbb{C}))} \to 0 \tag{5.7}$$
as $n \to \infty$.

Proof. Let $x_0 \in I \setminus Z_A$. Since $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}(d))$ as $n \to \infty$, there exists $n_0 \geq 1$ such that
$$\|A' - A_n'\|_{L^1(I,M_d(\mathbb{C}))} \leq \frac{r}{32}\|A(x_0)\|_2 \quad \text{if } n \geq n_0. \tag{5.8}$$
By (5.2), there exists $n_1 \geq n_0$ such that
$$\|A(x_0) - A_n(x_0)\|_2 \leq \frac{r}{8}\|A(x_0)\|_2 \quad \text{if } n \geq n_1, \tag{5.9}$$
and hence, in particular,
$$\frac{1}{2}\|A(x_0)\|_2 \leq \|A_n(x_0)\|_2 \leq \frac{3}{2}\|A(x_0)\|_2. \tag{5.10}$$

Choose a open subinterval $J \subseteq I$ containing x_0 such that
$$\|A'\|_{L^1(J,M_d(\mathbb{C}))} \leq \frac{r}{32}\,\|A(x_0)\|_2. \tag{5.11}$$
For $x \in J$ we have
$$|\|A(x)\|_2 - \|A(x_0)\|_2| \leq \|A(x) - A(x_0)\|_2 \leq \|A'\|_{L^1(J,M_d(\mathbb{C}))} \leq \frac{r}{32}\,\|A(x_0)\|_2,$$
and thus
$$\frac{1}{2}\|A(x_0)\|_2 \leq \|A(x)\|_2 \leq \frac{3}{2}\|A(x_0)\|_2. \tag{5.12}$$
For $x \in J$ we also have, by (5.8), (5.10), and (5.11),
$$\begin{aligned}|\|A_n(x)\|_2 - \|A_n(x_0)\|_2| &\leq \|A_n(x) - A_n(x_0)\|_2 \\ &\leq \|A_n'\|_{L^1(J,M_d(d))} \leq \frac{r}{16}\,\|A(x_0)\|_2 \leq \frac{r}{8}\,\|A_n(x_0)\|_2\end{aligned}$$
and thus
$$\frac{1}{2}\|A_n(x_0)\|_2 \leq \|A_n(x)\|_2 \leq \frac{3}{2}\|A_n(x_0)\|_2. \tag{5.13}$$

Therefore, $\underline{A} := A/\|A\|_2$ and $\underline{A}_n := A_n/\|A_n\|_2$ are well-defined on J and belong to $W^{1,q}(J, \mathrm{Herm}^0_T(d))$. By (4.16) and (5.9),
$$\|\underline{A}(x_0) - \underline{A}_n(x_0)\|_2 \leq \frac{2}{\|A(x_0)\|_2}\|A(x_0) - A_n(x_0)\|_2 \leq \frac{r}{4}.$$
Moreover, by (4.13) and (5.12), for $x \in J$,
$$\|\underline{A}'(x)\|_2 \leq 2\,\frac{\|A'(x)\|_2}{\|A(x)\|_2} \leq 4\,\frac{\|A'(x)\|_2}{\|A(x_0)\|_2}$$
and, by (5.12) and (5.10),
$$\|\underline{A}_n'(x)\|_2 \leq 2\,\frac{\|A_n'(x)\|_2}{\|A_n(x)\|_2} \leq 8\,\frac{\|A_n'(x)\|_2}{\|A(x_0)\|_2}.$$

Therefore, by (5.11),

$$\|\underline{A}'\|_{L^1(J,M_d(\mathbb{C}))} \le \frac{r}{8}$$

and, by (5.8) and (5.11),

$$\|\underline{A}_n'\|_{L^1(J,M_d(\mathbb{C}))} \le \frac{r}{2}.$$

Thus $\underline{A}(J) \subseteq B(\underline{A}(x_0), r/4)$ and $\underline{A}_n(J) \subseteq B(\underline{A}_n(x_0), r/2) \subseteq B(\underline{A}(x_0), 3r/4)$. By (4.4), there exists $i \in \{1,\ldots,s\}$ such that

$$\underline{A}(J) \subseteq \mathcal{V}_i \quad \text{and} \quad \underline{A}_n(J) \subseteq \mathcal{V}_i$$

for all $n \ge n_1$. Thus (1) is proved.

By (4.8) and (4.10), we have the simultaneous unitary block-diagonalization (5.5) as well as (5.6) on J for $n \ge n_1$. Let us check that B, B_n, C, C_n are of class $W^{1,q}$ on J and satisfy (5.7). To this end, we suppress the subscript and just write $U : \mathcal{V} \to U_d(\mathbb{C})$ for $U_i : \mathcal{V}_i \to U_d(\mathbb{C})$. By Lemma 4.8, pointwise on J,

$$\|U^*(\underline{A})A\,U(\underline{A}) - U^*(\underline{A}_n)A_n\,U(\underline{A}_n)\|_2 \le C\,\|A - A_n\|_2$$

and, using also (5.12), almost everywhere on J,

$$\begin{aligned}
&\|(U^*(\underline{A})A\,U(\underline{A}))' - (U^*(\underline{A}_n)A_n\,U(\underline{A}_n))'\|_2 \\
&\qquad \le C\Big(\|A' - A_n'\|_2 + (\|A'\|_2 + \|A_n'\|_2)\|A\|_2^{-1}\|A - A_n\|_2\Big), \\
&\qquad \le C\Big(\|A' - A_n'\|_2 + 2\,\|A(x_0)\|_2^{-1}\,(\|A'\|_2 + \|A_n'\|_2)\|A - A_n\|_2\Big),
\end{aligned}$$

where C is a universal constant depending only on d. This implies

$$\|U^*(\underline{A})A\,U(\underline{A}) - U^*(\underline{A}_n)A_n\,U(\underline{A}_n)\|_{L^q(J,M_d(\mathbb{C}))} \le C\,\|A - A_n\|_{L^q(J,M_d(\mathbb{C}))}$$

and

$$\begin{aligned}
&\|(U^*(\underline{A})A\,U(\underline{A}))' - (U^*(\underline{A}_n)A_n\,U(\underline{A}_n))'\|^q_{L^q(J,M_d(\mathbb{C}))} \\
&\quad \le (2C)^q\,\|A' - A_n'\|^q_{L^q(J,M_d(\mathbb{C}))} \\
&\qquad + \Big(\frac{8C}{\|A(x_0)\|_2}\Big)^q \big(\|A'\|^q_{L^q(J,M_d(\mathbb{C}))} + \|A_n'\|^q_{L^q(J,M_d(\mathbb{C}))}\big)\|A - A_n\|^q_{L^\infty(J,M_d(\mathbb{C}))}.
\end{aligned}$$

By Corollary 5.3, we conclude (5.7). This completes the proof. □

By Lemma 5.5, Lemma 5.8, and the induction hypothesis, for each $x_0 \in I \setminus Z_A$ there is an open interval J with $x_0 \in J \subseteq I \setminus Z_A$ such that

$$\|\mathcal{E}(A)' - \mathcal{E}(A_n)'\|_{L^q(J,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Thus, in view of Lemma 5.7, there is a countable cover $\{F_i\}_{i\ge 1}$ of I by measurable sets such that, for each F_i,

$$\|\mathcal{E}(A)' - \mathcal{E}(A_n)'\|_{L^q(F_i,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

In view of Lemma 2.3 (applied to $f_n := \|\mathcal{E}(A)' - \mathcal{E}(A_n)'\|_2^q$), the desired assertion (5.4) will follow from the following lemma.

Lemma 5.9. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Assume that $A_n \to A$ in $W^{1,q}(I, \mathrm{Herm}(d))$ as $n \to \infty$. Then $\{\|\mathcal{E}(A)' - \mathcal{E}(A_n)'\|_2^q : n \ge 1\} \subseteq L^1(I)$ is uniformly integrable.*

Proof. Since $A_n' \to A'$ in $L^q(I, M_d(\mathbb{C}))$ as $n \to \infty$, the set $\{\|A'\|_2^q + \|A_n'\|_2^q : n \geq 1\} \subseteq L^1(I)$ is uniformly integrable, by Vitali's convergence theorem 2.2. Thus, by de la Vallée Poussin's criterion 2.1, there exists a nonnegative increasing function G on $[0, \infty)$ such that $G(t)/t \to \infty$ as $t \to \infty$ and

$$\sup_{n \geq 1} \int_I G(2^q(\|A'\|_2^q + \|A_n'\|_2^q))\, dx < \infty.$$

By (3.2) or (3.4), almost everywhere in I,

$$\|\mathcal{E}(A)' - \mathcal{E}(A_n)'\|_2^q \leq 2^q(\|A'\|_2^q + \|A_n'\|_2^q)$$

so that the assertion follows, by de la Vallée Poussin's criterion 2.1 again. □

This completes the proof of Proposition 5.6 and thus of Theorem 5.1.

5.3. **Almost everywhere convergence of the derivatives of the eigenvalues.** The following theorem is a version of Theorem 1.7 for $m = 1$.

Theorem 5.10. *Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $C^{0,1}(\overline{I}, \mathrm{Herm}(d))$ as $n \to \infty$. Then, for almost every $x \in I$,*

$$\mathcal{E}(A_n)'(x) \to \mathcal{E}(A)'(x) \quad n \to \infty. \tag{5.14}$$

Let us prove Theorem 5.10 following the proof of Theorem 5.1 and indicating the required modifications.

In analogy to Lemma 5.5, it is easy to conclude the following lemma which allows us to assume $\mathrm{Tr}(A) = \mathrm{Tr}(A_n) = 0$ for all $n \geq 1$.

Lemma 5.11. *Let $I \subseteq \mathbb{R}$ be a bounded open interval. Assume that $A_n \to A$ in $C^{0,1}(\overline{I}, \mathrm{Herm}(d))$ as $n \to \infty$. Then:*

(1) *$\tilde{A}_n \to \tilde{A}$ in $C^{0,1}(\overline{I}, \mathrm{Herm}_T(d))$ as $n \to \infty$.*

(2) *Moreover,*

$$\mathcal{E}(A_n)' \to \mathcal{E}(A)' \qquad \text{almost everywhere in } I$$

if and only if

$$\mathcal{E}(\tilde{A}_n)' \to \mathcal{E}(\tilde{A})' \qquad \text{almost everywhere in } I$$

as $n \to \infty$.

We prove (5.14) by induction on d. If $d = 1$ the assertion is trivially true. So assume that $d \geq 2$.

On the zero set Z_A of A, we have the following lemma.

Lemma 5.12. *Let $I \subseteq \mathbb{R}$ be a bounded open interval and assume that $A_n \to A$ in $C^{0,1}(\overline{I}, \mathrm{Herm}(d))$ as $n \to \infty$. Then, for almost every $x \in Z_A$,*

$$\mathcal{E}(A_n)'(x) \to 0 \qquad \text{as } n \to \infty.$$

Proof. By Proposition 3.2, for every $n \ge 1$,

$$\sup_{(x,y)\in Z_A^{<2>}} \frac{\|\mathcal{E}(A_n)(x) - \mathcal{E}(A_n)(y)\|_2}{|x-y|} \le \sup_{(x,y)\in Z_A^{<2>}} \frac{\|A_n(x) - A_n(y)\|_2}{|x-y|}.$$

By Lemma 2.5 and Lemma 2.6,

$$\sup_{(x,y)\in Z_A^{<2>}} \frac{\|A_n(x) - A_n(y)\|_2}{|x-y|} \to \sup_{(x,y)\in Z_A^{<2>}} \frac{\|A(x) - A(y)\|_2}{|x-y|} = 0 \quad \text{as } n \to \infty.$$

Let E be the set of $x \in \operatorname{acc}(Z_A)$, where the derivatives $\mathcal{E}(A_n)'(x)$ for all $n \ge 1$ exist. For each $x_0 \in E$ there is a sequence $x_k \to x_0$ with $x_k \in Z_A$ and $x_k \ne x_0$ for all $k \ge 1$. Thus, for fixed n,

$$\begin{aligned}\big\|\mathcal{E}(A_n)'(x_0)\big\|_2 &= \lim_{k\to\infty} \frac{\|\mathcal{E}(A_n)(x_0) - \mathcal{E}(A_n)(x_k)\|_2}{|x_0 - x_k|} \\ &\le \sup_{(x,y)\in Z_A^{<2>}} \frac{\|\mathcal{E}(A_n)(x) - \mathcal{E}(A_n)(y)\|_2}{|x-y|}.\end{aligned}$$

Hence we may conclude that, for each $x_0 \in E$,

$$\mathcal{E}(A_n)'(x_0) \to 0 \quad \text{as } n \to \infty.$$

This completes the proof, since E has full measure in Z_A. □

We need the following variant of Lemma 5.8.

Lemma 5.13. *Let $I \subseteq \mathbb{R}$ be a bounded open interval. Assume that $A_n \to A$ in $C^{0,1}(\overline{I}, \operatorname{Herm}_T(d))$ as $n \to \infty$. Let $x_0 \in I \setminus Z_A$. Then there exist an open interval J with $x_0 \in J \subseteq I \setminus Z_A$, $n_0 \ge 1$, and $i \in \{1, \ldots, s\}$ such that*

1. *for all $n \ge n_0$, the curves $\underline{A} := A/\|A\|_2$ and $\underline{A}_n := A_n/\|A_n\|_2$ belong to $C^{0,1}(\overline{J}, \operatorname{Herm}_T^0(d))$ and satisfy $\underline{A}(J), \underline{A}_n(J) \subseteq \mathcal{V}_i$;*
2. *on J and for all $n \ge n_0$, we have*
$$U_i^*(\underline{A}) A\, U_i(\underline{A}) = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad U_i^*(\underline{A}_n) A_n\, U_i(\underline{A}_n) = \begin{pmatrix} B_n & 0 \\ 0 & C_n \end{pmatrix},$$
where $B, B_n \in C^{0,1}(\overline{J}, \operatorname{Herm}(d_1))$ and $C, C_n \in C^{0,1}(\overline{J}, \operatorname{Herm}(d_2))$ with $d_1 + d_2 = d$;
3. *on J and for all $n \ge n_0$, we have*
$$\mathcal{E}(A) = (\mathcal{E}(B), \mathcal{E}(C)), \quad \mathcal{E}(A_n) = (\mathcal{E}(B_n), \mathcal{E}(C_n));$$
4. *we have*
$$\|B - B_n\|_{C^{0,1}(\overline{J}, M_{d_1}(\mathbb{C}))} \to 0, \quad \|C - C_n\|_{C^{0,1}(\overline{J}, M_{d_2}(\mathbb{C}))} \to 0$$
as $n \to \infty$.

Proof. The existence of J, n_0, and i such that $\underline{A}(J), \underline{A}_n(J) \subseteq \mathcal{V}_i$, for $n \ge n_0$, follows from Lemma 5.8 (since $C^{0,1}$-convergence entails $W^{1,q}$-convergence). The other assertions follow easily from (4.8), (4.10), and Lemma 4.8. □

By Lemma 5.11, Lemma 5.13, and the induction hypothesis, for each $x_0 \in I \setminus Z_A$ there is an open interval J with $x_0 \in J \subseteq I \setminus Z_A$ such that

$$\mathcal{E}(A_n)' \to \mathcal{E}(A)' \quad \text{almost everywhere in } J \text{ as } n \to \infty.$$

Together with Lemma 5.12, this implies

$$\mathcal{E}(A_n)' \to \mathcal{E}(A)' \quad \text{almost everywhere in } I \text{ as } n \to \infty$$

and hence completes the proof of Theorem 5.10.

5.4. **Stability of the condition number of matrices.** We finish this section with the proof of Theorem 1.19.

Recall that the condition number of a nonsingular matrix $A \in M_d(\mathbb{C})$ is the ratio of the largest by the smallest singular value:

$$\kappa(A) = \frac{\sigma_1(A)}{\sigma_d(A)}.$$

Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Suppose that $A_0 \in W^{1,q}(I, M_d(\mathbb{C}))$ and $\inf_{x \in I} \sigma_d(A_0(x)) > 0$. We must show that $\kappa(A) \in W^{1,q}(I,\mathbb{R})$ is well-defined in a neighborhood $\mathcal{U}$ of A_0 in $W^{1,q}(I, M_d(\mathbb{C}))$ and that the induced map $\kappa : \mathcal{U} \to W^{1,q}(I,\mathbb{R})$ is continuous.

By the continuity of $\sigma_* : W^{1,q}(I, M_d(\mathbb{C})) \to W^{1,q}(I, \mathbb{R}^d)$, due to Theorem 1.17, and by Lemma 5.2 and (1.11), there is a neighborhood $\mathcal{U}$ of A_0 in $W^{1,q}(I, M_d(\mathbb{C}))$ such that

$$\delta := \inf_{A \in \mathcal{U}} \inf_{x \in I} \sigma_d(A(x)) > 0.$$

Thus the map $\kappa : \mathcal{U} \to W^{1,q}(I,\mathbb{R})$ is well-defined.

To see continuity, let $A, B \in \mathcal{U}$. Then, in I,

$$(\kappa(A) - \kappa(B))\sigma_d(A)\sigma_d(B) = (\sigma_1(A) - \sigma_1(B))\sigma_d(B) - \sigma_1(B)(\sigma_d(A) - \sigma_d(B))$$

and, almost everywhere in I,

$$\begin{aligned}
(\kappa(A)' - \kappa(B)')\sigma_d(A)\sigma_d(B) &+ (\kappa(A) - \kappa(B))\sigma_d(A)'\sigma_d(B) \\
&+ (\kappa(A) - \kappa(B))\sigma_d(A)\sigma_d(B)' \\
= (\sigma_1(A)' - \sigma_1(B)')\sigma_d(B) &- \sigma_1(B)'(\sigma_d(A) - \sigma_d(B)) \\
&+ (\sigma_1(A) - \sigma_1(B))\sigma_d(B)' - \sigma_1(B)(\sigma_d(A)' - \sigma_d(B)').
\end{aligned}$$

We see that, in I,

$$|\kappa(A) - \kappa(B)| \le \frac{\|B\|_2}{\delta^2}\big(|\sigma_1(A) - \sigma_1(B)| + |\sigma_d(A) - \sigma_d(B)|\big)$$

and, almost everywhere in I,

$$\begin{aligned}
|\kappa(A)' - \kappa(B)'| \le \frac{1}{\delta^2}\Big(&\|B\|_2|\sigma_1(A)' - \sigma_1(B)'| + |\sigma_1(B)'||\sigma_d(A) - \sigma_d(B)| \\
&+ |\sigma_d(B)'||\sigma_1(A) - \sigma_1(B)| + \|B\|_2|\sigma_d(A)' - \sigma_d(B)'| \\
&+ \big(\|B\|_2|\sigma_d(A)'| + \|A\|_2|\sigma_d(B)'|\big)|\kappa(A) - \kappa(B)|\Big).
\end{aligned}$$

Now continuity of $\kappa : \mathcal{U} \to W^{1,q}(I,\mathbb{R})$ can be easily deduced from Theorem 1.17, in view of Lemma 5.2 and (1.11). The proof of Theorem 1.19 is complete.

6. Eigenvalue stability for Hermitian matrices: multiparameter case

In this section, we will prove Theorem 1.1, Theorem 1.5, Corollary 1.6 and Theorem 1.7.

6.1. Proof of Theorem 1.1.

The next Theorem 6.1 implies Theorem 1.1 because $W^{1,q}(U, \mathrm{Herm}(d))$ is first-countable.

Theorem 6.1. *Let $1 \le q < \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. Let $A_n \to A$ in $W^{1,q}(U, \mathrm{Herm}(d))$ as $n \to \infty$. Then $\mathcal{E}(A_n) \to \mathcal{E}(A)$ in $W^{1,q}(U, \mathbb{R}^d)$ as $n \to \infty$.*

Proof. Set $\lambda := \mathcal{E}(A)$ and $\lambda_n := \mathcal{E}(A_n)$. By (3.2), we have

$$\|\lambda - \lambda_n\|_{L^q(U,\mathbb{R}^d)} \le \|A - A_n\|_{L^q(U,M_d(\mathbb{C}))} \to 0 \quad \text{as } n \to \infty.$$

Hence we have to show that

$$\|\partial_j \lambda - \partial_j \lambda_n\|_{L^q(U,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty, \tag{6.1}$$

for all $1 \le j \le m$. It is enough to show that there is a subsequence (n_k) with this property.

Let us first assume that $U = I_1 \times \cdots \times I_m$ is an open box with sides parallel to the coordinate axes. Let $j = 1$. We may assume that A and A_n are absolutely continuous on almost all line segments in U parallel to the x_1-axis and $\partial_1 A \in L^q(U, \mathrm{Herm}(d))$. Let $x = (x_1, x')$ for $x' \in U' = I_2 \times \cdots \times I_m$. By Tonelli's theorem,

$$\int_{U'} \int_{I_1} \|A(x_1, x') - A_n(x_1, x')\|_2^q \, dx_1 \, dx' = \|A - A_n\|^q_{L^q(U,M_d(\mathbb{C}))}$$

and

$$\int_{U'} \int_{I_1} \|\partial_1 A(x_1, x') - \partial_1 A_n(x_1, x')\|_2^q \, dx_1 \, dx' = \|\partial_1 A - \partial_1 A_n\|^q_{L^q(U,M_d(\mathbb{C}))}.$$

Thus there is a subsequence (n_k) such that

$$\int_{I_1} \|A(x_1, x') - A_{n_k}(x_1, x')\|_2^q \, dx_1 \to 0$$

and

$$\int_{I_1} \|\partial_1 A(x_1, x') - \partial_1 A_{n_k}(x_1, x')\|_2^q \, dx_1 \to 0$$

for almost every $x' \in U'$ as $k \to \infty$. By Theorem 5.1,

$$F_{1,n_k}(x') := \int_{I_1} \|\partial_1 \lambda(x_1, x') - \partial_1 \lambda_{n_k}(x_1, x')\|_2^q \, dx_1 \to 0 \quad \text{as } k \to \infty$$

for almost every $x' \in U'$.

By Vitali's convergence theorem 2.2, $\|\partial_1 A - \partial_1 A_n\|_{L^q(U,M_d(\mathbb{C}))} \to 0$ as $n \to \infty$ implies that the set $\{\|\partial_1 A\|_2^q + \|\partial_1 A_n\|_2^q : n \ge 1\} \subseteq L^1(U)$ is uniformly integrable and thus, by de la Vallée Poussin's criterion 2.1, there is a nonnegative increasing convex function G on $[0, \infty)$ such that $G(t)/t \to \infty$ as $t \to \infty$ and

$$\sup_{n \ge 1} \int_U G(2^q(\|\partial_1 A(x)\|_2^q + \|\partial_1 A_n(x)\|_2^q)) \, dx < \infty.$$

We claim that $\{|I_1|^{-1}F_{1,n_k} : k \geq 1\}$ and thus $\{F_{1,n_k} : k \geq 1\} \subseteq L^1(U')$ is uniformly integrable. By Tonelli's theorem, Jensen's inequality, and (3.4),

$$\begin{aligned}\int_{U'} G(|I_1|^{-1}F_{1,n_k}(x'))\,dx' &= \int_{U'} G\Big(\int_{I_1} \|\partial_1\lambda(x_1,x') - \partial_1\lambda_{n_k}(x_1,x')\|_2^q \,\frac{dx_1}{|I_1|}\Big)\,dx' \\ &\leq \int_{U'}\int_{I_1} G(\|\partial_1\lambda(x_1,x') - \partial_1\lambda_{n_k}(x_1,x')\|_2^q)\,\frac{dx_1}{|I_1|}\,dx' \\ &\leq \frac{1}{|I_1|}\int_U G(2^q(\|\partial_1\lambda(x)\|_2^q + \|\partial_1\lambda_{n_k}(x)\|_2^q))\,dx \\ &\leq \frac{1}{|I_1|}\int_U G(2^q(\|\partial_1 A(x)\|_2^q + \|\partial_1 A_{n_k}(x)\|_2^q))\,dx\end{aligned}$$

is bounded by a constant independent of k, which implies the claim, again by de la Vallée Poussin's criterion 2.1.

By Vitali's convergence theorem 2.2 and Tonelli's theorem,

$$\|\partial_1\lambda - \partial_1\lambda_{n_k}\|_{L^q(U,M_d(\mathbb{C}))} \to 0 \quad \text{as } k \to \infty.$$

The reasoning for $2 \leq j \leq m$ is analogous. Thus (6.1) is proved, in the case $U = I_1 \times \cdots \times I_m$.

Let now U be a general open bounded subset of $\mathbb{R}^m$. Since U is a countable union of open bounded boxes V_i with sides parallel to the coordinate axes, (6.1) follows from Lemma 2.3 applied to $f_n := \|\partial_j\lambda - \partial_j\lambda_n\|_2^q$, noting that $\{f_n\}$ is uniformly integrable which can be seen as in the proof of Lemma 5.9. This ends the proof. □

6.2. **Proofs of Theorem 1.5 and Theorem 1.7.** First observe that Theorem 1.7 follows easily from Theorem 5.10 (applying the latter coordinate-wise).

As already pointed out, Theorem 1.5 is an immediate consequence of Theorem 1.1. Using Theorem 1.7, we deduce the following slightly stronger version.

Theorem 6.2. *Let $U \subseteq \mathbb{R}^m$ be open and bounded. Assume that $A_n \to A$ in $C^{0,1}(\overline{U}, \mathrm{Herm}(d))$ as $n \to \infty$. Then*

$$\|\mathcal{E}(A) - \mathcal{E}(A_n)\|_{L^\infty(U,\mathbb{R}^d)} \leq \|A - A_n\|_{L^\infty(U,M_d(\mathbb{C}))} \to 0 \tag{6.2}$$

and, for each $1 \leq j \leq m$ and all $1 \leq q < \infty$,

$$\|\partial_j(\mathcal{E}(A)) - \partial_j(\mathcal{E}(A_n))\|_{L^q(U,\mathbb{R}^d)} \to 0 \tag{6.3}$$

as $n \to \infty$.

Proof. First, (6.2) follows immediately from (3.2).

By (1.4),

$$|\mathcal{E}(A_n)|_{C^{0,1}(\overline{U},\mathbb{R}^d)} \leq |A_n|_{C^{0,1}(\overline{U},M_d(\mathbb{C}))} \leq L,$$

for a constant $L > 0$ independent of n. In particular, for each $1 \leq j \leq m$ and almost every $x \in U$,

$$\|\partial_j(\mathcal{E}(A_n))(x)\|_2 \leq L.$$

We conclude (6.3), by Theorem 1.7 and the dominated convergence theorem. □

6.3. **Proof of Corollary 1.6.** Corollary 1.6 is an immediate consequence of the following corollary of Theorem 6.1 and Example 7.1.

Corollary 6.3. *Let $U \subseteq \mathbb{R}^m$ be a bounded open Lipschitz domain. Let $A_n \to A$ in $C^{0,1}(\overline{U}, \operatorname{Herm}(d))$ as $n \to \infty$. Then, for each $0 < \alpha < 1$,*

$$\|\mathcal{E}(A) - \mathcal{E}(A_n)\|_{C^{0,\alpha}(\overline{U},\mathbb{R}^d)} \to 0 \quad \text{ as } n \to \infty.$$

Proof. The assertion follows from Theorem 6.1 and Morrey's inequality,

$$\|\mathcal{E}(A) - \mathcal{E}(A_n)\|_{C^{0,\alpha}(\overline{U},\mathbb{R}^d)} \le C\, \|\mathcal{E}(A) - \mathcal{E}(A_n)\|_{W^{1,q}(U,\mathbb{R}^d)},$$

where $\alpha = 1 - m/q$, $q > m$, and $C = C(d, m, q, U)$. □

7. Examples

Example 7.1 shows that the characteristic map $\mathcal{E} : C^{0,1}(\overline{U}, \operatorname{Herm}(d)) \to C^{0,1}(\overline{U}, \mathbb{R}^d)$ is not continuous. This example appeared in [PR24a, Examples 1.12 and 7.13]; we repeat it for the reader's convenience.

Example 7.1. The sequence $(A_n)_n$ of curves of symmetric 2×2 matrices

$$A_n(x) = \begin{pmatrix} \frac{1}{n} & x \\ x & -\frac{1}{n} \end{pmatrix}, \quad x \in \mathbb{R}, \tag{7.1}$$

converges to

$$A(x) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \quad x \in \mathbb{R},$$

uniformly in all derivatives on every compact interval. We have $\mathcal{E}(A_n) = (-a_n, a_n)$ and $\mathcal{E}(A) = (-a, a)$, where

$$a_n(x) := \sqrt{x^2 + \tfrac{1}{n^2}} \quad \text{ and } \quad a(x) := |x|.$$

Then $\mathcal{E}(A_n) \not\to \mathcal{E}(A)$ as $n \to \infty$ in the $C^{0,1}$ topology.

Indeed, for each bounded open interval $I \subseteq \mathbb{R}$ containing 0,

$$\begin{aligned} |a - a_n|_{C^{0,1}(\overline{I})} &\ge \sup_{0<x\in I} \Big| \frac{(a(x) - a_n(x)) - (a(0) - a_n(0))}{x} \Big| \\ &= \sup_{0<x\in I} \Big| \frac{x - \sqrt{x^2 + \frac{1}{n^2}} + \frac{1}{n}}{x} \Big| \ge \Big| \frac{\frac{1}{n} - \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} + \frac{1}{n}}{\frac{1}{n}} \Big| = 2 - \sqrt{2}, \end{aligned}$$

for large enough n.

Furthermore, observe that

$$a_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}}$$

tends pointwise to $a'(x) = \operatorname{sgn}(x)$ for all $x \neq 0$ but not uniformly on any neighborhood of 0:

$$a_n'(\pm\tfrac{1}{n}) = \pm\frac{1}{\sqrt{2}}.$$

This also violates the first conclusion of Corollary 1.4 for $q = \infty$.

The following Example 7.2 proves that, for no $1 \le q < \infty$, the characteristic map $\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,1}_q(\overline{U}, \mathbb{R}^d)$ is uniformly continuous.

Example 7.2. Let $\varphi_n(x) : I := (0,1) \to \mathbb{R}$ be the sawtooth function defined by

$$\varphi_n(x) = (-1)^k (x - \tfrac{k}{n}) + \varphi_n(\tfrac{k}{n}) \quad \text{ if } k = 0, \dots, n-1 \text{ and } \tfrac{k}{n} \le x \le \tfrac{k+1}{n}.$$

Then $|\varphi_n'| = 1$ almost everywhere in I. Moreover, $0 \le \varphi_n \le \frac{1}{n}$ in I and $\varphi_n \ge \frac{1}{2n}$ on a measurable subset E_n of I of measure $|E_n| = \frac{1}{2}$.

Consider

$$A_n(x) = \begin{pmatrix} \frac{1}{n} & \varphi_n(x) \\ \varphi_n(x) & -\frac{1}{n} \end{pmatrix}, \quad B_n(x) = \begin{pmatrix} \frac{1}{2n} & \varphi_n(x) \\ \varphi_n(x) & -\frac{1}{2n} \end{pmatrix}, \quad x \in I.$$

Then

$$\|A_n\|_{C^{0,1}(\overline{I}, M_2(\mathbb{C}))} = \frac{2}{n} + \sqrt{2}, \quad \|B_n\|_{C^{0,1}(\overline{I}, M_2(\mathbb{C}))} = \frac{\sqrt{5}}{\sqrt{2}n} + \sqrt{2},$$

are bounded and

$$\|A_n - B_n\|_{C^{0,1}(\overline{I}, M_2(\mathbb{C}))} = \frac{\sqrt{2}}{2n} \to 0 \quad \text{ as } n \to \infty.$$

The nonnegative eigenvalues a_n and b_n of A_n and B_n,

$$a_n = \sqrt{\varphi_n^2 + \tfrac{1}{n^2}}, \quad b_n = \sqrt{\varphi_n^2 + \tfrac{1}{4n^2}},$$

are Lipschitz and satisfy, almost everywhere in I,

$$\begin{aligned} |a_n' - b_n'| &= \varphi_n |\varphi_n'| \Big| \frac{n}{\sqrt{n^2\varphi_n^2 + 1}} - \frac{2n}{\sqrt{4n^2\varphi_n^2+1}} \Big| \\ &= \varphi_n \frac{3n}{\sqrt{n^2\varphi_n^2+1}\sqrt{4n^2\varphi_n^2+1}\big(2\sqrt{n^2\varphi_n^2+1} + \sqrt{4n^2\varphi_n^2+1}\big)}. \\ &\ge \frac{n\,\varphi_n}{6}, \end{aligned} \tag{7.2}$$

since $\varphi_n \le \frac{1}{n}$ so that the denominator is bounded by

$$\sqrt{2}\sqrt{5}(2\sqrt{2} + \sqrt{5}) = 4\sqrt{5} + 5\sqrt{2} \le 18.$$

Thus, almost everywhere in E_n,

$$|a_n' - b_n'| \ge \frac{1}{2n} \cdot \frac{n}{6} = \frac{1}{12}$$

and, consequently,

$$\|a_n' - b_n'\|^q_{L^q(I)} \ge \|a_n' - b_n'\|^q_{L^q(E_n)} \ge \frac{1}{12^q}|E_n| = \frac{1}{2 \cdot 12^q},$$

for every $1 \le q < \infty$, showing that *the map* $\mathcal{E} : C^{0,1}(\overline{I}, \mathrm{Herm}(2)) \to C^{0,1}_q(\overline{I}, \mathbb{R}^2)$ *is not uniformly continuous.*

The next Example 7.3 shows that, for no $0 < \alpha < 1$, the characteristic map $\mathcal{E} : C^{0,1}(\overline{U}, \mathrm{Herm}(d)) \to C^{0,\alpha}(\overline{U}, \mathbb{R}^d)$ is uniformly continuous. In contrast to Example 7.2, the curves of symmetric matrices are smooth but their Lipschitz constants are unbounded.

Example 7.3. Let $\alpha \in (0,1)$ and set $r = \frac{\alpha}{1-\alpha}$. Consider

$$A_n(x) = \begin{pmatrix} \frac{1}{n^r} & nx \\ nx & -\frac{1}{n^r} \end{pmatrix}, \quad B_n(x) = \begin{pmatrix} \frac{1}{2n^r} & nx \\ nx & -\frac{1}{2n^r} \end{pmatrix}, \quad x \in \mathbb{R}.$$

Then

$$A_n - B_n = \begin{pmatrix} \frac{1}{2n^r} & 0 \\ 0 & -\frac{1}{2n^r} \end{pmatrix}$$

tends to zero uniformly in all derivatives as $n \to \infty$. The nonnegative eigenvalues of A_n and B_n are

$$a_n(x) = \sqrt{n^2x^2 + \tfrac{1}{n^{2r}}} \quad \text{and} \quad b_n(x) = \sqrt{n^2x^2 + \tfrac{1}{4n^{2r}}}.$$

For each bounded open interval $I \subseteq \mathbb{R}$ containing 0,

$$\begin{aligned} |a_n - b_n|_{C^{0,\alpha}(\overline{I})} &\ge \sup_{0<x\in I} \frac{|(a_n(x) - b_n(x)) - (a_n(0) - b_n(0))|}{x^\alpha} \\ &= \sup_{0<x\in I} \frac{\left|\sqrt{n^2x^2 + \frac{1}{n^{2r}}} - \sqrt{n^2x^2 + \frac{1}{4n^{2r}}} - \frac{1}{n^r} + \frac{1}{2n^r}\right|}{x^\alpha} \end{aligned}$$

and setting $x = n^{-1/(1-\alpha)}$,

$$\begin{aligned} &\ge \frac{\left|\sqrt{\frac{1}{n^{2r}} + \frac{1}{n^{2r}}} - \sqrt{\frac{1}{n^{2r}} + \frac{1}{4n^{2r}}} - \frac{1}{n^r} + \frac{1}{2n^r}\right|}{\frac{1}{n^r}} \\ &= \frac{\sqrt{5}}{2} + \frac{1}{2} - \sqrt{2} \end{aligned}$$

for large enough n, showing that *the map* $\mathcal{E} : C^{0,1}(\overline{I}, \mathrm{Herm}(2)) \to C^{0,\alpha}(\overline{I}, \mathbb{R}^2)$ *is not uniformly continuous.*

It also follows that $\mathcal{E} : C^{0,1}(\overline{I}, \mathrm{Herm}(2)) \to C^{0,1}_q(\overline{I}, \mathbb{R}^2)$ is not uniformly continuous, for every $1 < q < \infty$, by Morrey's inequality.

As discussed in Section 1.3, it would be desirable to find effective moduli of continuity for restrictions of the characteristic map $\mathcal{E}$ to (relatively) compact subspaces of $C^{0,1}(\overline{U}, \mathrm{Herm}(d))$, e.g., the subset K defined in (1.10). The following Example 7.4 excludes α-Hölder continuity of $\mathcal{E}|_K : K \to C^{0,1}_q(\overline{U}, \mathbb{R}^d)$ for a certain range of α depending on q.

Example 7.4. Let A_n, for $n \ge 1$, be the smooth curves of symmetric matrices defined in (7.1) and a_n the nonnegative eigenvalue of A_n. For $x > 0$, we find (by a computation similar to (7.2))

$$|a_n'(x) - a_{2n}'(x)| = \frac{3nx}{\sqrt{n^2x^2+1}\sqrt{4n^2x^2+1}(\sqrt{4n^2x^2+1} + 2\sqrt{n^2x^2+1})}.$$

If $x \in (0, \frac{1}{n})$, then the denominator is bounded by 18. Thus, for fixed $1 \le q < \infty$,

$$\|a_n' - a_{2n}'\|^q_{L^q((0,1))} \ge \|a_n' - a_{2n}'\|^q_{L^q((0,\frac{1}{n}))} \ge \frac{n^q}{6^q} \int_0^{1/n} x^q \, dx = \frac{1}{6^q(q+1)n},$$

so that

$$\|a_n' - a_{2n}'\|_{L^q((0,1))} \ge \frac{1}{6(q+1)^{1/q}\, n^{1/q}},$$

while

$$\|A_n - A_{2n}\|_{L^\infty(\mathbb{R},M_2(\mathbb{C}))} = \frac{\sqrt{2}}{2n} \quad \text{and} \quad A_n' - A_{2n}' \equiv 0.$$

This implies that, *for no* $\alpha \in (q^{-1}, 1]$*, the map* $\mathcal{E} : C^{0,1}(\overline{I}, \mathrm{Herm}(2)) \to C^{0,1}_q(\overline{I}, \mathbb{R}^2)$ *is* α*-Hölder continuous, where* $I \subseteq \mathbb{R}$ *is any bounded open interval containing* 0*.*

8. d-valued Sobolev functions

The aim of this section is to set up the background and tools for Theorem 1.8 and its proof. We follow [PR24b, Section 3].

8.1. **Unordered d-tuples of complex numbers.** The symmetric group S_d acts on $\mathbb{C}^d$ by permuting the coordinates,

$$\sigma z = \sigma(z_1, \ldots, z_d) := (z_{\sigma(1)}, \ldots, z_{\sigma(d)}), \quad \sigma \in \mathrm{S}_d,\ z \in \mathbb{C}^d,$$

and thus induces an equivalence relation. The equivalence class of $z = (z_1, \ldots, z_d)$ is the *unordered tuple* $[z] = [z_1, \ldots, z_d]$. Let us consider the set

$$\mathcal{A}_d(\mathbb{C}) := \{[z] : z \in \mathbb{C}^d\}$$

of unordered complex d-tuples and equip it with the metric

$$\mathbf{d}_2([z], [w]) := \min_{\sigma \in \mathrm{S}_d} \|z - \sigma w\|_2.$$

Then $(\mathcal{A}_d(\mathbb{C}), \mathbf{d}_2)$ is a complete metric space and the induced map $[\ ] : \mathbb{C}^d \to \mathcal{A}_d(\mathbb{C})$ is Lipschitz with $\mathrm{Lip}([\ ]) = 1$.

The element $[z_1, \ldots, z_d]$ of $\mathcal{A}_d(\mathbb{C})$ can also be represented by the sum $\sum_{i=1}^d [\![z_i]\!]$, where $[\![z_i]\!]$ denotes the Dirac mass at $z_i \in \mathbb{C}$. If normalized, i.e., $\frac{1}{d}\sum_{i=1}^d [\![z_i]\!]$, then, in this picture, $\frac{1}{\sqrt{d}}\mathbf{d}_2$ is induced by the L^2-based Wasserstein metric on the space of probability measures on $\mathbb{C}$. (For this reason, in [PR24b, Section 3], the factor $\frac{1}{\sqrt{d}}$ was integrated in the definition of $\mathbf{d}_2$.)

8.2. **d-valued Sobolev functions.** Due to Almgren [Alm00], see also [DLS11], there exist an integer $N = N(d)$ and an injective Lipschitz mapping

$$\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$$

with $\mathrm{Lip}(\Delta) \le 1$ and Lipschitz inverse $\Delta|^{-1}_{\Delta(\mathcal{A}_d(\mathbb{C}))}$ with $\mathrm{Lip}(\Delta|^{-1}_{\Delta(\mathcal{A}_d(\mathbb{C}))}) \le C(d)$. Moreover, there is a Lipschitz retraction of $\mathbb{R}^N$ onto $\Delta(\mathcal{A}_d(\mathbb{C}))$.

Following Almgren, the bi-Lipschitz embedding Δ can be used to define Sobolev spaces of $\mathcal{A}_d(\mathbb{C})$-valued functions: for open $U \subseteq \mathbb{R}^m$ and $1 \le q \le \infty$ set

$$W^{1,q}(U, \mathcal{A}_d(\mathbb{C})) := \{f : U \to \mathcal{A}_d(\mathbb{C}) : \Delta \circ f \in W^{1,q}(U, \mathbb{R}^N)\}.$$

For an equivalent intrinsic definition of $W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$, see [DLS11, Definition 0.5 and Theorem 2.4]. Then $W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$ carries the metric

$$\rho^{1,q}_\Delta(f, g) := \|\Delta \circ f - \Delta \circ g\|_{W^{1,q}(U,\mathbb{R}^N)} \tag{8.1}$$

which makes it a complete metric space (where functions that coincide almost everywhere are identified); see [PR24b, Lemma 3.1].

We will see in Theorem 8.11 that, for $1 \le q < \infty$, the topology on $W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$ induced by $\rho^{1,q}_\Delta$ is independent of the choice of the Almgren embedding Δ.

Remark 8.1. In [Alm00] and [DLS11], $W^{1,q}(U, \mathcal{A}_d(\mathbb{R}^\ell))$ is studied for arbitrary ℓ.

8.3. **Almgren's embedding.** Let us recall Almgren's construction of Δ for $\mathcal{A}_d(\mathbb{C})$.

Definition 8.2 (Almgren map)**.** We say that

$$H : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^d$$

is an *Almgren map* if there is a unit complex number $\theta \in \mathbb{C}$ such that $H([z])$ is an array of d real numbers $\eta(z_i) := \mathrm{Re}(\theta z_i)$ arranged in increasing order, i.e.,

$$H([z]) = H([z_1, \dots, z_d]) = (\eta(z_{\sigma(1)}), \dots, \eta(z_{\sigma(d)}),$$

where $\sigma \in \mathrm{S}_d$ is chosen such that $\eta(z_{\sigma(1)}) \le \eta(z_{\sigma(2)}) \le \cdots \le \eta(z_{\sigma(d)})$. We also say that H is the Almgren map associated to the real linear form η.

By Almgren's combinatorial lemma (see e.g. [DLS11, Lemma 2.3]) there exists $\alpha = \alpha(d) > 0$ and a finite set of linear forms $\Lambda = \{\eta_1, \dots \eta_h\}$, where $\eta_l(z) := \mathrm{Re}(\theta_l z)$ for unit complex numbers θ_l, with the following property: given any set of d^2 complex numbers, $\{z_1, \dots, z_{d^2}\} \subseteq \mathbb{C}$, there exists $\eta_l \in \Lambda$ such that

$$|\eta_l(z_k)| \ge \alpha |z_k| \quad \text{for all } k \in \{1, \dots, d^2\}. \tag{8.2}$$

For instance, we may take $h = 2d^2+1$ and $\{\eta_1, \dots, \eta_l\}$ induced by the set $\{\theta_1, \dots \theta_h\}$ of all h-th roots of unity.

Let H_l denote the Almgren map associated to $\eta_l \in \Lambda$. The *Almgren embedding* $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$, $N = dh$, is then defined by

$$\Delta([z]) = h^{-1/2}(H_1([z]), \dots, H_h([z])). \tag{8.3}$$

The property (8.2) guarantees that Δ is as required in Section 8.2; see e.g. [DLS11, Section 2.1.2].

8.4. **The space $\mathcal{A}_d(\mathbb{R})$.** Consider the subspace $\mathcal{A}_d(\mathbb{R}) := \{[x] : x \in \mathbb{R}^d\}$ of $\mathcal{A}_d(\mathbb{C})$. For $x \in \mathbb{R}^d$, let $x^\uparrow \in \mathbb{R}^d$ be the representative of the equivalence class $[x]$ with increasingly ordered coordinates. Clearly, $x^\uparrow$ only depends on $[x]$ and thus we have an injective map $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$. It is a right-inverse of $[\ ] : \mathbb{R}^d \to \mathcal{A}_d(\mathbb{R})$.

Lemma 8.3 ([PR24a, Lemma 7.1])**.** *We have*

$$\mathbf{d}_2([x],[y]) = \|x^\uparrow - y^\uparrow\|_2, \quad x, y \in \mathbb{R}^d.$$

In particular, $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$ *and* $[\ ] : \mathbb{R}^d \to \mathcal{A}_d(\mathbb{R})$ *are Lipschitz maps.*

It is easy to check that the map $H = (\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$ is an Almgren map (for $\mathbb{R}$ instead of $\mathbb{C}$, associated to the real linear form $\eta = \mathrm{id}$) and (8.2) is trivially true with $\Lambda = \{\mathrm{id}\}$. Thus $\Delta = H = (\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$ is an Almgren embedding.

Consequently, we can equip the space

$$W^{1,q}(U, \mathcal{A}_d(\mathbb{R})) = \{f : U \to \mathcal{A}_d(\mathbb{R}) : f^\uparrow \in W^{1,q}(U, \mathbb{R}^d)\}$$

with the metric

$$\rho^{1,q}_\uparrow(f,g) := \|f^\uparrow - g^\uparrow\|_{W^{1,q}(U,\mathbb{R}^d)}.$$

Thus Theorem 1.1 can be seen as a special case of Theorem 1.8.

8.5. **Curves of class $W^{1,q}$ in $\mathcal{A}_d(\mathbb{C})$.** Let $I \subseteq \mathbb{R}$ be an open bounded interval. We recall an equivalent definition of $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ (see [DLS11, Definition 0.5]) which is independent of Almgren's embedding.

Definition 8.4 (Intrinsic definition)**.** A measurable function $f : I \to \mathcal{A}_d(\mathbb{C})$ belongs to $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ $(1 \le q \le \infty)$ if there exists a function $\varphi \in L^q(I, \mathbb{R}_{\ge 0})$ such that

(i) $x \mapsto \mathbf{d}_2(f(x), T) \in W^{1,q}(I)$ for all $T \in \mathcal{A}_d(\mathbb{C})$;

(ii) $|(\mathbf{d}_2(f, T))'| \le \varphi$ almost everywhere in I for all $T \in \mathcal{A}_d(\mathbb{C})$.

The minimal function $\tilde{\varphi}$ fulfilling (ii), that is,

$$\tilde{\varphi} \le \varphi \quad \text{almost everywhere for every other } \varphi \text{ satisfying (ii),}$$

is measurable and we denote it by $|Df|$.

Proposition 8.5 ([DLS11, Proposition 1.2])**.** *Let $f \in W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$. Then,*

(*a*) *$f \in AC(I, \mathcal{A}_d(\mathbb{C}))$ and, moreover, $f \in C^{0,1-\frac{1}{q}}(I, \mathcal{A}_d(\mathbb{C}))$ for $q > 1$;*

(*b*) *there exists a parameterization $f_1, \ldots, f_d \in W^{1,q}(I, \mathbb{C})$ of f, i.e.,*

$$f = [\![f_1]\!] + \cdots + [\![f_d]\!],$$

such that $|Df_i| = |f_i'| \le |Df|$ almost everywhere.

Actually, the proof of Proposition 8.5 in [DLS11] implies that $f \in W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ belongs to $AC^q(I, \mathcal{A}_d(\mathbb{C}))$ in the sense of Section 2.4. In the situation of Proposition 8.5, we will always mean without further mention that f and f_i are the continuous representatives.

Following [DLS11, Definition 1.9], we can also define the differential Df.

Definition 8.6 (Differential)**.** Let $f = \sum_i [\![f_i]\!] : I \to \mathcal{A}_d(\mathbb{C})$ and $x_0 \in I$. We say that f is *differentiable* at x_0 if there exist d complex numbers L_i satisfying:

(i) $\mathbf{d}_2(f(x), T_{x_0} f(x)) = o(|x - x_0|)$, where

$$T_{x_0} f(x) := \sum_i [\![f_i(x_0) + L_i \cdot (x - x_0)]\!];$$

(ii) $L_i = L_j$ if $f_i(x_0) = f_j(x_0)$.

The d-valued map $T_{x_0} f$ is called the *first-order approximation* of f at x_0. We denote L_i by $Df_i(x_0)$ and the point $Df(x_0) := \sum_i [\![Df_i(x_0)]\!] \in \mathcal{A}_d(\mathbb{C})$ is called the *differential* of f at x_0.

By Definition 8.6(ii), the notation is consistent (see [DLS11, Remark 1.11]): if $g_1, \ldots, g_d$ is another parameterization of f, f is differentiable at x_0, and $\sigma \in \mathrm{S}_d$ is such that $g_i(x_0) = f_{\sigma(i)}(x_0)$ for all $1 \le i \le d$, then $Dg_i(x_0) = Df_{\sigma(i)}(x_0)$.

By Proposition 8.5, every $f \in W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ is differentiable almost everywhere and, if $f_1, \ldots, f_d \in W^{1,q}(I, \mathbb{C})$ is a parameterization of f, then $Df = \sum_i [\![f_i']\!]$ almost everywhere; see [PR24b, Section 3.4].

8.6. **A distance notion on $W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$.** The following definition is taken from [PR24b, Definition 3.6].

Definition 8.7 (The distance $\mathbf{d}_E^{1,q}$)**.** Let $f,g \in W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$ and let

$$f = [\![f_1]\!] + \cdots + [\![f_d]\!], \quad g = [\![g_1]\!] + \cdots + [\![g_d]\!]$$

be parameterizations of f, g with $f_i, g_i \in W^{1,q}(I,\mathbb{C})$ as in Proposition 8.5. Set

$$\mathbf{s}_0(f,g)(x) := \mathbf{d}_2(f(x), g(x)).$$

Fix an arbitrary ordering of the elements of S_d. For $x \in I$, let

$$\tau(x) := \min\Big\{\tau \in \mathrm{S}_d : \Big(\sum_i |f_i(x) - g_{\tau(i)}(x)|^2\Big)^{1/2} = \mathbf{d}_2(f(x), g(x))\Big\}.$$

For $x \in I$ such that $Df(x) = \sum_i [\![Df_i(x)]\!]$ and $Dg(x) = \sum_i [\![Dg_i(x)]\!]$ exist in the sense of Definition 8.6, set

$$\mathbf{s}_1(f,g)(x) := \max \Big(\sum_i |Df_i(x) - Dg_{\tau(x)(i)}(x)|^2\Big)^{1/2}, \tag{8.4}$$

where the maximum is taken over all orderings of S_d. By the remarks above, $\mathbf{s}_1(f,g)(x)$ is defined for almost every $x \in I$ and independent of the choices of parameterizations $f_1,\ldots,f_d$ and $g_1,\ldots,g_d$ of f and g.

For each measurable subset $E \subseteq I$, we define

$$\mathbf{d}_E^{1,q}(f,g) := \|\mathbf{s}_0(f,g)\|_{L^\infty(E)} + \|\mathbf{s}_1(f,g)\|_{L^q(E)}$$

which is justified, since the functions $\mathbf{s}_i(f,g) : I \to \mathbb{R}$, for $i = 0,1$, are Borel measurable, by [PR24b, Lemma 3.7].

Lemma 8.8 ([PR24b, Lemma 3.8])**.** *Let $I \subseteq \mathbb{R}$ be a bounded open interval and $E \subseteq I$ a measurable set. Let $f,g \in W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$. Then:*

(1) $\mathbf{d}_E^{1,q}(f,f) = 0$.

(2) $\mathbf{d}_E^{1,q}(f,g) = 0$ *implies* $f = g$ *on* E.

(3) $\mathbf{d}_E^{1,q}(f,g) = \mathbf{d}_E^{1,q}(g,f)$.

In particular, $\mathbf{d}_I^{1,q}$ is a semimetric on $W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$.

8.7. **Convergence in $W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$.** There is a notion of *weak convergence* in $W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$, see [DLS11, Definition 2.9], which is not appropriate for our purpose. We introduce a stronger notion of convergence based on the semimetric $\mathbf{d}_I^{1,q}$.

Definition 8.9 (Strong convergence)**.** Let $f_n, f \in W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$. We say that f_n *converges to f in* $W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$ as $n \to \infty$, and we write $f_n \to f$, if

$$\mathbf{d}_I^{1,q}(f,f_n) \to 0 \quad \text{as } n \to \infty.$$

Strong convergence in $W^{1,q}(I,\mathcal{A}_d(\mathbb{C}))$ is equivalent to convergence with respect to the topology induced by the metric $\rho_\Delta^{1,q}$ (see (8.1)), for some, equivalently every, Almgren embedding Δ:

Theorem 8.10 ([PR24b, Theorem 3.11]). *Let $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$ be an Almgren embedding. Then $f_n \to f$ in $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ as $n \to \infty$ if and only if*

$$\rho_\Delta^{1,q}(f, f_n) \to 0 \quad \text{as } n \to \infty.$$

In particular, the topology induced by the metric $\rho_\Delta^{1,q}$ on $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ does not depend on the choice of the Almgren embedding Δ. We shall see below that this is also true in several variables.

8.8. **The topology on $W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$ induced by $\rho_\Delta^{1,q}$ is independent of Δ.** The following theorem is an improved version of [PR24b, Theorem 10.2].

Theorem 8.11. *Let $\Delta^i : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^{N_i}$, for $i = 1, 2$, be two Almgren embeddings. Let $U \subseteq \mathbb{R}^m$ be open and bounded and $1 \le q < \infty$. Let $f, f_n \in W^{1,q}(U, \mathcal{A}_d(\mathbb{C}))$, for $n \ge 1$. Then,*

$$\|\Delta^1 \circ f - \Delta^1 \circ f_n\|_{W^{1,q}(U,\mathbb{R}^{N_1})} \to 0$$

if and only if

$$\|\Delta^2 \circ f - \Delta^2 \circ f_n\|_{W^{1,q}(U,\mathbb{R}^{N_2})} \to 0$$

as $n \to \infty$.

Proof. The map

$$\Delta^2 \circ (\Delta^1)|_{\Delta^1(\mathcal{A}_d(\mathbb{C}))}^{-1} : \Delta^1(\mathcal{A}_d(\mathbb{C})) \to \mathbb{R}^{N_2}$$

is Lipschitz and has a Lipschitz extension Γ to all of $\mathbb{R}^{N_1}$. Thus (see [ADM90]) superposition with Γ defines a bounded map from $W^{1,q}(U, \mathbb{R}^{N_1})$ to $W^{1,q}(U, \mathbb{R}^{N_2})$, where, setting $C := \operatorname{Lip}(\Gamma)$, for $F \in W^{1,q}(U, \mathbb{R}^{N_1})$,

$$\|\Gamma \circ F\|_2 \le C\, \|F\|_2,$$

because $\Gamma(0) = 0$, and

$$\|\nabla(\Gamma \circ F)\|_2 \le C\, \|\nabla F\|_2 \tag{8.5}$$

almost everywhere in U.

Set $F^i := \Delta^i \circ f$ and $F_n^i := \Delta^i \circ f_n$, for $i = 1, 2$. Assume that

$$\|F^1 - F_n^1\|_{W^{1,q}(U,\mathbb{R}^{N_1})} \to 0 \quad \text{as } n \to \infty. \tag{8.6}$$

Then $\{\|F^1\|_2^q + F_n^1\|_2^q : n \ge 1\}$ and $\{\|\partial_j F^1\|_2^q + \|\partial_j F_n^1\|_2^q : n \ge 1\}$, for $1 \le j \le m$, are uniformly integrable subsets of $L^1(U)$. Since $F^2 = \Gamma \circ F^1$ and $F_n^2 = \Gamma \circ F_n^1$, we may conclude similarly as in the proof of Lemma 5.9 (using (8.5)) that the sets $\{\|F^2 - F_n^2\|_2^q : n \ge 1\}$ and $\{\|\partial_j F^2 - \partial_j F_n^2\|_2^q : n \ge 1\}$, for $1 \le j \le m$, are uniformly integrable.

Let us assume that $U = I_1 \times \cdots \times I_m$ is an open bounded box with sides parallel to the coordinate axes. Let $j = 1$. For $x' \in U' = I_2 \times \cdots \times I_m$ and $i = 1, 2$, consider

$$A_n^i(x') = \int_{I_1} \left\|\partial_1 F^i(x_1, x') - \partial_1 F_n^i(x_1, x')\right\|_2^q \, dx_1,$$

$$B_n^i(x') = \int_{I_1} \left\|F^i(x_1, x') - F_n^i(x_1, x')\right\|_2^q \, dx_1.$$

By (8.6) and Tonelli's theorem,

$$\int_{U'} A_n^1(x')\,dx' \to 0 \quad \text{and} \quad \int_{U'} B_n^1(x')\,dx' \to 0 \quad \text{as } n \to \infty.$$

Thus there is a subsequence (n_k) such that $A^1_{n_k}(x') \to 0$ and $B^1_{n_k}(x') \to 0$ for almost every $x' \in U'$ as $k \to \infty$. For each such x', Theorem 8.10 implies that $A^2_{n_k}(x') \to 0$ and $B^2_{n_k}(x') \to 0$ as $k \to \infty$.

Tonelli's theorem and uniform integrability of the sets $\{\|F^2 - F^2_n\|_2^q : n \ge 1\}$ and $\{\|\partial_j F^2 - \partial_j F^2_n\|_2^q : n \ge 1\}$, for $1 \le j \le m$, imply, invoking de la Vallée Poussin's criterion 2.1, that $\{A^2_n : n \ge 1\}$ and $\{B^2_n : n \ge 1\}$ are uniformly integrable. (See the proof of Theorem 6.1.)

Then Vitali's convergence theorem 2.2 and Tonelli's theorem yield that

$$\|F^2 - F^2_{n_k}\|_{L^q(U,\mathbb{R}^{N_2})} \to 0 \quad \text{and} \quad \|\partial_1 F^2 - \partial_1 F^2_{n_k}\|_{L^q(U,\mathbb{R}^{N_2})} \to 0$$

as $k \to \infty$. Since the partial derivatives ∂_j, for $2 \le j \le m$, can be treated in the same way, we have showed that there is a subsequence (n_k) such that

$$\|F^2 - F^2_{n_k}\|_{W^{1,q}(U,\mathbb{R}^{N_2})} \to 0 \quad \text{as } k \to \infty$$

which implies

$$\|F^2 - F^2_{n}\|_{W^{1,q}(U,\mathbb{R}^{N_2})} \to 0 \quad \text{as } n \to \infty,$$

in the case that $U = I_1 \times \cdots \times I_m$.

The case that U is a general open bounded subset of $\mathbb{R}^m$ follows from Lemma 2.3 (applied to $f_n := \|F^2 - F^2_n\|_2^q$ and $f_n := \|\partial_j F^2 - \partial_j F^2_n\|_2^q$), since U is a countable union of bounded open boxes with sides parallel to the coordinate axes. □

9. The characteristic map for normal matrices

We are ready to introduce the characteristic map for normal matrices.

Proposition 9.1. *Let $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$ be an Almgren embedding. Let $1 \le q \le \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. Then the map*

$$\mathcal{E}_u : W^{1,q}(U, \mathrm{Norm}(d)) \to W^{1,q}(U, \mathcal{A}_d(\mathbb{C})), \quad A \mapsto \Lambda \circ A,$$

is well-defined and bounded, satisfying

$$\|\Delta(\mathcal{E}_u(A)(x))\|_2 \le C\,\|A(x)\|_2 \quad \text{and} \quad \|\nabla(\Delta(\mathcal{E}_u(A)(x)))\|_2 \le C\,\|\nabla A(x)\|_2$$

for almost every $x \in U$, where C is the Lipschitz constant of Δ.

Proof. The map $\Delta \circ \Lambda : \mathrm{Norm}(d) \to \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$ is Lipschitz with Lipschitz constant $C = \mathrm{Lip}(\Delta)$, by Proposition 3.3. It admits a Lipschitz extension $L : M_d(\mathbb{C}) \to \mathbb{R}^N$ preserving the Lipschitz constant C, by Kirszbraun's theorem.

It is well-known (see [ADM90]) that superposition with L defines a bounded map from $W^{1,q}(U, M_d(\mathbb{C}))$ to $W^{1,q}(U, \mathbb{R}^N)$, where

$$\|L \circ A\|_2 \le C\,\|A\|_2,$$

because $L(0) = \Delta(\Lambda(0)) = 0$, and

$$\|\nabla(L \circ A)\|_2 \le C\,\|\nabla A\|_2$$

almost everywhere in U. This implies the assertion. □

Let us recall an important relationship between the metric speed of an AC-curve in $\mathcal{A}_d(\mathbb{C})$ and the derivative of any parameterization.

Lemma 9.2 ([PR24b, Lemma 11.1])**.** *Let $\lambda : I \to \mathbb{C}^d$ be an absolutely continuous curve and $\gamma : I \to \mathcal{A}_d(\mathbb{C})$ defined by $\gamma(x) = [\lambda(x)]$, for $x \in I$. Then the metric speed of γ is given by*

$$|\dot{\gamma}|(x) = \|\lambda'(x)\|_2 \quad \text{for almost every } x \in I.$$

On an interval I, we can express the boundedness of $\mathcal{E}_u : W^{1,q}(I, \mathrm{Norm}(d)) \to W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$ in terms of $\mathbf{s}_0$ and $\mathbf{s}_1$, introduced in Definition 8.7.

Proposition 9.3. *Let $1 \le q < \infty$ and $I \subseteq \mathbb{R}$ a bounded open interval. Then, for $A \in W^{1,q}(I, \mathrm{Norm}(d))$,*

$$\|\mathbf{s}_0(\mathcal{E}_u(A), [0])\|_{L^\infty(I)} \le |I|^{-1/q} \|A\|_{L^q(I,M_d(\mathbb{C}))} + |I|^{1-1/q} \|A'\|_{L^q(I,M_d(\mathbb{C}))}, \tag{9.1}$$

$$\|\mathbf{s}_1(\mathcal{E}_u(A), [0])\|_{L^q(I)} \le \|A'\|_{L^q(I,M_d(\mathbb{C}))}. \tag{9.2}$$

Proof. Let $A \in W^{1,q}(I, \mathrm{Norm}(d))$. Let $\lambda \in W^{1,q}(I, \mathbb{C}^d)$ be a parameterization of $\Lambda := \mathcal{E}_u(A)$ (see Proposition 8.5). By (5.1), for all $x \in I$,

$$\|\lambda(x)\|_2 = \|A(x)\|_2 \le |I|^{-1/q} \|A\|_{L^q(I,M_d(\mathbb{C}))} + |I|^{1-1/q} \|A'\|_{L^q(I,M_d(\mathbb{C}))},$$

which implies (9.1). By Lemma 9.2 and (3.3), for almost every $x \in I$,

$$\|\lambda'(x)\|_2 = |\dot{\Lambda}|(x) \le \|A'(x)\|_2,$$

entailing (9.2). □

10. Eigenvalue stability for normal matrices: one-parameter case

In view of Theorem 8.10, the following theorem is a version of Theorem 1.8 for $m = 1$. Recall the semimetric $\mathbf{d}_I^{1,q}$ on $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$, introduced in Definition 8.7.

Theorem 10.1. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Norm}(d))$ as $n \to \infty$. Then*

$$\mathbf{d}_I^{1,q}(\mathcal{E}_u(A), \mathcal{E}_u(A_n)) \to 0 \quad \text{as } n \to \infty.$$

Theorem 10.1 and Theorem 8.10 yield the following corollary.

Corollary 10.2. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$ be an Almgren embedding. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Norm}(d))$ as $n \to \infty$. Then*

$$\|\Delta \circ \mathcal{E}_u(A) - \Delta \circ \mathcal{E}_u(A_n)\|_{W^{1,q}(I,\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

The proof of Theorem 10.1 comprises Section 10.1 and Section 10.2.

10.1. **Preliminary observations and reductions.** Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval and assume that $A_n \to A$ in $W^{1,q}(I, \operatorname{Norm}(d))$ as $n \to \infty$.

By Proposition 9.1, $\Lambda := \mathcal{E}_u(A)$ and $\Lambda_n := \mathcal{E}_u(A_n)$ belong to $W^{1,q}(I, \mathcal{A}_d(\mathbb{C}))$. Let $\lambda = (\lambda_1, \ldots, \lambda_d) \in W^{1,q}(I, \mathbb{C}^d)$ and $\lambda_n = (\lambda_{n,1}, \ldots, \lambda_{n,d}) \in W^{1,q}(I, \mathbb{C}^d)$ be parameterizations of Λ and Λ_n, respectively; see Proposition 8.5. Then, for $1 \le i \le d$, $n \ge 1$, and almost every $x \in I$,

$$D\lambda_i(x) = \lambda_i'(x) \quad \text{and} \quad D\lambda_{n,i}(x) = \lambda_{n,i}'(x);$$

see Definition 8.6.

By Proposition 3.3 and Lemma 5.2,

$$\begin{aligned}\mathbf{d}_2(\Lambda(x), \Lambda_n(x)) &\le \|A(x) - A_n(x)\|_2 \\ &\le |I|^{-1/q} \|A - A_n\|_{L^q(I, M_d(\mathbb{C}))} + |I|^{1-1/q} \|A' - A_n'\|_{L^q(I, M_d(\mathbb{C}))}\end{aligned}$$

so that

$$\|\mathbf{s}_0(\Lambda, \Lambda_n)\|_{L^\infty(I)} = \|\mathbf{d}_2(\Lambda, \Lambda_n)\|_{L^\infty(I)} \to 0 \quad \text{as } n \to \infty. \tag{10.1}$$

Thus, to complete the proof of Theorem 10.1, it suffices to show that

$$\widehat{\mathbf{d}}^{1,q}_I(\Lambda, \Lambda_n) \to 0 \quad \text{as } n \to \infty, \tag{10.2}$$

where, for each measurable set $E \subseteq I$, we define

$$\widehat{\mathbf{d}}^{1,q}_E(\Lambda, \Lambda_n) := \|\mathbf{s}_1(\Lambda, \Lambda_n)\|_{L^q(E)};$$

see (8.4) for the definition of $\mathbf{s}_1$.

With $A \in W^{1,q}(I, \operatorname{Norm}(d))$ we associate $\tilde{A} := A - \frac{1}{d} \operatorname{Tr}(A) \mathbb{I}_d \in W^{1,q}(I, \operatorname{Norm}_T(d))$.

Lemma 10.3. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \operatorname{Norm}(d))$ as $n \to \infty$. Then:*

(1) *$\tilde{A}_n \to \tilde{A}$ in $W^{1,q}(I, \operatorname{Norm}_T(d))$ as $n \to \infty$.*

Let $\Lambda, \Lambda_n, \tilde{\Lambda}, \tilde{\Lambda}_n : I \to \mathcal{A}_d(\mathbb{C})$ be the curves of unordered eigenvalues of $A, A_n, \tilde{A}, \tilde{A}_n$, respectively. Then:

(2) *$\widehat{\mathbf{d}}^{1,q}_I(\Lambda, \Lambda_n) \to 0$ if and only if $\widehat{\mathbf{d}}^{1,q}_I(\tilde{\Lambda}, \tilde{\Lambda}_n) \to 0$ as $n \to \infty$.*

Proof. (1) This is clear since $A_n \to A$ in $W^{1,q}(I, \operatorname{Norm}(d))$ implies $\operatorname{Tr}(A_n) \to \operatorname{Tr}(A)$ in $W^{1,q}(I, \mathbb{C})$ as $n \to \infty$.

(2) By (10.1) and (1), $\|\mathbf{d}_2(\Lambda, \Lambda_n)\|_{L^\infty(I)} \to 0$ as well as $\|\mathbf{d}_2(\tilde{\Lambda}, \tilde{\Lambda}_n)\|_{L^\infty(I)} \to 0$ as $n \to \infty$. Assume that $\widehat{\mathbf{d}}^{1,q}_I(\Lambda, \Lambda_n) \to 0$ as $n \to \infty$. By Theorem 8.10, we have

$$\|\Delta \circ \Lambda - \Delta \circ \Lambda_n\|_{W^{1,q}(I, \mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty,$$

for every Almgren embedding $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$. Let $H : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^d$ be an Almgren map with associated real linear form η (see Definition 8.2). Then

$$\begin{aligned}H \circ \Lambda - H \circ \tilde{\Lambda} &= \tfrac{1}{d} \eta(\operatorname{Tr}(A))(1, 1, \ldots, 1), \\ H \circ \Lambda_n - H \circ \tilde{\Lambda}_n &= \tfrac{1}{d} \eta(\operatorname{Tr}(A_n))(1, 1, \ldots, 1).\end{aligned}$$

Thus $\|H \circ \tilde{\Lambda} - H \circ \tilde{\Lambda}_n\|_{W^{1,q}(I, \mathbb{R}^d)} \to 0$ and, consequently, in view of (8.3),

$$\|\Delta \circ \tilde{\Lambda} - \Delta \circ \tilde{\Lambda}_n\|_{W^{1,q}(I, \mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty,$$

which implies $\mathbf{d}_I^{1,q}(\tilde\Lambda,\tilde\Lambda_n)\to 0$ and hence $\widehat{\mathbf{d}}_I^{1,q}(\tilde\Lambda,\tilde\Lambda_n)\to 0$, again by Theorem 8.10. The opposite direction follows from the same arguments. ☐

By Lemma 10.3, we may assume that all matrices A, A_n are traceless.

10.2. **Proof of (10.2).** We will proceed by induction on d. If $d=1$ then

$$\widehat{\mathbf{d}}_I^{1,q}(\Lambda,\Lambda_n)=\|A'-A_n'\|_{L^q(I,M_1(\mathbb{C}))}\to 0 \quad \text{as } n\to\infty.$$

So assume $d\ge 2$.

On the zero set Z_A of A we have the following lemma.

Lemma 10.4. *Let $1\le q<\infty$. Let $I\subseteq\mathbb{R}$ be a bounded open interval. Assume that $A_n\to A$ in $W^{1,q}(I,\operatorname{Norm}(d))$ as $n\to\infty$. Let $\Lambda:=\mathcal{E}_u(A)$ and $\Lambda_n:=\mathcal{E}_u(A_n)$. Then*

$$\widehat{\mathbf{d}}_{Z_A}^{1,q}(\Lambda,\Lambda_n)\to 0 \quad \textit{as } n\to\infty.$$

Proof. Let $\lambda,\lambda_n\in W^{1,q}(I,\mathbb{C}^d)$ be parameterizations of Λ,Λ_n, respectively; see Proposition 8.5. Let E be the set of $x\in\operatorname{acc}(Z_A)$, where the derivatives $A'(x)$, $\lambda'(x)$ and $A_n'(x)$, $\lambda_n'(x)$ for all $n\ge 1$ exist. For each $x\in E$, $A'(x)=0$, $\lambda'(x)=0$, and, by Lemma 9.2 and (3.3),

$$\big\|\lambda_n'(x)\big\|_2=|\dot\Lambda_n|(x)\le\|A_n'(x)\|_2.$$

Since E has full measure in Z_A, we thus get

$$\begin{aligned}\widehat{\mathbf{d}}_{Z_A}^{1,q}([\lambda],[\lambda_n])&=\|\mathbf{s}_1([\lambda],[\lambda_n])\|_{L^q(Z_A)}=\big\|\|\lambda_n'\|_2\big\|_{L^q(Z_A)}\\&\le\big\|\|A_n'\|_2\big\|_{L^q(Z_A)}=\|A'-A_n'\|_{L^q(Z_A,M_d(\mathbb{C}))}\end{aligned}$$

which implies the assertion. ☐

Let us fix a uniform unitary block-diagonalization $(\{U_i:\mathcal{V}_i\to U_d(\mathbb{C})\}_{i=1}^s,r,\chi)$ for $\operatorname{Norm}_T^0(d)$; see Definition 4.5.

Lemma 10.5. *Let $1\le q<\infty$. Let $I\subseteq\mathbb{R}$ be a bounded open interval. Assume that $A_n\to A$ in $W^{1,q}(I,\operatorname{Norm}_T(d))$ as $n\to\infty$. Let $x_0\in I\setminus Z_A$. Then there exist an open interval J with $x_0\in J\subseteq I\setminus Z_A$, $n_0\ge 1$, and $i\in\{1,\dots,s\}$ such that*

1. *for all $n\ge n_0$, the curves $\underline{A}:=A/\|A\|_2$ and $\underline{A}_n:=A_n/\|A_n\|_2$ belong to $W^{1,q}(J,\operatorname{Norm}_T^0(d))$ and satisfy $\underline{A}(J),\underline{A}_n(J)\subseteq\mathcal{V}_i$;*
2. *on J and for all $n\ge n_0$, we have*
$$U_i^*(\underline{A})A\,U_i(\underline{A})=\begin{pmatrix}B&0\\0&C\end{pmatrix},\quad U_i^*(\underline{A}_n)A_n\,U_i(\underline{A}_n)=\begin{pmatrix}B_n&0\\0&C_n\end{pmatrix},\tag{10.3}$$
where $B,B_n\in W^{1,q}(J,\operatorname{Norm}(d_1))$ and $C,C_n\in W^{1,q}(J,\operatorname{Norm}(d_2))$ with $d_1+d_2=d$;
3. *for all $x\in J$ and $n\ge n_0$, we have*
$$\big|\|A_n(x)\|_2\,\mu(x)-\|A(x)\|_2\,\nu_n(x)\big|>\chi\,\|A(x)\|_2\|A_n(x)\|_2\tag{10.4}$$
for all eigenvalues $\mu(x)$ of $B(x)$ and all eigenvalues $\nu_n(x)$ of $C_n(x)$, where B and C_n are defined by (10.3);

(4) *we have*

$$\|B - B_n\|_{W^{1,q}(J,M_{d_1}(\mathbb{C}))} \to 0, \quad \|C - C_n\|_{W^{1,q}(J,M_{d_2}(\mathbb{C}))} \to 0$$

as $n \to \infty$.

Proof. The proof is almost identical to the one of Lemma 5.8. (In particular, (5.8)–(5.13) hold and we will use them in the proof of Lemma 10.6 below.) Item (3) follows from (4.9). □

The unitary block-diagonalization (10.3) will permit us to use the induction hypothesis.

Consider the parameterizations $\lambda, \lambda_n \in W^{1,q}(I, \mathbb{C}^d)$ of $\Lambda = \mathcal{E}_u(A)$ and $\Lambda_n = \mathcal{E}_u(A_n)$. In the situation of Lemma 10.5, by reordering (independently of x) if necessary, we may assume that $\Lambda^1 = [\lambda_1(x), \ldots, \lambda_{d_1}(x)]$ is the unordered d_1-tuple of the eigenvalues of $B(x)$ and $\Lambda^2 = [\lambda_{d_1+1}(x), \ldots, \lambda_d(x)]$ is the unordered d_2-tuple of the eigenvalues of $C(x)$, for $x \in J$. Analogously, we may assume that $\Lambda^1_n = [\lambda_{n,1}(x), \ldots, \lambda_{n,d_1}(x)]$ is the unordered d_1-tuple of the eigenvalues of $B_n(x)$ and $\Lambda^2_n = [\lambda_{n,d_1+1}(x), \ldots, \lambda_{n,d}(x)]$ is the unordered d_2-tuple of the eigenvalues of $C_n(x)$, for $x \in J$ and $n \ge n_0$. Since the reordering is independent of x, λ and λ_n remain parameterizations in $W^{1,q}(J, \mathbb{C}^d)$ of Λ and Λ_n on J, respectively.

By Lemma 10.3, Lemma 10.5, and the induction hypothesis,

$$\widehat{\mathbf{d}}^{1,q}_J(\Lambda^1, \Lambda^1_n) \to 0 \quad \text{and} \quad \widehat{\mathbf{d}}^{1,q}_J(\Lambda^2, \Lambda^2_n) \to 0 \quad \text{as } n \to \infty,$$

where $\widehat{\mathbf{d}}^{1,q}_J([\Lambda^j], [\Lambda^j_n])$ is interpreted in dimension d_j, for $j = 1, 2$. Thanks to the following lemma, this implies

$$\widehat{\mathbf{d}}^{1,q}_J(\Lambda, \Lambda_n) \to 0 \quad \text{as } n \to \infty. \tag{10.5}$$

Lemma 10.6. *In the situation of Lemma 10.5, after possibly increasing n_0, for all $x \in J$ and $n \ge n_0$, the following holds. If $\tau \in \mathrm{S}_d$ satisfies*

$$\|\lambda(x) - \tau\lambda_n(x)\|_2 = \mathbf{d}_2([\lambda(x)], [\lambda_n(x)]) \tag{10.6}$$

then τ respects (10.3) in the sense that $\tau(\{1, \ldots, d_1\}) = \{1, \ldots, d_1\}$ and (consequently) $\tau(\{d_1 + 1, \ldots, d\}) = \{d_1 + 1, \ldots, d\}$.

Proof. By (10.4), for all $x \in J$, $n \ge n_0$, $1 \le i \le d_1$, and $d_1 + 1 \le j \le d$,

$$\big|\|A_n(x)\|_2\, \lambda_i(x) - \|A(x)\|_2\, \lambda_{n,j}(x)\big| > \chi\, \|A(x)\|_2 \|A_n(x)\|_2. \tag{10.7}$$

By (5.10), (5.12), and (5.13),

$$\|A(x)\|_2 \|A_n(x)\|_2 \ge \frac{1}{8}\, \|A(x_0)\|_2^2$$

so that, by (10.7),

$$\begin{aligned} \frac{\chi}{8}\, \|A(x_0)\|_2^2 &< \big|\|A_n(x)\|_2\, \lambda_i(x) - \|A(x)\|_2\, \lambda_{n,j}(x)\big| \\ &\le \big|\|A_n(x)\|_2 - \|A(x)\|_2\big| |\lambda_i(x)| + |\lambda_i(x) - \lambda_{n,j}(x)| \|A(x)\|_2. \end{aligned}$$

By (5.12),

$$|\lambda_i(x)| \le \|A(x)\|_2 \le \frac{3}{2}\, \|A(x_0)\|_2.$$

Since $\|A - A_n\|_{L^\infty(I,M_d(\mathbb{C}))} \to 0$ as $n \to \infty$, by Corollary 5.3,

$$\big|\|A_n(x)\|_2 - \|A(x)\|_2\big| \le \frac{\chi}{24}\,\|A(x_0)\|_2$$

for all $x \in J$ if n is large enough. Thus

$$\frac{\chi}{8}\,\|A(x_0)\|_2^2 < \frac{\chi}{16}\,\|A(x_0)\|_2^2 + \frac{3}{2}\,\|A(x_0)\|_2|\lambda_i(x) - \lambda_{n,j}(x)|$$

which implies

$$\frac{\chi}{24}\,\|A(x_0)\|_2 < |\lambda_i(x) - \lambda_{n,j}(x)|, \tag{10.8}$$

for all $x \in J$ and all sufficiently large n, say for all $n \ge n_1$ where $n_1 \ge n_0$.

By (10.1), there exists $n_2 \ge n_1$ such that

$$\mathbf{d}_2([\lambda(x)], [\lambda_n(x)]) < \frac{\chi}{24}\,\|A(x_0)\|_2, \tag{10.9}$$

for all $x \in J$ and $n \ge n_2$.

Assume that $\tau \in \mathrm{S}_d$ is such that $\tau(\{1, \ldots, d_1\}) \ne \{1, \ldots, d_1\}$. Then there exist i, j with $1 \le i \le d_1 < j \le d$ and $\tau(i) = j$.

Fix $x \in J$ and $n \ge n_2$ and suppose that τ satisfies (10.6). By (10.8) and (10.9),

$$\begin{aligned}\frac{\chi}{24}\,\|A(x_0)\|_2 < |\lambda_i(x) - \lambda_{n,j}(x)| &\le \|\lambda(x) - \tau\lambda_n(x)\|_2 \\ &= \mathbf{d}_2([\lambda(x)], [\lambda_n(x)]) < \frac{\chi}{24}\,\|A(x_0)\|_2,\end{aligned}$$

a contradiction. The lemma is proved. □

By Lemma 10.4 and (10.5), the interval I can be covered by countably many measurable sets F_i such that

$$\widehat{\mathbf{d}}^{1,q}_{F_i}(\Lambda, \Lambda_n) \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.3 (applied to $f_n = \mathbf{s}_1(\Lambda, \Lambda_n)^q$), we conclude (10.2). Indeed, $\{\mathbf{s}_1(\Lambda, \Lambda_n)^q : n \ge 1\} \subseteq L^1(I)$ is uniformly integrable which can be checked following the arguments in the proof of Lemma 5.9 and noting that

$$\mathbf{s}_1(\Lambda, \Lambda_n) \le \|\lambda'\|_2 + \|\lambda_n'\|_2 \le \|A'\|_2 + \|A_n'\|_2,$$

almost everywhere in I, by Lemma 9.2 and (3.3).

This completes the induction and hence the proof of Theorem 10.1.

10.3. **Variants of Theorem 10.1.** Theorem 10.7 and Theorem 10.8 are versions of Theorem 1.13 and Theorem 1.14 for $m = 1$, respectively.

Theorem 10.7. *Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \mathrm{Norm}(d))$ as $n \to \infty$. Assume that $\lambda, \lambda_n \in W^{1,q}(I, \mathbb{C}^d)$ are parameterizations of the eigenvalues of A, A_n, respectively, and that $\lim_{n\to\infty} \lambda_n(x_0) = \lambda(x_0)$, for some $x_0 \in I$. Then $\lambda_n \to \lambda$ in $L^\infty(I, \mathbb{C}^d)$ if and only if $\lambda_n' \to \lambda'$ in $L^q(I, \mathbb{C}^d)$ as $n \to \infty$.*

Proof. Let us assume that $\lambda_n \to \lambda$ in $L^\infty(I, \mathbb{C}^d)$ as $n \to \infty$. The proof (by induction on d) that then $\lambda_n' \to \lambda'$ in $L^q(I, \mathbb{C}^d)$ as $n \to \infty$ follows from the arguments in the proof of Theorem 10.1 with slight modifications. It is easy to adjust Lemma 10.3 and Lemma 10.4, while Lemma 10.5 can be used unchanged. In the situation of Lemma 10.5, there exist $\mu, \mu_n \in W^{1,q}(J, \mathbb{C}^{d_1})$ and $\nu, \nu_n \in W^{1,q}(J, \mathbb{C}^{d_2})$ such that

$$\lambda = (\mu, \nu) \quad \text{and} \quad \lambda_n = (\mu_n, \nu_n). \tag{10.10}$$

Since $\lambda_n \to \lambda$ in $L^\infty(I, \mathbb{C}^d)$, we have $\mu_n \to \mu$ in $L^\infty(I, \mathbb{C}^{d_1})$ and $\nu_n \to \nu$ in $L^\infty(I, \mathbb{C}^{d_2})$ as $n \to \infty$. The induction hypothesis implies that there is countable cover $\{F_i\}$ of I by measurable sets such that, for all F_i,

$$\|\lambda' - \lambda_n'\|_{L^q(F_i, \mathbb{C}^d)} \to 0 \quad \text{as } n \to \infty$$

which implies $\lambda_n' \to \lambda'$ in $L^q(I, \mathbb{C}^d)$, thanks to Lemma 2.3.

Conversely, assume that $\lambda_n' \to \lambda'$ in $L^q(I, \mathbb{C}^d)$ as $n \to \infty$. Then, for $x \in I$,

$$\begin{aligned} \|\lambda(x) - \lambda_n(x)\|_2 &= \Big\| \lambda(x_0) - \lambda_n(x_0) + \int_{x_0}^x \lambda'(t) - \lambda_n'(t)\, dt \Big\|_2 \\ &\le \|\lambda(x_0) - \lambda_n(x_0)\|_2 + \|\lambda' - \lambda_n'\|_{L^1(I, \mathbb{C}^d)} \\ &\le \|\lambda(x_0) - \lambda_n(x_0)\|_2 + |I|^{1-1/q} \|\lambda' - \lambda_n'\|_{L^q(I, \mathbb{C}^d)} \end{aligned}$$

which implies $\lambda_n \to \lambda$ in $L^\infty(I, \mathbb{C}^d)$ as $n \to \infty$. □

Theorem 10.8. *Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $C^{0,1}(\overline{I}, \operatorname{Norm}(d))$ as $n \to \infty$. Assume that $\lambda, \lambda_n : I \to \mathbb{C}^d$ are continuous (thus Lipschitz) parameterizations of the eigenvalues of A, A_n, respectively, and that $\lim_{n\to\infty} \lambda_n(x_0) = \lambda(x_0)$, for some $x_0 \in I$. Then $\lambda_n \to \lambda$ in $L^\infty(I, \mathbb{C}^d)$ if and only if $\lambda_n' \to \lambda'$ almost everywhere in I as $n \to \infty$.*

Proof. Every continuous parameterization λ, λ_n of the eigenvalues of A, A_n, respectively, is actually Lipschitz, by [Rai13, Proposition 6.3], (3.3), and Lemma 9.2.

If $\lambda_n' \to \lambda'$ almost everywhere in I, then $\lambda_n' \to \lambda'$ in $L^q(I, \mathbb{C}^d)$ as $n \to \infty$, by the dominated convergence theorem. So $\lambda_n \to \lambda$ in $L^\infty(I, \mathbb{C}^d)$ as $n \to \infty$, by Theorem 10.7.

Assume that $\lambda_n \to \lambda$ in $L^\infty(I, \mathbb{C}^d)$ as $n \to \infty$. We see that then $\lambda_n' \to \lambda'$ almost everywhere as $n \to \infty$ by adjusting the arguments in the proof of Theorem 10.1. It is easy to adjust Lemma 10.3 and Lemma 10.5. We replace Lemma 10.4 by Lemma 10.9. Here we have (10.10) with $\mu, \mu_n \in C^{0,1}(\overline{J}, \mathbb{C}^{d_1})$ and $\nu, \nu_n \in C^{0,1}(\overline{J}, \mathbb{C}^{d_2})$. Hence, $\mu_n \to \mu$ in $L^\infty(I, \mathbb{C}^{d_1})$ and $\nu_n \to \nu$ in $L^\infty(I, \mathbb{C}^{d_2})$ as $n \to \infty$. By the induction hypothesis and Lemma 10.9, we may conclude that $\lambda_n' \to \lambda'$ almost everywhere in I as $n \to \infty$. □

Lemma 10.9. *Let $I \subseteq \mathbb{R}$ be a bounded open interval and assume that $A_n \to A$ in $C^{0,1}(\overline{I}, \operatorname{Norm}(d))$ as $n \to \infty$. Let $\lambda_n \in C^{0,1}(\overline{I}, \mathbb{C}^d)$ be a parameterization of $\Lambda_n = \mathcal{E}_u(A_n)$. Then, for almost every $x \in Z_A$,*

$$\lambda_n'(x) \to 0 \quad \text{as } n \to \infty. \tag{10.11}$$

Proof. This is a simple modification of the proof of Lemma 5.12. By Proposition 3.3, Lemma 2.5, and Lemma 2.6,

$$\sup_{(x,y)\in Z_A^{<2>}} \frac{\mathbf{d}_2(\Lambda_n(x),\Lambda_n(y))}{|x-y|} \le \sup_{(x,y)\in Z_A^{<2>}} \frac{\|A_n(x)-A_n(y)\|_2}{|x-y|} \to 0$$

as $n \to \infty$. Let E be the set of $x \in \operatorname{acc}(Z_A)$, where the derivatives $\lambda_n'(x)$ for all $n \ge 1$ exist. For each $x_0 \in E$ there is a sequence $x_k \to x_0$ with $x_k \in Z_A$ and $x_k \ne x_0$ for all $k \ge 1$. Thus, by Lemma 9.2, for fixed n,

$$\|\lambda_n'(x_0)\|_2 = |\dot{\Lambda}_n|(x_0) = \lim_{k\to\infty} \frac{\mathbf{d}_2(\lambda_n(x_0),\lambda_n(x_k))}{|x_0-x_k|} \le \sup_{(x,y)\in Z_A^{<2>}} \frac{\mathbf{d}_2(\lambda_n(x),\lambda_n(y))}{|x-y|}.$$

This implies (10.11), since E has full measure in Z_A. □

10.4. **Proofs of Corollary 1.9 and Corollary 1.10.** Let $1 \le q < \infty$. Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $A_n \to A$ in $W^{1,q}(I, \operatorname{Norm}(d))$ as $n \to \infty$. Let $\lambda, \lambda_n \in W^{1,q}(I, \mathbb{C}^d)$ be parameterizations of the eigenvalues of A, A_n, respectively.

Fix an ordering of S_d and let $\tau(x) \in \mathrm{S}_d$ be as in Definition 8.7. Then

$$\big|\|\lambda'\|_2 - \|\lambda_n'\|_2\big| = \big|\|\lambda'\|_2 - \|\tau\lambda_n'\|_2\big| \le \|\lambda' - \tau\lambda_n'\|_2 \le \mathbf{s}_1([\lambda],[\lambda_n])$$

almost everywhere in I. By Theorem 10.1, this easily implies Corollary 1.10.

Thanks to Lemma 9.2, Corollary 1.9 follows from Corollary 1.10 and (10.1).

11. Eigenvalue stability for normal matrices: multiparameter case

In this section, we prove Theorem 1.8, Corollary 1.12, Theorem 1.13, and Theorem 1.14.

11.1. **Proof of Theorem 1.8.** Let $1 \le q < \infty$. Let $U \subseteq \mathbb{R}^m$ be open and bounded. Let $\Delta : \mathcal{A}_d(\mathbb{C}) \to \mathbb{R}^N$ be an Almgren embedding. Let $A_n \to A$ in $W^{1,q}(U, \operatorname{Norm}(d))$ as $n \to \infty$. Set $\Lambda := \mathcal{E}_u(A), \Lambda_n := \mathcal{E}_u(A_n) : U \to \mathcal{A}_d(\mathbb{C})$ and $F := \Delta \circ \Lambda, F_n := \Delta \circ \Lambda_n : U \to \mathbb{R}^N$. We have to show that

$$\|F - F_n\|_{W^{1,q}(U,\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

By (3.3), for $x \in U$,

$$\|F(x) - F_n(x)\|_2 \le \operatorname{Lip}(\Delta)\, \mathbf{d}_2(\Lambda(x),\Lambda_n(x)) \le \operatorname{Lip}(\Delta)\, \|A(x) - A_n(x)\|_2$$

so that

$$\|F - F_n\|_{L^q(U,\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

It remains to show

$$\|\partial_j F - \partial_j F_n\|_{L^q(U,\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty, \tag{11.1}$$

for all $1 \le j \le m$. It is enough to prove that there is a subsequence (n_k) with this property.

Let us first assume that $U = I_1 \times \cdots \times I_m$ is an open box with sides parallel to the coordinate axes. Let $j = 1$. As in the proof of Theorem 6.1, we conclude that there is a subsequence (n_k) such that

$$\int_{I_1} \|A(x_1,x') - A_{n_k}(x_1,x')\|_2^q \, dx_1 \to 0$$

and

$$\int_{I_1} \|\partial_1 A(x_1,x') - \partial_1 A_{n_k}(x_1,x')\|_2^q \, dx_1 \to 0$$

for almost every $x' \in U' = I_2 \times \cdots \times I_m$ as $k \to \infty$. By Corollary 10.2,

$$G_{1,n_k}(x') := \int_{I_1} \|\partial_1 F(x_1,x') - \partial_1 F_{n_k}(x_1,x')\|_2^q \, dx_1 \to 0 \quad \text{as } k \to \infty$$

for almost every $x' \in U'$.

Using Proposition 9.1 instead of Proposition 3.5, we see as in the proof of Theorem 6.1, that the set $\{G_{1,n_k} : k \ge 1\} \subseteq L^1(U')$ is uniformly integrable. So Vitali's convergence theorem 2.2 and Tonelli's theorem imply (11.1) for $j = 1$, in the case that $U = I_1 \times \cdots \times I_m$. For $2 \le j \le m$, the reasoning is analogous.

Let now U be a general open bounded subset of $\mathbb{R}^m$. Then (11.1) follows from Lemma 2.3 (applied to $f_n := \|\partial_j F - \partial_j F_n\|_2^q$), since $\{f_n\}$ is uniformly integrable which can be seen in analogy to the proof of Lemma 5.9, using Proposition 9.1. This ends the proof of Theorem 1.8.

11.2. **Proof of Corollary 1.12.** The corollary follows from Theorem 1.8 and Morrey's inequality,

$$\|\Delta \circ \mathcal{E}(A) - \Delta \circ \mathcal{E}(A_n)\|_{C^{0,\alpha}(\overline{U},\mathbb{R}^N)} \le C\, \|\Delta \circ \mathcal{E}(A) - \Delta \circ \mathcal{E}(A_n)\|_{W^{1,q}(U,\mathbb{R}^N)},$$

where $\alpha = 1 - m/q$, $q > m$, and $C = C(m,N,q,U)$.

11.3. **Proof of Theorem 1.13.** We adapt the proof of (11.1) with F, F_n replaced by λ, λ_n. Assume first that $U = I_1 \times \cdots \times I_m$ and $j = 1$. By Theorem 10.7, we may conclude that there is a subsequence (n_k) of (n) such that

$$G_{1,n_k}(x') := \int_{I_1} \|\partial_1 \lambda(x_1,x') - \partial_1 \lambda_{n_k}(x_1,x')\|_2^q \, dx_1 \to 0 \quad \text{as } k \to \infty$$

for almost every $x' \in U'$. It is easy to check (as before) that $\{G_{1,n_k} : k \ge 1\}$ is uniformly integrable. So the assertion follows from Vitali's convergence theorem 2.2 and Tonelli's theorem. The case of general U then follows from Lemma 2.3.

11.4. **Proof of Theorem 1.14.** It follows easily from Theorem 10.8 (applied coordinate-wise).

12. An application for compact self-adjoint operators

Let H be a Hilbert space. A bounded operator $A \in L(H)$ is *compact* if for each bounded sequence $(v_n) \subseteq H$ the image (Av_n) contains a convergent subsequence.

The set $K(H)$ of all compact operators $A \in L(H)$ form a closed linear subspace of $L(H)$ and thus a Banach space endowed with the operator norm.

The *resolvent set* of A is by definition the set of $z \in \mathbb{C}$ such that $A - z$ is invertible with *resolvent* $R(z) := (A-z)^{-1} \in L(H)$. The resolvent set $P(A)$ is an open subset of $\mathbb{C}$, its complement is the *spectrum* of A. As A is compact, the spectrum of A is a countable set which accumulates at zero, and zero is its only accumulation point. Every nonzero point in the spectrum is an eigenvalue of A with finite multiplicity. See [Kat76, III.6.26].

Let $A \in K(H)$ be a self-adjoint nonnegative compact operator. Let

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots$$

denote its decreasingly ordered eigenvalues. Then, by the Courant–Fischer–Weyl min-max principle (see e.g. [GGK03, Chapter IV, Theorem 9.1] or [RS80, Theorem XIII.1]),

$$\lambda_i = \min_{\substack{M \subseteq H \\ \dim M = i-1}} \max_{\substack{\|x\|=1 \\ x \perp M}} \langle Ax, x \rangle. \tag{12.1}$$

As a consequence we get the following version of Weyl's theorem (see e.g. [GGK03, Chapter IV, Section 4.9]).

Proposition 12.1. *Let $A, B \in K(H)$ be self-adjoint nonnegative compact operators. Then, for all $i \geq 1$,*

$$|\lambda_i(A) - \lambda_i(B)| \leq \|A - B\|_{\mathrm{op}}.$$

Proof. Let $x \in H$ with $\|x\| = 1$. Then

$$|\langle Ax, x\rangle - \langle Bx, x\rangle| = |\langle (A-B)x, x\rangle| \leq \|A - B\|_{\mathrm{op}}$$

which means that

$$\langle Ax, x\rangle \leq \langle Bx, x\rangle + \|A - B\|_{\mathrm{op}}$$

and

$$\langle Bx, x\rangle \leq \langle Ax, x\rangle + \|A - B\|_{\mathrm{op}}.$$

Applying (12.1), we conclude that

$$\lambda_i(A) \leq \lambda_i(B) + \|A - B\|_{\mathrm{op}} \quad \text{and} \quad \lambda_i(B) \leq \lambda_i(A) + \|A - B\|_{\mathrm{op}}$$

which gives the statement. □

Let $U \subseteq \mathbb{R}^m$ be a bounded open set. Consider a Lipschitz family $A \in C^{0,1}(\overline{U}, K(H))$ of compact operators such that each $A(x)$, for $x \in U$, is self-adjoint. Fix $x_0 \in U$. Let λ be an eigenvalue of $A(x_0)$ of multiplicity d. Let Γ be a simple closed C^1 curve in the resolvent set $P(A(x_0))$ enclosing only λ among all eigenvalues of $A(x_0)$. Then there is an open neighborhood V of x_0 such that Γ lies in $P(A(x))$, for $x \in V$, and $A(x)$, for $x \in V$, has precisely d eigenvalues (counted with multiplicities) in the interior of Γ; see [Kat76, IV.2.16, IV.3.15, IV.3.16]. Moreover, for $x \in V$ and z close to any $z_0 \in P(A(x_0))$, the operator $A(x) - z$ is invertible with resolvent $R(A(x), z) = (A(x) - z)^{-1} \in L(H)$. We conclude that the map

$$(x, z) \mapsto R(A(x), z) = (A(x) - z)^{-1} \in L(H)$$

is of class $C^{0,1}$ in x and holomorphic in z, since inversion is analytic on $L(H)$. Consequently,

$$\Pi : x \mapsto -\frac{1}{2\pi i} \int_\Gamma R(A(x), z)\, dz = -\frac{1}{2\pi i} \int_\Gamma (A(x) - z)^{-1}\, dz \in L(H)$$

defines a Lipschitz family Π of projections onto the direct sum of eigenspaces of the corresponding eigenvalues in the interior of Γ. We have that $A(x)$ commutes with $\Pi(x)$, and the spectrum of $A(x)$ inside Γ coincides with the spectrum of $A(x)\Pi(x) = \Pi(x)A(x)\Pi(x) \in L(\Pi(x)H)$; see [Kat76, III.6.17].

The image of $x \mapsto \Pi(x)$ describes a d-dimensional Lipschitz subbundle of the trivial bundle $U \times H \to U$. Indeed, choose $v_1, \ldots, v_d \in H$ such that the vectors $\Pi(x_0)v_i$

span $\Pi(x_0)H$. This remains true locally for x near x_0. By the Gram–Schmidt orthonormalization procedure (which is real analytic), we obtain a local orthonormal Lipschitz frame of the bundle. In this frame, $\Pi(x)A(x)\Pi(x)$ is given by a Hermitian $d \times d$ matrix parameterized in a Lipschitz way by x.

For the reader's convenience, we restate Theorem 1.21:

Theorem 12.2. *Let $U \subseteq \mathbb{R}^m$ be a bounded open set. Let H be a Hilbert space and $A, A_n \in C^{0,1}(\overline{U}, K(H))$, for $n \ge 1$, Lipschitz families of compact self-adjoint nonnegative operators. Assume that $A(x)$ is positive definite for all $x \in U$ and $A_n \to A$ in $C^{0,1}(\overline{U}, K(H))$ as $n \to \infty$. Then:*

(1) *The decreasingly ordered eigenvalues $\lambda_i(A)$ and $\lambda_i(A_n)$, for all i and large enough n, belong to $C^{0,1}(\overline{U})$.*

(2) *For all $1 \le j \le m$ and almost every $x \in U$,*
$$\lim_{n\to\infty} \partial_j(\lambda_i(A_n))(x) = \partial_j(\lambda_i(A))(x), \quad i = 1, 2, \dots$$

(3) *For every $1 \le q < \infty$,*
$$\lim_{n\to\infty} \|\lambda_i(A) - \lambda_i(A_n)\|_{W^{1,q}(U)} = 0, \quad i = 1, 2, \dots$$

Proof. (1) Since $A(x)$ is positive definite for all $x \in U$, all eigenvalues $\lambda_i(A(x))$ are positive for all $x \in U$. By Proposition 12.1, this also holds for A_n if n is large enough. Then Proposition 12.1 implies that each $\lambda_i(A)$ and $\lambda_i(A_n)$, for large enough n, is Lipschitz and belongs to $C^{0,1}(\overline{U})$.

(2) Fix $x_0 \in U$ and $i \ge 1$. Let Γ be a simple closed C^1 curve in $P(A(x_0))$ enclosing only $\lambda_i(A(x_0))$ among all eigenvalues of $A(x_0)$. As above, by [Kat76, IV.2.16, IV.3.15, IV.3.16], we see that there is an open neighborhood V of x_0 and $n_0 \ge 1$ such that Γ lies in $P(A(x))$ and $P(A_n(x))$, for $x \in V$ and $n \ge n_0$, and $A(x)$ and $A_n(x)$, for $x \in V$ and $n \ge n_0$, have precisely d eigenvalues (counted with multiplicities) in the interior of Γ. By Proposition 12.1, $\lambda_i(A_n(x))$ is among those eigenvalues, if n_0 is large enough.

The discussion before the theorem shows that the d eigenvalues of $A(x)$ as well as the d eigenvalues of $A_n(x)$ in the interior of Γ are precisely the d eigenvalues of $B(x) := \Pi(x)A(x)\Pi(x)$ and $B_n(x) := \Pi_n(x)A_n(x)\Pi_n(x)$, given by a Lipschitz family of Hermitian $d \times d$ matrices, respectively. Moreover, $B_n \to B$ in $C^{0,1}(\overline{V}, \mathrm{Herm}(d))$ as $n \to \infty$, after slightly shrinking V if necessary. Thus, Theorem 1.7 implies that, for all $1 \le j \le m$ and almost every $x \in J$, we have
$$\lim_{n\to\infty} \partial_j(\lambda_i(A_n))(x) = \partial_j(\lambda_i(A))(x).$$
Since U can be covered by countably many such open subsets V, this holds for almost every $x \in U$.

(3) By Proposition 12.1, for each $i \ge 1$ and $n \ge 1$,
$$|\lambda_i(A_n)|_{C^{0,1}(\overline{U})} \le |A_n|_{C^{0,1}(\overline{U}, K(H))} \le L,$$
for a positive constant L independent of n and i. Thus, for each $1 \le j \le m$ and almost every $x \in U$,
$$|\partial_j(\lambda_i(A_n))(x)| \le L.$$

Consequently, in view of (2), the dominated convergence theorem implies that, for each $1 \le j \le m$ and every $1 \le q < \infty$,

$$\|\partial_j(\lambda_i(A)) - \partial_j(\lambda_i(A_n))\|_{L^q(U)} \to 0 \quad \text{as } n \to \infty.$$

By Proposition 12.1, we have

$$\|\lambda_i(A) - \lambda_i(A_n)\|_{L^\infty(U)} \to 0 \quad \text{as } n \to \infty$$

and (3) is proved. □

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Adam Parusiński: Université Côte d'Azur, CNRS, LJAD, UMR 7351, 06108 Nice, France

Email address: adam.parusinski@univ-cotedazur.fr

Armin Rainer: Faculty of Mathematics and Geoinformation, Institute for Statistics and Mathematical Methods in Economics, E105-04, TU Wien, Wiedner Hauptstrasse 8, 1040 Vienna, Austria

Email address: armin.rainer@tuwien.ac.at