CONTINUITY OF THE SOLUTION MAP FOR HYPERBOLIC POLYNOMIALS

ADAM PARUSIŃSKI AND ARMIN RAINER

Abstract. Hyperbolic polynomials are monic real-rooted polynomials. By Bronshtein's theorem, the increasingly ordered roots of a hyperbolic polynomial of degree d with $C^{d-1,1}$ coefficients are locally Lipschitz and the solution map "coefficients-to-roots" is bounded. We prove continuity of this solution map from hyperbolic polynomials of degree d with C^d coefficients to their increasingly ordered roots with respect to the C^d structure on the source space and the Sobolev $W^{1,q}$ structure, for all $1 \le q < \infty$, on the target space. Continuity fails for $q = \infty$. As a consequence, we obtain continuity of the local surface area of the roots as well as local lower semicontinuity of the area of the zero sets of hyperbolic polynomials. We also discuss applications for the eigenvalues of Hermitian matrices and singular values.

Contents

Date: February 2, 2026.
2020 *Mathematics Subject Classification.* 26C05, 26C10, 26A16, 30C15, 46E35, 47A55, 47H30.
Key words and phrases. Hyperbolic polynomials, regularity of the roots, continuity of the solution map, Bronshtein's theorem, Tschirnhausen (trans-)form, continuity of surface area, eigenvalues of Hermitian matrices, singular values.
This research was funded in part by the Austrian Science Fund (FWF) DOI 10.55776/P32905 and DOI 10.55776/PAT1381823.

1. Introduction

Determining the optimal regularity of the roots of polynomials whose coefficients depend smoothly on parameters is a much studied problem with a long history. It has important applications in various fields such as partial differential equations and perturbation theory.

The subject started with Rellich's work [Rel37] on the analytic perturbation theory of linear operators. Bronshtein [Bro80] proved Gevrey well-posedness of the hyperbolic Cauchy problem with multiple characteristics using his result [Bro79] on the Lipschitz continuity of the roots of hyperbolic polynomials. Spagnolo [Spa00], motivated by his analysis of certain systems of pseudo-differential equations, conjectured that the roots of smooth curves of (not necessarily hyperbolic) polynomials admit absolutely continuous parameterizations. This conjecture was proved and the optimal Sobolev regularity of the roots was established in a series of papers by Parusiński and Rainer [PR16, PR18, PR20a], after the optimal result for radicals had been obtained by Ghisi and Gobbino [GG13]. For a more comprehensive account of the history of the problem and its ramifications, e.g., in the perturbation theory of linear operators, we refer to the recent survey article [PR25].

In this paper, we focus on the class of monic *hyperbolic polynomials* for which the regularity problem has a special flavor; the general case of monic complex polynomials is treated in [PR24]. A monic real polynomial of degree d is called *hyperbolic* if all its d roots (counted with multiplicities) are real. Hyperbolic polynomials appear naturally as the characteristic polynomials of Hermitian matrices for instance.

We refer by *Bronshtein's theorem* to the statement that any continuous system of the roots of a $C^{d-1,1}$ family of hyperbolic polynomials of degree d is actually locally Lipschitz continuous (i.e. $C^{0,1}$). In general, this is optimal. Bronshtein [Bro79] originally proved a version for C^d curves of hyperbolic polynomials of degree d (under these assumptions the roots can be represented by differentiable functions). Bronshtein's rather dense proof is hard to follow. Wakabayashi [Wak86] gave a complex analytic proof of a more general Hölder version of Bronshtein's theorem, which had been announced by Ohya and Tarama [OT86]; it was later proved by Tarama [Tar06] following Bronshtein's original approach. Kurdyka and Păunescu [KP08] used resolution of singularities to deduce local Lipschitz continuity of the roots of hyperbolic polynomials with real analytic coefficients. A simple proof of Bronshtein's theorem, based on the splitting principle, which also established explicit uniform bounds for the Lipschitz constants of the roots in terms of the $C^{d-1,1}$ norms of the coefficients, was given by Parusiński and Rainer [PR15]. We will recall this version in Theorem 4.1.

Bronshtein's theorem gives rise to a bounded *solution map* that takes hyperbolic polynomials of degree d with $C^{d-1,1}$ coefficients to $C^{0,1}$ systems of their roots. This will be made precise below.

The purpose of this paper is to investigate the continuity of the solution map and thus answer a question of Antonio Lerario.

1.1. **Hyperbolic polynomials and the solution map.** A monic polynomial of degree d,

$$P_{\mathsf{a}}(Z) = Z^d + \sum_{j=1}^d a_j Z^{d-j} \in \mathbb{R}[Z],$$

is called *hyperbolic* if all its d roots are real. In the following, we will identify the polynomial P_{a} with its coefficient vector $\mathsf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Then the set of all hyperbolic polynomials of degree d is identified with the image of the map $\sigma = (\sigma_1, \ldots, \sigma_d) : \mathbb{R}^d \to \mathbb{R}^d$, where

$$\sigma_j(x_1, \ldots, x_d) = (-1)^j \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}$$

is the j-th elementary symmetric function (up to sign). By the Tarski–Seidenberg theorem, $\sigma(\mathbb{R}^d)$ is a closed semialgebraic subset of $\mathbb{R}^d$ which we equip with the trace topology. We denote this space by $\mathrm{Hyp}(d)$ and call it the *space of hyperbolic polynomials of degree d.*

For $\mathsf{a} \in \mathrm{Hyp}(d)$, let $\lambda_1^{\uparrow}(\mathsf{a}) \le \cdots \le \lambda_d^{\uparrow}(\mathsf{a})$ denote the increasingly ordered roots of P_{a}. Then

$$\lambda^{\uparrow} = (\lambda_1^{\uparrow}, \ldots, \lambda_d^{\uparrow}) : \mathrm{Hyp}(d) \to \mathbb{R}^d$$

is a continuous map, see [AKLM98, Lemma 4.1] or, alternatively, [PR24, Lemma 6.4] combined with Lemma 7.1.

Let $U \subseteq \mathbb{R}^m$ be open. Let $C^{d-1,1}(U, \mathrm{Hyp}(d))$ denote the space of all $C^{d-1,1}$ maps $\mathsf{a} : U \to \mathbb{R}^d$ such that $\mathsf{a}(U) \subseteq \mathrm{Hyp}(d)$. Thus $\mathsf{a} \in C^{d-1,1}(U, \mathrm{Hyp}(d))$ amounts to a hyperbolic polynomial P_{a} of degree d whose coefficients are $C^{d-1,1}$ functions defined on U. We equip $C^{d-1,1}(U, \mathrm{Hyp}(d))$ with the trace topology of the natural Fréchet topology on $C^{d-1,1}(U, \mathbb{R}^d)$. Note that $C^{d-1,1}(U, \mathrm{Hyp}(d))$ is a closed nonlinear subset of $C^{d-1,1}(U, \mathbb{R}^d)$. Then Bronshtein's theorem (see Theorem 4.1) implies that the *solution map*

$$\mathcal{S} := (\lambda^{\uparrow})_* : C^{d-1,1}(U, \mathrm{Hyp}(d)) \to C^{0,1}(U, \mathbb{R}^d), \quad \mathsf{a} \mapsto \lambda^{\uparrow} \circ \mathsf{a}, \tag{1.1}$$

is well-defined and bounded (i.e., it maps bounded sets to bounded sets).

1.2. **The main results.** We will see in Example 1.12 that the solution map $\mathcal{S} : C^{d-1,1}(U, \mathrm{Hyp}(d)) \to C^{0,1}(U, \mathbb{R}^d)$ is *not* continuous: the natural topology on the target $C^{0,1}(U, \mathbb{R}^d)$ is too strong.

However, the solution map $\mathcal{S}$ becomes continuous if we restrict it to $C^d(U, \mathrm{Hyp}(d))$, carrying the trace topology of the natural Fréchet topology on $C^d(U, \mathbb{R}^d)$, and relax the topology on the target space: for $1 \le q < \infty$, let $C_q^{0,1}(U, \mathbb{R}^d)$ denote the set $C^{0,1}(U, \mathbb{R}^d)$ equipped with the trace topology of the inclusion in Sobolev space $C^{0,1}(U, \mathbb{R}^d) \to W^{1,q}_{\mathrm{loc}}(U, \mathbb{R}^d)$. See Section 2 for precise definitions of the function spaces.

The following theorem, which is our main result, solves Open Problem 3.8 in [PR25].

Theorem 1.1. *Let $U \subseteq \mathbb{R}^m$ be open. The solution map*

$$\mathcal{S} : C^d(U, \mathrm{Hyp}(d)) \to C_q^{0,1}(U, \mathbb{R}^d), \quad \mathsf{a} \mapsto \lambda^{\uparrow} \circ \mathsf{a},$$

is continuous, for all $1 \le q < \infty$.

As a corollary, we find that the solution map on $C^d(U, \mathrm{Hyp}(d))$ is continuous into the Hölder space $C^{0,\alpha}(U, \mathbb{R}^d)$, carrying its natural topology, for all $0 < \alpha < 1$.

Corollary 1.2. *Let $U \subseteq \mathbb{R}^m$ be open. The solution map*

$$\mathcal{S} : C^d(U, \mathrm{Hyp}(d)) \to C^{0,\alpha}(U, \mathbb{R}^d), \quad \mathsf{a} \mapsto \lambda^\uparrow \circ \mathsf{a},$$

is continuous, for all $0 < \alpha < 1$, but not for $\alpha = 1$.

The essential work for the proof of Theorem 1.1 happens in dimension $m = 1$ of the parameter space. The passage from one to several parameters is rather easy. The following is the main technical result of the paper.

Theorem 1.3. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(I, \mathrm{Hyp}(d))$, i.e., for each relatively compact open interval $I_1 \Subset I$,*

$$\|\mathsf{a} - \mathsf{a}_n\|_{C^d(\overline{I}_1, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{1.2}$$

Then $\{\mathcal{S}(\mathsf{a}_n) : n \ge 1\}$ is a bounded set in $C^{0,1}(I, \mathbb{R}^d)$ (with respect to its natural topology) and, for each relatively compact open interval $I_0 \Subset I$ and each $1 \le q < \infty$,

$$\|\mathcal{S}(\mathsf{a}) - \mathcal{S}(\mathsf{a}_n)\|_{W^{1,q}(I_0, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{1.3}$$

The proof of Theorem 1.3 is based on the dominated convergence theorem. The domination follows from Bronshtein's theorem which we recall in Theorem 4.1. We will show in Theorem 5.1 that, for almost every $x \in I$,

$$\mathcal{S}(\mathsf{a}_n)'(x) \to \mathcal{S}(\mathsf{a})'(x) \quad \text{as } n \to \infty.$$

To this end, we will develop a version of Bronshtein's theorem at a single point, see Theorem 4.7.

In Section 8, we prove a refinement of Theorem 1.3 in which the assumption that $\mathsf{a}_n \to \mathsf{a}$ in C^d as $n \to \infty$ can be weakened to convergence in C^p, where p is the (uniform) maximal multiplicity of the roots of P_a.

Note that by Egorov's theorem [Ego11] we may conclude that $\mathcal{S}(\mathsf{a}_n)' \to \mathcal{S}(\mathsf{a})'$ almost uniformly on I as $n \to \infty$, i.e., for each $\epsilon > 0$ there exists a measurable subset $E \subseteq I$ with $|E| < \epsilon$ such that $\mathcal{S}(\mathsf{a}_n)' \to \mathcal{S}(\mathsf{a})'$ uniformly on $I \setminus E$. In general, the convergence is not uniform on the whole interval I; see Example 1.12.

For later reference, we state a simple consequence of Theorem 1.3. Here $\|x\|_2$ denotes the 2-norm of $x \in \mathbb{R}^d$ and $\|f\|_{L^q(I_0,\mathbb{R}^d)} := \big\|\|f\|_2\big\|_{L^q(I_0)}$, see Section 1.6.

Corollary 1.4. *Let $I \subseteq \mathbb{R}$ be an open interval and $I_0 \Subset I$ a relatively compact open subinterval. If $\mathsf{a}_n \to \mathsf{a}$ in $C^d(I, \mathrm{Hyp}(d))$ as $n \to \infty$, then*

$$\big\|\|\mathcal{S}(\mathsf{a})'\|_2 - \|\mathcal{S}(\mathsf{a}_n)'\|_2\big\|_{L^q(I_0)} \to 0 \quad \text{as } n \to \infty,$$

and

$$\|\mathcal{S}(\mathsf{a}_n)'\|_{L^q(I_0,\mathbb{R}^d)} \to \|\mathcal{S}(\mathsf{a})'\|_{L^q(I_0,\mathbb{R}^d)} \quad \text{as } n \to \infty,$$

for all $1 \le q < \infty$.

Proof. Let us set $\lambda := \mathcal{S}(\mathsf{a})$ and $\lambda_n := \mathcal{S}(\mathsf{a}_n)$. Then

$$\begin{aligned}\big|\|\lambda'\|_{L^q(I_0,\mathbb{R}^d)} - \|\lambda_n'\|_{L^q(I_0,\mathbb{R}^d)}\big| &= \Big|\big\|\|\lambda'\|_2\big\|_{L^q(I_0)} - \big\|\|\lambda_n'\|_2\big\|_{L^q(I_0)}\Big| \\ &\le \big\|\|\lambda'\|_2 - \|\lambda_n'\|_2\big\|_{L^q(I_0)} \le \big\|\|\lambda' - \lambda_n'\|_2\big\|_{L^q(I_0)} = \|\lambda' - \lambda_n'\|_{L^q(I_0,\mathbb{R}^d)}\end{aligned}$$

so that the assertions follow from (1.3). □

It would be interesting to have quantitative versions of the continuity results.

Question 1.5. *Are the continuous solution maps $\mathcal{S}$ in Theorem 1.1 and Corollary 1.2 uniformly continuous and, if yes, is there an effective modulus of continuity?*

We have the following quantitative result in the case that all roots are simple. The monic hyperbolic polynomials P_{a} of degree d with d simple roots are in one-to-one correspondence with the points a in the interior $\operatorname{Hyp}^\circ(d)$ of $\operatorname{Hyp}(d)$.

Theorem 1.6. *Let $U \subseteq \mathbb{R}^m$ be open and $k \ge 1$. The solution map*

$$\mathcal{S}^\circ : C^k(U, \operatorname{Hyp}^\circ(d)) \to C^k(U, \mathbb{R}^d), \quad \mathsf{a} \mapsto \lambda^\uparrow \circ \mathsf{a},$$

is locally Lipschitz continuous: let $U_0 \Subset U$ and $V_0 \Subset \operatorname{Hyp}^\circ(d)$ be relatively compact open convex sets and B a bounded subset of $C^k(\overline{U}_0, V_0)$. Then, for all $\mathsf{a}_1, \mathsf{a}_2 \in B$,

$$\|\mathcal{S}^\circ(\mathsf{a}_1) - \mathcal{S}^\circ(\mathsf{a}_2)\|_{C^k(\overline{U}_0,\mathbb{R}^d)} \le C\,\|\mathsf{a}_1 - \mathsf{a}_2\|_{C^k(\overline{U}_0,\mathbb{R}^d)},$$

where $C = C(d, k, B, V_0)$.

We do not know if the continuity results in Theorem 1.1, Corollary 1.2, Theorem 1.3, and Corollary 1.4 still hold for the solution map $\mathcal{S}$ on $C^{d-1,1}(U, \operatorname{Hyp}(d))$ (instead of $C^d(U, \operatorname{Hyp}(d))$).

Question 1.7. *Is the solution map $\mathcal{S} : C^{d-1,1}(U, \operatorname{Hyp}(d)) \to C^{0,1}_q(U, \mathbb{R}^d)$ continuous, for $1 \le q < \infty$? Is the solution map $\mathcal{S} : C^{d-1,1}(U, \operatorname{Hyp}(d)) \to C^{0,\alpha}(U, \mathbb{R}^d)$ continuous, for $0 \le \alpha < 1$?*

In the proof of Theorem 1.3, we need the convergence of the coefficient vectors in C^d only on the accumulation points of the preimage under a of the discriminant locus. If this preimage is the union of an open set and a set of measure zero, then for (1.3) it is enough that $\mathsf{a}_n \to \mathsf{a}$ in $C^{d-1,1}$. Thus, for a potential counterexample a has to meet the discriminant locus in a Cantor-like set with positive measure.

Remark 1.8. If the coefficients are of class C^d, as in the setting of Theorem 1.3, the roots of P_{a_n} can be chosen as C^1 functions $\lambda_{n,1}, \ldots, \lambda_{n,d} : I \to \mathbb{R}$, see [COP12] or [PR15, Theorem 2.4]. This choice is not necessarily unique. Also, such a choice imposes an order on the roots that may change when the parameter changes. Therefore, if the parameter space is a circle and not an interval, a consistent choice may not be possible; see e.g. Remark 7.15. Moreover, in general, the roots $\lambda_{n,1}, \ldots, \lambda_{n,d}$ do not converge to a differentiable system of the roots of P_{a} (even just pointwise) as $n \to \infty$, see Example 1.12.

1.3. **Applications.** We will give several applications of our continuity results by highlighting, in particular, several consequences for stability under perturbations.

1.3.1. *Relation to the results for general polynomials.* In Section 7.1, we will interpret the results for hyperbolic polynomials as special and stronger versions of the general theorems of [PR24]. In the general case of a complex polynomial P_{a} of degree d the coefficient vector a is an arbitrary element of $\mathbb{C}^d$. Then it is natural to consider the *unordered* d-tuple of roots because there is no canonical choice of a parameterization of the roots by continuous functions. If the parameter space has dimension ≥ 2, then continuous selections of the roots might not even exist. In contrast to the hyperbolic case, the general theorems of [PR24] are only valid in $W^{1,q}$, for $1 \le q < d/(d-1)$.

1.3.2. *Continuity of the area of the solution map.* In Section 7.2, we will deduce from Theorem 1.1 that, if $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$, where $U \subseteq \mathbb{R}^m$ is open, then the Jacobian $|J(\mathcal{S}(\mathsf{a}_n))|$ of $\mathcal{S}(\mathsf{a}_n)$ converges to the Jacobian $|J(\mathcal{S}(\mathsf{a}))|$ of $\mathcal{S}(\mathsf{a})$ in L^q_{loc}, for all $1 \le q < \infty$ (see Corollary 7.5). Combining this with the area formula, we conclude that the surface area of the graph of each single root $\mathcal{S}(\mathsf{a}_n)_j = \lambda^\uparrow_j \circ \mathsf{a}_n$, for $1 \le j \le d$, converges locally to the surface area of the graph of $\mathcal{S}(\mathsf{a})_j$ (see Corollary 7.7).

As a consequence, we find that the area of the zero sets of C^d families of hyperbolic polynomials of degree d locally has a lower semicontinuity property:

Corollary 1.9. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. For any relatively compact open $U_0 \Subset U$, consider the zero sets*

$$\begin{aligned} Z &= \{(x,y) \in U_0 \times \mathbb{R} : P_{\mathsf{a}(x)}(y) = 0\} \quad \text{and} \\ Z_n &= \{(x,y) \in U_0 \times \mathbb{R} : P_{\mathsf{a}_n(x)}(y) = 0\}, \quad n \ge 1. \end{aligned} \tag{1.4}$$

Then

$$\liminf_{n\to\infty} \mathcal{H}^m(Z_n) \ge \mathcal{H}^m(Z).$$

Here $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure. Corollary 1.9 will be restated and proved in Corollary 7.8.

Without hyperbolicity, the area of the real zero set is generally not semicontinuous: e.g., for the intersections Z_t of Whitney's umbrella $\{(x,y,z) \in \mathbb{R}^3 : x^2 - y^2 z = 0\}$ with the planes $\{z = t\}$ and the cylinder $\{x^2 + y^2 < 1\}$ we have

$$\mathcal{H}^1(Z_t) = \begin{cases} 0 & \text{if } t < 0, \\ 2 & \text{if } t = 0, \\ 4 & \text{if } t > 0. \end{cases}$$

1.3.3. *Approximation by hyperbolic polynomials with simple roots.* In Section 7.3, combining our results with a lemma of Wakabayashi [Wak86], we will obtain the following approximation result (Corollary 7.10): for each hyperbolic polynomial P_{a}, where $\mathsf{a} \in C^d(U, \mathrm{Hyp}(d))$, there exists a sequence $(\mathsf{a}_n)_{n\ge1} \subseteq C^d(U, \mathrm{Hyp}(d))$ such that

- $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$;
- all roots of $P_{\mathsf{a}_n(x)}$ are simple for all $x \in U$ and all $n \ge 1$;
- $\mathcal{S}(\mathsf{a}_n) \in C^d(U, \mathbb{R}^d)$, for all $n \ge 1$, and $\mathcal{S}(\mathsf{a}_n) \to \mathcal{S}(\mathsf{a})$ in $C^{0,1}_q(U, \mathbb{R}^d)$, for all $1 \le q < \infty$, as $n \to \infty$;

- for each relatively compact open $U_0 \subseteq U$, defining the zero sets Z and Z_n as in (1.4), the limit $\lim_{n\to\infty} \mathcal{H}^m(Z_n)$ exists and satisfies
$$\lim_{n\to\infty} \mathcal{H}^m(Z_n) \ge \mathcal{H}^m(Z).$$

1.3.4. *Perturbation theory for Hermitian matrices.* In Section 7.4, we will apply our results to the eigenvalues of Hermitian matrices. Ordering the eigenvalues increasingly, induces a continuous map
$$\lambda^\uparrow : \mathrm{Herm}(d) \to \mathbb{R}^d$$
on the real vector space $\mathrm{Herm}(d)$ of complex Hermitian $d \times d$ matrices. By Weyl's perturbation theorem (see Proposition 7.11), we obtain a bounded map
$$\mathcal{E} := (\lambda^\uparrow)_* : C^{0,1}(U, \mathrm{Herm}(d)) \to C^{0,1}(U, \mathbb{R}^d), \quad A \mapsto \lambda^\uparrow \circ A.$$

The continuity results for hyperbolic polynomials imply the following result.

Theorem 1.10. *Let $U \subseteq \mathbb{R}^m$ be open. Then the map*
$$\mathcal{E} : C^d(U, \mathrm{Herm}(d)) \to C^{0,1}_q(U, \mathbb{R}^d), \quad A \mapsto \lambda^\uparrow \circ A,$$
is continuous, for all $1 \le q < \infty$, and the map
$$\mathcal{E} : C^d(U, \mathrm{Herm}(d)) \to C^{0,\alpha}(U, \mathbb{R}^d), \quad A \mapsto \lambda^\uparrow \circ A,$$
is continuous, for all $0 < \alpha < 1$.

Theorem 1.10 will be proved in Corollary 7.12. The map $\mathcal{E}$ is not continuous with respect to the $C^{0,1}$ topology on the target space, as will be seen in Example 7.13 which is based on Example 1.12.

Given that the map $\mathcal{E}$ is defined and bounded on $C^{0,1}(U, \mathrm{Herm}(d))$, it is natural to ask whether in Theorem 1.10 one can replace C^d by C^1:

Question 1.11. *Is the map $\mathcal{E} : C^1(U, \mathrm{Herm}(d)) \to C^{0,1}_q(U, \mathbb{R}^d)$ continuous, for $1 \le q < \infty$? Is $\mathcal{E} : C^1(U, \mathrm{Herm}(d)) \to C^{0,\alpha}(U, \mathbb{R}^d)$ continuous, for $0 < \alpha < 1$?*

We will prove in Proposition 7.14 that the answer to Question 1.11 is affirmative in the case $d = 2$.

1.3.5. *Singular values.* In Section 7.5, we will obtain an analogue of Theorem 1.10 for the singular values (ordered by size) of C^{2d} families of general complex $D \times d$ matrices with $d \le D$ (see Corollary 7.16). As in Question 1.11, it is natural to ask whether C^{2d} can actually be replaced by C^1.

1.4. **On the optimality of the results.** The following example shows that the solution map $\mathcal{S} : C^d(I, \mathrm{Hyp}(d)) \to C^{0,1}(I, \mathbb{R}^d)$, where $I \subseteq \mathbb{R}$ is an open interval, is not continuous with respect to the $C^{0,1}$ topology on the target space.

Example 1.12. Let $g(x) := x^2$ and $g_n(x) := x^2 + 1/n^2$, $n \ge 1$. Then, for all $k \in \mathbb{N}$ and each bounded open interval $I \subseteq \mathbb{R}$, $\|g - g_n\|_{C^k(\overline{I})} = 1/n^2 \to 0$ as $n \to \infty$. Let f and f_n be the positive square roots of g and g_n, respectively: $f(x) := |x|$ and $f_n(x) := \sqrt{x^2 + 1/n^2}$. Then, for each bounded open interval $I \subseteq \mathbb{R}$ containing 0,
$$|f - f_n|_{C^{0,1}(\overline{I})} \ge \sup_{0<x\in I} \Big| \frac{(f(x) - f_n(x)) - (f(0) - f_n(0))}{x} \Big|$$

$$= \sup_{0<x\in I} \Big| \frac{x - \sqrt{x^2 + \frac{1}{n^2}} + \frac{1}{n}}{x} \Big| \ge \Big| \frac{\frac{1}{n} - \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} + \frac{1}{n}}{\frac{1}{n}} \Big| = 2 - \sqrt{2},$$

for large enough n. Observe that

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}}$$

tends pointwise to $f'(x) = \operatorname{sgn}(x)$ for all $x \ne 0$ but not uniformly on any neighborhood of 0:

$$f_n'(\pm\tfrac{1}{n}) = \pm\frac{1}{\sqrt{2}}.$$

This also violates the first conclusion of Corollary 1.4 for $q = \infty$.

Notice that this example also shows that not every continuous (thus $C^{0,1}$) system of the roots of g is the limit of a continuous system of the roots of g_n: each continuous system of the roots of g_n tends to $\pm|x|$, none to $\pm x$. See Remark 1.8.

In the example, the hyperbolic polynomial $Z^2 = g(x)$ with double root at $x = 0$ is approximated by the hyperbolic polynomials $Z^2 = g_n(x)$ with simple roots for all x. We will see in Corollary 7.10 that such an approximation is always possible.

1.5. **Structure of the paper.** We fix notation and recall facts on function spaces in Section 2 and provide the necessary background on hyperbolic polynomials in Section 3. In Section 4, we recall Bronshtein's theorem in Theorem 4.1 and prove a version of it at a single point in Theorem 4.7. The latter provides bounds for the derivatives of the roots that are crucial for the proof of Theorem 1.3 which is carried out in Section 5. In Section 6, we generalize Theorem 1.3 to several variables in Theorem 6.1 which allows as to complete the proofs of Theorem 1.1 and Corollary 1.2; also Theorem 1.6 is proved in Section 6. Section 7 is dedicated to the applications; in particular, it contains the proofs of Corollary 1.9 and Theorem 1.10. Finally, Section 8 presents a refinement of Theorem 1.3, namely Theorem 8.2, in the case that the maximal multiplicity of the roots is smaller than the degree.

1.6. **Notation.** The m-dimensional Lebesgue measure in $\mathbb{R}^m$ is denoted by $\mathcal{L}^m$. If not stated otherwise, 'measurable' means 'Lebesgue measurable' and 'almost everywhere' means 'almost everywhere with respect to Lebesgue measure'. For measurable $E \subseteq \mathbb{R}^m$, we usually write $|E| = \mathcal{L}^m(E)$. We will also use the k-dimensional Hausdorff measure $\mathcal{H}^k$.

For $1 \le p \le \infty$, $\|x\|_p$ denotes the p-norm of $x \in \mathbb{R}^d$. If $f : E \to \mathbb{R}^d$, for measurable $E \subseteq \mathbb{R}^m$, is a measurable map, then we set

$$\|f\|_{L^p(E,\mathbb{R}^d)} := \big\| \|f\|_2 \big\|_{L^p(E)}.$$

In the following, a set is called *countable* if it is either finite or has the cardinality of $\mathbb{N}$.

To avoid confusion, coefficient vectors of hyperbolic polynomials are written in *sans serif* type. For example, the coefficient vector $\mathsf{a}_n = (a_{n,1}, a_{n,2}, \ldots, a_{n,d})$, indexed by $n \in \mathbb{N}$, is notationally distinguished from the scalar a_n, which denotes the n-th component of the coefficient vector a.

We use the notation $C(d,\ldots)$ to denote a constant that depends only on $d,\ldots$; its value may change from line to line.

2. Function spaces

Let us fix notation and recall background on the function spaces used in this paper.

2.1. Hölder–Lipschitz spaces. Let $U \subseteq \mathbb{R}^m$ be open and $k \in \mathbb{N}$. Then $C^k(U)$ is the space of k-times continuously differentiable real valued functions, equipped with its natural Fréchet topology. If U is bounded, then $C^k(\overline{U})$ denotes the space of all $f \in C^k(U)$ such that each $\partial^\alpha f$, $0 \le |\alpha| \le k$, has a continuous extension to the closure $\overline{U}$. Endowed with the norm

$$\|f\|_{C^k(\overline{U})} := \max_{|\alpha|\le k} \sup_{x\in U} |\partial^\alpha f(x)|$$

it is a Banach space. For $0 < \gamma \le 1$, we consider the Hölder–Lipschitz seminorm

$$|f|_{C^{0,\gamma}(\overline{U})} := \sup_{x,y\in U,\, x\ne y} \frac{|f(x)-f(y)|}{\|x-y\|_2^\gamma}.$$

For $k \in \mathbb{N}$ and $0 < \gamma \le 1$, we have the Banach space

$$C^{k,\gamma}(\overline{U}) := \{f \in C^k(\overline{U}) : \|f\|_{C^{k,\gamma}(\overline{U})} < \infty\},$$

where

$$\|f\|_{C^{k,\gamma}(\overline{U})} := \|f\|_{C^k(\overline{U})} + \max_{|\alpha|=k} |\partial^\alpha f|_{C^{0,\gamma}(\overline{U})}.$$

We write $C^{k,\gamma}(U)$ for the space of C^k functions on U that belong to $C^{k,\gamma}(\overline{V})$ for each relatively compact open $V \Subset U$, and endow $C^{k,\gamma}(U)$ with its natural Fréchet topology.

2.2. Lebesgue spaces. Let $U \subseteq \mathbb{R}^m$ be open and $1 \le p \le \infty$. We denote by $L^p(U)$ the Lebesgue space with respect to the m-dimensional Lebesgue measure $\mathcal{L}^m$, and $\|\cdot\|_{L^p(U)}$ is the corresponding L^p-norm. We will also use the space $L^p_{\mathrm{loc}}(U)$ of measurable functions $f : U \to \mathbb{R}$ satisfying $\|f\|_{L^p(K)} < \infty$ for all compact subsets $K \subseteq U$. For Lebesgue measurable sets $E \subseteq \mathbb{R}^m$ we also write $|E| = \mathcal{L}^m(E)$. We remark that for continuous functions $f : U \to \mathbb{R}$ we have (and use interchangeably) $\|f\|_{L^\infty(U)} = \|f\|_{C^0(\overline{U})}$.

2.3. Sobolev spaces. For $k \in \mathbb{N}$ and $1 \le q \le \infty$, we consider the Sobolev space

$$W^{k,q}(U) := \{f \in L^q(U) : \partial^\alpha f \in L^q(U) \text{ for } |\alpha| \le k\},$$

where $\partial^\alpha f$ are distributional derivatives. Endowed with the norm

$$\|f\|_{W^{k,q}(U)} := \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{L^q(U)}$$

it is a Banach space. We will also use

$$W^{k,q}_{\mathrm{loc}}(U) := \{f \in L^q_{\mathrm{loc}}(U) : \partial^\alpha f \in L^q_{\mathrm{loc}}(U) \text{ for } |\alpha| \le k\}$$

and endow this space with its natural topology.

2.4. **A result on composition.** In the following proposition we use the norm

$$\|f\|_{C^k(\overline{U},\mathbb{R}^\ell)} := \max_{0\le j\le k}\sup_{x\in U}\|d^j f(x)\|_{L_j(\mathbb{R}^m,\mathbb{R}^\ell)}$$

on the space $C^k(\overline{U},\mathbb{R}^\ell) := (C^k(\overline{U},\mathbb{R}))^\ell$, where $U \subseteq \mathbb{R}^m$ and $L_j(\mathbb{R}^m,\mathbb{R}^\ell)$ is the space of j-linear maps with j arguments in $\mathbb{R}^m$ and values in $\mathbb{R}^\ell$.

Proposition 2.1. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^\ell$ be open, bounded, and convex. Let $\psi \in C^{k+1}(\overline{V},\mathbb{R}^p)$. Then*

$$\psi_* : C^k(\overline{U},V) \to C^k(\overline{U},\mathbb{R}^p), \quad \varphi \mapsto \psi\circ\varphi,$$

is well-defined and continuous. More precisely, for φ_1,φ_2 in a bounded subset B of $C^k(\overline{U},V)$,

$$\|\psi_*(\varphi_1)-\psi_*(\varphi_2)\|_{C^k(\overline{U},\mathbb{R}^p)} \le C\,\|\psi\|_{C^{k+1}(\overline{V},\mathbb{R}^p)}\|\varphi_1-\varphi_2\|_{C^k(\overline{U},\mathbb{R}^\ell)},$$

where $C = C(k,B)$.

A short proof of this result can be found in [PR24, Appendix A.2].

3. Hyperbolic polynomials

In this section, we recall basic facts on hyperbolic polynomials that will be used below. The exposition follows [PR15] and [PR25]. For the convenience of the reader and to keep the paper largely self-contained, we include details where this does not substantially interrupt the flow.

3.1. **Tschirnhausen form.** We say that a monic polynomial

$$P_{\mathsf{a}}(Z) = Z^d + \sum_{j=1}^d a_j Z^{d-j}$$

is in *Tschirnhausen form* if $a_1 = 0$. Every polynomial P_{a} can be put in Tschirnhausen form by the substitution

$$P_{\tilde{\mathsf{a}}}(Z) = P_{\mathsf{a}}(Z - \tfrac{a_1}{d}) = Z^d + \sum_{j=2}^d \tilde a_j Z^{d-j},$$

which is called the *Tschirnhausen transformation*. For clarity, we consistently equip the coefficients of polynomials in Tschirnhausen form with a 'tilde'. Note that

$$\tilde a_j = \sum_{i=0}^j C_i a_i a_1^{j-i}, \quad 2 \le j \le d, \tag{3.1}$$

where $a_0 = 1$ and the C_i are universal constants independent of a. For a polynomial $P_{\tilde{\mathsf{a}}}$ in Tschirnhausen form with coefficient vector $\tilde{\mathsf{a}} = (0,\tilde a_2,\dots,\tilde a_d)$ we have

$$-2\tilde a_2 = \lambda_1^2 + \cdots + \lambda_d^2, \tag{3.2}$$

where $\lambda_1,\dots,\lambda_d$ is an enumeration of the roots of $\tilde{\mathsf{a}}$. Consequently, for a hyperbolic polynomial $P_{\tilde{\mathsf{a}}}$ in Tschirnhausen form,

$$\tilde a_2 \le 0.$$

Recall that the coefficients (up to their sign) are the elementary symmetric polynomials in the roots, by Vieta's formulas.

Lemma 3.1 ([PR25, Lemma 2.4])**.** *The coefficients of a hyperbolic polynomial* $P_{\tilde{\mathsf{a}}}$ *in Tschirnhausen form satisfy*

$$|\tilde{a}_j|^{1/j} \le \sqrt{2}\,|\tilde{a}_2|^{1/2}, \quad j = 2, \dots, d.$$

Proof. This follows easily from the Newton identities

$$j\sigma_j = \sum_{i=1}^{j} (-1)^{i-1} \sigma_{j-i} s_i, \quad d \ge j \ge 1,$$

between the Newton polynomials $s_k = \lambda_1^k + \cdots + \lambda_d^k$ and the elementary symmetric polynomials σ_k, observing that $|s_k|^{1/k} \le |s_2|^{1/2}$, for $2 \le k \le d$, by a well-known relation between the p-norms. □

As a consequence, $\tilde{\mathsf{a}} = (0, \tilde{a}_2, \tilde{a}_3, \dots, \tilde{a}_d) = 0$ if and only if $\tilde{a}_2 = 0$.

Definition 3.2 (Spaces of hyperbolic polynomials)**.** Let $\mathrm{Hyp}_T(d)$ denote the space of monic hyperbolic polynomials of degree d in Tschirnhausen form and $\mathrm{Hyp}_T^0(d)$ the compact subspace of polynomials $P_{\tilde{\mathsf{a}}}$ with $\tilde{a}_2 = -1$, i.e.,

$$\mathrm{Hyp}_T(d) = \{\tilde{\mathsf{a}} \in \mathrm{Hyp}(d) : \tilde{a}_1 = 0\},$$
$$\mathrm{Hyp}_T^0(d) = \{\tilde{\mathsf{a}} \in \mathrm{Hyp}_T(d) : \tilde{a}_2 = -1\}.$$

3.2. **Splitting.** Let us recall a simple consequence of the inverse function theorem.

Lemma 3.3 (E.g. [PR25, Lemma 2.5])**.** *Let* $P_{\mathsf{a}} = P_{\mathsf{b}} P_{\mathsf{c}}$*, where* P_{b} *and* P_{c} *are monic real polynomials without common (complex) root. Then we have* $P = P_{\mathsf{b}(P)} P_{\mathsf{c}(P)}$ *for analytic mappings* $P \mapsto \mathsf{b}(P) \in \mathbb{R}^{\deg P_{\mathsf{b}}}$ *and* $P \mapsto \mathsf{c}(P) \in \mathbb{R}^{\deg P_{\mathsf{c}}}$*, defined for* P *near* P_{a} *in* $\mathbb{R}^{\deg P_{\mathsf{a}}}$*, with the given initial values.*

Proof. The product $P_{\mathsf{a}} = P_{\mathsf{b}} P_{\mathsf{c}}$ defines on the coefficients a polynomial map φ such that $\mathsf{a} = \varphi(\mathsf{b}, \mathsf{c})$. Its Jacobian determinant equals the resultant of P_{b} and P_{c} which is nonzero, by assumption. Thus φ can be inverted locally, by the inverse function theorem. □

Let $P_{\tilde{\mathsf{a}}} \in \mathrm{Hyp}_T(d)$ be such that $\tilde{\mathsf{a}} \ne 0$, equivalently, $\tilde{a}_2 \ne 0$. Then the polynomial

$$Q_{\underline{\mathsf{a}}}(Z) := |\tilde{a}_2|^{-d/2} P_{\tilde{\mathsf{a}}}(|\tilde{a}_2|^{1/2} Z) = Z^d - Z^{d-2} + \sum_{j=3}^{d} |\tilde{a}_2|^{-j/2} \tilde{a}_j Z^{d-j}$$

belongs to $\mathrm{Hyp}_T^0(d)$. By Lemma 3.3, we have a splitting

$$Q_{\underline{\mathsf{a}}} = Q_{\underline{\mathsf{b}}} Q_{\underline{\mathsf{c}}},$$

on some open neighborhood $\underline{U} \subseteq \mathbb{R}^d$ of $\underline{\mathsf{a}}$ such that $d_{\mathsf{b}} := \deg Q_{\underline{\mathsf{b}}} < d$, $d_{\mathsf{c}} := \deg Q_{\underline{\mathsf{c}}} < d$, and

$$\underline{b}_i = \psi_i(|\tilde{a}_2|^{-3/2} \tilde{a}_3, \dots, |\tilde{a}_2|^{-d/2} \tilde{a}_d), \quad i = 1, \dots, \deg Q_{\underline{\mathsf{b}}},$$

where ψ_i are real analytic functions; likewise for $\underline{c}_i$. If $Q_{\underline{\mathsf{a}}}$ is hyperbolic, then also $Q_{\underline{\mathsf{b}}}$ and $Q_{\underline{\mathsf{c}}}$ are hyperbolic. If $\underline{\lambda}_1 \le \cdots \le \underline{\lambda}_d$ are the roots of $Q_{\underline{\mathsf{a}}}$, then we may assume that, on $\underline{U} \cap \mathrm{Hyp}_T^0(d)$, $\underline{\lambda}_1 \le \cdots \le \underline{\lambda}_{d_{\mathsf{b}}}$ are the roots of $Q_{\underline{\mathsf{b}}}$ and $\underline{\lambda}_{d_{\mathsf{b}}+1} \le \cdots \le \underline{\lambda}_d$ are the roots of $Q_{\underline{\mathsf{c}}}$; this follows from continuity of the map $\lambda^\uparrow$ and the simple topology of $\mathrm{Hyp}_T^0(d)$ (induced by the embedding in $\mathbb{R}^d$); cf. [PR25, Theorem 8.1].

The splitting $Q_{\underline{\mathsf{a}}} = Q_{\underline{\mathsf{b}}} Q_{\underline{\mathsf{c}}}$ induces a splitting

$$P_{\tilde{\mathsf{a}}} = P_{\mathsf{b}} P_{\mathsf{c}}$$

on an open neighborhood $\tilde U$ of $\tilde{\mathsf{a}}$, where

$$b_i = |\tilde a_2|^{i/2} \psi_i(|\tilde a_2|^{-3/2} \tilde a_3, \ldots, |\tilde a_2|^{-d/2} \tilde a_d), \quad i = 1, \ldots, \deg P_{\mathsf{b}}. \tag{3.3}$$

The coefficients $\tilde b_i$ of $P_{\tilde{\mathsf{b}}}$, resulting from P_{b} by the Tschirnhausen transformation, have an analogous representation, i.e.,

$$\tilde b_i = |\tilde a_2|^{i/2} \tilde\psi_i(|\tilde a_2|^{-3/2} \tilde a_3, \ldots, |\tilde a_2|^{-d/2} \tilde a_d), \quad i = 1, \ldots, \deg P_{\mathsf{b}}. \tag{3.4}$$

Shrinking $\tilde U$ slightly, we may assume that all partial derivatives of all orders of the real analytic functions ψ_i and $\tilde\psi_i$ are bounded on $\tilde U$.

Furthermore, since the roots of $P_{\tilde{\mathsf{a}}}$ are given by $\lambda_j := |\tilde a_2|^{1/2} \cdot \underline{\lambda}_j$, for $1 \le j \le d$, we have that, on $\tilde U \cap \mathrm{Hyp}_T(d)$, $\lambda_1 \le \cdots \le \lambda_{d_{\mathsf{b}}}$ are the roots of P_{b} and $\lambda_{d_{\mathsf{b}}+1} \le \cdots \le \lambda_d$ are the roots of P_{c}.

Lemma 3.4 ([PR25, Lemma 3.13])**.** *In this situation, we have* $|\tilde b_2| \le 4\,|\tilde a_2|$.

Proof. Using (3.2) and $|b_1| \le \sum_{j=1}^{d_{\mathsf{b}}} |\lambda_j| \le \sqrt{d_{\mathsf{b}}}\,\big(\sum_{j=1}^{d_{\mathsf{b}}} \lambda_j^2\big)^{1/2}$, we find

$$2\,|\tilde b_2| = \sum_{j=1}^{d_{\mathsf{b}}} \Big(\lambda_j + \frac{b_1}{d_{\mathsf{b}}}\Big)^2 = \sum_{j=1}^{d_{\mathsf{b}}} \lambda_j^2 + \frac{2b_1}{d_{\mathsf{b}}} \sum_{j=1}^{d_{\mathsf{b}}} \lambda_j + \frac{b_1^2}{d_{\mathsf{b}}} \le (1+2+1) \sum_{j=1}^{d_{\mathsf{b}}} \lambda_j^2 \le 8\,|\tilde a_2|.$$

□

3.3. **Universal splitting.** For each $d \ge 2$ fix the following data. Choose a finite cover of $\mathrm{Hyp}_T^0(d)$ by open sets $\underline{U}_1, \ldots, \underline{U}_s$ such that on each $\underline{U}_i$ we have a splitting $Q_{\underline{\mathsf{a}}} = Q_{\underline{\mathsf{b}}} Q_{\underline{\mathsf{c}}}$ and, consequently, a splitting $P_{\tilde{\mathsf{a}}} = P_{\mathsf{b}} P_{\mathsf{c}}$ as above together with analytic functions ψ_i and $\tilde\psi_i$, and we fix this splitting. As seen above, we may assume that the roots $\lambda_1 \le \cdots \le \lambda_d$ of $P_{\tilde{\mathsf{a}}}$ are labelled such that $\lambda_1 \le \cdots \le \lambda_{d_{\mathsf{b}}}$ are the roots of P_{b} and $\lambda_{d_{\mathsf{b}}+1} \le \cdots \le \lambda_d$ are the roots of P_{c}.

By the Lebesgue covering lemma, there exists $\delta > 0$ such that each subset of $\mathrm{Hyp}_T^0(d)$ of diameter less that δ is contained in some $\underline{U}_i$. Choose $r \in (0, \min\{\delta/2, 1\})$. Then for each $\underline{\mathsf{p}} \in \mathrm{Hyp}_T^0(d)$ there exists $1 \le i \le s$ with

$$B(\underline{\mathsf{p}}, r) \cap \mathrm{Hyp}_T^0(d) \subseteq \underline{U}_i \cap \mathrm{Hyp}_T^0(d). \tag{3.5}$$

Definition 3.5 (Universal splitting)**.** We refer to this data as a *universal splitting of hyperbolic polynomials of degree d in Tschirnhausen form* and to r as the *radius of the splitting.*

4. Bronshtein's theorem and a variant at a single point

We recall Bronshtein's theorem in Theorem 4.1. We shall need a version at a single point with a suitable bound for the derivative of the roots. This version is given in Theorem 4.7.

4.1. **Bronshtein's theorem.** The following result is a version of Bronshtein's theorem [Bro79] with uniform bounds due to [PR15], see also [PR25, Theorem 3.2].

Theorem 4.1. *Let $I \subseteq \mathbb{R}$ be an open interval and $\mathsf{a} \in C^{d-1,1}(I, \operatorname{Hyp}(d))$. Then any continuous root $\lambda \in C^0(I)$ of P_a is locally Lipschitz and, for any pair of relatively compact open intervals $I_0 \Subset I_1 \Subset I$,*

$$|\lambda|_{C^{0,1}(\overline{I}_0)} \le C \max_{1 \le j \le d} \|a_j\|^{1/j}_{C^{d-1,1}(\overline{I}_1)}, \tag{4.1}$$

with $C = C(d) \max\{\delta^{-1}, 1\}$, where $\delta := \operatorname{dist}(I_0, \mathbb{R} \setminus I_1)$.

A multiparameter version follows easily; see [PR15] and [PR25, Theorem 3.4].

Note that Wakabayashi [Wak86] proved a Hölder version of Theorem 4.1 (without uniform bounds of the type (4.1)), see also Tarama [Tar06] for a different proof.

4.2. **Reclusive points.** The local version of Bronshtein's theorem, Theorem 4.7, holds at all points of I except for a countable subset of points, which we call reclusive points. A point $x \in I$ is reclusive if either all the roots of $P_{\tilde{\mathsf{a}}(x)}$ are zero and x is isolated for this property, or it satisfies a similar condition for one of the local factors of $P_{\tilde{\mathsf{a}}(x)}$, see Definition 4.4 for a precise formulation.

Definition 4.2 (Zero sets)**.** Let $\tilde{\mathsf{a}} : I \to \operatorname{Hyp}_T(d) \subseteq \mathbb{R}^d$. We consider the zero set

$$Z_{\tilde{\mathsf{a}}} := \{x \in I : \tilde{\mathsf{a}}(x) = 0\} = \{x \in I : \text{ all roots of } P_{\mathsf{a}(x)} \text{ coincide}\}$$

which coincides with $Z_{\tilde{a}_2} = \{x \in I : \tilde{a}_2(x) = 0\}$, by Lemma 3.1. (For notational simplicity, we will generally use $Z_{\tilde{\mathsf{a}}}$.) We write $\operatorname{acc}(Z_{\tilde{\mathsf{a}}})$ and $\operatorname{iso}(Z_{\tilde{\mathsf{a}}}) := Z_{\tilde{\mathsf{a}}} \setminus \operatorname{acc}(Z_{\tilde{\mathsf{a}}})$ for the sets of accumulation points and isolated points of $Z_{\tilde{\mathsf{a}}}$, respectively.

Lemma 4.3. *Let $\tilde{\mathsf{a}} : I \to \operatorname{Hyp}_T(d) \subseteq \mathbb{R}^d$. Then*

$$Z_{\tilde{\mathsf{a}}} = \{x \in I : \text{ all roots of } P_{\tilde{\mathsf{a}}(x)} \text{ vanish}\}.$$

Proof. Since $Z_{\tilde{\mathsf{a}}} = Z_{\tilde{a}_2}$, this is immediate from (3.2). □

Let $I \subseteq \mathbb{R}$ be an open interval and $\tilde{\mathsf{a}} \in C^{d-1,1}(I, \operatorname{Hyp}_T(d))$; recall that this means $\tilde{\mathsf{a}} \in C^{d-1,1}(I, \mathbb{R}^d)$ and $\tilde{\mathsf{a}}(I) \subseteq \operatorname{Hyp}_T(d)$. Let $x_0 \in I$ be such that $\tilde{a}_2(x_0) \ne 0$. Then not all roots of $P_{\tilde{\mathsf{a}}(x_0)}$ coincide and hence $P_{\tilde{\mathsf{a}}}$ splits in a neighborhood of x_0. We may assume that it is a *full splitting*, i.e., if $\{\lambda_1, \ldots, \lambda_k\}$ are the distinct roots of $P_{\tilde{\mathsf{a}}(x_0)}$ with multiplicities $\{m_1, \ldots, m_k\}$ then

$$P_{\tilde{\mathsf{a}}} = P_{\mathsf{b}_1} P_{\mathsf{b}_2} \cdots P_{\mathsf{b}_k} \quad \text{ in a neighborhood of } x_0, \tag{4.2}$$

where $\deg P_{\mathsf{b}_j} = m_j$ and $P_{\mathsf{b}_j(x_0)}(Z) = (Z - \lambda_j)^{m_j}$, for all $1 \le j \le k$. Note that the full splitting is unique up to the order of the factors. Since the Tschirnhausen transformation $\mathsf{b}_j \rightsquigarrow \tilde{\mathsf{b}}_j$ effects a shift of the roots by $b_{j,1}/m_j = -\lambda_j$, we have $x_0 \in Z_{\tilde{\mathsf{b}}_j}$, for all $1 \le j \le k$.

Definition 4.4 (Reclusive points)**.** Let $\tilde{\mathsf{a}} \in C^{d-1,1}(I, \operatorname{Hyp}_T(d))$. We say that $x_0 \in I$ is *reclusive for* $\tilde{\mathsf{a}}$ if

- $x_0 \in \operatorname{iso}(Z_{\tilde{\mathsf{a}}})$,
- or $x_0 \notin Z_{\tilde{\mathsf{a}}}$ and $x_0 \in \operatorname{iso}(Z_{\tilde{\mathsf{b}}_j})$ for some $j \in \{1 \ldots, k\}$, where we refer to the full splitting (4.2).

Note that, by Lemma 4.3, x_0 is an isolated point of $Z_{\tilde{\mathsf{b}}_j}$ if and only if x_0 is an isolated point of

$$E_{\mathsf{b}_j} := \{x : \text{ all roots of } P_{\mathsf{b}_j(x)} \text{ coincide}\}.$$

Lemma 4.5. *Let $I \subseteq \mathbb{R}$ be an open interval and $\tilde{\mathsf{a}} \in C^{d-1,1}(I, \operatorname{Hyp}_T(d))$. Let $x_0 \in I$ be such that $\tilde{a}_2(x_0) \neq 0$ and assume that x_0 is not reclusive for $\tilde{\mathsf{a}}$. If $P_{\tilde{\mathsf{a}}} = P_{\mathsf{b}} P_{\mathsf{c}}$ is any splitting near x_0, then x_0 is not reclusive for $\tilde{\mathsf{b}}$ and $\tilde{\mathsf{c}}$ (which result from b and c by the Tschirnhausen transformation).*

Proof. After possibly reordering the factors in (4.2), we may assume that, in a neighborhood of x_0,

$$P_{\mathsf{b}} = P_{\mathsf{b}_1} \cdots P_{\mathsf{b}_j} \quad \text{ and } \quad P_{\mathsf{c}} = P_{\mathsf{b}_{j+1}} \cdots P_{\mathsf{b}_k}.$$

The Tschirnhausen transformation $\mathsf{b} \rightsquigarrow \tilde{\mathsf{b}}$ effects a shift on all roots of P_{b} by $b_1/\deg P_{\mathsf{b}}$ and retains the splitting,

$$P_{\tilde{\mathsf{b}}} = P_{\hat{\mathsf{b}}_1} \cdots P_{\hat{\mathsf{b}}_j}.$$

It follows that $E_{\hat{\mathsf{b}}_i} = E_{\mathsf{b}_i}$ for all $1 \le i \le j$. Suppose for contradiction that x_0 is reclusive for $\tilde{\mathsf{b}}$. If x_0 is an isolated point of $Z_{\tilde{\mathsf{b}}}$, then $j = 1$, by Lemma 4.3, and hence x_0 is reclusive for $\tilde{\mathsf{a}}$. If $x_0 \notin Z_{\tilde{\mathsf{b}}}$ and there is $i \in \{1, \ldots, j\}$ such that x_0 is an isolated point of $E_{\hat{\mathsf{b}}_i} = E_{\mathsf{b}_i}$, then again x_0 is reclusive for $\tilde{\mathsf{a}}$. Since we assumed that x_0 is not reclusive for $\tilde{\mathsf{a}}$, we conclude that x_0 is not reclusive for $\tilde{\mathsf{b}}$.

The proof that x_0 is not reclusive for $\tilde{\mathsf{c}}$ is analogous. □

Lemma 4.6. *Let $I \subseteq \mathbb{R}$ be an open interval and $\tilde{\mathsf{a}} \in C^{d-1,1}(I, \operatorname{Hyp}_T(d))$. The set of all $x_0 \in I$ that are reclusive for $\tilde{\mathsf{a}}$ is countable.*

Proof. Let $\lambda := \mathcal{S}(\tilde{\mathsf{a}})$. Then λ is a curve in $\{y \in \mathbb{R}^d : y_1 \le y_2 \le \cdots \le y_d\}$. For $1 \le i < d$, let $\ell_i(y) := y_{i+1} - y_i$. If $x_0 \in I$ is reclusive for $\tilde{\mathsf{a}}$, then there exist $1 \le i_1 < \cdots < i_k < d$ such that x_0 is an isolated point of

$$\{x \in I : \ell_{i_j}(\lambda(x)) = 0 \text{ for all } 1 \le j \le k\}.$$

The set of isolated points of the latter set is countable. The statement follows. □

4.3. **A version of Bronshtein's theorem at a single point.** For $x_0 \in \mathbb{R}$ and $r > 0$, let $I(x_0, r)$ denote the open interval centered at x_0 with radius r,

$$I(x_0, r) := \{x \in \mathbb{R} : |x - x_0| < r\}.$$

Its closure is denoted by $\overline{I}(x_0, r)$.

Theorem 4.7. *Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde{\mathsf{a}} \in C^{d-1,1}(\overline{I}(x_0, \delta), \operatorname{Hyp}_T(d))$. Assume that x_0 is not reclusive for $\tilde{\mathsf{a}}$. Let $\lambda \in C^0(I(x_0, \delta))$ be a continuous root of $P_{\tilde{\mathsf{a}}}$ and assume that $\lambda'(x_0)$ exists. Then*

$$|\lambda'(x_0)| \le C(d)\, A(\delta),$$

where

$$\begin{aligned} A(\delta) &:= 6 \max\{A_1(\delta), A_2(\delta)\}, \qquad (4.3)\\ A_1(\delta) &:= \max\big\{\delta^{-1} |\tilde{a}_2(x_0)|^{1/2}, |\tilde{a}_2'|^{1/2}_{C^{0,1}(\overline{I}(x_0,\delta))}\big\}, \end{aligned}$$

$$A_2(\delta) := \max_{2\le j\le d} \big\{ |\tilde a_j^{(d-1)}|_{C^{0,1}(\overline{I}(x_0,\delta))} \cdot \|\tilde a_2\|^{(d-j)/2}_{L^\infty(I(x_0,\delta))} \big\}^{1/d}.$$

Here $\tilde a_2'$ denotes the first derivative of $\tilde a_2$ and $\tilde a_j^{(d-1)}$ the derivative of order $d-1$ of $\tilde a_j$.

The proof follows the general strategy of the proof of Theorem 4.1 in [PR15] and [PR25], but some modifications are required. Before we prove Theorem 4.7 let us recall two important tools.

4.4. **Local Glaeser inequality.** Glaeser's inequality [Gla63] gives Bronshtein's theorem in the simplest nontrivial case: for nonnegative C^1 functions f on $\mathbb{R}$ with $f'' \in L^\infty(\mathbb{R})$ we have

$$f'(x)^2 \le 2\, f(x) \|f''\|_{L^\infty(\mathbb{R})}, \quad x \in \mathbb{R}.$$

We need a local version.

Lemma 4.8 ([PR25, Lemma 3.14])**.** *Let $I \subseteq \mathbb{R}$ be an open bounded interval. Let $f \in C^{1,1}(\overline{I})$ satisfy $f \ge 0$ or $f \le 0$ on I. Assume that $x_0 \in I$ satisfies $f(x_0) \ne 0$ and let $M > 0$ be such that $I_0 := I(x_0, M^{-1}|f(x_0)|^{1/2}) \subseteq I$. Then*

$$|f'(x_0)| \le (M + M^{-1}|f'|_{C^{0,1}(\overline{I}_0)})|f(x_0)|^{1/2}.$$

Therefore, if additionally $|f'|_{C^{0,1}(\overline{I}_0)} \le M^2$, then

$$|f'(x_0)| \le 2M\,|f(x_0)|^{1/2}.$$

It should be added that, for a function $f \in C^{1,1}(\overline{I})$ satisfying $f \ge 0$ or $f \le 0$ on I, we have that $f(x_0) = 0$ implies $f'(x_0) = 0$, so that the conclusion of the lemma also holds trivially at zeros x_0 of f.

Proof. Suppose that $f \ge 0$; otherwise consider $-f$. Thus $f(x_0) > 0$ and

$$0 \le f(x_0 + h) = f(x_0) + f'(x_0)h + \int_0^1 f'(x_0 + hs) - f'(x_0)\, ds \cdot h.$$

Setting $h := \pm M^{-1}|f(x_0)|^{1/2}$, implies the lemma. □

4.5. **Interpolation.** Let us recall an interpolation inequality for intermediate derivatives.

Lemma 4.9 ([PR25, Lemma 3.16])**.** *Let $f \in C^{m,1}(\overline{I})$, where $I \subseteq \mathbb{R}$ is a bounded open interval. Then, for $1 \le j \le m$,*

$$|f^{(j)}(x)| \le C(m)\, |I|^{-j} \big(\|f\|_{L^\infty(I)} + |f^{(m)}|_{C^{0,1}(\overline{I})} |I|^{m+1} \big), \quad x \in I.$$

Proof. Fix $x \in I$. Then $[x, x + |I|/2)$ or $(x - |I|/2, x]$ is contained in I. Let y be a point in the respective interval. By Taylor's formula,

$$\Big| \sum_{j=0}^m \frac{f^{(j)}(x)}{j!} (y-x)^j \Big|$$
$$= \Big| f(y) - (y-x)^m \int_0^1 \frac{(1-t)^{m-1}}{(m-1)!} (f^{(m)}(x + t(y-x)) - f^{(m)}(x))\, dt \Big|$$

$$\leq \|f\|_{L^\infty(I)} + |I|^{m+1} |f^{(m)}|_{C^{0,1}(\overline{I})}.$$

This implies the lemma in view of the following fact: if a polynomial $T(x) = a_0 + a_1 x + \cdots + a_m x^m \in \mathbb{C}[x]$ satisfies $|T(x)| \leq A$ for $x \in [0, B] \subseteq \mathbb{R}$, then

$$|a_j| \leq C(m)\, A B^{-j}, \quad 0 \leq j \leq m.$$

Indeed, for $A = B = 1$ this follows easily, by comparison with the interpolation polynomial for the equidistant points $0 = x_0 < x_1 < \cdots < x_m = 1$. In the general case, consider $A^{-1} T(Bx)$. □

4.6. **Proof of Theorem 4.7.** The rest of the section is devoted to the proof of Theorem 4.7. We may assume that $d \geq 2$, since Theorem 4.7 is trivially true for $d = 1$. The following definition will prove convenient for the inductive proof based on the splitting.

We will work on intervals centered at x_0 whose radius depends on the size of $\tilde{a}_2(x_0)$. More precisely, assuming that $\tilde{a}_2(x_0) \neq 0$ we set, for any constant $A > 0$,

$$\mathbf{I}(x_0, A) := I(x_0, A^{-1} |\tilde{a}_2(x_0)|^{1/2}) \tag{4.4}$$

and denote by $\overline{\mathbf{I}}(x_0, A)$ the closure of $\mathbf{I}(x_0, A)$.

Definition 4.10 ($C^{d-1,1}$-admissible data)**.** Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde{\mathsf{a}} \in C^{d-1,1}(\overline{I}(x_0, \delta), \mathrm{Hyp}_T(d))$ be such that $\tilde{a}_2(x_0) \neq 0$. Let $A > 0$ be a constant. We say that $(\tilde{\mathsf{a}}, x_0, \delta, A)$ is $C^{d-1,1}$-*admissible* if the following holds:

(1) $\mathbf{I}(x_0, A) \subseteq I(x_0, \delta)$.

(2) For all $x \in \mathbf{I}(x_0, A)$,
$$\frac{1}{2} \leq \frac{\tilde{a}_2(x)}{\tilde{a}_2(x_0)} \leq 2. \tag{4.5}$$

(3) For all $2 \leq j \leq d$,
$$|\tilde{a}_j^{(d-1)}|_{C^{0,1}(\overline{\mathbf{I}}(x_0, A))} \leq A^d\, |\tilde{a}_2(x_0)|^{(j-d)/2}. \tag{4.6}$$

(4) For all $2 \leq j \leq d$, $1 \leq k \leq d-1$, and $x \in \mathbf{I}(x_0, A)$,
$$|\tilde{a}_j^{(k)}(x)| \leq A^k\, |\tilde{a}_2(x_0)|^{(j-k)/2}. \tag{4.7}$$

Lemma 4.11. *Let $x_0 \in \mathbb{R}$ and $A, \delta > 0$. Let $\tilde{\mathsf{a}} \in C^{d-1,1}(\overline{I}(x_0, \delta), \mathrm{Hyp}_T(d))$ be such that $\tilde{a}_2(x_0) \neq 0$. Assume that*

- $\mathbf{I}(x_0, A) \subseteq I(x_0, \delta)$,
- (4.6) *holds, and*
- (4.7) *holds for $2 \leq j \leq d$ and $k \geq j$.*

Then there is a constant $C(d) \geq 1$ such that $(\tilde{\mathsf{a}}, x_0, \delta, C(d)A)$ is $C^{d-1,1}$-admissible.

Proof. We first observe that we have $|\tilde{a}_2'|_{C^{0,1}(\overline{\mathbf{I}}(x_0, A))} \leq A^2$. Indeed, if $d = 2$ this is immediate from (4.6) and if $d \geq 3$ then it follows from (4.7) with $j = k = 2$. By Lemma 4.8, we conclude that

$$|\tilde{a}_2'(x_0)| \leq 2A\, |\tilde{a}_2(x_0)|^{1/2}.$$

Thus, for $x \in \mathbf{I}(x_0, 6A)$,

$$\begin{aligned}|\tilde{a}_2(x) - \tilde{a}_2(x_0)| &\le |\tilde{a}_2'(x_0)||x - x_0| + |\tilde{a}_2'|_{C^{0,1}(\overline{\mathbf{I}}(x_0,A))}\, |x - x_0|^2 \\ &\le \frac{1}{3}\,|\tilde{a}_2(x_0)| + \frac{1}{36}\,|\tilde{a}_2(x_0)| < \frac{1}{2}\,|\tilde{a}_2(x_0)|,\end{aligned}$$

implying (4.5) on $\mathbf{I}(x_0, 6A)$.

Finally, we check that (4.7) also holds for $k < j$, for A replaced by $C(d)A$, where $C(d) \ge 1$ is a suitable constant (note that $\mathbf{I}(x_0, C(d)A) \subseteq \mathbf{I}(x_0, A)$). By Lemma 4.9, for $1 \le k \le j - 1$ and $x \in \mathbf{I}(x_0, 6A)$,

$$\begin{aligned}|\tilde{a}_j^{(k)}(x)| &\le C(d)\, |\mathbf{I}(x_0, 6A)|^{-k} \big(\|\tilde{a}_j\|_{L^\infty(\mathbf{I}(x_0,6A))} + |\tilde{a}_j^{(j-1)}|_{C^{0,1}(\overline{\mathbf{I}}(x_0,6A))}\, |\mathbf{I}(x_0, 6A)|^j\big) \\ &\le C(d)\, (3A)^k |\tilde{a}_2(x_0)|^{-k/2} \big(2^j\, |\tilde{a}_2(x_0)|^{j/2} + A^j \cdot (3A)^{-j} |\tilde{a}_2(x_0)|^{j/2}\big) \\ &\le \tilde{C}(d)\, A^k\, |\tilde{a}_2(x_0)|^{(j-k)/2},\end{aligned}$$

since $|\tilde{a}_j(x)| \le (\sqrt{2}\,|\tilde{a}_2(x)|^{1/2})^j \le 2^j\, |\tilde{a}_2(x_0)|^{j/2}$, by Lemma 3.1 and (4.5) on $\mathbf{I}(x_0, 6A)$, and $|\tilde{a}_j^{(j-1)}|_{C^{0,1}(\overline{\mathbf{I}}(x_0,6A))} \le A^j$, by (4.6) or (4.7) for $k = j$. □

Lemma 4.12. *Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde{\mathsf{a}} \in C^{d-1,1}(\overline{I}(x_0, \delta), \mathrm{Hyp}_T(d))$ be such that $\tilde{a}_2(x_0) \ne 0$. Let $A(\delta)$ be defined by (4.3). Then $(\tilde{\mathsf{a}}, x_0, \delta, C(d)A(\delta))$ is $C^{d-1,1}$-admissible for some constant $C(d) \ge 1$.*

Proof. (1) By (4.3), $A(\delta) \ge A_1(\delta) \ge \delta^{-1}|\tilde{a}_2(x_0)|^{1/2}$ and thus

$$\mathbf{I}(x_0, A(\delta)) \subseteq \mathbf{I}(x_0, A_1(\delta)) \subseteq I(x_0, \delta). \tag{4.8}$$

(2) By Lemma 4.8 and the definition of $A_1(\delta)$,

$$|\tilde{a}_2'(x_0)| \le 2A_1(\delta)\, |\tilde{a}_2(x_0)|^{1/2}.$$

Then, for $x \in \mathbf{I}(x_0, 6A_1(\delta))$, we find (as in the proof of Lemma 4.11)

$$|\tilde{a}_2(x) - \tilde{a}_2(x_0)| \le \frac{1}{2}|\tilde{a}_2(x_0)|,$$

using (4.8) and the definition of $A_1(\delta)$, which implies (4.5) with $A = A(\delta)$.

(3) By the definition of $A_2(\delta)$, (4.6) with $A = A(\delta)$ is clear.

(4) By Lemma 3.1 and (4.5), we have $|\tilde{a}_j(x)| \le 2^j\, |\tilde{a}_2(x_0)|^{j/2}$, for $x \in \mathbf{I}(x_0, A(\delta))$. In conjunction with (4.6), it implies (4.7) with $A = C(d)A(\delta)$, by Lemma 4.9. We clearly may assume that $C(d) \ge 1$ so that $\mathbf{I}(x_0, C(d)A(\delta)) \subseteq \mathbf{I}(x_0, A(\delta))$. □

Lemma 4.13. *Let $(\tilde{\mathsf{a}}, x_0, \delta, A)$ be $C^{d-1,1}$-admissible. Then the functions $\underline{a}_j := |\tilde{a}_2|^{-j/2}\tilde{a}_j$, $2 \le j \le d$, are well-defined and of class $C^{d-1,1}$ on $\mathbf{I}(x_0, A)$ and they satisfy*

$$|\underline{a}_j^{(d-1)}|_{C^{0,1}(\overline{\mathbf{I}}(x_0,A))} \le C(d)\, A^d\, |\tilde{a}_2(x_0)|^{-d/2}, \quad 2 \le j \le d, \tag{4.9}$$

$$|\underline{a}_j^{(k)}(x)| \le C(d)\, A^k\, |\tilde{a}_2(x_0)|^{-k/2}, \quad 2 \le j \le d,\ 1 \le k \le d-1,\ x \in \mathbf{I}(x_0, A). \tag{4.10}$$

Proof. By (4.5) and (3.2), $|\tilde{a}_2| > 0$ on $\mathbf{I}(x_0, A)$. Thus the functions $\underline{a}_j$ are well-defined and of class $C^{d-1,1}$ on $\mathbf{I}(x_0, A)$.

Let $1 \le s \le d-1$ and $r \in \mathbb{R}$. Then Faà di Bruno's formula implies

$$\partial^s \big(\tilde a_2^r\big) = \sum_{\ell \ge 1}^{s} \sum_{\gamma \in \Gamma(\ell,s)} c_{\gamma,\ell,r}\, \tilde a_2^{r-\ell} \tilde a_2^{(\gamma_1)} \cdots \tilde a_2^{(\gamma_\ell)} \tag{4.11}$$

where $\Gamma(\ell,s) = \{\gamma \in \mathbb{N}_{>0}^\ell : |\gamma| = s\}$ and

$$c_{\gamma,\ell,r} = \frac{s!}{\ell!\gamma!} r(r-1)\cdots(r-\ell+1).$$

Thus, by (4.5) and (4.7), for $x \in \mathbf{I}(x_0,A)$,

$$\begin{aligned}
|\partial^s\big(\tilde a_2^r\big)(x)| &\le \sum_{\ell \ge 1}^{s} \sum_{\gamma \in \Gamma(\ell,s)} |c_{\gamma,\ell,r}|\, |\tilde a_2^{r-\ell}(x)| |\tilde a_2^{(\gamma_1)}(x)| \cdots |\tilde a_2^{(\gamma_\ell)}(x)| \\
&\le \sum_{\ell \ge 1}^{s} \sum_{\gamma \in \Gamma(\ell,s)} |c_{\gamma,\ell,r}|\, 2^{r-\ell} |\tilde a_2(x_0)|^{r-\ell}\, A^s\, |\tilde a_2(x_0)|^{\ell - s/2} \\
&= A^s\, |\tilde a_2(x_0)|^{r-s/2} \sum_{\ell \ge 1}^{s} \sum_{\gamma \in \Gamma(\ell,s)} |c_{\gamma,\ell,r}|\, 2^{r-\ell}.
\end{aligned} \tag{4.12}$$

Consequently, by the Leibniz formula, (4.5), and (4.7),

$$|\underline{a}_j^{(k)}(x)| \le \sum_{s=0}^{k} \binom{k}{s} |\partial^s(|\tilde a_2|^{-j/2})(x)| |\tilde a_j^{(k-s)}(x)| \le C(d)\, A^k\, |\tilde a_2(x_0)|^{-k/2},$$

for $1 \le k \le d-1$ and $x \in \mathbf{I}(x_0,A)$, that is (4.10).

To see (4.9) it suffices to repeat the above argument, using that, for functions $f_1,\ldots,f_m$ on an interval I,

$$\Big|\prod_{i=1}^m f_i\Big|_{C^{0,1}(\overline I)} \le \sum_{i=1}^m |f_i|_{C^{0,1}(\overline I)} \prod_{j \ne i} \|f_j\|_{L^\infty(I)}$$

and

$$|f^r|_{C^{0,1}(\overline I)} \le |r|\, \|f^{r-1}\|_{L^\infty(I)} \|f'\|_{L^\infty(I)},$$

if f is differentiable. □

Proposition 4.14. *Let $(\tilde{\mathsf a}, x_0, \delta, A)$ be $C^{d-1,1}$-admissible. Then there exist $\delta_1 > 0$ and a constant $C(d) > 1$ such that the following holds. There is a splitting*

$$P_{\tilde{\mathsf a}} = P_{\mathsf b} P_{\mathsf c}, \qquad \text{on } I(x_0,\delta_1),$$

where $P_{\mathsf b}$ and $P_{\mathsf c}$ are monic hyperbolic polynomials of degree $< d$ with coefficients in $C^{d-1,1}(\overline I(x_0,\delta_1))$. We have, for all $1 \le i \le \deg P_{\mathsf b}$,

$$|b_i^{(d-1)}|_{C^{0,1}(\overline I(x_0,\delta_1))} \le C(d) A^d\, |\tilde a_2(x_0)|^{(i-d)/2}. \tag{4.13}$$

$$|b_i^{(k)}(x)| \le C(d) A^k\, |\tilde a_2(x_0)|^{(i-k)/2}, \quad 1 \le k \le d-1,\ x \in I(x_0,\delta_1), \tag{4.14}$$

If, after Tschirnhausen transformation, $\tilde b_2(x_0) \ne 0$, then $(\tilde{\mathsf b}, x_0, \delta_1, C(d)A)$ is $C^{d-1,1}$-admissible. The analogous statements hold for $\tilde{\mathsf c}$.

Proof. Consider the continuous bounded (cf. Lemma 3.1 and Lemma 4.13) curve

$$\underline{\mathsf{a}} := (0, -1, \underline{a}_3, \ldots, \underline{a}_d) : \mathbf{I}(x_0, A) \to \mathrm{Hyp}^0_T(d) \subseteq \mathbb{R}^d,$$

where $\underline{a}_j := |\tilde{a}_2|^{-j/2}\tilde{a}_j$. Then, by (4.10), there exists $C_1 = C_1(d)$ such that

$$\|\underline{\mathsf{a}}'(x)\|_2 \le C_1 A\, |\tilde{a}_2(x_0)|^{-1/2}, \quad x \in \mathbf{I}(x_0, A). \tag{4.15}$$

We may assume that $C_1 > 1$. Let $0 < r < 1$ be the radius of the splitting (see Definition 3.5) and define

$$\delta_1 := \frac{|\tilde{a}_2(x_0)|^{1/2}\, r}{C_1 A}.$$

Then $I(x_0, \delta_1) \subseteq \mathbf{I}(x_0, A)$ and $\underline{\mathsf{a}}(I(x_0, \delta_1)) \subseteq B(\underline{\mathsf{a}}(x_0), r)$, by (4.15). Consequently, we have a splitting

$$P_{\tilde{\mathsf{a}}} = P_{\mathsf{b}} P_{\mathsf{c}}, \quad \text{ on } I(x_0, \delta_1),$$

by (3.5).

Next we check (4.13) and (4.14). By (3.3),

$$b_i = |\tilde{a}_2|^{i/2} \cdot \psi_i \circ \underline{\mathsf{a}}. \tag{4.16}$$

We claim that, for $1 \le s \le d-1$ and $x \in \overline{I}(x_0, \delta_1)$,

$$|(\psi_i \circ \underline{\mathsf{a}})^{(s)}(x)| \le C(d)\, A^s\, |\tilde{a}_2(x_0)|^{-s/2}. \tag{4.17}$$

We have

$$\begin{aligned}
(\psi_i \circ \underline{\mathsf{a}})' &= \sum_{j=1}^{d} ((\partial_j \psi_i) \circ \underline{\mathsf{a}}) \cdot \underline{a}_j', \\
(\psi_i \circ \underline{\mathsf{a}})^{(s)} &= \sum_{j=1}^{d} \partial^{s-1}\big(((\partial_j \psi_i) \circ \underline{\mathsf{a}}) \cdot \underline{a}_j'\big) \\
&= \sum_{j=1}^{d} \sum_{k=0}^{s-1} \binom{s-1}{k} ((\partial_j \psi_i) \circ \underline{\mathsf{a}})^{(k)} \underline{a}_j^{(s-k)}.
\end{aligned}$$

For $s = 1$ the claim (4.17) follows from (4.10). For $s \ge 2$ the claim follows by induction and (4.10).

Now (4.14) is a consequence of the Leibniz formula, (4.12), (4.16), and (4.17). To see (4.13) we proceed analogously, combining (4.9) with the observations at the end of the proof of Lemma 4.13.

Suppose that $\tilde{b}_2(x_0) \ne 0$ and let us show that $(\tilde{\mathsf{b}}, x_0, \delta_1, C(d)A)$, for a suitable constant $C(d) > 1$, is $C^{d-1,1}$-admissible. Set

$$B := \frac{2C_1 A}{r}.$$

By Lemma 3.4, we have

$$|\tilde{b}_2(x_0)| \le 4\, |\tilde{a}_2(x_0)| \tag{4.18}$$

which implies

$$B^{-1} |\tilde{b}_2(x_0)|^{1/2} \le \frac{|\tilde{a}_2(x_0)|^{1/2}\, r}{C_1 A} = \delta_1,$$

so that $\mathbf{J}(x_0, B) := I(x_0, B^{-1}|\tilde{b}_2(x_0)|^{1/2}) \subseteq I(x_0, \delta_1)$. From (4.13) and (4.14), we easily get the same bounds for $\tilde{b}_i$ instead of b_i (by means of (3.1)). By (4.18), we may replace $\tilde{a}_2(x_0)$ by $\tilde{b}_2(x_0)$ on the right-hand side of these estimates if $k \ge i$ (note that $d > i$). Now it suffices to invoke Lemma 4.11.

The same arguments yield the analogous statement about $\tilde{\mathsf{c}}$. □

Proposition 4.15. *Let $(\tilde{\mathsf{a}}, x_0, \delta, A)$ be $C^{d-1,1}$-admissible and assume that x_0 is not reclusive for $\tilde{\mathsf{a}}$. If $\lambda \in C^0(I(x_0,\delta))$ is a root of $P_{\tilde{\mathsf{a}}}$ and $\lambda'(x_0)$ exists, then*

$$|\lambda'(x_0)| \le C(d)A. \tag{4.19}$$

Proof. By assumption, $\tilde{a}_2(x_0) \ne 0$ and hence $d \ge 2$. By Proposition 4.14, there exists $\delta_1 > 0$ such that there is a splitting $P_{\tilde{\mathsf{a}}} = P_{\mathsf{b}} P_{\mathsf{c}}$ on $I(x_0, \delta_1)$. We may assume that λ is a root of P_{b} and hence

$$\lambda(x) = -\frac{b_1(x)}{\deg P_{\mathsf{b}}} + \mu(x), \quad x \in I(x_0,\delta_1), \tag{4.20}$$

where μ is a continuous root of $P_{\tilde{\mathsf{b}}}$ and $\mu'(x_0)$ exists (since we assumed that $\lambda'(x_0)$ exists). By (4.14) for $i = k = 1$, we have

$$|b_1'(x_0)| \le C(d)A. \tag{4.21}$$

By Lemma 4.5, x_0 is not reclusive for $\tilde{\mathsf{b}}$, since x_0 is not reclusive for $\tilde{\mathsf{a}}$.

Let us now prove the proposition by induction on d.

If $d = 2$, then $\deg P_{\mathsf{b}} = 1$ and $\tilde{\mathsf{b}} \equiv 0$. Thus $\lambda(x) = -b_1(x)$ for $x \in I(x_0,\delta_1)$ so that (4.21) gives (4.19).

Assume that $d \ge 3$. If $\tilde{b}_2(x_0) \ne 0$, then $(\tilde{\mathsf{b}}, x_0, \delta_1, C(d)A)$ is $C^{d-1,1}$-admissible, by Proposition 4.14. By the induction hypothesis,

$$|\mu'(x_0)| \le C(d)A.$$

Thus (4.19) follows from (4.20) and (4.21).

If $\tilde{b}_2(x_0) = 0$, then x_0 (being not reclusive for $\tilde{\mathsf{b}}$) is an accumulation point of $Z_{\tilde{\mathsf{b}}}$. Consequently, $\mu'(x_0) = 0$ and (4.19) again follows from (4.20) and (4.21). □

Proof of Theorem 4.7. Let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let $\tilde{\mathsf{a}} \in C^{d-1,1}(\overline{I}(x_0,\delta), \mathrm{Hyp}_T(d))$. Assume that x_0 is not reclusive for $\tilde{\mathsf{a}}$. Let $\lambda \in C^0(I(x_0,\delta))$ be a continuous root of $P_{\tilde{\mathsf{a}}}$ and assume that $\lambda'(x_0)$ exists.

If $\tilde{a}_2(x_0) \ne 0$, then, by Lemma 4.12, $(\tilde{\mathsf{a}}, x_0, \delta, C(d)A(\delta))$ is $C^{d-1,1}$-admissible, where $A(\delta)$ is defined by (4.3) and $C(d) \ge 1$. Then Proposition 4.15 yields

$$|\lambda'(x_0)| \le C(d)A(\delta).$$

If $\tilde{a}_2(x_0) = 0$, then x_0 (being not reclusive for $\tilde{\mathsf{a}}$) is an accumulation point of $Z_{\tilde{\mathsf{a}}}$. Hence $\lambda'(x_0) = 0$ and the assertion is trivially true. □

5. Proof of the main technical result

The goal of this section is the proof of Theorem 1.3.

Let $I \subseteq \mathbb{R}$ be an open interval. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(I, \mathrm{Hyp}(d))$, i.e., for each relatively compact open interval $I_1 \Subset I$,

$$\|\mathsf{a} - \mathsf{a}_n\|_{C^d(\overline{I}_1,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

It follows from Theorem 4.1 that the set $\{\mathcal{S}(\mathsf{a}_n) : n \geq 1\}$ is bounded in $C^{0,1}(I, \mathbb{R}^d)$. We must show that, for each relatively compact open interval $I_0 \Subset I$ and each $1 \leq q < \infty$,

$$\|\mathcal{S}(\mathsf{a}) - \mathcal{S}(\mathsf{a}_n)\|_{W^{1,q}(I_0,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

5.1. **Strategy of the proof.** The proof of (5.1) is subdivided into three steps. The first two steps are dedicated to the proof of

$$\|\mathcal{S}(\mathsf{a})' - \mathcal{S}(\mathsf{a}_n)'\|_{L^q(I_0,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty, \tag{5.2}$$

for $1 \leq q < \infty$, using the dominated convergence theorem. In the third step, we show that

$$\|\mathcal{S}(\mathsf{a}) - \mathcal{S}(\mathsf{a}_n)\|_{L^\infty(I_0,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{5.3}$$

Then (5.2) and (5.3) imply (5.1).

Step 1. We claim that the sequence of derivatives $\mathcal{S}(\mathsf{a}_n)'$ is dominated almost everywhere on I_0 by a positive constant.

To see this, fix $I_0 \Subset I_1 \Subset I$. By the assumption of Theorem 1.3, $\{\mathsf{a}_n|_{I_1} : n \geq 1\}$ is a bounded subset of $C^{d-1,1}(\overline{I}_1, \mathbb{R}^d)$. By Theorem 4.1, the derivative of $\mathcal{S}(\mathsf{a}_n)$ exists almost everywhere in I_0 and satisfies

$$\|\mathcal{S}(\mathsf{a}_n)'\|_{L^\infty(I_0,\mathbb{R}^d)} \leq C \sup_{n\geq 1} \max_{1\leq j\leq d} \|a_{n,j}\|^{1/j}_{C^{d-1,1}(\overline{I}_1)} =: B < \infty.$$

This implies the claim.

Step 2. The aim of this step is to prove the following result.

Theorem 5.1. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(I, \mathrm{Hyp}(d))$ as $n \to \infty$. Then, for almost every $x \in I$,*

$$\mathcal{S}(\mathsf{a}_n)'(x) \to \mathcal{S}(\mathsf{a})'(x) \quad \textit{as } n \to \infty.$$

By Step 1 and 2, the dominated convergence theorem yields that (5.2) holds, for each relatively compact open interval $I_0 \Subset I$ and each $1 \leq q < \infty$.

Step 3. Now we show (5.3). Fix $x_0 \in I_0$. We have

$$\mathcal{S}(\mathsf{a}_n)(x_0) = \lambda^\uparrow(\mathsf{a}_n(x_0)) \to \lambda^\uparrow(\mathsf{a}(x_0)) = \mathcal{S}(\mathsf{a})(x_0) \quad \text{as } n \to \infty, \tag{5.4}$$

since the map $\lambda^\uparrow : \mathrm{Hyp}(d) \to \mathbb{R}^d$ is continuous (cf. [AKLM98, Lemma 4.1]). For arbitrary $x \in I_0$,

$$\begin{aligned}\|\mathcal{S}(\mathsf{a})(x) - \mathcal{S}(\mathsf{a}_n)(x)\|_2 &= \Big\|\mathcal{S}(\mathsf{a})(x_0) - \mathcal{S}(\mathsf{a}_n)(x_0) + \int_{x_0}^x \mathcal{S}(\mathsf{a})'(t) - \mathcal{S}(\mathsf{a}_n)'(t)\, dt\Big\|_2 \\ &\leq \|\mathcal{S}(\mathsf{a})(x_0) - \mathcal{S}(\mathsf{a}_n)(x_0)\|_2 + \|\mathcal{S}(\mathsf{a})' - \mathcal{S}(\mathsf{a}_n)'\|_{L^1(I_0,\mathbb{R}^d)}.\end{aligned}$$

Thus, (5.2) and (5.4) imply (5.3).

Remark 5.2. Alternatively, (5.3) is a consequence of [PR24, Corollary 6.5] and Lemma 7.1; for this argument it is actually enough that $\mathsf{a}_n \to \mathsf{a}$ in $C^0(\overline{I}_0, \mathrm{Hyp}(d))$ as $n \to \infty$.

Therefore, in order to prove Theorem 1.3 it remains to show Theorem 5.1. The rest of the section is devoted to the proof of Theorem 5.1.

5.2. **On the zero set of $\tilde{\mathsf{a}}$.** Recall that $\mathrm{acc}(Z_{\tilde{\mathsf{a}}})$ denotes the set of accumulation points of the zero set $Z_{\tilde{\mathsf{a}}}$ of $\tilde{\mathsf{a}} : I \to \mathrm{Hyp}_T(d)$. By Lemma 3.1, $Z_{\tilde{\mathsf{a}}} = Z_{\tilde{a}_2}$.

Lemma 5.3. *Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^d(I, \mathrm{Hyp}_T(d))$ as $n \to \infty$. Then, for almost every $x_0 \in Z_{\tilde{\mathsf{a}}}$,*

$$\mathcal{S}(\tilde{\mathsf{a}}_n)'(x_0) \to 0 \qquad \textit{as } n \to \infty. \tag{5.5}$$

Proof. We will show that (5.5) holds for all $x_0 \in J$, where

$$\begin{aligned} J := \mathrm{acc}(Z_{\tilde{\mathsf{a}}}) \cap \bigcap_{n \ge 1} \{x \in I : x \text{ is not reclusive for } \tilde{\mathsf{a}}_n\} \\ \cap \bigcap_{n \ge 1} \{x \in I : \mathcal{S}(\tilde{\mathsf{a}}_n)'(x) \text{ exists}\}. \end{aligned}$$

The set J has full measure in $Z_{\tilde{\mathsf{a}}}$, by Lemma 4.6 and Rademacher's theorem.

Fix $x_0 \in J \subseteq \mathrm{acc}(Z_{\tilde{\mathsf{a}}})$. Then $\tilde{a}_j^{(k)}(x_0) = 0$ for all $2 \le j \le d$ and $0 \le k \le d$, by Rolle's theorem. Let $\epsilon > 0$ be fixed. By continuity, there exists $\delta > 0$ such that $I(x_0, \delta) \Subset I$ and

$$\|\tilde{a}_j\|_{C^d(\overline{I}(x_0,\delta))} \le \frac{\epsilon^j}{2}, \quad 2 \le j \le d. \tag{5.6}$$

Since $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^d(I, \mathrm{Hyp}_T(d))$ as $n \to \infty$, there exists $n_0 \ge 1$ such that, for $n \ge n_0$,

$$\|\tilde{a}_j - \tilde{a}_{n,j}\|_{C^d(\overline{I}(x_0,\delta))} \le \frac{\epsilon^j}{2}, \quad 2 \le j \le d, \tag{5.7}$$

and

$$|\tilde{a}_{n,2}(x_0)| \le \delta^2 \epsilon^2. \tag{5.8}$$

By (5.6) and (5.7), for $n \ge n_0$ and $2 \le j \le d$,

$$\|\tilde{a}_{n,j}\|_{C^d(\overline{I}(x_0,\delta))} \le \|\tilde{a}_j\|_{C^d(\overline{I}(x_0,\delta))} + \|\tilde{a}_j - \tilde{a}_{n,j}\|_{C^d(\overline{I}(x_0,\delta))} \le \epsilon^j. \tag{5.9}$$

Since $x_0 \in J$ is not reclusive for $\tilde{\mathsf{a}}_n$ and $\mathcal{S}(\tilde{\mathsf{a}}_n)'(x_0)$ exists, we may apply Theorem 4.7 to $\tilde{\mathsf{a}}_n$ and conclude

$$\|\mathcal{S}(\tilde{\mathsf{a}}_n)'(x_0)\|_2 \le C(d)\, A(\delta),$$

where $A(\delta)$ is defined in (4.3) with $\tilde{\mathsf{a}}$ replaced by $\tilde{\mathsf{a}}_n$. By (5.8) and (5.9),

$$A(\delta) \le 6\, \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\mathcal{S}(\tilde{\mathsf{a}}_n)'(x_0) \to 0 \quad \text{as } n \to \infty.$$

The proof is complete. □

5.3. **Admissible data.** At points x_0 with $\tilde{a}_2(x_0) \neq 0$ we have a splitting of $P_{\tilde{\mathsf{a}}}$ and we may use induction on the degree. The following definition is a preparation for the induction argument.

Let us recall (from (4.4)) that

$$\mathbf{I}(x_0, A) := I(x_0, A^{-1}|\tilde{a}_2(x_0)|^{1/2}).$$

Definition 5.4 (C^d-admissible data)**.** Let $I_1 \subseteq \mathbb{R}$ be an open bounded interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{\mathsf{a}} \in C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$. Let $A > 0$ be a constant. We say that $(\tilde{\mathsf{a}}, I_0, I_1, A)$ is *C^d-admissible* if, for every $x_0 \in I_0 \setminus \{x : \tilde{a}_2(x) = 0\}$, the following holds:

(1) $\mathbf{I}(x_0, A) \subseteq I_1$.

(2) For all $x \in \mathbf{I}(x_0, A)$,
$$\frac{1}{2} \le \frac{\tilde{a}_2(x)}{\tilde{a}_2(x_0)} \le 2. \tag{5.10}$$

(3) For all $2 \le j \le d$, $1 \le k \le d$, and $x \in \mathbf{I}(x_0, A)$,
$$|\tilde{a}_j^{(k)}(x)| \le A^k\, |\tilde{a}_2(x_0)|^{(j-k)/2}. \tag{5.11}$$

Note that if we take $I_1 := I(x_0, \delta)$, let I_0 shrink to the point x_0, assume $\tilde{a}_2(x_0) \neq 0$, and use $C^{d-1,1}$- instead of C^d-regularity, we recover the notion from Definition 4.10.

Lemma 5.5. *Let $I_1 \subseteq \mathbb{R}$ be a bounded open interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$ as $n \to \infty$. Set*

$$A := 6 \max\{A_1, A_2\}, \tag{5.12}$$

where, using $\tilde{a}_{0,j} = \tilde{a}_j$ for convenience and $\delta := \mathrm{dist}(I_0, \mathbb{R} \setminus I_1)$,

$$A_1 := \sup_{n \ge 0} \max \big\{\delta^{-1} \|\tilde{a}_{n,2}\|^{1/2}_{L^\infty(I_1)}, |\tilde{a}'_{n,2}|^{1/2}_{C^{0,1}(\overline{I}_1)}\big\},$$
$$A_2 := \sup_{n \ge 0} \max_{2 \le j \le d} \big\{\|\tilde{a}^{(d)}_{n,j}\|_{L^\infty(I_1)} \cdot \|\tilde{a}_{n,2}\|^{(d-j)/2}_{L^\infty(I_1)}\big\}^{1/d}.$$

Then $(\tilde{\mathsf{a}}, I_0, I_1, C(d)A)$ and $(\tilde{\mathsf{a}}_n, I_0, I_1, C(d)A)$, for $n \ge 1$, are C^d-admissible, for some constant $C(d) \ge 1$.

Proof. Fix $n \ge 0$ and $x_0 \in I_0 \setminus \{x : \tilde{a}_{n,2}(x) = 0\}$. By the definition of A_1, we have $\mathbf{I}(x_0, A_1) \subseteq I_1$. By Lemma 4.8,

$$|\tilde{a}'_{n,2}(x_0)| \le 2A_1\, |\tilde{a}_{n,2}(x_0)|^{1/2}$$

which entails (as in the proof of Lemma 4.11) that (5.10) holds on $\mathbf{I}(x_0, 6A_1)$. By the definition of A_2, (5.11) holds for $k = d$. By Lemma 3.1 and (5.10), we have $|\tilde{a}_j(x)| \le 2^j\, |\tilde{a}_2(x_0)|^{j/2}$, for $x \in \mathbf{I}(x_0, A)$. Thus, (5.11) follows from Lemma 4.9. □

In the following, we will use $\mathbf{I}(x_0, A)$ as well as its counterpart for a_n instead of a, that is

$$\mathbf{I}_n(x_0, A) := I(x_0, A^{-1}|\tilde{a}_{n,2}(x_0)|^{1/2}). \tag{5.13}$$

5.4. **Towards a simultaneous splitting.** Our next goal is to show that, if $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$ as $n \to \infty$ and $(\tilde{\mathsf{a}}, I_0, I_1, A)$ and $(\tilde{\mathsf{a}}_n, I_0, I_1, A)$, for $n \ge 1$, are C^d-admissible, then $P_{\tilde{\mathsf{a}}}$ and $P_{\tilde{\mathsf{a}}_n}$, for n large enough, admit a simultaneous splitting; see Definition 5.6.

Let $I_1 \subseteq \mathbb{R}$ be a bounded open interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$, i.e.,

$$\|\tilde{\mathsf{a}} - \tilde{\mathsf{a}}_n\|_{C^d(\overline{I}_1,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty. \tag{5.14}$$

Assume that $(\tilde{\mathsf{a}}, I_0, I_1, A)$ and $(\tilde{\mathsf{a}}_n, I_0, I_1, A)$, for $n \ge 1$, are C^d-admissible for some $A > 0$.

Fix $x_0 \in I_0 \setminus \{x : \tilde{a}_2(x) = 0\}$. By (5.14), there is $n_0 \ge 1$ such that

$$||\tilde{a}_2(x_0)|^{1/2} - |\tilde{a}_{n,2}(x_0)|^{1/2}| < \frac{1}{2}\,|\tilde{a}_2(x_0)|^{1/2}, \quad n \ge n_0,$$

and hence

$$\frac{1}{2} < \frac{|\tilde{a}_{n,2}(x_0)|^{1/2}}{|\tilde{a}_2(x_0)|^{1/2}} < \frac{3}{2}, \quad n \ge n_0. \tag{5.15}$$

So, for $n \ge n_0$,

$$\mathbf{I}(x_0, 2A) \subseteq \mathbf{I}_n(x_0, A) \subseteq \mathbf{I}(x_0, 2A/3), \tag{5.16}$$

where $\mathbf{I}_n(x_0, A)$ is defined in (5.13). Since $(\tilde{\mathsf{a}}_n, I_0, I_1, A)$, for $n \ge 1$, is C^d-admissible and thanks to (5.16) we see that, for $n \ge n_0$,

$$\mathbf{I}(x_0, 2A) \subseteq I_1, \tag{5.17}$$

$$\frac{1}{2} \le \frac{\tilde{a}_{n,2}(x)}{\tilde{a}_{n,2}(x_0)} \le 2, \quad x \in \mathbf{I}(x_0, 2A), \tag{5.18}$$

$$|\tilde{a}_{n,j}^{(k)}(x)| \le A^k\,|\tilde{a}_{n,2}(x_0)|^{(j-k)/2}, \quad 2 \le j \le d,\ 1 \le k \le d,\ x \in \mathbf{I}(x_0, 2A). \tag{5.19}$$

Consider the C^d curves

$$\underline{\mathsf{a}} := (0, -1, \underline{a}_3, \ldots, \underline{a}_d) : \mathbf{I}(x_0, 2A) \to \mathrm{Hyp}_T^0(d) \subseteq \mathbb{R}^d,$$
$$\underline{\mathsf{a}}_n := (0, -1, \underline{a}_{n,3}, \ldots, \underline{a}_{n,d}) : \mathbf{I}(x_0, 2A) \to \mathrm{Hyp}_T^0(d) \subseteq \mathbb{R}^d, \quad n \ge n_0,$$

where $\underline{a}_j := |\tilde{a}_2|^{-j/2}\tilde{a}_j$ and $\underline{a}_{n,j} := |\tilde{a}_{n,2}|^{-j/2}\tilde{a}_{n,j}$. Then, by the proof of Lemma 4.13, we conclude that there is a constant

$$C_1 = C_1(d) > 1 \tag{5.20}$$

such that, for $x \in \mathbf{I}(x_0, 2A)$,

$$\|\underline{\mathsf{a}}'(x)\|_2 \le C_1 A\,|\tilde{a}_2(x_0)|^{-1/2} \quad \text{and} \quad \|\underline{\mathsf{a}}_n'(x)\|_2 \le C_1 A\,|\tilde{a}_{n,2}(x_0)|^{-1/2}.$$

Let $0 < r < 1$ be the radius of the splitting (see Definition 3.5) and define

$$J_1 := \mathbf{I}(x_0, 4C_1A/r) = I(x_0, \tfrac{r}{4C_1A}|\tilde{a}_2(x_0)|^{1/2}).$$

Then $\underline{\mathsf{a}}(J_1) \subseteq B(\underline{\mathsf{a}}(x_0), r/4)$ and $\underline{\mathsf{a}}_n(J_1) \subseteq B(\underline{\mathsf{a}}_n(x_0), r/2)$, using (5.15). By (5.14), there is $n_1 \ge n_0$ such that

$$\|\underline{\mathsf{a}}(x_0) - \underline{\mathsf{a}}_n(x_0)\|_2 < \frac{r}{4}, \quad n \ge n_1. \tag{5.21}$$

Consequently, $B(\underline{\mathsf{a}}_n(x_0), r/2)$ is contained in $B(\underline{\mathsf{a}}(x_0), 3r/4)$, for $n \ge n_1$.

In view of (3.5) and Definition 3.5, we have splittings on J_1,

$$P_{\tilde{\mathsf{a}}} = P_{\mathsf{b}} P_{\mathsf{c}} \quad \text{and} \quad P_{\tilde{\mathsf{a}}_n} = P_{\mathsf{b}_n} P_{\mathsf{c}_n}, \quad n \ge n_1, \tag{5.22}$$

with the following properties:

(1) $d_{\mathsf{b}} := \deg P_{\mathsf{b}} = \deg P_{\mathsf{b}_n}$, for all $n \ge n_1$, and $d_{\mathsf{b}} < d$.

(2) There exist bounded analytic functions $\psi_1, \ldots, \psi_{d_{\mathsf{b}}}$ with bounded partial derivatives of all orders such that, for $x \in J_1$ and $1 \le i \le d_{\mathsf{b}}$,

$$b_i(x) = |\tilde{a}_2(x)|^{i/2}\, \psi_i(\underline{\mathsf{a}}(x)),$$
$$b_{n,i}(x) = |\tilde{a}_{n,2}(x)|^{i/2}\, \psi_i(\underline{\mathsf{a}}_n(x)), \quad n \ge n_1.$$

The same is true for the second factors P_{c} and P_{c_n}.

Definition 5.6 (Simultaneous splitting)**.** We say that the family $\{P_{\tilde{\mathsf{a}}}\} \cup \{P_{\tilde{\mathsf{a}}_n}\}_{n \ge n_1}$ has a *simultaneous splitting on an interval* J_1 if (5.22) and the above properties (1) and (2) are satisfied.

Note that, applying the Tschirnhausen transformation to P_{b} and P_{b_n} and by (3.1), we find bounded analytic functions $\tilde{\psi}_1, \ldots, \tilde{\psi}_{d_{\mathsf{b}}}$ with bounded partial derivatives of all orders such that, for $x \in J_1$ and $1 \le i \le d_{\mathsf{b}}$,

$$\tilde{b}_i(x) = |\tilde{a}_2(x)|^{i/2}\, \tilde{\psi}_i(\underline{\mathsf{a}}(x)),$$
$$\tilde{b}_{n,i}(x) = |\tilde{a}_{n,2}(x)|^{i/2}\, \tilde{\psi}_i(\underline{\mathsf{a}}_n(x)), \quad n \ge n_1.$$

Lemma 5.7. *We have $\mathsf{b}_n \to \mathsf{b}$ and $\tilde{\mathsf{b}}_n \to \tilde{\mathsf{b}}$ in $C^d(\overline{J}_1, \mathbb{R}^{d_{\mathsf{b}}})$ as $n \to \infty$.*

Proof. By (5.10) and (5.18), $|\tilde{a}_2|^{1/2}, |\tilde{a}_{n,2}|^{1/2} \in C^d(\overline{J}_1)$ and $\underline{\mathsf{a}}, \underline{\mathsf{a}}_n \in C^d(\overline{J}_1, \mathbb{R}^d)$, for $n \ge n_0$, and the assertion follows from Proposition 2.1. □

Summarizing, we have the following proposition.

Proposition 5.8. *Let $I_1 \subseteq \mathbb{R}$ be a bounded open interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$ as $n \to \infty$. Assume that $(\tilde{\mathsf{a}}, I_0, I_1, A)$ and $(\tilde{\mathsf{a}}_n, I_0, I_1, A)$, for $n \ge 1$, are C^d-admissible for some $A > 0$. Let $x_0 \in I_0 \setminus \{x : \tilde{a}_2(x) = 0\}$. Then the following holds:*

(1) *There exist an interval J_1 containing x_0 and $n_0 \ge 1$ such that the family $\{P_{\tilde{\mathsf{a}}}\} \cup \{P_{\tilde{\mathsf{a}}_n}\}_{n \ge n_0}$ has a simultaneous splitting (5.22) on J_1.*

(2) *For the factors in the simultaneous splitting (5.22), $\mathsf{b}_n \to \mathsf{b}$ and $\tilde{\mathsf{b}}_n \to \tilde{\mathsf{b}}$ in $C^d(\overline{J}_1, \mathbb{R}^{d_{\mathsf{b}}})$ as $n \to \infty$.*

(3) *There exist a relatively compact open subinterval $J_0 \Subset J_1$ containing x_0 and a constant $C = C(d) > 1$ such that $(\tilde{\mathsf{b}}, J_0, J_1, CA)$ and $(\tilde{\mathsf{b}}_n, J_0, J_1, CA)$, for $n \ge n_0$, are C^d-admissible.*

The properties (2) and (3) also hold for $\mathsf{b}, \mathsf{b}_n, \tilde{\mathsf{b}}, \tilde{\mathsf{b}}_n$ replaced by $\mathsf{c}, \mathsf{c}_n, \tilde{\mathsf{c}}, \tilde{\mathsf{c}}_n$.

Proof. (1) This was proved above.

(2) Lemma 5.7.

(3) Set $J_0 := \mathbf{I}(x_0, 8C_1A/r) = I(x_0, \frac{r}{8C_1A}|\tilde a_2(x_0)|^{1/2})$, where C_1 is the constant from (5.20). Then clearly $J_0 \Subset J_1$.

Fix $x_1 \in J_0 \setminus \{x : \tilde b_2(x) = 0\}$. Then

$$|x_1 - x_0| < \frac{r}{8C_1A}|\tilde a_2(x_0)|^{1/2}.$$

By Lemma 3.4 and (5.10),

$$|\tilde b_2(x_1)|^{1/2} \le 2\,|\tilde a_2(x_1)|^{1/2} \le 2\sqrt{2}\,|\tilde a_2(x_0)|^{1/2}.$$

Setting

$$B := \frac{16\sqrt{2}\,C_1A}{r}$$

we have

$$B^{-1}\,|\tilde b_2(x_1)|^{1/2} \le \frac{r}{8C_1A}\,|\tilde a_2(x_0)|^{1/2},$$

and hence

$$\mathbf{J}(x_1, B) := I(x_1, B^{-1}\,|\tilde b_2(x_1)|^{1/2}) \subseteq I(x_0, \tfrac{r}{4C_1A}|\tilde a_2(x_0)|^{1/2}) = J_1.$$

One checks, exactly as in the proof of Proposition 4.14, that

$$|\tilde b_i^{(k)}(x)| \le C(d)\,A^k\,|\tilde a_2(x_0)|^{(i-k)/2},$$

for all $2 \le i \le d_{\mathsf{b}}$, $1 \le k \le d$, and $x \in J_1$. If $k \ge i$, we may replace $\tilde a_2(x_0)$ by $\tilde b_2(x_1)$ on the right-hand side. Thus, we may conclude that $(\tilde{\mathsf{b}}, J_0, J_1, CA)$ is C^d-admissible, for a suitable constant $C = C(d) > 1$, by the proof of Lemma 4.11.

To see that also $(\tilde{\mathsf{b}}_n, J_0, J_1, CA)$ is C^d-admissible, fix $x_1 \in J_0 \setminus \{x : \tilde b_{n,2}(x) = 0\}$. By Lemma 3.4, (5.15), and (5.18),

$$|\tilde b_{n,2}(x_1)|^{1/2} \le 2\,|\tilde a_{n,2}(x_1)|^{1/2} \le 2\sqrt{2}\,|\tilde a_{n,2}(x_0)|^{1/2} \le 3\sqrt{2}\,|\tilde a_2(x_0)|^{1/2}.$$

Hence, using

$$B := \frac{24\sqrt{2}\,C_1A}{r},$$

we find

$$\mathbf{J}_n(x_1, B) := I(x_1, B^{-1}\,|\tilde b_{n,2}(x_1)|^{1/2}) \subseteq I(x_0, \tfrac{r}{4C_1A}|\tilde a_2(x_0)|^{1/2}) = J_1.$$

The rest follows in the same way as described above. □

5.5. **The induction argument.**

Proposition 5.9. *Let $I_1 \subseteq \mathbb{R}$ be a bounded open interval and $I_0 \Subset I_1$ a relatively compact open subinterval. Let $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^d(\overline{I}_1, \mathrm{Hyp}_T(d))$ as $n \to \infty$. Assume that $(\tilde{\mathsf{a}}, I_0, I_1, A)$ and $(\tilde{\mathsf{a}}_n, I_0, I_1, A)$, for $n \ge 1$, are C^d-admissible for some $A > 0$. Then, for almost every $x \in I_0$,*

$$\mathcal{S}(\tilde{\mathsf{a}}_n)'(x) \to \mathcal{S}(\tilde{\mathsf{a}})'(x) \quad \text{as } n \to \infty. \tag{5.23}$$

Proof. We proceed by induction on d. The base case is trivial, since Z is the unique polynomial in Tschirnhausen form of degree 1. Let us assume that $d \ge 2$ and that the statement is true for monic hyperbolic polynomials of degree $\le d-1$.

If $x \in \mathrm{acc}(Z_{\tilde{\mathsf{a}}})$ and $\mathcal{S}(\tilde{\mathsf{a}})'(x)$ exists, then $\mathcal{S}(\tilde{\mathsf{a}})'(x) = 0$. Thus, by Lemma 5.3, it is enough to show that (5.23) holds for almost every $x \in I_0 \setminus \{x : \tilde a_2(x) = 0\}$.

Fix $x_0 \in I_0 \setminus \{x : \tilde a_2(x) = 0\}$. By Proposition 5.8, there exist intervals $J_1 \Supset J_0 \ni x_0$, $n_0 \ge 1$, and $C = C(d) > 1$ such that the family $\{P_{\tilde{\mathsf a}}\} \cup \{P_{\tilde{\mathsf a}_n}\}_{n \ge n_0}$ has a simultaneous splitting (5.22) on J_1, $(\tilde{\mathsf b}, J_0, J_1, CA)$ and $(\tilde{\mathsf b}_n, J_0, J_1, CA)$, for $n \ge n_0$, are C^d-admissible, and $\mathsf b_n \to \mathsf b$ and $\tilde{\mathsf b}_n \to \tilde{\mathsf b}$ in $C^d(\overline{J}_1, \mathbb{R}^{d_{\mathsf b}})$ as $n \to \infty$.

We may assume that, for $x \in J_1$,

$$\mu(x) := (\lambda_1^{\uparrow}(\tilde{\mathsf a}(x)), \lambda_2^{\uparrow}(\tilde{\mathsf a}(x)), \ldots, \lambda_{d_{\mathsf b}}^{\uparrow}(\tilde{\mathsf a}(x)))$$

is the increasingly ordered root vector of $P_{\mathsf b(x)}$ and, for $n \ge n_0$,

$$\mu_n(x) := (\lambda_1^{\uparrow}(\tilde{\mathsf a}_n(x)), \lambda_2^{\uparrow}(\tilde{\mathsf a}_n(x)), \ldots, \lambda_{d_{\mathsf b}}^{\uparrow}(\tilde{\mathsf a}_n(x)))$$

is the increasingly ordered root vector of $P_{\mathsf b_n(x)}$; see Definition 3.5. Then

$$\mu(x) + \tfrac{1}{d_{\mathsf b}}(b_1(x), \ldots, b_1(x)) \quad \text{and} \quad \mu_n(x) + \tfrac{1}{d_{\mathsf b}}(b_{n,1}(x), \ldots, b_{n,1}(x))$$

are the corresponding root vectors for $P_{\tilde{\mathsf b}(x)}$ and $P_{\tilde{\mathsf b}_n(x)}$, respectively. By induction hypothesis and since $b'_{n,1}(x) \to b'_1(x)$ as $n \to \infty$, we have

$$\mu'_n(x) \to \mu'(x) \quad \text{as } n \to \infty,$$

for almost every $x \in J_0$.

Treating the second factors $P_{\mathsf c}$ and $P_{\mathsf c_n}$ analogously, we conclude that (5.23) holds for almost every $x \in J_0$.

The set $I_0 \setminus \{x : \tilde a_2(x) = 0\}$ can be covered by the open intervals J_0 and this cover admits a countable subcover. This ends the proof. □

5.6. **Proof of Theorem 5.1.** Let $I \subseteq \mathbb{R}$ be an open interval. Let $\mathsf a_n \to \mathsf a$ in $C^d(I, \mathrm{Hyp}(d))$ as $n \to \infty$. The Tschirnhausen transformation effects a shift of $\mathcal{S}(\mathsf a)$ by $\frac{1}{d}(a_1, \ldots, a_1)$ and of $\mathcal{S}(\mathsf a_n)$ by $\frac{1}{d}(a_{n,1}, \ldots, a_{n,1})$. The new coefficients are polynomials in the old ones, see (3.1). Hence we may assume that the polynomials are all in Tschirnhausen form (by Proposition 2.1). Then Theorem 5.1 follows from Lemma 5.5 and Proposition 5.9.

This also completes the proof of Theorem 1.3.

Remark 5.10. We need C^d convergence in Lemma 5.3. For all other arguments, it would be enough to work in the class $C^{d-1,1}$.

6. Proofs of the main results

In this section, we will deduce Theorem 1.1 and Corollary 1.2 from Theorem 1.3. We also prove Theorem 1.6.

6.1. **A multiparameter version.** The following theorem is a multiparameter version of Theorem 1.3.

Theorem 6.1. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf a_n \to \mathsf a$ in $C^d(U, \mathrm{Hyp}(d))$, i.e., for each relatively compact open subset $U_1 \Subset U$,*

$$\|\mathsf a - \mathsf a_n\|_{C^d(\overline{U}_1, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Then $\{\mathcal{S}(\mathsf a_n) : n \ge 1\}$ is a bounded set in $C^{0,1}(U, \mathbb{R}^d)$ and, for each relatively compact open subset $U_0 \Subset U$ and each $1 \le q < \infty$,

$$\|\mathcal{S}(\mathsf a) - \mathcal{S}(\mathsf a_n)\|_{W^{1,q}(U_0, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Proof. Let us assume that U_0 is an open box $U_0 = I_1 \times \cdots \times I_m$ with sides parallel to the coordinate axes. Set $\lambda := \mathcal{S}(\mathsf{a})$ and $\lambda_n := \mathcal{S}(\mathsf{a}_n)$. Let $x = (x_1, x')$ and for $x' \in U_0' = I_2 \times \cdots \times I_m$ consider

$$A_n(x') := \int_{I_1} \|\partial_1 \lambda(x_1, x') - \partial_1 \lambda_n(x_1, x')\|_2^q \, dx_1.$$

Then $A_n(x') \to 0$ as $n \to \infty$, by Theorem 1.3. The boundedness of $\{\lambda_n : n \ge 1\}$ in $C^{0,1}(U, \mathbb{R}^d)$ is a consequence of Bronshtein's theorem 4.1. It implies that $|\partial_1 \lambda - \partial_1 \lambda_n|$ is dominated on U_0 by an integrable function. By Fubini's theorem,

$$\int_{U_0} \|\partial_1 \lambda(x) - \partial_1 \lambda_n(x)\|_2^q \, dx = \int_{U_0'} A_n(x') \, dx'.$$

By the dominated convergence theorem, we conclude that

$$\int_{U_0} \|\partial_1 \lambda(x) - \partial_1 \lambda_n(x)\|_2^q \, dx \to 0 \quad \text{as } n \to \infty.$$

In an analogous way, one sees that $\|\partial_j \lambda - \partial_j \lambda_n\|_{L^q(U_0, \mathbb{R}^d)} \to 0$ as $n \to \infty$, for each $1 \le j \le m$.

We may conclude that $\|\lambda - \lambda_n\|_{L^\infty(U_0, \mathbb{R}^d)} \to 0$ as $n \to \infty$ from the fact that this is true component-wise (see Step 3 in Section 5.1).

For general U_0, we observe that there are finitely many open boxes as before that are relatively compact in U and cover U_0. This ends the proof. □

6.2. **Proof of Theorem 1.1.** It is clear that Theorem 6.1 implies Theorem 1.1 because $C^d(U, \mathrm{Hyp}(d))$ is first-countable.

6.3. **Proof of Corollary 1.2.** Corollary 1.2 is an immediate consequence of the following corollary of Theorem 6.1 and Example 1.12.

Corollary 6.2. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Then, for each relatively compact open set $U_0 \Subset U$ and each $0 < \alpha < 1$,*

$$\|\mathcal{S}(\mathsf{a}) - \mathcal{S}(\mathsf{a}_n)\|_{C^{0,\alpha}(\overline{U}_0, \mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Proof. Again we may assume that U_0 is a box (and hence has Lipschitz boundary). Then the assertion follows from Theorem 6.1 and Morrey's inequality,

$$\|\mathcal{S}(\mathsf{a}) - \mathcal{S}(\mathsf{a}_n)\|_{C^{0,\alpha}(\overline{U}_0, \mathbb{R}^d)} \le C \, \|\mathcal{S}(\mathsf{a}) - \mathcal{S}(\mathsf{a}_n)\|_{W^{1,q}(U_0, \mathbb{R}^d)},$$

where $\alpha = 1 - m/q$, $q > m$, and $C = C(m, q, U_0)$. □

6.4. **Proof of Theorem 1.6.** The restriction $\psi := \lambda^\uparrow|_{\mathrm{Hyp}^\circ(d)} : \mathrm{Hyp}^\circ(d) \to \mathbb{R}^d$ is real analytic, by Lemma 3.3. Thus Theorem 1.6 is a consequence of Proposition 2.1, observing that $\mathcal{S}^\circ = \psi_*$ and that $\|\psi\|_{C^{k+1}(\overline{V}_0, \mathbb{R}^d)}$ depends only on d, k, and V_0.

7. Applications

In this section, we give several applications of our results. In Section 7.1, we clarify their relation to the continuity results for the solution map of general polynomials obtained in [PR24]. In Section 7.2, we deduce that locally the surface area of the graphs of the roots of hyperbolic polynomials is continuous and conclude local lower semicontinuity of the area of the zero sets of hyperbolic polynomials. In Section 7.3, we prove a theorem on approximation by hyperbolic polynomials with all roots simple. Finally, we obtain continuity results for the eigenvalues of Hermitian matrices, in Section 7.4, and for singular values, in Section 7.5.

7.1. **Relation to the results for general polynomials.** The case of general complex (not necessarily hyperbolic) polynomials is treated in [PR24] which builds on the results of [PR16, PR18]. The crucial difference is that in general there is no canonical choice of a continuous ordered d-tuple of the complex roots. Even worse, if the parameter space is at least two-dimensional, then a parameterization of the roots by continuous functions might not exist; but there exist parameterizations by functions of bounded variation, see [PR20a]. Therefore the continuity results in [PR24] are formulated in terms of the *unordered* d-tuple of the roots.

Let us compare the results obtained in this paper with the ones of [PR24]. To this end, we investigate the metric space $\mathcal{A}_d(\mathbb{R})$ of unordered d-tuples of real numbers. It is a simple instance of the space $\mathcal{A}_d(\mathbb{R}^m)$ considered in [Alm00] and [DLS11].

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $[x] = [x_1, \ldots, x_d]$ be the corresponding unordered d-tuple, i.e., the equivalence class (or orbit) of x with respect to the action of the symmetric group S_d on $\mathbb{R}^d$ by permutation of the coordinates:

$$\sigma x := (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(d)}), \quad \sigma \in \mathrm{S}_d,\ x \in \mathbb{R}^d.$$

The set $\mathcal{A}_d(\mathbb{R}) := \{[x] : x \in \mathbb{R}^d\}$ with the distance

$$\mathbf{d}([x],[y]) := \min_{\sigma \in \mathrm{S}_d} \frac{1}{\sqrt{d}} \|x - \sigma y\|_2$$

is a complete metric space. If we identify the elements of $\mathcal{A}_d(\mathbb{R})$ with formal sums $\frac{1}{d}\sum_{i=1}^d [\![x_i]\!]$, where $[\![x_i]\!]$ denotes the Dirac mass of $x_i \in \mathbb{R}$, then $\mathbf{d}$ is induced by the L^2 based Wasserstein metric on the space of probability measures on $\mathbb{R}$.

For $x \in \mathbb{R}^d$, let $x^\uparrow \in \mathbb{R}^d$ be the representative of the equivalence class $[x]$ with increasingly ordered coordinates. Clearly, $x^\uparrow$ only depends on $[x]$ and thus we have an injective map $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$. It is a right-inverse of $[\ ] : \mathbb{R}^d \to \mathcal{A}_d(\mathbb{R})$.

Lemma 7.1. *We have*

$$\mathbf{d}([x],[y]) = \frac{1}{\sqrt{d}} \|x^\uparrow - y^\uparrow\|_2, \quad x, y \in \mathbb{R}^d.$$

In particular, $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$ *and* $[\ ] : \mathbb{R}^d \to \mathcal{A}_d(\mathbb{R})$ *are Lipschitz maps.*

Proof. Evidently,

$$\mathbf{d}([x],[y]) = \mathbf{d}([x^\uparrow],[y^\uparrow]) = \min_{\sigma \in \mathrm{S}_d} \frac{1}{\sqrt{d}} \|x^\uparrow - \sigma y^\uparrow\|_2 \le \frac{1}{\sqrt{d}} \|x^\uparrow - y^\uparrow\|_2.$$

Thus the assertion will follow from the claim that $\|x^\uparrow - y^\uparrow\|_2 \le \|x^\uparrow - y\|_2$, for all $x, y \in \mathbb{R}^d$. For $d = 2$, the claim is equivalent to

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 \le (x_1 - y_2)^2 + (x_2 - y_1)^2$$

whenever $x_1 \le x_2$ and $y_1 \le y_2$. By a simple computation, it is further equivalent to the true statement $(x_2 - x_1)(y_2 - y_1) \ge 0$. The general case follows from the fact that any permutation is a finite composite of transpositions. □

By Lemma 7.1, the map $(\ )^\uparrow : \mathcal{A}_d(\mathbb{R}) \to \mathbb{R}^d$ satisfies the conditions of an Almgren embedding (as defined in [PR24] following [Alm00] and [DLS11]). Thus Theorem 1.3 can be interpreted as a special version of the general theorem [PR24, Theorem 1.1] with the important difference that $\mathcal{S}(\mathsf{a}_n) \to \mathcal{S}(\mathsf{a})$ in $W^{1,q}_{\mathrm{loc}}$ as $n \to \infty$, see (1.3), holds for each $1 \le q < \infty$, while in the general result the corresponding fact is valid only for $1 \le q < d/(d-1)$.

For the next theorem, which is a stronger version of [PR24, Theorem 1.3] in the hyperbolic case, we need to recall the notions of metric speed and q-energy.

Definition 7.2 (Metric speed and q-energy)**.** Let $I \subseteq \mathbb{R}$ be an open interval and $\mathsf{a} \in C^d(I, \mathrm{Hyp}(d))$. Consider the Lipschitz curve $\Lambda(x) := [\mathcal{S}(\mathsf{a})(x)]$, for $x \in I$, in the metric space $\mathcal{A}_d(\mathbb{R})$. Then (see [AGS08]) the limit

$$|\dot\Lambda|(x) := \lim_{h \to 0} \frac{\mathbf{d}(\Lambda(x+h), \Lambda(x))}{|h|}$$

exists for almost every $x \in I$ and is called the *metric speed* of Λ at x. The *q-energy* of Λ on a subinterval $I_0 \subseteq I$ is defined by

$$\mathcal{E}_{q,I_0}(\Lambda) := \int_{I_0} \big(|\dot\Lambda|(x)\big)^q \, dx.$$

Theorem 7.3. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(I, \mathrm{Hyp}(d))$ as $n \to \infty$. Set $\Lambda := [\mathcal{S}(\mathsf{a})]$ and $\Lambda_n := [\mathcal{S}(\mathsf{a}_n)]$. Then, for each relatively compact open interval $I_0 \Subset I$,*

$$\|\mathbf{d}(\Lambda, \Lambda_n)\|_{L^\infty(I_0)} \to 0 \quad \text{as } n \to \infty, \tag{7.1}$$

$$\big\||\dot\Lambda| - |\dot\Lambda_n|\big\|_{L^q(I_0)} \to 0 \quad \text{as } n \to \infty, \tag{7.2}$$

$$\big|\mathcal{E}_{q,I_0}(\Lambda) - \mathcal{E}_{q,I_0}(\Lambda_n)\big| \to 0 \quad \text{as } n \to \infty, \tag{7.3}$$

for each $1 \le q < \infty$.

Proof. First, (7.1) is a consequence of (5.3) and Lemma 7.1. By [PR24, Lemma 11.1],

$$|\dot\Lambda|(x) = \frac{1}{\sqrt{d}} \|\mathcal{S}(\mathsf{a})'(x)\|_2 \quad \text{and} \quad |\dot\Lambda_n|(x) = \frac{1}{\sqrt{d}} \|\mathcal{S}(\mathsf{a}_n)'(x)\|_2$$

for almost every $x \in I$. Thus, (7.2) and (7.3) follow from Corollary 1.4. □

7.2. **Continuity of the area of the solution map.** Let us first expand Corollary 1.4.

Corollary 7.4. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Let $R \in \mathbb{R}[X_1, \ldots, X_{dm}]$ be any real polynomial in the $d \cdot m$ variables $X_1, \ldots, X_{dm}$. Set $\lambda = (\lambda_1, \ldots, \lambda_d) := \mathcal{S}(\mathsf{a})$ and $\lambda_n = (\lambda_{n,1}, \ldots, \lambda_{n,d}) := \mathcal{S}(\mathsf{a}_n)$, for $n \geq 1$. Then, for each relatively compact open subset $U_0 \Subset U$ and each $1 \leq q < \infty$,*

$$\Big\| R\Big((\partial_i \lambda_j)_{\substack{1\le i\le m\\ 1\le j\le d}}\Big) - R\Big((\partial_i \lambda_{n,j})_{\substack{1\le i\le m\\ 1\le j\le d}}\Big)\Big\|_{L^q(U_0)} \to 0 \quad \text{as } n \to \infty, \tag{7.4}$$

and consequently,

$$\Big\| R\Big((\partial_i \lambda_{n,j})_{\substack{1\le i\le m\\ 1\le j\le d}}\Big)\Big\|_{L^q(U_0)} \to \Big\| R\Big((\partial_i \lambda_j)_{\substack{1\le i\le m\\ 1\le j\le d}}\Big)\Big\|_{L^q(U_0)} \quad \text{as } n \to \infty. \tag{7.5}$$

Proof. Clearly, (7.5) is a consequence of (7.4).

Let us prove (7.4). It is enough to show the assertion for monomials R. Let us proceed by induction on the degree ℓ of the monomial R. For $\ell = 1$, the assertion follows from Theorem 6.1 in view of

$$\|\partial_i \lambda_j - \partial_i \lambda_{n,j}\|_{L^q(U_0)} \leq \|\partial_i \lambda - \partial_i \lambda_n\|_{L^q(U_0, \mathbb{R}^d)}.$$

If $\ell \geq 2$, then, by Hölder's inequality,

$$\begin{aligned}
&\|\partial_{i_1}\lambda_{j_1} \cdots \partial_{i_\ell}\lambda_{j_\ell} - \partial_{i_1}\lambda_{n,j_1} \cdots \partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\quad \leq \|\partial_{i_1}\lambda_{j_1} \cdots \partial_{i_\ell}\lambda_{j_\ell} - \partial_{i_1}\lambda_{j_1} \cdots \partial_{i_{\ell-1}}\lambda_{j_{\ell-1}} \cdot \partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\qquad + \|\partial_{i_1}\lambda_{j_1} \cdots \partial_{i_{\ell-1}}\lambda_{j_{\ell-1}} \cdot \partial_{i_\ell}\lambda_{n,j_\ell} - \partial_{i_1}\lambda_{n,j_1} \cdots \partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\quad \leq \|\partial_{i_1}\lambda_{j_1} \cdots \partial_{i_{\ell-1}}\lambda_{j_{\ell-1}}\|_{L^\infty(U_0)} \|\partial_{i_\ell}\lambda_{j_\ell} - \partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^q(U_0)} \\
&\qquad + \|\partial_{i_1}\lambda_{j_1} \cdots \partial_{i_{\ell-1}}\lambda_{j_{\ell-1}} - \partial_{i_1}\lambda_{n,j_1} \cdots \partial_{i_{\ell-1}}\lambda_{n,j_{\ell-1}}\|_{L^q(U_0)} \|\partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^\infty(U_0)}
\end{aligned}$$

which tends to zero as $n \to 0$, by the induction hypothesis, because

$$\|\partial_{i_1}\lambda_{j_1} \cdots \partial_{i_{\ell-1}}\lambda_{j_{\ell-1}}\|_{L^\infty(U_0)} \leq C \quad \text{and} \quad \|\partial_{i_\ell}\lambda_{n,j_\ell}\|_{L^\infty(U_0)} \leq C$$

for a constant $C > 0$ independent of n and i_k, j_k, by Bronshtein's theorem (see Theorem 4.1). □

Let $f : U \to \mathbb{R}^d$ be a Lipschitz map, where $U \subseteq \mathbb{R}^m$ is open. We recall that the *Jacobian* $|Jf|$ of f is the square root of the sum of the squares of the determinants of the $k \times k$ minors with $k = \min\{m, d\}$ of the Jacobian matrix

$$(\partial_i f_j)_{\substack{1\le i\le m\\ 1\le j\le d}},$$

which exists almost everywhere, by Rademacher's theorem.

Corollary 7.5. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Then, for each relatively compact open subset $U_0 \Subset U$ and each $1 \leq q < \infty$,*

$$\big\| |J(\mathcal{S}(\mathsf{a}))| - |J(\mathcal{S}(\mathsf{a}_n))| \big\|_{L^q(U_0)} \to 0 \quad \text{as } n \to \infty,$$

and consequently,

$$\big\| |J(\mathcal{S}(\mathsf{a}_n))| \big\|_{L^q(U_0)} \to \big\| |J(\mathcal{S}(\mathsf{a}))| \big\|_{L^q(U_0)} \quad \text{as } n \to \infty.$$

Proof. Let $M_1, \dots, M_p$ and $M_{n,1}, \dots, M_{n,p}$ denote the determinants of all the $k \times k$ minors with $k = \min\{m, d\}$ of the Jacobian matrices of $\mathcal{S}(\mathsf{a})$ and $\mathcal{S}(\mathsf{a}_n)$, respectively. Fix $1 \le q < \infty$. Then, by Hölder's inequality,

$$\big\| |J(\mathcal{S}(\mathsf{a}))| - |J(\mathcal{S}(\mathsf{a}_n))| \big\|_{L^q(U_0)} \le |U_0|^{1/(2q)} \big\| |J(\mathcal{S}(\mathsf{a}))| - |J(\mathcal{S}(\mathsf{a}_n))| \big\|_{L^{2q}(U_0)}$$

and

$$\big\| |J(\mathcal{S}(\mathsf{a}))| - |J(\mathcal{S}(\mathsf{a}_n))| \big\|_{L^{2q}(U_0)}^{2q} = \big\| \big(\sum_i M_i^2\big)^{1/2} - \big(\sum_i M_{n,i}^2\big)^{1/2} \big\|_{L^{2q}(U_0)}^{2q}$$
$$\le \big\| \big| \sum_i M_i^2 - \sum_i M_{n,i}^2 \big|^{1/2} \big\|_{L^{2q}(U_0)}^{2q} = \big\| \sum_i M_i^2 - \sum_i M_{n,i}^2 \big\|_{L^q(U_0)}^{q}.$$

Now it suffices to apply Corollary 7.4. □

Next we will combine Corollary 7.5 with the area and the coarea formula (see e.g. [EG92]) which we recall for the convenience of the reader.

Let $f : \mathbb{R}^m \to \mathbb{R}^d$ be Lipschitz and let $E \subseteq \mathbb{R}^m$ be Lebesgue measurable. The *area formula* states that, if $m \le d$, then

$$\int_E |Jf|\, dx = \int_{\mathbb{R}^d} \mathcal{H}^0(E \cap f^{-1}(y))\, d\mathcal{H}^m(y).$$

The *coarea formula* posits that, if $m \ge d$, then

$$\int_E |Jf|\, dx = \int_{\mathbb{R}^d} \mathcal{H}^{m-d}(E \cap f^{-1}(y))\, dy.$$

Recall that $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure, in particular, $\mathcal{H}^0$ is the counting measure.

Corollary 7.6. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. Set $\lambda := \mathcal{S}(\mathsf{a})$ and $\lambda_n := \mathcal{S}(\mathsf{a}_n)$, for $n \ge 1$.*

(1) *If $m \le d$, then for each relatively compact open subset $U_0 \Subset U$,*
$$\int_{\mathbb{R}^d} \mathcal{H}^0(U_0 \cap \lambda_n^{-1}(y))\, d\mathcal{H}^m(y) \to \int_{\mathbb{R}^d} \mathcal{H}^0(U_0 \cap \lambda^{-1}(y))\, d\mathcal{H}^m(y)$$
as $n \to \infty$.

(2) *If $m > d$, then for each relatively compact open subset $U_0 \Subset U$,*
$$\int_{\mathbb{R}^d} \mathcal{H}^{m-d}(U_0 \cap \lambda_n^{-1}(y))\, dy \to \int_{\mathbb{R}^d} \mathcal{H}^{m-d}(U_0 \cap \lambda^{-1}(y))\, dy$$
as $n \to \infty$.

Proof. This is an immediate consequence of Corollary 7.5 (for $q = 1$) and the area and coarea formula. □

We can also conclude that the surface area of the graphs of all the single roots $\mathcal{S}(\mathsf{a}_n)_j = \lambda_j^{\uparrow} \circ \mathsf{a}_n$, for $1 \le j \le d$, is locally convergent as $n \to \infty$.

Corollary 7.7. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. For each $1 \le j \le d$ and for each relatively compact open subset $U_0 \Subset U$, the surface area of the graph of $\lambda_{n,j} := \mathcal{S}(\mathsf{a}_n)_j$ converges to the surface area of the graph of $\lambda_j := \mathcal{S}(\mathsf{a})_j$ as $n \to \infty$: if $\overline{\lambda}_{n,j}(x) := (x, \lambda_{n,j}(x))$ and $\overline{\lambda}_j(x) := (x, \lambda_j(x))$ denote the corresponding graph mappings, then*

$$\mathcal{H}^m(\overline{\lambda}_{n,j}(U_0)) \to \mathcal{H}^m(\overline{\lambda}_j(U_0)) \quad \text{as } n \to \infty.$$

Proof. We have

$$|J\overline{\lambda}_j| = \Big(1 + \sum_{i=1}^m (\partial_i \lambda_j)^2\Big)^{1/2} \quad \text{and} \quad |J\overline{\lambda}_{n,j}| = \Big(1 + \sum_{i=1}^m (\partial_i \lambda_{n,j})^2\Big)^{1/2}.$$

As in the proof of Corollary 7.5, we have

$$\begin{aligned}
\Big\| \Big(1 + \sum_{i=1}^m (\partial_i \lambda_j)^2\Big)^{1/2} - \Big(1 + \sum_{i=1}^m (\partial_i \lambda_{n,j})^2\Big)^{1/2} \Big\|_{L^2(U_0)}^2 \\
\le \Big\| \Big| \sum_{i=1}^m (\partial_i \lambda_j)^2 - \sum_{i=1}^m (\partial_i \lambda_{n,j})^2 \Big|^{1/2} \Big\|_{L^2(U_0)}^2 \\
= \Big\| \sum_{i=1}^m (\partial_i \lambda_j)^2 - \sum_{i=1}^m (\partial_i \lambda_{n,j})^2 \Big\|_{L^1(U_0)}.
\end{aligned}$$

So the assertion follows from Corollary 7.4 and the area formula. □

It follows that the area of the zero sets of C^d families of hyperbolic polynomials of degree d locally has a lower semicontinuity property; for the reader's convenience, we restate Corollary 1.9:

Corollary 7.8. *Let $U \subseteq \mathbb{R}^m$ be open. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \mathrm{Hyp}(d))$ as $n \to \infty$. For any relatively compact open $U_0 \Subset U$, consider the zero sets*

$$\begin{aligned}
Z &= \{(x, y) \in U_0 \times \mathbb{R} : P_{\mathsf{a}(x)}(y) = 0\} \quad \text{and} \\
Z_n &= \{(x, y) \in U_0 \times \mathbb{R} : P_{\mathsf{a}_n(x)}(y) = 0\}, \quad n \ge 1.
\end{aligned}$$

Then

$$\liminf_{n \to \infty} \mathcal{H}^m(Z_n) \ge \mathcal{H}^m(Z).$$

Proof. Set $\lambda := \mathcal{S}(\mathsf{a})$. For $i = 2, \ldots, d$, let $E_i := \{x \in U_0 : \lambda_{i-1}(x) = \lambda_i(x)\}$. Then, using the notation of Corollary 7.7,

$$\mathcal{H}^m(Z) = \mathcal{H}^m(\overline{\lambda}_1(U_0)) + \sum_{i=2}^d \mathcal{H}^m(\overline{\lambda}_i(U_0 \setminus E_i)). \tag{7.6}$$

Analogously, setting $\lambda_n := \mathcal{S}(\mathsf{a}_n)$ and $E_{n,i} := \{x \in U_0 : \lambda_{n,i-1}(x) = \lambda_{n,i}(x)\}$, we have

$$\mathcal{H}^m(Z_n) = \mathcal{H}^m(\overline{\lambda}_{n,1}(U_0)) + \sum_{i=2}^d \mathcal{H}^m(\overline{\lambda}_{n,i}(U_0 \setminus E_{n,i})). \tag{7.7}$$

By the continuity of $\lambda^\uparrow : \mathrm{Hyp}(d) \to \mathbb{R}^d$, for each $i = 2, \ldots, d$ and each $x \in U_0$,

$$\limsup_{n \to \infty} \mathbf{1}_{E_{n,i}}(x) \le \mathbf{1}_{E_i}(x). \tag{7.8}$$

By Theorem 5.1 (applied coordinate by coordinate), we have that $|J\overline{\lambda}_{n,i}| \to |J\overline{\lambda}_i|$ as $n \to \infty$ almost everywhere in U_0. By Bronshtein's theorem (Theorem 4.1), there is a constant $B > 0$ such that $\big\| |J\overline{\lambda}_{n,i}| \big\|_{L^\infty(U_0)} \le B$ for all $n \ge 1$. Thus, by the area formula and the reverse Fatou lemma,

$$\begin{aligned}
\limsup_{n\to\infty} \mathcal{H}^m(\overline{\lambda}_{n,i}(E_{n,i})) &= \limsup_{n\to\infty} \int_{U_0} \mathbf{1}_{E_{n,i}} |J\overline{\lambda}_{n,i}|\, dx \\
&\le \int_{U_0} \limsup_{n\to\infty} \big(\mathbf{1}_{E_{n,i}} |J\overline{\lambda}_{n,i}|\big)\, dx \\
&\le \int_{U_0} \limsup_{n\to\infty} \mathbf{1}_{E_{n,i}} \cdot \limsup_{n\to\infty} |J\overline{\lambda}_{n,i}|\, dx \\
&\le \int_{E_i} |J\overline{\lambda}_i|\, dx = \mathcal{H}^m(\overline{\lambda}_i(E_i)),
\end{aligned}$$

where we used (7.8) in the last inequality.

Together with (7.6), (7.7), and Corollary 7.7, this gives

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{H}^m(Z_n) &= \liminf_{n\to\infty} \Big(\sum_{i=1}^d \mathcal{H}^m(\overline{\lambda}_{n,i}(U_0)) - \sum_{i=2}^d \mathcal{H}^m(\overline{\lambda}_{n,i}(E_{n,i})) \Big) \\
&= \sum_{i=1}^d \liminf_{n\to\infty} \mathcal{H}^m(\overline{\lambda}_{n,i}(U_0)) - \sum_{i=2}^d \limsup_{n\to\infty} \mathcal{H}^m(\overline{\lambda}_{n,i}(E_{n,i})) \\
&\ge \sum_{i=1}^d \mathcal{H}^m(\overline{\lambda}_i(U_0)) - \sum_{i=2}^d \mathcal{H}^m(\overline{\lambda}_i(E_i)) = \mathcal{H}^m(Z)
\end{aligned}$$

which ends the proof. □

7.3. **Approximation by hyperbolic polynomials with all roots distinct.** We recall a lemma of Wakabayshi [Wak86] which extends an observation of Nuij [Nui68].

Lemma 7.9 ([Wak86, Lemma 2.2]). *Let $P_{\mathsf{a}} \in \operatorname{Hyp}(d)$ and set*

$$P_{\mathsf{a},s}(Z) := (1 + s\tfrac{\partial}{\partial Z})^{d-1} P_{\mathsf{a}}(Z), \quad s \in \mathbb{R}. \tag{7.9}$$

Then $P_{\mathsf{a},s} \in \operatorname{Hyp}(d)$ for all $s \in \mathbb{R}$ and there are positive constants $c_i = c_i(d)$, $i = 1, 2$, such that, if $\lambda_1^{\uparrow}(\mathsf{a}, s) \le \cdots \le \lambda_d^{\uparrow}(\mathsf{a}, s)$ denote the increasingly ordered roots of $P_{\mathsf{a},s}$, then

$$\lambda_j^{\uparrow}(\mathsf{a}, s) - \lambda_{j-1}^{\uparrow}(\mathsf{a}, s) \ge c_1|s|, \quad \text{for } s \in \mathbb{R} \text{ and } 2 \le j \le d, \tag{7.10}$$

and

$$0 < \pm(\lambda_j^{\uparrow}(\mathsf{a}) - \lambda_j^{\uparrow}(\mathsf{a}, s)) \le c_2|s|, \quad \text{for } \pm s > 0 \text{ and } 1 \le j \le d. \tag{7.11}$$

In conjunction with our findings, Lemma 7.9 leads to the following approximation result.

Corollary 7.10. *Let $U \subseteq \mathbb{R}^m$ be open and $\mathsf{a} \in C^d(U, \operatorname{Hyp}(d))$. There exists a sequence $(\mathsf{a}_n)_{n\ge1} \subseteq C^d(U, \operatorname{Hyp}(d))$ with the following properties:*

(1) *$\mathsf{a}_n \to \mathsf{a}$ in $C^d(U, \operatorname{Hyp}(d))$ as $n \to \infty$;*

(2) $\mathcal{S}(\mathsf{a}_n)_1(x) < \mathcal{S}(\mathsf{a}_n)_2(x) < \cdots < \mathcal{S}(\mathsf{a}_n)_d(x)$ *for all* $x \in U$ *and all* $n \geq 1$*;*

(3) $\mathcal{S}(\mathsf{a}_n) \in C^d(U, \mathbb{R}^d)$ *for all* $n \geq 1$*;*

(4) $\mathcal{S}(\mathsf{a}_n) \to \mathcal{S}(\mathsf{a})$ *in* $C_q^{0,1}(U, \mathbb{R}^d)$*, for all* $1 \leq q < \infty$*, as* $n \to \infty$*;*

(5) *for any relatively compact open* $U_0 \Subset U$*, consider the zero sets*

$$Z = \{(x,y) \in U_0 \times \mathbb{R} : P_{\mathsf{a}(x)}(y) = 0\} \quad \text{and}$$
$$Z_n = \{(x,y) \in U_0 \times \mathbb{R} : P_{\mathsf{a}_n(x)}(y) = 0\}, \quad n \geq 1.$$

Then $\lim_{n\to\infty} \mathcal{H}^m(Z_n)$ *exists and*

$$\lim_{n\to\infty} \mathcal{H}^m(Z_n) \geq \mathcal{H}^m(Z).$$

Proof. Let $(s_n)_{n\geq 1}$ be any positive sequence of reals that tends to 0. Consider the polynomial $P_{\mathsf{a}(x),s_n}$ (defined in (7.9)), where $x \in U$, and let $\mathsf{a}_n(x)$ be its coefficient vector. Then, by Lemma 7.9, $\mathsf{a}_n \in C^d(U, \mathrm{Hyp}(d))$, for $n \geq 1$. We will show that the sequence $(\mathsf{a}_n)_{n\geq 1}$ has the desired properties.

(1) This is clear by the definition (7.9) and since $s_n \to 0$ as $n \to \infty$.

(2) follows from (7.10) and the fact that $s_n > 0$ for all $n \geq 1$.

(3) For fixed $x \in U$, $\frac{\partial}{\partial Z} P_{\mathsf{a}_n(x)}(Z)$ does not vanish at any root of $P_{\mathsf{a}_n(x)}$, by (2). So, by the implicit function theorem, the roots of $P_{\mathsf{a}_n(x)}$ are of class C^d in a neighborhood of x. This implies (3).

(4) is a consequence of (1) and Theorem 1.1.

(5) Using the notation of Corollary 7.7, for each $n \geq 1$, the set Z_n is the union of the graphs $\overline{\lambda}_{n,j}(U_0)$ of the single roots $\lambda_{n,j}|_{U_0} = \mathcal{S}(\mathsf{a}_n)_j|_{U_0}$, for $1 \leq j \leq d$, and these graphs are pairwise disjoint, by (2). Thus, by (1) and Corollary 7.7,

$$\lim_{n\to\infty} \mathcal{H}^m(Z_n) = \lim_{n\to\infty} \sum_{j=1}^d \mathcal{H}^m(\overline{\lambda}_{n,j}(U_0)) = \sum_{j=1}^d \mathcal{H}^m(\overline{\lambda}_j(U_0)) \geq \mathcal{H}^m(Z).$$

For the inequality at the end, note that the union $Z = \bigcup_{j=1}^d \overline{\lambda}_j(U_0)$ is not necessarily disjoint; see also Corollary 7.8. □

7.4. **Perturbation theory for Hermitian matrices.** Let $\mathrm{Herm}(d)$ denote the real vector space of complex Hermitian $d \times d$ matrices. With $A \in \mathrm{Herm}(d)$ we associate its increasingly ordered eigenvalues $\lambda_1^\uparrow(A) \leq \lambda_2^\uparrow(A) \leq \cdots \leq \lambda_d^\uparrow(A)$ and thus obtain a continuous map

$$\lambda^\uparrow = (\lambda_1^\uparrow, \ldots, \lambda_d^\uparrow) : \mathrm{Herm}(d) \to \mathbb{R}^d. \tag{7.12}$$

Proposition 7.11 (Weyl's perturbation theorem [Wey12]; see e.g. [Bha97, III.2.6])**.** *Let* $A, B \in \mathrm{Herm}(d)$*. Then*

$$\|\lambda^\uparrow(A) - \lambda^\uparrow(B)\|_\infty \leq \|A - B\|, \tag{7.13}$$

where $\|A - B\|$ *denotes the operator norm of* $A - B$*.*

In [Bha97], the result is stated for eigenvalue vectors with decreasing eigenvalues, but reversing the order evidently leaves the left-hand side of (7.13) unchanged.

As a consequence of Proposition 7.11, the map (7.12) induces a bounded map

$$\mathcal{E} := (\lambda^{\uparrow})_* : C^{0,1}(\overline{I}, \mathrm{Herm}(d)) \to C^{0,1}(\overline{I}, \mathbb{R}^d), \quad A \mapsto \lambda^{\uparrow} \circ A, \tag{7.14}$$

which takes Lipschitz curves of Hermitian matrices to Lipschitz curves of their increasingly ordered eigenvalues. The Lipschitz constants satisfy

$$|\mathcal{E}(A)|_{C^{0,1}(\overline{I},\mathbb{R}^d)} \le |A|_{C^{0,1}(\overline{I},\mathrm{Herm}(d))}, \tag{7.15}$$

if $\mathrm{Herm}(d)$ is endowed with the operator norm and $\mathbb{R}^d$ with the maximum norm. This remains true if we replace the interval I by a bounded open set $U \subseteq \mathbb{R}^m$.

The following corollary includes Theorem 1.10.

Corollary 7.12. *Let $U \subseteq \mathbb{R}^m$ be open. Then:*

(1) *The map*
$$\mathcal{E} : C^d(U, \mathrm{Herm}(d)) \to C^{0,1}_q(U, \mathbb{R}^d), \quad A \mapsto \lambda^{\uparrow} \circ A,$$
is continuous, for all $1 \le q < \infty$.

(2) *The map*
$$\mathcal{E} : C^d(U, \mathrm{Herm}(d)) \to C^{0,\alpha}(U, \mathbb{R}^d), \quad A \mapsto \lambda^{\uparrow} \circ A,$$
is continuous, for all $0 < \alpha < 1$.

(3) *If $A_n \to A$ in $C^d(U, \mathrm{Herm}(d))$ as $n \to \infty$, then over each relatively compact open subset $U_0 \Subset U$ the surface area of the graph of $\mathcal{E}(A_n)_j$ converges to the surface area of the graph of $\mathcal{E}(A)_j$ as $n \to \infty$, for each $1 \le j \le d$.*

Proof. (1) We have the commuting diagram

$$\begin{array}{ccc} C^d(U,\mathrm{Herm}(d)) & \xrightarrow{\quad\mathcal{E}\quad} & C^{0,1}_q(U,\mathbb{R}^d) \\ & {\scriptstyle\mathcal{P}}\searrow \qquad \nearrow{\scriptstyle\mathcal{S}} & \\ & C^d(U,\mathrm{Hyp}(d)) & \end{array}$$

where $\mathcal{P}$ sends A to its characteristic polynomial P_A. The coefficients of P_A are given by polynomials in the entries of A. Thus, Proposition 2.1 implies that the map $\mathcal{P}$ is continuous. Consequently, $\mathcal{E} = \mathcal{S} \circ \mathcal{P}$ is continuous by Theorem 1.1.

(2) This follows similarly from the continuity of $\mathcal{P}$ and Corollary 1.2.

(3) Use Corollary 7.7 and the continuity of $\mathcal{P}$. □

The following example shows that $\mathcal{E}$ is not continuous with respect to the $C^{0,1}$ topology on the target space.

Example 7.13. The sequence $(A_n)_n$ of curves of symmetric 2×2 matrices

$$A_n(x) = \begin{pmatrix} \frac{1}{n} & x \\ x & -\frac{1}{n} \end{pmatrix}, \quad x \in \mathbb{R},$$

converges to

$$A(x) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \quad x \in \mathbb{R},$$

uniformly on every compact interval in all derivatives. We have

$$\mathcal{E}(A_n)(x) = \Big(- \sqrt{x^2 + \frac{1}{n^2}}, \sqrt{x^2 + \frac{1}{n^2}} \Big)$$

and

$$\mathcal{E}(A)(x) = (-|x|, |x|).$$

Hence, Example 1.12 shows that the Lipschitz constant of $\mathcal{E}(A) - \mathcal{E}(A_n)$ on each bounded open interval containing 0 is bounded below by $2 - \sqrt{2}$ which shows that $\mathcal{E}(A_n) \not\to \mathcal{E}(A)$ in the $C^{0,1}$ topology.

Given that the map $\mathcal{E}$ is defined and bounded on $C^{0,1}(U, \operatorname{Herm}(d))$ (see (7.14) and (7.15)), it is natural to ask whether in Corollary 7.12 one can replace C^d by C^1, see Question 1.11.

If $d = 2$, this is indeed the case as evidenced in the following proposition.

Proposition 7.14. *Let $U \subseteq \mathbb{R}^m$ be open. Then:*

(1) *The map*

$$\mathcal{E} : C^1(U, \operatorname{Herm}(2)) \to C^{0,1}_q(U, \mathbb{R}^2), \quad A \mapsto \lambda^\uparrow \circ A,$$

is continuous, for all $1 \le q < \infty$.

(2) *The map*

$$\mathcal{E} : C^1(U, \operatorname{Herm}(2)) \to C^{0,\alpha}(U, \mathbb{R}^2), \quad A \mapsto \lambda^\uparrow \circ A,$$

is continuous, for all $0 < \alpha < 1$.

(3) *If $A_n \to A$ in $C^1(U, \operatorname{Herm}(2))$ as $n \to \infty$, then over each relatively compact open subset $U_0 \Subset U$ the surface area of the graph of $\mathcal{E}(A_n)_j$ converges to the surface area of the graph of $\mathcal{E}(A)_j$ as $n \to \infty$, for $j = 1, 2$.*

Proof. It suffices to prove (1) in the case $m = 1$. Then the multiparameter version of (1) as well as (2) and (3) follow by the arguments given in detail for hyperbolic polynomials, if one uses Proposition 7.11 instead of Bronshtein's theorem 4.1.

Let us show (1) for $m = 1$. We may assume that the trace of A vanishes, by replacing A by $A - \frac{1}{2}\operatorname{tr}(A)\mathbb{I}$. Thus, we have

$$A = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$, and

$$\lambda^\uparrow(A) = \big(- \sqrt{a^2 + b^2 + c^2}, \sqrt{a^2 + b^2 + c^2} \big).$$

Let us assume that $a, b, c \in C^1(I, \mathbb{R})$, where $I \subseteq \mathbb{R}$ is an open interval. Then

$$I \ni x \mapsto \sqrt{a(x)^2 + b(x)^2 + c(x)^2} =: \mu(x)$$

is locally Lipschitz, differentiable almost everywhere, and

$$|\mu'(x_0)| \le \sup_{x \in \overline{I}_1} \|A'(x)\|_2 = \sqrt{2} \sup_{x \in \overline{I}_1} \sqrt{a'(x)^2 + b'(x)^2 + c'(x)^2}, \tag{7.16}$$

for each relatively compact open subinterval $I_1 \Subset I$ and each $x_0 \in I_1$ where $\mu'(x_0)$ exists, by (7.15).

Suppose that

$$A_n = \begin{pmatrix} a_n & b_n + ic_n \\ b_n - ic_n & -a_n \end{pmatrix} \longrightarrow A = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \quad \text{in } C^1(I, \mathrm{Herm}(2))$$

as $n \to \infty$. It suffices to prove that

$$\|\mu - \mu_n\|_{W^{1,q}(I_1)} \to 0 \quad \text{as } n \to \infty, \tag{7.17}$$

for each relatively compact open subinterval $I_1 \Subset I$ and all $1 \le q < \infty$, where

$$\mu_n := \sqrt{a_n^2 + b_n^2 + c_n^2}.$$

By (7.13), we have

$$\|\mu - \mu_n\|_{L^\infty(I_1)} \to 0 \quad \text{as } n \to \infty,$$

for each relatively compact open subinterval $I_1 \Subset I$.

We claim that, for almost every $x \in I$,

$$\mu_n'(x) \to \mu'(x) \quad \text{as } n \to \infty.$$

This is clear on the set $\Omega := \{x \in I : a(x)^2 + b(x)^2 + c(x)^2 \ne 0\}$: for each $x_0 \in \Omega$, the derivative $\mu'(x_0)$ exists, and, by assumption, $a(x_0)^2 + b(x_0)^2 + c(x_0)^2 \ne 0$ if n is large enough so that also $\mu_n'(x_0)$ exists and $\mu_n'(x_0) \to \mu'(x_0)$.

Now consider $Z := \{x \in I : a(x)^2 + b(x)^2 + c(x)^2 = 0\}$ and the set $\mathrm{acc}(Z)$ of accumulation points of Z. Note that a', b', and c' vanish on $\mathrm{acc}(Z)$. Fix $x_0 \in \mathrm{acc}(Z)$ and $\epsilon > 0$. By continuity, there exists $\delta > 0$ such that $\overline{I}(x_0, \delta) \Subset I$ and

$$\sup_{x \in \overline{I}(x_0,\delta)} \sqrt{a'(x)^2 + b'(x)^2 + c'(x)^2} \le \frac{\epsilon}{2}.$$

As $A_n \to A$ in $C^1(I, \mathrm{Herm}(2))$, there is $n_0 \ge 1$ such that, for all $n \ge n_0$,

$$\sup_{x \in \overline{I}(x_0,\delta)} \sqrt{a_n'(x)^2 + b_n'(x)^2 + c_n'(x)^2} \le \epsilon.$$

If $\mu_n'(x_0)$ exists, then we conclude, by (7.16), that

$$|\mu_n'(x_0)| \le \sqrt{2}\,\epsilon, \quad n \ge n_0.$$

This implies the claim, since the set of accumulation points of Z, where all μ_n and μ are differentiable, has full measure in Z and μ' vanishes on this set.

Now the dominated convergence theorem implies that

$$\|\mu' - \mu_n'\|_{L^q(I_1)} \to 0 \quad \text{as } n \to \infty,$$

for each relatively compact open subinterval $I_1 \Subset I$ and all $1 \le q < \infty$, completing the proof of (7.17). □

Remark 7.15. Corollary 7.12 has an evident analogue for skew-Hermitian matrices which simply follows from the fact that a $d \times d$ matrix A is skew-Hermitian if and only if iA is Hermitian. The eigenvalues of A and iA just differ by multiplication by i.

On the other hand, there is no consistent continuous choice of the eigenvalues of unitary $d \times d$ matrices. Consider, for example, the curve of unitary matrices

$$A(x) = \begin{pmatrix} 0 & e^{2\pi i x} \\ 1 & 0 \end{pmatrix}, \quad x \in \mathbb{R},$$

with the eigenvalues $\lambda_\pm(x) = \pm e^{\pi i x}$. Even though $\lambda_\pm : \mathbb{R} \to (\mathbb{S}^1)^2$ is continuous, there is no continuous choice of the eigenvalues $\mathbb{S}^1 \to (\mathbb{S}^1)^2$ of the curve of unitary matrices induced by A on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ (because $\lambda_\pm(0) = \pm 1 \ne \mp 1 = \lambda_\pm(1)$).

In this case, and more generally for normal matrices, the general continuity results of [PR24] apply. For the perturbation theory of normal matrices, see [Rai13], [PR20b], and the survey [PR25].

7.5. **Singular values.** Let us consider the vector space $M_{D,d}(\mathbb{C})$ of complex $D \times d$ matrices, where $d \le D$. The singular values of $A \in M_{D,d}(\mathbb{C})$ are the nonnegative square roots of the eigenvalues of the Hermitian matrix A^*A, usually ordered decreasingly

$$\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_d(A) \ge 0.$$

This defines a map $\sigma = (\sigma_1, \ldots, \sigma_d) : M_{D,d}(\mathbb{C}) \to \mathbb{R}^d$.

Let us consider the real vector space $M_{D,d}(\mathbb{C}) \times \mathbb{R}$ and the homogeneous polynomial of degree $2d$,

$$f(A, r) := \det(r^2\, \mathbb{I} - A^*A), \quad (A, r) \in M_{D,d}(\mathbb{C}) \times \mathbb{R}.$$

Then f is *Gårding hyperbolic* with respect to the direction $(0, 1) \in M_{D,d}(\mathbb{C}) \times \mathbb{R}$ (see [BGLS01, Sec. 6]) which means by definition that all roots of the univariate polynomial

$$P_{A,r}(Z) := f((A, r) - Z\,(0, 1)) = \det((r - Z)^2\, \mathbb{I} - A^*A)$$

are real. Indeed, the roots of $P_{A,r}$ (in decreasing order) are

$$r + \sigma_1(A), r + \sigma_2(A), \ldots, r + \sigma_d(A), r - \sigma_d(A), \ldots, r - \sigma_1(A).$$

Hence, by Theorem 4.1, σ induces a bounded map

$$\sigma_* : C^{2d-1,1}(U, M_{D,d}(\mathbb{C})) \to C^{0,1}(U, \mathbb{R}^d), \quad A \mapsto \sigma \circ A,$$

where $U \subset \mathbb{R}^m$ is open. In general, this map is not continuous, which follows from Example 7.13, but we have the following result.

Corollary 7.16. *Let $U \subseteq \mathbb{R}^m$ be open. Then:*

(1) *The map*

$$\sigma_* : C^{2d}(U, M_{D,d}(\mathbb{C})) \to C^{0,1}_q(U, \mathbb{R}^d), \quad A \mapsto \sigma \circ A,$$

is continuous, for all $1 \le q < \infty$.

(2) *The map*

$$\sigma_* : C^{2d}(U, M_{D,d}(\mathbb{C})) \to C^{0,\alpha}(U, \mathbb{R}^d), \quad A \mapsto \sigma \circ A,$$

is continuous, for all $0 < \alpha < 1$.

(3) *If* $A_n \to A$ *in* $C^{2d}(U, M_{D,d}(\mathbb{C}))$ *as* $n \to \infty$, *then over each relatively compact open subset* $U_0 \Subset U$ *the surface area of the graph of* $\sigma_j(A_n)$ *converges to the surface area of the graph of* $\sigma_j(A)$ *as* $n \to \infty$, *for each* $1 \le j \le d$.

Proof. Similarly as in the proof of Corollary 7.12, we have, for each $r \in \mathbb{R}$, a continuous map $C^{2d}(U, M_{D,d}(\mathbb{C})) \to C^{2d}(U, \mathrm{Hyp}(d))$, $A \mapsto P_{A,r}$, which can be used to reduce the statements of the corollary to the corresponding one for hyperbolic polynomials. □

Observing that the Hermitian matrix

$$\mathbf{A} := \begin{pmatrix} 0 & \tilde A \\ \tilde A^* & 0 \end{pmatrix},$$

where $\tilde A$ is the $D \times D$ matrix resulting from A by adding $D - d$ columns consisting of zeros, has the eigenvalues

$$\sigma_1(A), \ldots, \sigma_d(A), 0, \ldots, 0, -\sigma_d(A), \ldots, -\sigma_1(A),$$

we conclude from (7.13) that, for $A, B \in M_{D,d}(\mathbb{C})$ and $1 \le i \le d$,

$$\begin{aligned} |\sigma_i(A) - \sigma_i(B)| &\le \|\mathbf{A} - \mathbf{B}\| \le \|\mathbf{A} - \mathbf{B}\|_2 = |\operatorname{tr}\big((\mathbf{A} - \mathbf{B})^*(\mathbf{A} - \mathbf{B})\big)|^{1/2} \\ &= |2 \operatorname{tr}\big((\tilde A - \tilde B)^*(\tilde A - \tilde B)\big)|^{1/2} = \sqrt{2}\, \|A - B\|_2. \end{aligned}$$

Consequently, the map

$$\sigma_* : C^{0,1}(U, M_{D,d}(\mathbb{C})) \to C^{0,1}(U, \mathbb{R}^d)$$

is well-defined and bounded. So, in analogy to Question 1.11, it is thus natural to ask whether in Corollary 7.16 one can replace the assumption C^{2d} by C^1.

8. Restricted multiplicity

In this section we prove a refinement of Theorem 1.3 which accounts for the case that the maximal multiplicity of the roots is smaller than the degree.

First we recall the following version of Bronshtein's theorem.

Theorem 8.1 ([PR15, Theorem 2.1])**.** *Let* $I \subseteq \mathbb{R}$ *be an open interval and* $\mathsf{a} \in C^{p-1,1}(I, \mathrm{Hyp}(d))$, *where* p *is the maximal multiplicity of the roots of* $P_{\mathsf{a}(x)}$, *for* $x \in I$. *Then any continuous root* $\lambda \in C^0(I)$ *of* P_{a} *is locally Lipschitz.*

If $p = d$, *then we have the bound* (4.1).

Assume $p < d$ *and suppose that* $P_{\tilde{\mathsf{a}}}$ *is in Tschirnhausen form. Let* $\lambda_1^\uparrow(x) \le \cdots \le \lambda_d^\uparrow(x)$ *be the increasingly ordered roots of* $P_{\tilde{\mathsf{a}}(x)}$, *for* $x \in I$, *and consider*

$$\alpha(x) := \frac{|\lambda_d^\uparrow(x) - \lambda_1^\uparrow(x)|}{\min_{1 \le i \le d-p} |\lambda_{i+p}^\uparrow(x) - \lambda_i^\uparrow(x)|} \quad \text{and} \quad \alpha_I := \sup_{x \in I} \alpha(x). \tag{8.1}$$

Then each continuous root λ of $P_{\tilde{a}}$ satisfies, for any pair of relatively compact open intervals $I_0 \Subset I_1 \Subset I$,

$$|\lambda|_{C^{0,1}(\overline{I}_0)} \le C(d)\, \alpha_{I_1}^{\frac{d-p}{p}} \max\Big\{\delta^{-1}\|\tilde{a}_2\|^{1/2}_{L^\infty(I_1)}, |\tilde{a}_2'|^{1/2}_{C^{0,1}(\overline{I}_1)}, \max_{i\le p}\Big(|\tilde{a}_i^{(p-1)}|_{C^{0,1}(\overline{I}_1)}\|\tilde{a}_2\|^{\frac{p-i}{2}}_{L^\infty(I_1)}\Big)^{1/p}, \max_{i>p}\Big(|\tilde{a}_i^{(p-1)}|_{C^{0,1}(\overline{I}_1)}\big(\min_{x\in\overline{I}_0}|\tilde{a}_2(x)|\big)^{\frac{p-i}{2}}\Big)^{1/p}\Big\},$$

where $\delta := \operatorname{dist}(I_0, \mathbb{R}\setminus I_1)$.

The next theorem generalizes Theorem 1.3.

Theorem 8.2. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $\mathsf{a}_n \to \mathsf{a}$ in $C^p(I, \mathrm{Hyp}(d))$ as $n \to \infty$, where p is the maximal multiplicity of the roots of $P_{\mathsf{a}(x)}$, for $x \in I$. If $p < d$ assume that, for each relatively compact open $I_1 \Subset I$,*

$$\alpha_{I_1} < \infty.$$

Then $\{\mathcal{S}(\mathsf{a}_n) : n \ge 1\}$ is a bounded set in $C^{0,1}(I, \mathbb{R}^d)$ and, for each relatively compact open interval $I_0 \Subset I$ and each $1 \le q < \infty$,

$$\|\mathcal{S}(\mathsf{a}) - \mathcal{S}(\mathsf{a}_n)\|_{W^{1,q}(I_0,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Proof. In view of Theorem 1.3, we may assume that $p < d$. Furthermore, we may assume that all polynomials are in Tschirnhausen form.

We first observe that, for each relatively compact open $I_1 \Subset I$,

$$\|\mathcal{S}(\tilde{\mathsf{a}}) - \mathcal{S}(\tilde{\mathsf{a}}_n)\|_{L^\infty(I_1,\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty, \tag{8.2}$$

as a consequence of [PR24, Corollary 6.5] and Lemma 7.1. (For this it is actually enough that $\tilde{\mathsf{a}}_n \to \tilde{\mathsf{a}}$ in $C^0(\overline{I}_1, \mathrm{Hyp}_T(d))$ as $n \to \infty$.)

Fix relatively compact open subintervals $I_0 \Subset I_1 \Subset I$. Then there exists $n_0 \ge 1$ such that for all $n \ge n_0$ the maximal multiplicity of the roots of $P_{\tilde{\mathsf{a}}_n}$ on I_1 is at most p. (If not this is violated on a sequence x_{n_k} in I_1, leading to a contradiction at an accumulation point of this sequence in $\overline{I}_1$, since $\lambda^\uparrow : \mathrm{Hyp}_T(d) \to \mathbb{R}^d$ is continuous.)

Consequently, the functions $\alpha_n : I_1 \to \mathbb{R}$ associated to $P_{\tilde{\mathsf{a}}_n}$ as in (8.1) are well-defined, for all $n \ge n_0$. By the assumption $\alpha_{I_1} < \infty$ and (8.2), $\alpha_n \to \alpha$ uniformly on I_1 as $n \to \infty$ and thus the sequence $\alpha_{n,I_1} := \sup_{x\in I_1} \alpha_n(x)$ is bounded.

Hence, by Theorem 8.1, the derivative of $\mathcal{S}(\mathsf{a}_n)$ exists almost everywhere in I_0 and is uniformly bounded on I_0 by a constant independent of n.

By Lemma 3.3 and Proposition 2.1, we can split $P_{\tilde{\mathsf{a}}}$ and $P_{\tilde{\mathsf{a}}_n}$, for large n, locally in factors of degrees at most p in a simultaneous way. This allows us to apply Theorem 5.1 in the case $d = p$ and conclude that, for almost every $x \in I_0$,

$$\mathcal{S}(\mathsf{a}_n)'(x) \to \mathcal{S}(\mathsf{a})'(x) \quad \text{as } n \to \infty.$$

Now it suffice to invoke the dominated convergence theorem to finish the proof. □

Acknowledgement. We are grateful to Antonio Lerario for posing the question about the continuity of the solution map.

This research was funded in part by the Austrian Science Fund (FWF) DOI 10.55776/P32905 and DOI 10.55776/PAT1381823. For open access purposes, the authors have applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

References

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Basel: Birkhäuser, 2008 (English).

[AKLM98] D. Alekseevsky, A. Kriegl, M. Losik, and P. W. Michor, *Choosing roots of polynomials smoothly*, Israel J. Math. **105** (1998), 203–233.

[Alm00] F. J. Almgren, Jr., *Almgren's big regularity paper*, World Scientific Monograph Series in Mathematics, vol. 1, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

[BGLS01] H. H. Bauschke, O. Güler, A. S. Lewis, and H. S. Sendov, *Hyperbolic polynomials and convex analysis*, Canad. J. Math. **53** (2001), no. 3, 470–488.

[Bha97] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, vol. 169, Springer-Verlag, New York, 1997.

[Bro79] M. D. Bronshtein, *Smoothness of roots of polynomials depending on parameters*, Sibirsk. Mat. Zh. **20** (1979), no. 3, 493–501, 690, English transl. in Siberian Math. J. **20** (1980), 347–352.

[Bro80] ———, *The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity*, Trudy Moskov. Mat. Obshch. **41** (1980), 83–99, English transl. in Trans. Moscow Math. Soc. **1**(1982), 87–103.

[COP12] F. Colombini, N. Orrù, and L. Pernazza, *On the regularity of the roots of hyperbolic polynomials*, Israel J. Math. **191** (2012), 923–944.

[DLS11] C. De Lellis and E. N. Spadaro, *Q-valued functions revisited*, Mem. Amer. Math. Soc. **211** (2011), no. 991, vi+79.

[EG92] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[Ego11] D. Th. Egoroff, *Sur les suites de fonctions mesurables.*, C. R. Acad. Sci., Paris **152** (1911), 244–246 (French).

[GG13] M. Ghisi and M. Gobbino, *Higher order Glaeser inequalities and optimal regularity of roots of real functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), no. 4, 1001–1021.

[Gla63] G. Glaeser, *Racine carrée d'une fonction différentiable*, Ann. Inst. Fourier (Grenoble) **13** (1963), no. 2, 203–210.

[KP08] K. Kurdyka and L. Paunescu, *Hyperbolic polynomials and multiparameter real-analytic perturbation theory*, Duke Math. J. **141** (2008), no. 1, 123–149.

[Nui68] W. Nuij, *A note on hyperbolic polynomials*, Math. Scand. **23** (1968), 69–72 (1969).

[OT86] Y. Ohya and S. Tarama, *Le problème de Cauchy à caractéristiques multiples dans la classe de Gevrey. I. Coefficients hölderiens en t*, Hyperbolic equations and related topics (Katata/Kyoto, 1984), Academic Press, Boston, MA, 1986, pp. 273–306.

[PR15] A. Parusiński and A. Rainer, *A new proof of Bronshtein's theorem*, J. Hyperbolic Differ. Equ. **12** (2015), no. 4, 671–688.

[PR16] ———, *Regularity of roots of polynomials*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **16** (2016), 481–517.

[PR18] ———, *Optimal Sobolev regularity of roots of polynomials*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 5, 1343–1387, doi:10.24033/asens.2376.

[PR20a] ———, *Selections of bounded variation for roots of smooth polynomials*, Sel. Math. New Ser. **26** (2020), no. 13, doi:10.1007/s00029-020-0538-z.

[PR24] ———, *On the continuity of the solution map for polynomials*, (2024), https://arxiv.org/abs/2410.01326.

[PR25] ———, *Perturbation theory of polynomials and linear operators*, Handbook of Geometry and Topology of Singularities, vol. VII, pp. 121–202, Springer, 2025, arXiv:2308.01299.

[PR20b] A. Parusiński and G. Rond, *Multiparameter perturbation theory of matrices and linear operators*, Transactions of the American Mathematical Society **373** (2020), no. 4, 2933–2948.

[Rai13] A. Rainer, *Perturbation theory for normal operators*, Trans. Amer. Math. Soc. **365** (2013), no. 10, 5545–5577.

[Rel37] F. Rellich, *Störungstheorie der Spektralzerlegung, I*, Math. Ann. **113** (1937), 600–619.

[Spa00] S. Spagnolo, *Local and semi-global solvability for systems of non-principal type*, Comm. Partial Differential Equations **25** (2000), no. 5-6, 1115–1141.

[Tar06] S. Tarama, *Note on the Bronshtein theorem concerning hyperbolic polynomials*, Sci. Math. Jpn. **63** (2006), no. 2, 247–285.

[Wak86] S. Wakabayashi, *Remarks on hyperbolic polynomials*, Tsukuba J. Math. **10** (1986), no. 1, 17–28.

[Wey12] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), 441–479 (German).

Adam Parusiński: Université Côte d'Azur, CNRS, LJAD, UMR 7351, 06108 Nice, France

Email address: `adam.parusinski@univ-cotedazur.fr`

Armin Rainer: Faculty of Mathematics and Geoinformation, Institute for Statistics and Mathematical Methods in Economics, E105-04, TU Wien, Wiedner Hauptstrasse 8, 1040 Vienna, Austria

Email address: `armin.rainer@tuwien.ac.at`