9th of May, 2025 Austrian Numerical Analysis Day

Unconditional convergence and optimal complexity of adaptive iteratively linearized FEM

slides

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Motivation and context

TU WIEN NumPDEs

MODELING

Partial Differential Equation (PDE) $u^\star \in \mathcal{X}$

estimator $\eta_\ell(u_\ell^\star)$
$\approx$
discretization error
$\|\!|u^\star - u_\ell^\star|\!\|$
→

DISCRETIZATION

Finite Element Method (FEM) $u_\ell^\star \in \mathcal{X}_\ell^p$

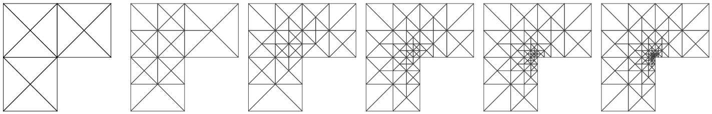
computational mesh $\mathcal{T}_\ell$
polynomial degree p

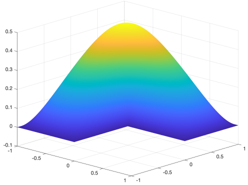
estimator $\zeta_\ell(u_\ell)$
$\approx$
solver error
$\|\!|u_\ell^\star - u_\ell|\!\|$
→

SOLVER

$u_\ell \in \mathcal{X}_\ell^p$

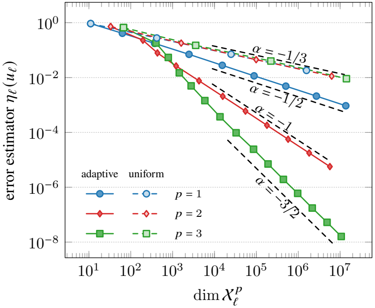
Example: Laplace problem $-\Delta u = 1$ in L-shaped domain with known exact solution
AFEM: SOLVE & ESTIMATE – MARK – REFINE

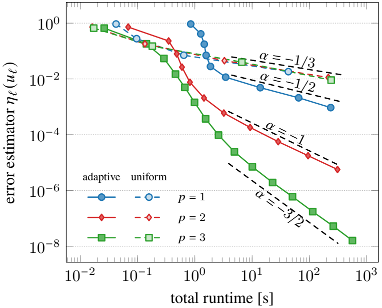




Goals: Optimal convergence wrt. dofs and time

1) Unconditional convergence: *The adaptive algorithm should converge for arbitrary choice of parameters.*





2) Best possible rate is $\alpha = p/d$, we want that AFEM produces it wrt:

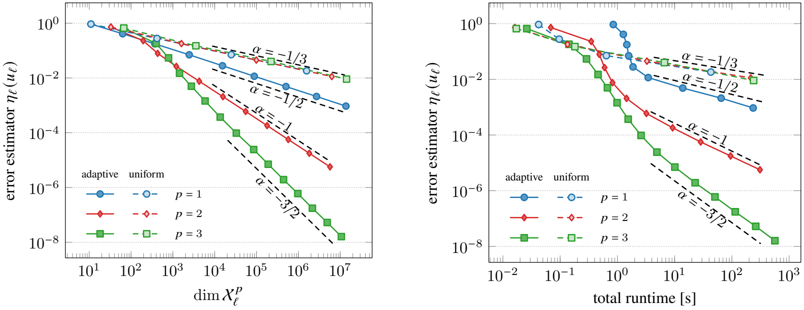
dofs: i.e., $\dim \mathcal{X}_\ell^p \eqsim \#\mathcal{T}_\ell$ for fixed p

$$\sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell(u_\ell) < \infty$$

overall computational work: assuming $\mathtt{work}(\mathcal{T}_{\ell'}) \eqsim \#\mathcal{T}_{\ell'}$

$$\sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big)^\alpha \eta_\ell(u_\ell) < \infty$$

Towards optimal complexity



Optimal complexity of AFEMs requires each of its modules to be realized in linear complexity:

- SOLVE is critical
- ESTIMATE ✓
- MARK (Stevenson 2007, Pfeiler-Praetorius 2020 for minimal cardinality marking) ✓
- REFINE (Binev-Dahmen-DeVore 2004, Stevenson 2008) ✓

Adaptive iteratively linearized FEM

Non-linear problem with energy structure

$\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ a polyhedral Lipschitz domain $\quad f \in L^2(\Omega) \quad \|\cdot\| := \|\nabla \cdot\|_{L^2(\Omega)}$

Scalar non-linearity: $M\,(t-s) \leq \mu(t^2)t - \mu(s^2)s \leq L\,(t-s) \quad \forall\, 0 \leq s \leq t$

and $\mu'(t) \leq 0 \quad \forall t \geq 0$

$$-\mathrm{div}\left(\mu(|\nabla u^\star|^2)\nabla u^\star\right) = f \quad \text{in } \Omega, \quad u^\star = 0 \quad \text{on } \partial\Omega$$

Weak formulation: find $u^\star \in H_0^1(\Omega)$ s.t.

$$\langle \mathcal{A}\, u^\star\,,\, v\rangle := \langle \mu(|\nabla u^\star|^2)\nabla u^\star\,,\, \nabla v\rangle_{L^2(\Omega)} = F(v) \quad \forall\, v \in H_0^1(\Omega)$$

- strongly monotone $\quad M\,\|u-v\|^2 \leq \langle \mathcal{A}u - \mathcal{A}v\,,\, u-v\rangle$
- Lipschitz continuous $\quad \langle \mathcal{A}u - \mathcal{A}v\,,\, w\rangle \leq L\,\|u-v\|\,\|w\|$

$\Longrightarrow$ existence and uniqueness of weak solution $u^\star \in H_0^1(\Omega)$

Energy setting

- energy functional $\quad \mathcal{E}(v) := \dfrac{1}{2}\displaystyle\int_\Omega \int_0^{|\nabla v(x)|^2} \mu(t)\,\mathrm{d}t\,\mathrm{d}x - F(v)$
- energy distance $\quad \mathbb{D}^2(u,v) := \mathcal{E}(v) - \mathcal{E}(u) \quad$ note: $\mathbb{D}^2(u^\star, v) \eqsim \|u^\star - v\|^2$

Zeidler: (1990)

Discrete problem, linearization, algebra

- $\mathcal{T}_\ell$ a simplicial triangulation of Ω and $p = 1 \implies \mathcal{X}_\ell^p := \{v_\ell \in H_0^1(\Omega) : v_\ell|_T \in \mathbb{P}_p(T)\ \forall T \in \mathcal{T}_\ell\}$
- seek $u_\ell^\star \in \mathcal{X}_\ell^p$ solution to

$$\langle \mathcal{A} u_\ell^\star , v_\ell \rangle := \langle \mu(|\nabla u_\ell^\star|^2)\nabla u_\ell^\star , \nabla v_\ell \rangle_{L^2(\Omega)} = F(v_\ell) \quad \forall\, v_\ell \in \mathcal{X}_\ell^p .$$

- note that: $\mathbb{D}^2(u_\ell^\star, v_\ell) \eqsim \|u_\ell^\star - v_\ell\|^2 \quad \forall v_\ell \in \mathcal{X}_\ell^p$

Idea: combine mesh-refinement (ℓ) with a linearization iteration (k) and algebraic solver (j) such that

$$u^\star \approx u_\ell^\star \approx u_\ell^{k,\star} \approx u_\ell^{k,j} \quad (\ell, k, j) \in \mathcal{Q} \subset \mathbb{N}_0^3$$

Kačanov linearization: the iteration mapping $\Phi_\ell : \mathcal{X}_\ell^p \to \mathcal{X}_\ell^p$ given as

$$\langle \mu(|\nabla u_\ell|^2)\, \nabla \Phi_\ell(u_\ell) , \nabla v_\ell \rangle_{L^2(\Omega)} = F(v_\ell) \quad \text{for all} \quad u_\ell, v_\ell \in \mathcal{X}_\ell$$

Parameter-free method

- contractive linearization: $q_{\rm lin}^\star \in (0,1)$ such that $\mathbb{D}^2(\Phi_\ell(u_\ell^{k,J}), u_\ell^\star) \le q_{\rm lin}^\star\, \mathbb{D}^2(u_\ell^{k,J}, u_\ell^\star) \quad \forall k \in \mathbb{N}_0$
- contractive algebra: $q_{\rm alg} \in (0,1)$ such that $\|\Phi_\ell(u_\ell^{k-1,J}) - u_\ell^{k,j}\| \le q_{\rm alg}\, \|\Phi_\ell(u_\ell^{k,J}) - u_\ell^{k,j-1}\| \quad \forall j \in \mathbb{N}_0$

Heid, Praetorius, Wihler: *Comput. Methods Appl. Math.*, 21 (2021)

Innerberger, Miraçi, Praetorius, Streitberger: *ESAIM Math. Model. Numer. Anal.*, 58 (2024)

Adaptive FEM with nested iterative solvers

TU WIEN NumPDEs

input: initial mesh $\mathcal{T}_0$, initial guess $u_0^{0,0}$, adaptivity parameters $0 < \theta \le 1$, $\lambda_{\text{lin}} > 0$

for each $\ell = 0, 1, 2, \ldots$ **repeat** (mesh-refinement loop)

SOLVE & ESTIMATE (linearization loop)

for $k = 1, 2, \ldots, K$, **repeat**

for $j = 1, 2, \ldots, J$, **repeat** (algebra loop)

compute $u_\ell^{k,j} \approx u_\ell^{k,\star}$ from the previous step $u_\ell^{k,j-1}$

compute the local indicators $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

until [algebra-criterion]

until $\mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k-1,J}) \le \lambda_{\text{lin}}\, \eta_\ell(u_\ell^{k,J})^2$

MARK choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^{K,J})^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^{K,J})^2$

REFINE $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

$u_{\ell+1}^{0,0} := u_\ell^{K,J}$

output: discrete solutions $u_\ell^{K,J}$ and corresponding estimators $\eta_\ell(u_\ell^{K,J})$

Termination of algebraic solver I

- the stopping criterion should guarantee that nested linearization-algebraic solver contracts in energy

$$\mathbb{D}^2(u_\ell^\star, u_\ell^{k,J}) \le q_{\mathrm{ctr}}\, \mathbb{D}^2(u_\ell^\star, u_\ell^{k-1,J}) \qquad 0 < q_{\mathrm{ctr}} < 1$$

Equilibration criterion [HPSV21]

- stop algebraic solver if $\|u_\ell^{k,J} - u_\ell^{k,J-1}\|^2 \le \lambda_{\mathrm{alg}} \left[\eta_\ell(u_\ell^{k,J})^2 + \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k,J-1}) \right]$
- stop linearization if $\mathbb{D}^2(u_\ell^{K,J}, u_\ell^{K-1,J}) \le \lambda_{\mathrm{lin}}\, \eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ full R-linear convergence for arbitrary λ_{lin} but sufficiently small λ_{alg}

Note: there exists $C_{\mathrm{nrg}}^\star > 0$ st. $C_{\mathrm{nrg}}^\star \|u_\ell^{k,\star} - u_\ell^{k-1,J}\|^2 \le \mathbb{D}^2(u_\ell^{k,\star}, u_\ell^{k-1,J})$

Energy-based criterion [MPS24+]

- enforce algorithmically $\|u_\ell^{k,J} - u_\ell^{k,J-1}\|^2 \lesssim \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k,J-1})$ **(parameter-free)**
- stop linearization if $\mathbb{D}^2(u_\ell^{K,J}, u_\ell^{K-1,J}) \le \lambda_{\mathrm{lin}}\, \eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ full R-linear convergence for arbitrary $\lambda_{\mathrm{lin}} > 0$

Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

Miraçi, Praetorius, Streitberger: arXiv: 2401.17778 (2024)

Termination of algebraic solver II

Recall: there exists $C^\star_{\mathrm{nrg}} > 0$ st. $C^\star_{\mathrm{nrg}} |\!|\!| u_\ell^{k,\star} - u_\ell^{k-1,J} |\!|\!|^2 \le \mathbb{D}^2(u_\ell^{k,\star}, u_\ell^{k-1,J})$

Energy-based criterion [MPS24+]

- Initialize threshold bounds $\alpha_{\min} = 100$, $J_{\max} = 1$, and reduction factor $\rho = 0.5$
- In each inner algebra loop, in addition to $u_\ell^{k,j}$ compute $\alpha_\ell^{k,j} := \mathbb{D}^2(u_\ell^{k,j}, u_\ell^{k-1,J}) / |\!|\!| u_\ell^{k,j} - u_\ell^{k-1,J} |\!|\!|^2$

  Until $\boxed{u_\ell^{k,j} = u_\ell^{k-1,J}}$ or $\boxed{\alpha_\ell^{k,j} \ge \alpha_{\min}}$ or $\boxed{\alpha_\ell^{k,j} > 0 \text{ and } j > J_{\max}}$
- If $J[\ell,k] > J_{\max}$, then update $J_{\max} \leftarrow J[\ell,k]$ and $\alpha_{\min} \leftarrow \rho\, \alpha_{\min}$

Proposition (uniform bound on algebraic steps)

- There exists an index $j_0 \in \mathbb{N}$ such that $J[\ell,k] \le j_0$ for all $(\ell,k,0) \in \mathcal{Q}$
- There exists $0 < C_{\mathrm{nrg}} < C^\star_{\mathrm{nrg}}$ such that

$$C_{\mathrm{nrg}} |\!|\!| u_\ell^{k,J} - u_\ell^{k-1,J} |\!|\!|^2 \le \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k-1,J}) \quad \text{for all } (\ell,k,0) \in \mathcal{Q} \text{ with } k \ge 1.$$

Proposition (energy-contraction of inexact linearization)

There exists $0 < q_{\mathrm{ctr}} \le 1 - 9\, C_{\mathrm{nrg}} (1 - q_{\mathrm{alg}})^2 M^2 / L^3 < 1$ such that $\mathbb{D}^2(u_\ell^\star, u_\ell^{k,J}) \le q_{\mathrm{ctr}}\, \mathbb{D}^2(u_\ell^\star, u_\ell^{k-1,J})$

Main results

A posteriori error control

Note that for each iterate u_ℓ^k computed by the adaptive algorithm, there holds

$$\underbrace{|\!|\!| u^\star - u_\ell^{k,j} |\!|\!|}_{\substack{\text{overall}\\ \text{error}}} \leq \underbrace{|\!|\!| u^\star - u_\ell^\star |\!|\!|}_{\substack{\text{discretization}\\ \text{error}}} + \underbrace{|\!|\!| u_\ell^\star - u_\ell^{k,\star} |\!|\!|}_{\substack{\text{linearization}\\ \text{error}}} + \underbrace{|\!|\!| u_{k,\ell}^\star - u_\ell^{k,j} |\!|\!|}_{\substack{\text{algebraic}\\ \text{error}}}$$

$$\overset{\text{reliability}}{\lesssim} \underbrace{\underbrace{\eta_\ell(u_\ell^\star)}_{\substack{\text{discretization}\\ \text{error estimator}}} + |\!|\!| u_\ell^\star - u_\ell^{k,\star} |\!|\!| + |\!|\!| u_\ell^\star - u_\ell^{k,j} |\!|\!|}_{\text{quasi-error } \mathrm{H}_\ell^{k,j}}$$

$$\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^{k,j}) + |\!|\!| u_\ell^\star - u_\ell^{k,\star} |\!|\!| + |\!|\!| u_\ell^\star - u_\ell^{k,j} |\!|\!|$$

$$\overset{\substack{\text{contractive}\\ \text{solvers}}}{\lesssim} \eta_\ell(u_\ell^k) + \underbrace{\mathbb{D}^2(u_\ell^{k,j}, u_\ell^{k-1,J})^{1/2}}_{\substack{\text{linearization}\\ \text{error estimator}}} + \underbrace{|\!|\!| u_\ell^{k,j-1} - u_\ell^{k,j} |\!|\!|}_{\substack{\text{algebraic}\\ \text{error estimator}}}$$

Main results I: full R-linear convergence

Theorem (full R-linear convergence of the quasi-error)

Consider arbitrary $0 < \theta \leq 1$, $\lambda_{\mathrm{lin}} > 0$. For all $(\ell', k', j'), (\ell, k, j) \in \mathcal{Q}$ with $|\ell', k', j'| > |\ell, k, j|$, there holds

$$\mathrm{H}_{\ell'}^{k',j'} \leq C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell',k',j'|-|\ell,k,j|}\, \mathrm{H}_{\ell}^{k,j}, \quad C_{\mathrm{lin}} > 0,\ 0 < q_{\mathrm{lin}} < 1$$

Contraction regardless of algorithmic step: mesh-refinement, linearization, or algebraic solver step

1. algebraic solver norm contraction $\quad \| u_\ell^{k,\star} - u_\ell^{k,j} \| \leq q_{\mathrm{alg}} \| u_\ell^{k,\star} - u_\ell^{k,j-1} \|$
2. linearization energy contraction $\quad \mathbb{D}^2(u_\ell^{k,\star}, u_\ell^\star) \leq q_{\mathrm{lin}}^\star\, \mathbb{D}^2(u_\ell^{k-1,J}, u_\ell^\star)$
3. reduction, Dörfler, nested iteration $\quad \eta_{\ell+1}(u_{\ell+1}^{K,J}) \leq q_\theta\, \eta_\ell(u_\ell^{K,J}) + C \| u_{\ell+1}^{K,J} - u_\ell^{K,J} \|$
4. energy orthogonality $\quad \mathbb{D}^2(u_\ell^\star, u^\star) = \mathbb{D}^2(u_{\ell+1}^\star, u^\star) + \mathbb{D}^2(u_\ell^\star, u_{\ell+1}^\star)$
5. tail summability $\quad \mathrm{H}_\ell^{k,j} \lesssim q_{\mathrm{lin}}^{|\ell,k,j|-|\ell',k',j'|}\, \mathrm{H}_{\ell'}^{k',j'} \iff \sum_{|\ell,k,j|>|\ell',k',j'|} \mathrm{H}_\ell^{k,j} \lesssim \mathrm{H}_{\ell'}^{k',j'}$

Corollary (unconditional convergence)

$$\| u^\star - u_\ell^{k,j} \| \lesssim \mathrm{H}_\ell^{k,j} \lesssim q_{\mathrm{lin}}^{|\ell,k,j|}\, \mathrm{H}_0^{0,0} \longrightarrow 0 \quad \text{as} \quad |\ell,k,j| \longrightarrow \infty$$

Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)
Bringmann, Feischl, Miraçi, Praetorius, Streitberger: *Comput. Math. Appl.*, 180 (2025)

Main results II: optimal complexity of AFEM

Theorem (optimal convergence with respect to overall computational cost)

Consider $\alpha > 0$ and $\|u^\star\|_{\mathbb{A}_\alpha} := \sup_{N \geq \#\mathcal{T}_0} N^\alpha \big[\min_{\#\mathcal{T}_{\mathrm{opt}} \leq N} \eta_{\mathrm{opt}}(u^\star_{\mathrm{opt}}) \big] < \infty.$

Let full R-linear convergence hold. For sufficiently small $0 < \theta < 1$ and $\lambda_{\mathrm{lin}} > 0$

$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_\alpha} \lesssim \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'| \leq |\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \mathrm{H}_\ell^{k,j} \lesssim \max\big\{ \|u^\star\|_{\mathbb{A}_\alpha}, \mathrm{H}_0^{0,0} \big\}$$

if $u^\star$ *can be approximated at rate* α *over dofs and there holds full R-linear convergence,*
then AFEM approximates $u^\star$ *at rate* α *with respect to overall computational cost.*

1. $0 < \theta \ll 1 \quad \Longrightarrow \quad$ *optimal rates* for AFEM with exact solver
2. $0 < \lambda_{\mathrm{lin}} \ll \theta \quad \Longrightarrow \quad u_\ell^{K,J} \approx u_\ell^\star$ and Dörfler marking is equivalent

Stevenson: *Found. Comput. Math.*, 7 (2007)

Carstensen, Feischl, Page, Praetorius: *Comput. Math. Appl.*, 67 (2014)

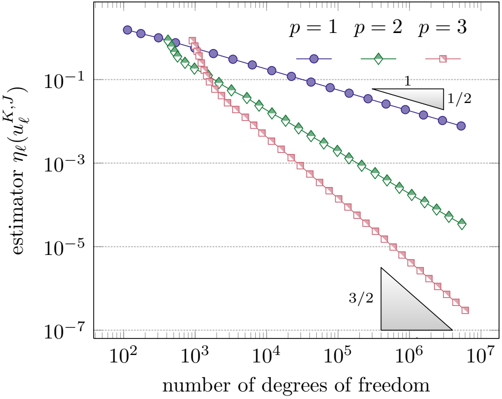
Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

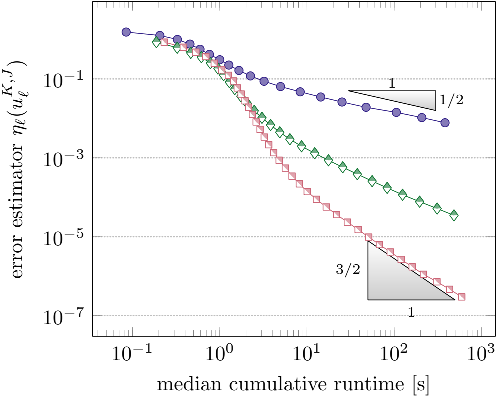
Numerical experiments

Optimality of AILFEM

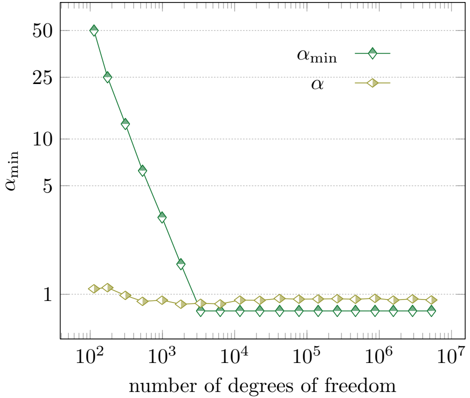
Test case: L-shaped domain and coefficient $\mu(t) = 1 + \exp(-t)$

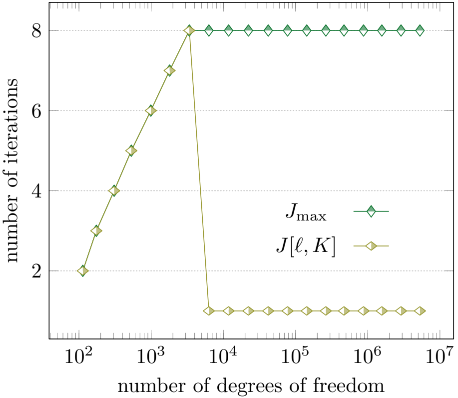
Parameters: $\theta = 0.5$ and $\lambda_{\text{lin}} = 0.7$





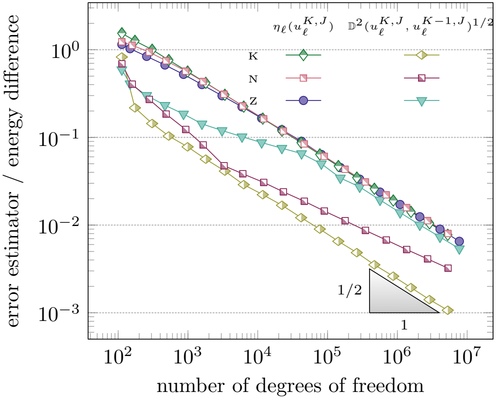
Parameter-less algebraic stopping criterion

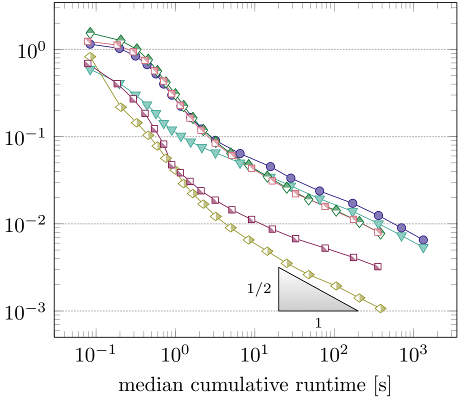




Other iteration alternatives: linearization

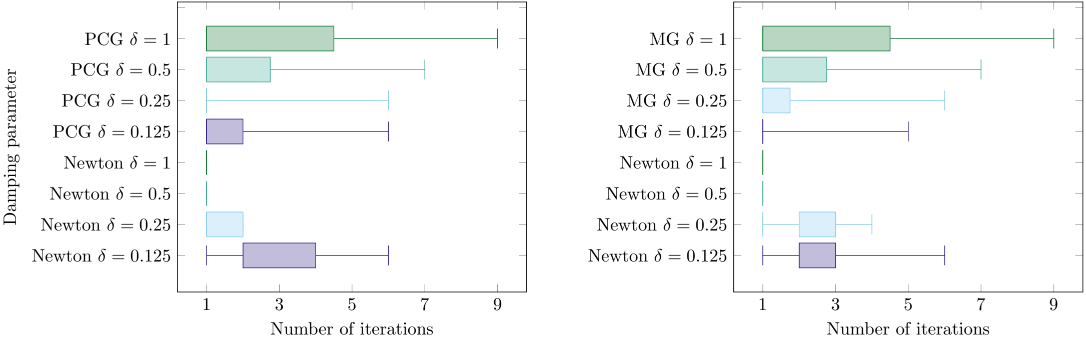
$\theta = 0.5 \quad \lambda_{\text{lin}} = 0.7$ N: Newton (damped $\delta = 1/2$) K: Kačanov Z: Zarantonello (damped $\delta = 1/6$)





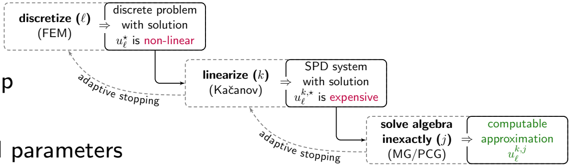
Other iteration alternatives: algebra

$\theta = 0.5 \quad \lambda_{\mathrm{lin}} = 0.7$



Take home messages

1. analysis of adaptive algorithms should focus rather on rates wrt. *complexity/time* than dofs
2. *linear complexity* and *contraction* of the iterative solvers is crucial
3. *nested* iterations and new *parameter-free* algebraic stopping criterion enforcing norm-energy equivalence
4. *reliability* via a posteriori error estimators is ensured
5. full R-linear convergence
   - holds for *arbitrary* adaptivity parameters
   - gives *contraction* regardless of algorithmic step
   - provides the equivalence *rates* = *complexity*
6. *optimal complexity* is ensured for sufficiently small parameters



Thank you for your attention!

Miraçi, Praetorius, Streitberger
Unconditional full linear convergence and optimal complexity of adaptive iteratively linearized FEM for nonlinear PDEs
Preprint, arXiv: 2401.17778 (2024)

Bringmann, Feischl, Miraçi, Praetorius, Streitberger
On full linear convergence and optimal complexity of adaptive FEM with inexact solver
Comput. Math. Appl., 180, DOI: 10.1016/j.camwa.2024.12.013 (2025)

Innerberger, Miraçi, Praetorius, Streitberger
hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs
ESAIM Math. Model. Numer. Anal., 58, DOI: 10.1051/m2an/2023104 (2024)

slides

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