09. May 2025 - Austrian Numerical Analysis Day 2025

Convergence of adaptive BEM driven by functional error estimates

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NumPDEs Workgroup on Numerics of PDEs

FWF Österreichischer Wissenschaftsfonds

Outline

Introduction

Functional estimates

Estimator-driven adaptive algorithm

Numerical experiments

Introduction

Model problem

Laplace equation

- $\Delta u^\star = 0$ in $\Omega \subset \mathbb{R}^d$, where $d \geq 2$
- $u^\star = g$ on $\Gamma := \partial\Omega$

Fundamental solution

- $G(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } d = 2 \\ \frac{1}{(d-2)|\mathbb{S}^{d-1}|} |x|^{2-d} & \text{if } d \geq 3 \end{cases}$

Single-layer potential

- $(\widetilde{V}\phi)(x) := (G * \phi)(x) = \displaystyle\int_\Gamma G(x-y)\phi(y)\,\mathrm{d}y$
- $V\phi := (\widetilde{V}\phi)|_\Gamma$

Boundary integral equation (BIE)

Properties of the single-layer potential

- $\widetilde{V} : H^{-1/2}(\Gamma) \to H^1(\Omega) \quad \Longrightarrow \quad V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$
- $\Delta(\widetilde{V}\phi) = 0$ for all $\phi \in H^{-1/2}(\Gamma)$
- V is elliptic

BEM ansatz

- $u^\star = \widetilde{V}\phi^\star$ with unknown $\phi^\star \in H^{-1/2}(\Gamma)$
- $g = V\phi^\star$ by taking trace on Γ
- Lax–Milgram: Existence and uniqueness of $\phi^\star$

- Any approximation $\phi_\ell \approx \phi^\star$ leads to $u_\ell := \widetilde{V}\phi_\ell \approx u^\star$ with $u_\ell|_\Gamma = V\phi_\ell$

Why BEM?

Advantages

- Mesh on the boundary only = dimension reduction
- For any approximation, $u_\ell = \widetilde{V}\phi_\ell$ is harmonic, i.e. $\Delta u_\ell = 0$ in the strong sense
- Can also treat exterior problems on unbounded domains

Challenge

- Solve for $\phi^\star$ instead of $u^\star$
- $\phi^\star$ has no immediate physical relevance

$\Longrightarrow$ Rather control $\|\nabla(u^\star - u_\ell)\|_{L^2(\Omega)}$ instead of $\|\phi^\star - \phi_\ell\|_{H^{-1/2}(\Gamma)}$

Functional estimates

Functional error identity

Theorem

- $u, v \in H^1(\Omega)$ harmonic, i.e., $\Delta u = \Delta v = 0$

$$\Longrightarrow \max_{\substack{\boldsymbol{\tau} \in H(\mathrm{div},\Omega) \\ \mathrm{div}\,\boldsymbol{\tau} = 0}} \big[2\langle (u-v)|_\Gamma \,,\, \boldsymbol{\tau} \cdot n \rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2 \big] = \|\nabla (u-v)\|_\Omega^2 = \min_{\substack{w \in H^1(\Omega) \\ w|_\Gamma = (u-v)|_\Gamma}} \|\nabla w\|_\Omega^2$$

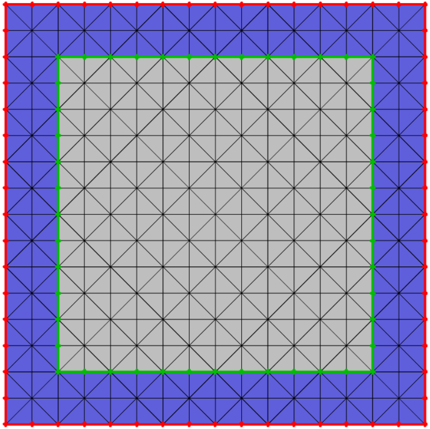
- Lower bound: $\|x\|_{\mathcal{H}}^2 = \max_{y \in \mathcal{H}} \big[2\langle x \,,\, y \rangle_{\mathcal{H}} - \|y\|_{\mathcal{H}}^2 \big]$ by abstract variational argument

- Upper bound: Energy minimization property of harmonic functions (Dirichlet principle)

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Computable bounds

- Goal: Find functions $\boldsymbol{\tau}_\ell$ and w_ℓ s.t.
  - $\boldsymbol{\tau}_\ell \in H(\mathrm{div}, \Omega)$, $\mathrm{div}\, \boldsymbol{\tau}_\ell = 0$
  - $w_\ell \in H^1(\Omega)$, $w_\ell|_\Gamma = (u^\star - u_\ell)|_\Gamma$
  - $\boldsymbol{\tau}_\ell, w_\ell$ easily computable
- Idea: Solve problems on strip domain ω_ℓ
  - Mesh $\mathcal{T}_\ell^\omega$ on ω_ℓ
  - $\omega_\ell \subset \Omega$ and $\Gamma \subset \partial\omega_\ell$
  - ω_ℓ varies within adaptive algorithm



Computable lower bound

Auxiliary problem

- Raviart–Thomas space $\mathcal{RT}_*^{q-1}(\mathcal{T}_\ell^\omega) \subset \{\boldsymbol{\sigma} \in H(\mathrm{div}, \omega_\ell) \mid \boldsymbol{\sigma} \cdot n = 0$ on $\partial\omega_\ell \setminus \Gamma\}$
- Piecewise polynomials $\mathcal{P}^{q-1}(\mathcal{T}_\ell^\omega) \subset L^2(\omega_\ell)$

- Compute FE-solution $(\boldsymbol{\tau}_\ell^\star, p_\ell^\star) \in \mathcal{RT}_*^{q-1}(\mathcal{T}_\ell^\omega) \times \mathcal{P}^{q-1}(\mathcal{T}_\ell^\omega)$ such that

$$
\begin{aligned}
\langle \boldsymbol{\tau}_\ell^\star \,,\, \boldsymbol{\sigma}_\ell \rangle_{\omega_\ell} + \langle \mathrm{div}\, \boldsymbol{\sigma}_\ell \,,\, p_\ell^\star \rangle_{\omega_\ell} &= \langle g - V\phi_\ell \,,\, \boldsymbol{\sigma}_\ell \cdot n \rangle_\Gamma \qquad \forall \boldsymbol{\sigma}_\ell \in \mathcal{RT}_*^{q-1}(\mathcal{T}_\ell^\omega) \\
\langle \mathrm{div}\, \boldsymbol{\tau}_\ell^\star \,,\, q_\ell \rangle_{\omega_\ell} &= 0 \qquad \forall q_\ell \in \mathcal{P}^{q-1}(\mathcal{T}_\ell^\omega)
\end{aligned}
$$

- Note: $2\langle g - V\phi_\ell \,,\, \boldsymbol{\tau}_\ell^\star \cdot n \rangle_\Gamma - \|\boldsymbol{\tau}_\ell^\star\|_{\omega_\ell}^2 = \|\boldsymbol{\tau}_\ell^\star\|_{\omega_\ell}^2$ and $\mathrm{div}\, \boldsymbol{\tau}_\ell^\star = 0$

$\Longrightarrow \quad \|\boldsymbol{\tau}_\ell^\star\|_{\omega_\ell} = \|\boldsymbol{\tau}_\ell^\star\|_\Omega \le \|\nabla(u^\star - u_\ell)\|_\Omega$

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Computable upper bound I

- Problem: Discrete functions w_ℓ cannot satisfy $w_\ell|_\Gamma = g - V\phi_\ell$ exactly
- Solution: Include data oscillations
  - Employ Scott–Zhang projection $J_\ell\colon H^1(\Gamma) \mapsto \mathcal{S}^q(\mathcal{T}_\ell^\omega|_\Gamma)$
  - Assume additional regularity $g \in H^1(\Gamma)$:

$$C_{\mathsf{osc}}^{-1}\|(1-J_\ell)(g-V\phi_\ell)\|_{H^{1/2}(\Gamma)} \quad \leq \quad \|h_\ell^{1/2}\nabla_\Gamma(1-J_\ell)(g-V\phi_\ell)\|_{L^2(\Gamma)} \quad =: \quad \mathrm{osc}_\ell$$

$$\Longrightarrow \quad \min_{\substack{w\in H^1(\Omega)\\ w|_\Gamma = g - V\phi_\ell}} \|\nabla w\|_\Omega \quad \leq \quad \min_{\substack{w\in H^1(\Omega)\\ w|_\Gamma = J_\ell(g - V\phi_\ell)}} \|\nabla w\|_\Omega + C_{\mathsf{osc}}\mathrm{osc}_\ell$$

Aurada, Feischl, Führer, Karkulik, Praetorius: *Applied Numerical Mathematics*, 95 (2015)

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Computable upper bound II

Auxiliary problem

- FEM space $\mathcal{S}^q_*(\mathcal{T}^\omega_\ell) \subset \{w \in H^1(\omega_\ell) \mid w = 0 \text{ on } \partial\omega_\ell \setminus \Gamma\}$
- Compute FE-solution $w^\star_\ell \in \mathcal{S}^q_*(\mathcal{T}^\omega_\ell)$ such that

$$\begin{aligned} \langle \nabla w^\star_\ell \,,\, \nabla v_\ell \rangle_{\omega_\ell} &= 0 \qquad\qquad \forall v_\ell \in \mathcal{S}^q_0(\mathcal{T}^\omega_\ell) \\ w^\star_\ell|_\Gamma &= J_\ell(g - V\phi_\ell) \end{aligned}$$

- Set $\eta_\ell := \|\nabla w^\star_\ell\|_{\omega_\ell} = \|\nabla w^\star_\ell\|_{\Omega}$

$\Longrightarrow \quad \|\nabla(u^\star - u_\ell)\|_\Omega \leq \eta_\ell + C_{\mathsf{osc}} \mathrm{osc}_\ell$

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numerische Mathematik*, 147 (2021)

Estimator-driven adaptive algorithm

Galerkin BEM

Galerkin discretization

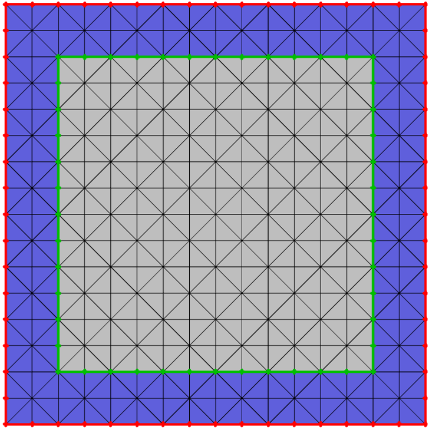
- Mesh $\mathcal{T}_\ell$ of Ω
- Boundary mesh $\mathcal{T}_\ell^\Gamma$ induced by $\mathcal{T}_\ell$
- Galerkin BEM: Find $\phi_\ell^\star \in \mathcal{P}^p(\mathcal{T}_\ell^\Gamma)$ s.t.

$$\langle V\phi_\ell^\star\,,\,\psi_\ell\rangle_\Gamma = \langle g\,,\,\psi_\ell\rangle_\Gamma \qquad \forall \psi_\ell \in \mathcal{P}^p(\mathcal{T}_\ell^\Gamma)$$

- Lax–Milgram: Existence and uniqueness of $\phi_\ell^\star$

Strip domain

- Strip domain ω_ℓ : k-patch of Γ w.r.t. $\mathcal{T}_\ell$
- Compromise :
  - Small k for cheaper computations
  - Large k for more accurate estimator
- For experiments : $k \in \{1, \ldots, 6\}$



Adaptive algorithm

- Input: Initial mesh $\mathcal{T}_0$, marking parameter $0 < \theta \leq 1$, polynomial degrees $p, q \in \mathbb{N}$

Iterate until $\eta_\ell + \mathrm{osc}_\ell \leq \varepsilon$

1. Extract boundary mesh $\mathcal{T}_\ell^\Gamma$, strip domain ω_ℓ and strip mesh $\mathcal{T}_\ell^\omega$ from $\mathcal{T}_\ell$
2. Compute $\phi_\ell^\star$ by Galerkin BEM
3. Compute discretized residual $J_\ell(g - V\phi_\ell^\star)$ and data oscillations $\mathrm{osc}_\ell(\partial T \cap \Gamma) =: \mathrm{osc}_\ell(T)$
4. Compute FE-solution $w_\ell^\star \in \mathcal{S}_*^q(\mathcal{T}_\ell^\omega)$ and local error indicators $\eta_\ell(T) := \|\nabla w_\ell^\star\|_T$
5. Choose minimal $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ s.t. $\theta \sum\limits_{T\in\mathcal{T}_\ell} [\eta_\ell(T)^2 + \mathrm{osc}_\ell(T)^2] \leq \sum\limits_{T\in\mathcal{M}_\ell} [\eta_\ell(T)^2 + \mathrm{osc}_\ell(T)^2]$
6. Refine at least all elements in $\mathcal{M}_\ell$

- Output: Fully reconstructed solution $u_\ell^\star$ if $\eta_\ell + \mathrm{osc}_\ell$ sufficiently small

Theorem

$$C_{\text{eff}}^{-1}\eta_\ell \quad \leq \quad \|\nabla(u^\star - u_\ell^\star)\|_\Omega \quad \leq \quad \eta_\ell + C_{\text{osc}}\,\mathrm{osc}_\ell \quad \xrightarrow{\ell\to\infty} \quad 0.$$

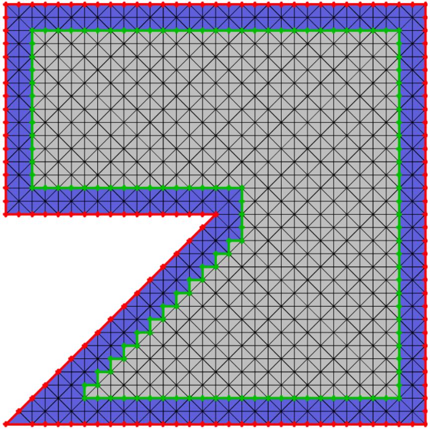
- Convergence of Galerkin schemes: $\phi_\ell^\star \to \phi_\infty^\star$ in $H^{-1/2}(\Gamma) \implies u_\ell^\star \to u_\infty^\star$ in $H^1(\Omega)$
- Challenge: Show that $\phi_\infty^\star = \phi^\star$ and $u_\infty^\star = u^\star$
- $\mathrm{osc}_\ell \to 0$
- Use *a priori convergence* $w_\ell^\star \to w_\infty^\star$ to show that $\eta_\ell \to 0$ (since ω_ℓ varies)

Babuska, Vogelius: *Numerische Mathematik*, 44 (1984)

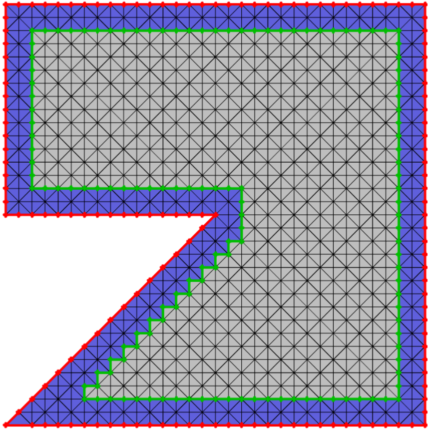
Numerical experiments

Z-shaped domain

- Z-shaped domain Ω
- $u^\star(x,y) = r^{4/7}\cos(4\varphi/7)$
- Lowest-order BEM $(p=0)$
- P1-FEM and P2-FEM $(q=1,2)$
- Strip depths $k \in \{1,2,3,4,5,6\}$

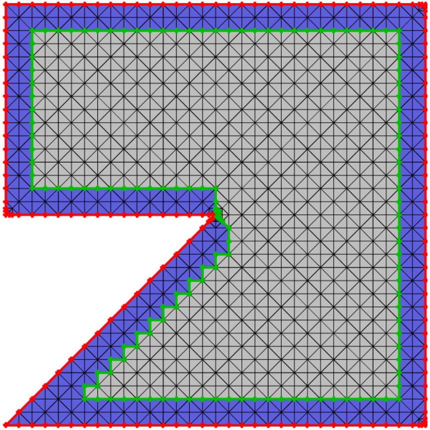


Z-shaped domain, generated meshes



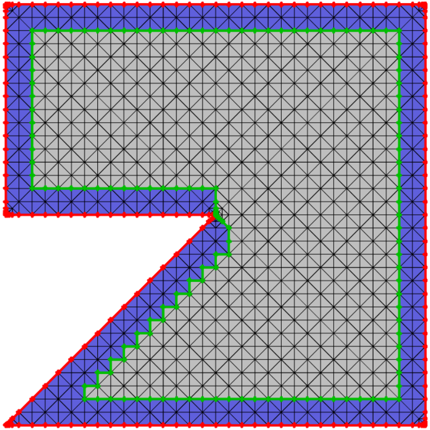
$\ell = 0$

Z-shaped domain, generated meshes



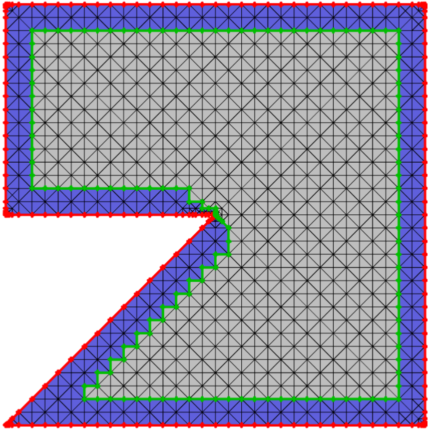
$\ell = 10$

Z-shaped domain, generated meshes



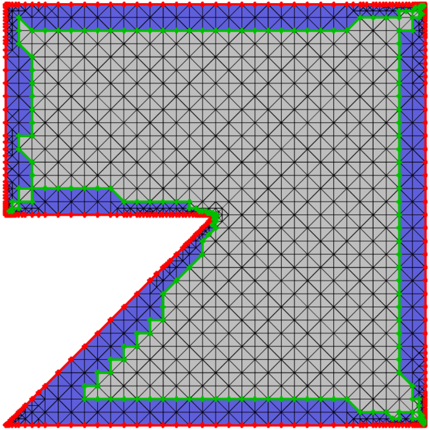
$\ell = 20$

Z-shaped domain, generated meshes



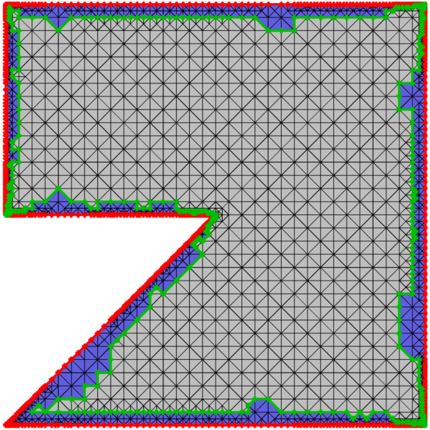
$\ell = 30$

Z-shaped domain, generated meshes



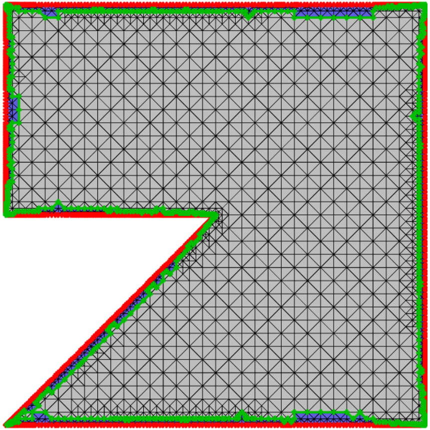
$\ell = 40$

Z-shaped domain, generated meshes



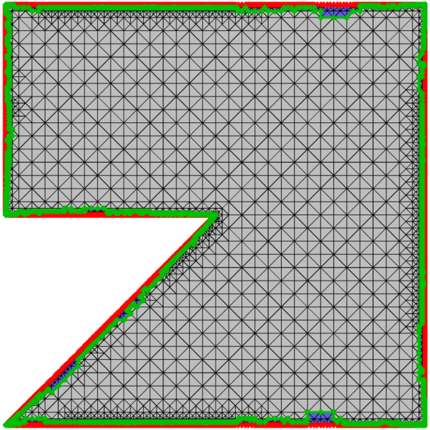
$\ell = 50$

Z-shaped domain, generated meshes



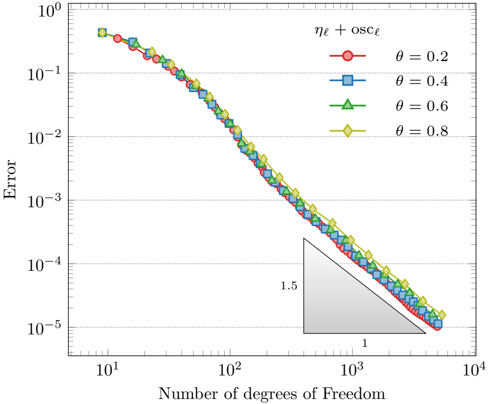
$\ell = 60$

Z-shaped domain, generated meshes

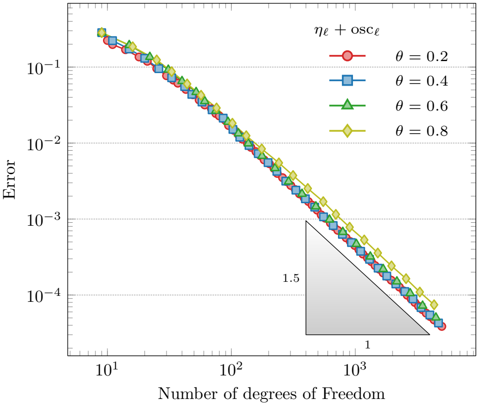


$\ell = 70$

Z-shaped domain, convergence rates



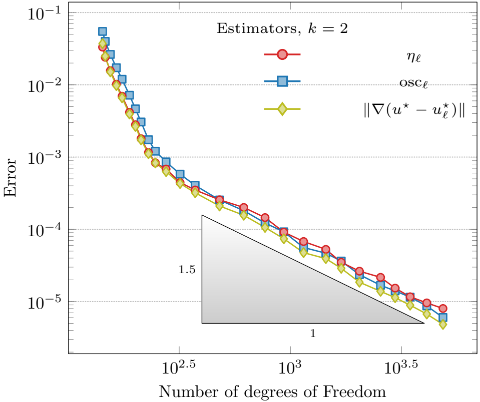
$q = 1$



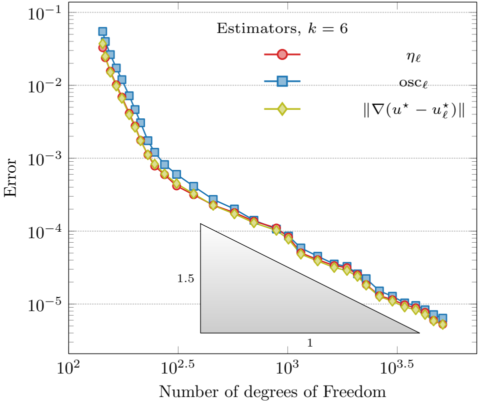
$q = 2$

- Convergence rates $\quad p = 0 \quad k = 3$

Z-shaped domain, accuracy

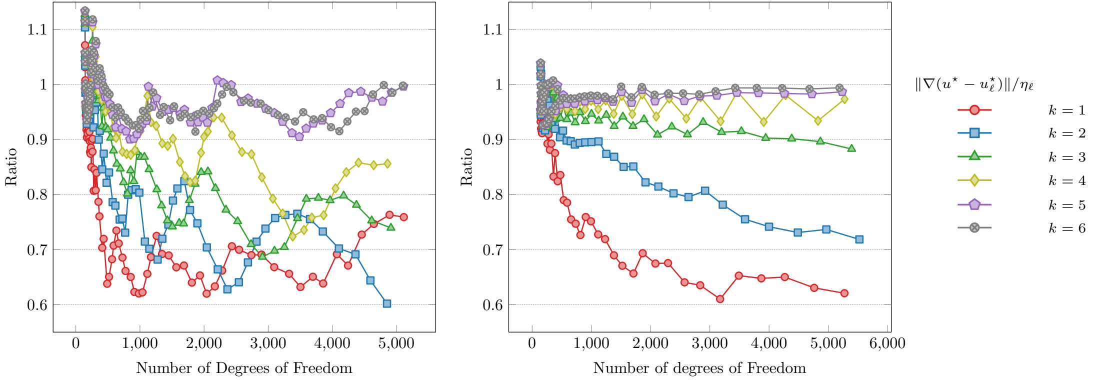


$q = 1$



$q = 2$

Z-shaped domain, efficiency indices I



$q = 1$ $\qquad$ $q = 2$

- Efficiency indices $\quad p = 0 \quad \theta = 0.5$

Summary

- Control $\|\nabla(u^\star - u_\ell)\|_\Omega$ instead of $\|\phi^\star - \phi_\ell\|_{H^{-1/2}(\Gamma)}$
- Functional error estimates for BEM: $\tau_\ell \leq \|\nabla(u^\star - u_\ell)\|_\Omega \leq \eta_\ell + C_{\mathsf{osc}}\mathrm{osc}_\ell$
- Scheme does not require computation of $u_\ell = \widetilde{V}\phi_\ell$
- Independent of approximation $\phi_\ell \approx \phi^\star$
- Adaptive algorithm guarantees convergence for Galerkin BEM
- Future work: Exterior domains, Iterative solver, Matrix compression

Thank you for your attention!

Kurz, Pauly, Praetorius, Repin, Sebastian

Functional a posteriori error estimates for boundary element methods

Numerische Mathematik, 147 (2021)

Freiszlinger, Pauly, Praetorius

Convergence of adaptive boundary element methods driven by functional a posteriori error estimates

(2025)

Strip domain

- Difficulty: Strip domain ω_ℓ varies with each step of algorithm
  - Usual tools for convergence analysis do not apply
  - Not clear if $(w_\ell)_{\ell\in\mathbb{N}}$ bounded in $H^1(\Omega)$
- Solution: Find convergent sequence $(\widehat{g}_\ell)_{\ell\in\mathbb{N}}$ of liftings of $J_\ell(g - V\phi_\ell^\star)$

Theorem

There exists $\widehat{g}_\ell \in \mathcal{S}^q(\mathcal{T}_\ell^\omega)$ s.t.

$$\begin{aligned} \widehat{g}_\ell|_\Gamma &= J_\ell(g - V\phi_\ell^\star) \\ \widehat{g}_\ell|_{\partial\omega_\ell\setminus\Gamma} &= 0 \\ \lim_{\ell\to\infty} \widehat{g}_\ell &= \widehat{g}_\infty \quad \textit{in } H^1(\Omega) \end{aligned}$$

A–priori convergence

- Goal: Show that $(w_\ell)_{\ell \in \mathbb{N}}$ converges in $H^1(\Omega)$
- Consequence: $\|\nabla w_\ell\|_{T_\ell} \leq \|\nabla w_\infty^\star\|_{T_\ell} + \|\nabla(w_\ell^\star - w_\infty^\star)\|_\Omega \xrightarrow{\ell \to \infty} 0$ if $|T_\ell| \to 0$

Theorem

There exists $w_\infty^\star \in H^1(\Omega)$ s.t.

$$\lim_{\ell \to \infty} w_\ell^\star = w_\infty^\star \quad \textit{in } H^1(\Omega)$$

- Main argument: $w_\ell^\star = w_{\ell,0}^\star + \widehat{g}_\ell$ where $w_{\ell,0}^\star \in \mathcal{S}_0^q(\mathcal{T}_\ell^\omega)$ solves

$$\langle \nabla w_{\ell,0}^\star \,,\, \nabla v_\ell \rangle_{\omega_\ell} = -\langle \nabla \widehat{g}_\ell \,,\, \nabla v_\ell \rangle_{\omega_\ell} \qquad \forall v_\ell \in \mathcal{S}_0^q(\mathcal{T}_\ell^\omega)$$