Unconditional full R-linear convergence and optimal complexity of AFEM for nonlinear PDEs

Poster:

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Motivation

The goal of this adaptive algorithm is to approximate the solution $u^\star$ of a nonlinear PDE with *optimal convergence rates* at *(quasi-)minimal computational cost* (and hence computation time), i.e., **optimal complexity**. Given an initial mesh $\mathcal{T}_0$, we say that $u^\star$ can be approximated at rate $s > 0$, i.e., $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} < \infty$, if there exist (theoretical) optimal refinements of $\mathcal{T}_0$ along which the (quasi-)error decreases with rate s. Crucially, we focus on proving that the adaptive algorithm is unconditionally convergent regardless of the user-chosen parameters. Next, we verify optimal complexity for sufficiently small adaptivity parameters.

mesh-adaptivity (ℓ)	linearization (k)	algebraic solver (j)
✓ discretize problem	✓ linearize discrete problem	✓ solve linearized system inexactly
✓ employ adaptive FEM	✓ employ Kačanov iteration	✓ employ local hp-robust MG
✗ nonlinear discrete problem	✗ SPD problem is expensive	✓ computable approximation $u_\ell^{k,j} \approx u_\ell^{k,\star}$
✗ unavailable exact solution $u_\ell^\star \approx u^\star$	✗ unavailable exact solution $u_\ell^{k,\star} \approx u_\ell^\star$	

Setting: $\mathsf{X} := H_0^1(\Omega)$ with $|||\cdot||| := \|\nabla \cdot\|_{L^2(\Omega)}$, $F \in H^{-1}(\Omega)$ right-hand side

Nonlinearity: $\mu \in C^1(\mathbb{R}_{\geq 0})$ monotonically decreasing with linear growth

Energy: $\mathsf{E}(v) := \frac{1}{2}\int_\Omega \int_0^{|\nabla v|^2} \mu(t)\,\mathrm{d}t\,\mathrm{d}x - F(v)$ for $v \in \mathsf{X}$

Energy minimization: Find $u^\star \in \mathsf{X}$ with $\mathsf{E}(u^\star) = \min_{v \in \mathsf{X}} \mathsf{E}(v)$ equivalent to

Euler–Lagrange equation: Find $u^\star \in \mathsf{X}$ such that

$$\langle \mu(|\nabla u^\star|^2)\,\nabla u^\star, \nabla v\rangle = F(v) \quad \text{for all } v \in \mathsf{X}$$

Adaptive algorithm

Approach: adaptive mesh refinement, linearization, algebraic solver

Input: initial mesh $\mathcal{T}_0$, $u_0^{0,0} \in \mathsf{X}_0 := \{v \in \mathsf{X} \colon \forall T \in \mathcal{T}_0,\ v|_T \in \mathbb{P}_1(T)\}$, $0 < \theta \leq 1$, $0 < \lambda_{\mathsf{lin}}$, $0 < \rho < 1$, $J_{\mathsf{max}} := 1$, $0 < \alpha_{\mathsf{min}}$

for $\ell = 0, 1, 2, \dots$ repeat

SOLVE & ESTIMATE

for $k = 1, 2, \dots, K$ repeat

Define $u_\ell^{k,0} := u_\ell^{k-1,J}$ and seek $u_\ell^{k,\star} \in \mathsf{X}_\ell$ such that

$$\langle \mu(|\nabla u_\ell^{k-1,J}|^2)\,\nabla u_\ell^{k,\star}, \nabla v_\ell\rangle = F(v_\ell) \quad \text{for all } v_\ell \in \mathsf{X}_\ell$$

for $j = 1, 2, \dots, J$ repeat

compute $u_\ell^{k,j} := u_\ell^{k,j-1} + \texttt{update}$ by solver from [IMPS2024]

compute $\eta_\ell(T, u_\ell^{k,j})$ for all $T \in \mathcal{T}_\ell$

compute $\alpha_\ell^{k,j} := \big(\mathsf{E}(u_\ell^{k-1,J}) - \mathsf{E}(u_\ell^{k,j})\big) / |||u_\ell^{k,j} - u_\ell^{k-1,J}|||^2$.

until $\boxed{\alpha_\ell^{k,j} \geq \alpha_{\min}}$ or $\boxed{u_\ell^{k,j} = u_\ell^{k-1,J}}$ or $\boxed{\alpha_\ell^{k,j} > 0 \text{ and } j > J_{\max}}$

(enforce norm-energy equivalence)

if $J[\ell,k] > J_{\max}$, then update $J_{\max} \hookleftarrow J[\ell,k]$ and $\alpha_{\min} \hookleftarrow \rho\,\alpha_{\min}$.

until $|||u_\ell^{k,J} - u_\ell^{k-1,J}||| \leq \lambda_{\mathsf{lin}}\,\eta_\ell(u_\ell^{k,J})$ **(equilibrate linearization error)**

MARK choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of quasi-minimal cardinality such that

$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^{K,J})^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^{K,J})^2$$

REFINE $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^{0,0} := u_\ell^{K,J}$ **(nested iteration)**

Output: approximations $u_\ell^{k,j}$; estimators $\eta_\ell(u_\ell^{k,j})$; index set Q with counter

$$|\ell,k,j| := \#\{(\ell',k',j') \in \mathsf{Q} \colon u_{\ell'}^{k',j'} \text{ computed not later than } u_\ell^{k,j}\}$$

Important consequences

Norm-energy estimate: For any adaptivity parameters $0 < \theta \leq 1$ and $0 < \lambda_{\mathsf{lin}}$, there exist C_{nrg} and $j_{\mathsf{nrg}} \in \mathbb{N}_0$ such that

$$0 < C_{\mathsf{nrg}}\,|||u_\ell^{k,j} - u_\ell^{k-1,J}|||^2 \leq \mathsf{E}(u_\ell^{k-1,J}) - \mathsf{E}(u_\ell^{k,j}) \quad \text{for all } j \text{ with } j_{\mathsf{nrg}} \leq j$$

Uniform bound on algebraic solver steps: There exists $j_0 \in \mathbb{N}$ such that for all $J[\ell,k] \leq j_0$ for all $(\ell,k,0) \in \mathsf{Q}$

Main results

Unconditional full R-linear convergence: For any parameters $0 < \theta \leq 1$ and $0 < \lambda_{\mathsf{lin}}$, the quasi-error

$$\mathsf{H}_\ell^{k,j} := \underbrace{\eta_\ell(u_\ell^\star)}_{\text{discretization error}} + \underbrace{|||u_\ell^\star - u_\ell^{k,\star}|||}_{\text{linearization error}} + \underbrace{|||u_\ell^{k,\star} - u_\ell^{k,j}|||}_{\text{algebraic error}}$$

satisfies

$$\mathsf{H}_\ell^{k,j} \leq C_{\mathsf{lin}}\, q_{\mathsf{lin}}^{|\ell,k,j| - |\ell',k',j'|}\, \mathsf{H}_{\ell'}^{k',j'} \quad \text{for all } (\ell',k',j') \leq (\ell,k,j) \in \mathsf{Q}$$

Rate = complexity: Convergence rate s with respect to the degrees of freedom is equivalent to convergence rate s with respect to the computational cost

$$\sup_{(\ell,k,j)\in\mathsf{Q}} (\#\mathcal{T}_\ell)^s\, \mathsf{H}_\ell^{k,j} \simeq \sup_{(\ell,k,j)\in\mathsf{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathsf{Q} \\ |\ell',k',j'| \leq |\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \mathsf{H}_\ell^{k,j}$$

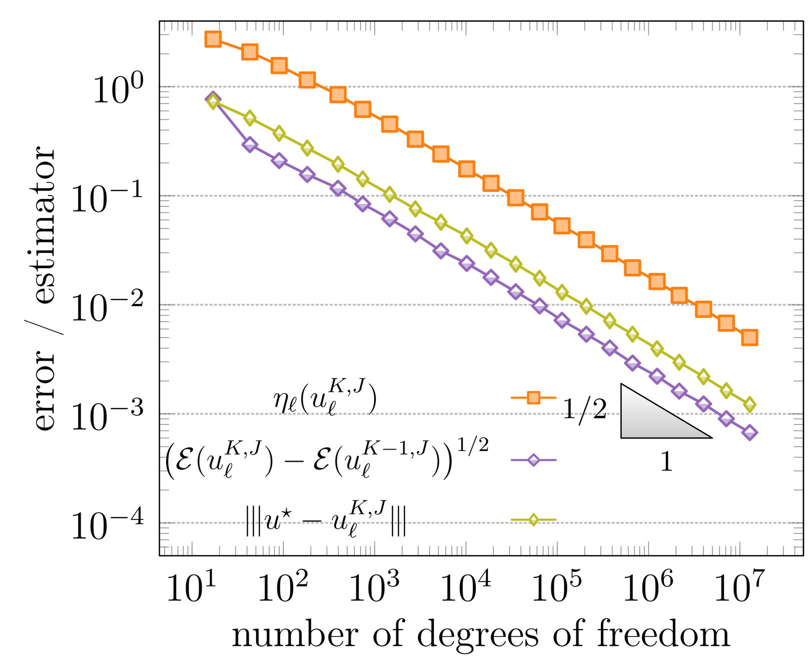
Cost-optimality: Optimal convergence rate s with respect to the computational cost for sufficiently small adaptivity parameters θ, λ_{lin}

$$\sup_{(\ell,k,j)\in\mathsf{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathsf{Q} \\ |\ell',k',j'| \leq |\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \mathsf{H}_\ell^{k,j} \lesssim \max\{\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)}, \mathsf{H}_0^{0,0}\}$$

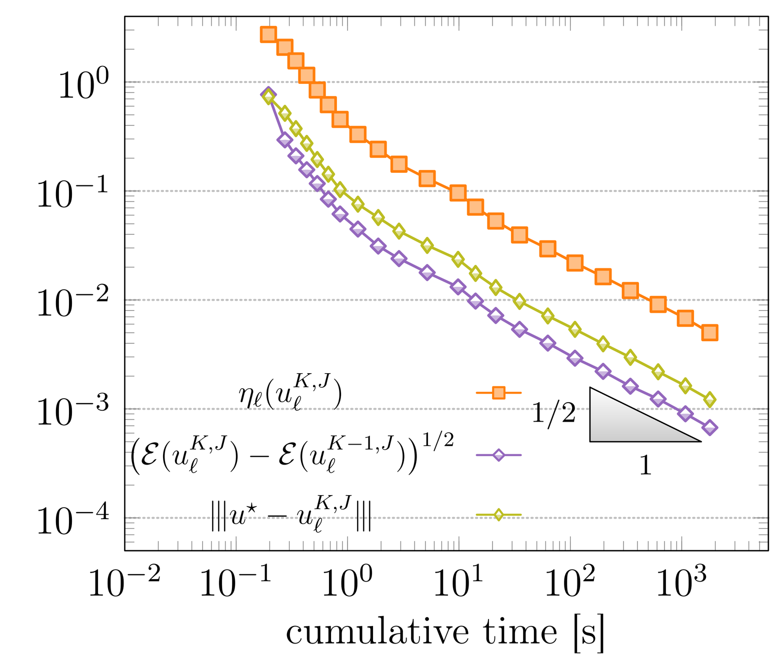
Numerical experiments

Lowest-order conforming FEM for the solution of

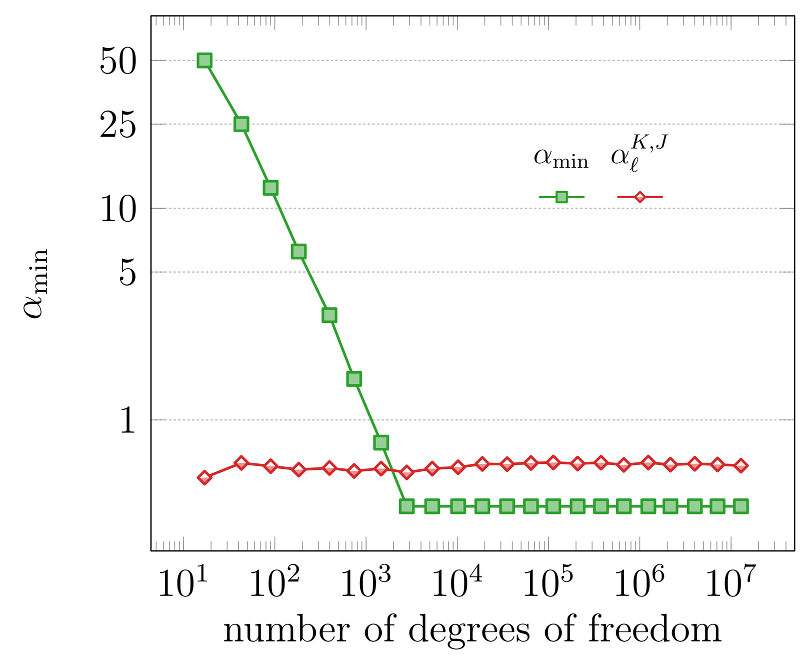
$$-\operatorname{div}\big(\nabla u^\star + \exp(-|\nabla u^\star|^2)\nabla u^\star\big) = 1 \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega$$



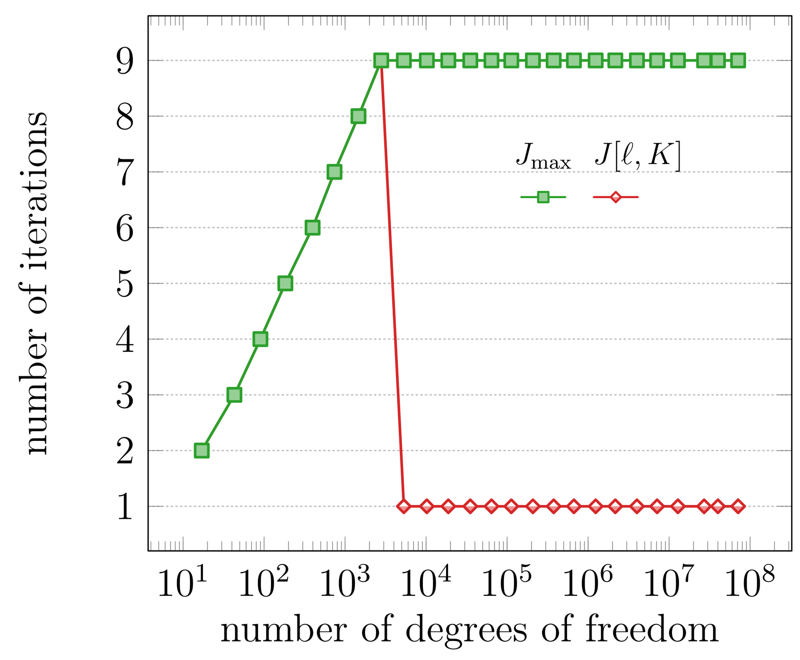
(a) Optimal rates



(b) Optimal complexity



(c) Norm-energy equivalence constant



(d) Uniform bounded iteration numbers

References

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Michael Innerberger, Ani Miraçi, Dirk Praetorius, Julian Streitberger: *hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs.* ESAIM: Math. Model. Numer. Anal. 58, 247–272 (2024)

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FWF Österreichischer Wissenschaftsfonds

SFB PDE Taming Complexity in Partial Differential Systems