AFEM for fractional PDEs

Adaptive Algorithm

- initial mesh $\mathcal{T}_0$

For all $\ell = 0, 1, 2, 3, \ldots$ iterate

1. SOLVE: compute discrete solution u_ℓ for mesh $\mathcal{T}_\ell$
2. ESTIMATE: compute some indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. MARK: mark a precentage of the largest indicators (Dörfler marking)
4. REFINE: refine (at least) all marked elements to obtain $\mathcal{T}_{\ell+1}$ (NVB)

Adaptivity for FL

question: which error estimator for $(-\Delta)^s$?

weighted residual error estimator

$$\eta_\ell(T) := h_\ell^s(T)\,\|(-\Delta)^s u_\ell - f\|_{L^2(T)}$$

- reliable: $\|u - u_\ell\|_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell \quad := \left(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2\right)^{1/2}$
- however: not well defined for all $s \in (0,1)$!
  - in general: $(-\Delta)^s u_\ell \notin L^2(\Omega)$ for $s \geq 3/4$

Adaptivity for FL

question: which error estimator for $(-\Delta)^s$?

weighted residual error estimator

$$\eta_\ell(T) := h_\ell^s(T)\, \|(-\Delta)^s u_\ell - f\|_{L^2(T)}$$

- reliable: $\|u - u_\ell\|_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell \quad := \Big(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2}$
- however: not well defined for all $s \in (0,1)$!
  - in general: $(-\Delta)^s u_\ell \notin L^2(\Omega)$ for $s \geq 3/4$

Adaptivity for FL

question: which error estimator for $(-\Delta)^s$?

weighted residual error estimator

$$\eta_\ell(T) := h_\ell^s(T)\,\|(-\Delta)^s u_\ell - f\|_{L^2(T)}$$

- reliable: $\|u - u_\ell\|_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell \quad := \Big(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2}$
- however: not well defined for all $s \in (0,1)$!
  - in general: $(-\Delta)^s u_\ell \notin L^2(\Omega)$ for $s \geq 3/4$

Adaptivity for FL

question: which error estimator for $(-\Delta)^s$?

weighted residual error estimator

$$\eta_\ell(T) := h_\ell^s(T)\,\|(-\Delta)^s u_\ell - f\|_{L^2(T)}$$

- reliable: $\|u - u_\ell\|_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell \quad := \Big(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2}$
- however: not well defined for all $s \in (0,1)$!
  - in general: $(-\Delta)^s u_\ell \notin L^2(\Omega)$ for $s \geq 3/4$

Adaptivity for FL

question: which error estimator for $(-\Delta)^s$?

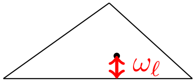
weighted residual error estimator

$$\eta_\ell(T) := h_\ell^s(T)\,\|(-\Delta)^s u_\ell - f\|_{L^2(T)}$$

- reliable: $\|u - u_\ell\|_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell \quad := \Big(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2\Big)^{1/2}$
- however: not well defined for all $s \in (0,1)$!
  - in general: $(-\Delta)^s u_\ell \notin L^2(\Omega)$ for $s \geq 3/4$

Modified Error Estimator

- blow-up at skeleton can be measured by powers of distance function
- $\omega_\ell \in L^\infty(\Omega)$ ... local distance function, $\omega_\ell|_T(x) := \mathrm{dist}(x, \partial T)$



Lemma

For any $\beta > 2s - 3/2$:

$$\omega_\ell^\beta (-\Delta)^s u_\ell \in L^2(\Omega)$$

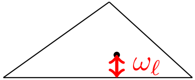
- case: $1/2 < s < 1$, choose $\beta = s - 1/2$, modified error estimator with $\widetilde{h}_\ell^s := h_\ell^{s-\beta} \omega_\ell^\beta$

$$\eta_\ell(T) := \left\| \widetilde{h}_\ell^s ((-\Delta)^s u_\ell - f) \right\|_{L^2(T)}$$

- well-defined, reliable $\|u - u_\ell\|^2_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell^2 := \sum_{T \in \mathcal{T}_h} \eta_\ell(T)^2$

Modified Error Estimator

- blow-up at skeleton can be measured by powers of distance function
- $\omega_\ell \in L^\infty(\Omega)$ … local distance function, $\omega_\ell|_T(x) := \mathrm{dist}(x, \partial T)$



Lemma

For any $\beta > 2s - 3/2$:

$$\omega_\ell^\beta (-\Delta)^s u_\ell \in L^2(\Omega)$$

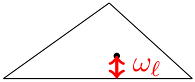
- case: $1/2 < s < 1$, choose $\beta = s - 1/2$, modified error estimator with $\widetilde{h}_\ell^s := h_\ell^{s-\beta}\omega_\ell^\beta$

$$\eta_\ell(T) := \left\| \widetilde{h}_\ell^s((-\Delta)^s u_\ell - f) \right\|_{L^2(T)}$$

- well-defined, reliable $\|u - u_\ell\|^2_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell^2 := \sum_{T \in \mathcal{T}_h} \eta_\ell(T)^2$

Modified Error Estimator

- blow-up at skeleton can be measured by powers of distance function
- $\omega_\ell \in L^\infty(\Omega)$ ... local distance function, $\omega_\ell|_T(x) := \mathrm{dist}(x, \partial T)$



Lemma

For any $\beta > 2s - 3/2$:

$$\omega_\ell^\beta (-\Delta)^s u_\ell \in L^2(\Omega)$$

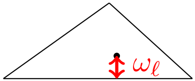
- case: $1/2 < s < 1$, choose $\beta = s - 1/2$, modified error estimator with $\widetilde{h}_\ell^s := h_\ell^{s-\beta} \omega_\ell^\beta$

$$\eta_\ell(T) := \left\| \widetilde{h}_\ell^s ((-\Delta)^s u_\ell - f) \right\|_{L^2(T)}$$

- well-defined, reliable $\|u - u_\ell\|^2_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell^2 := \sum_{T \in \mathcal{T}_h} \eta_\ell(T)^2$

Modified Error Estimator

- blow-up at skeleton can be measured by powers of distance function
- $\omega_\ell \in L^\infty(\Omega)$ ... local distance function, $\omega_\ell|_T(x) := \mathrm{dist}(x, \partial T)$



Lemma

For any $\beta > 2s - 3/2$:

$$\omega_\ell^\beta (-\Delta)^s u_\ell \in L^2(\Omega)$$

- case: $1/2 < s < 1$, choose $\beta = s - 1/2$, modified error estimator with $\widetilde{h}_\ell^s := h_\ell^{s-\beta} \omega_\ell^\beta$

$$\eta_\ell(T) := \left\| \widetilde{h}_\ell^s ((-\Delta)^s u_\ell - f) \right\|_{L^2(T)}$$

- well-defined, reliable $\|u - u_\ell\|^2_{\widetilde{H}^s(\Omega)} \lesssim \eta_\ell^2 := \sum_{T \in \mathcal{T}_h} \eta_\ell(T)^2$

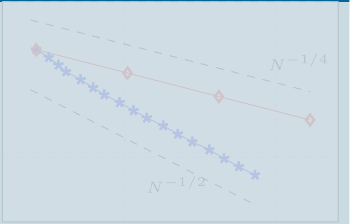
Main Theorem on h-Adaptive Algorithms

Theorem

$\Longrightarrow$ $\exists q \in (0,1) C > 0 : \forall \ell, n \geq 0 \quad \eta_{\ell+n} \leq C\, q^n\, \eta_\ell$

- $\mathcal{T}_N := \{\mathcal{T} \in \mathtt{refine}(\mathcal{T}_0) \ : \ \#\mathcal{T} \leq N\} \cup \{\mathcal{T}_0\}$
- $r > 0$ *arbitrary, marking parameter suff. small*

$\Longrightarrow$ $\sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^r \eta_\ell \simeq \sup_{N>0} \big(N^r \min_{\mathcal{T}_{\text{opt}} \in \mathcal{T}_N} \eta_{\text{opt}} \big)$



$\Longrightarrow$ if problem allows rate N^{-r}, AFEM reproduces rate

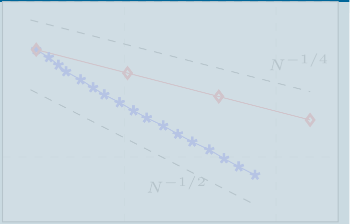
Main Theorem on h-Adaptive Algorithms

Theorem

$\Longrightarrow \quad \exists q \in (0,1)\, C > 0 : \forall \ell, n \geq 0 \quad \eta_{\ell+n} \leq C\, q^n\, \eta_\ell$

- $\mathcal{T}_N := \{\mathcal{T} \in \mathtt{refine}(\mathcal{T}_0) \ : \ \#\mathcal{T} \leq N\} \cup \{\mathcal{T}_0\}$
- $r > 0$ *arbitrary, marking parameter suff. small*

$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^r \eta_\ell \simeq \sup_{N>0} \left(N^r \min_{\mathcal{T}_{\text{opt}} \in \mathcal{T}_N} \eta_{\text{opt}}\right)$



$\Longrightarrow$ if problem allows rate N^{-r}, AFEM reproduces rate

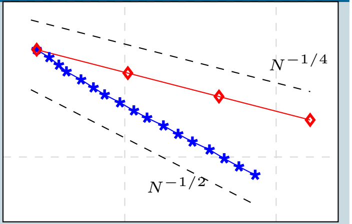
Main Theorem on h-Adaptive Algorithms

Theorem

$\Longrightarrow$ $\exists q \in (0,1) C > 0 : \forall \ell, n \geq 0 \quad \eta_{\ell+n} \leq C\, q^n\, \eta_\ell$

- $\mathcal{T}_N := \{\mathcal{T} \in \mathtt{refine}(\mathcal{T}_0) \ : \ \#\mathcal{T} \leq N\} \cup \{\mathcal{T}_0\}$
- $r > 0$ *arbitrary, marking parameter suff. small*

$\Longrightarrow$ $\sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^r \eta_\ell \simeq \sup_{N>0} \left(N^r \min_{\mathcal{T}_{\text{opt}} \in \mathcal{T}_N} \eta_{\text{opt}} \right)$



$\Longrightarrow$ if problem allows rate N^{-r}, AFEM reproduces rate

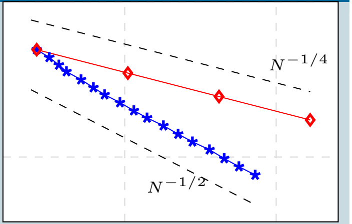
Main Theorem on h-Adaptive Algorithms

Theorem

$\Longrightarrow \quad \exists q \in (0,1) C > 0 : \forall \ell, n \geq 0 \quad \eta_{\ell+n} \leq C\, q^n\, \eta_\ell$

- $\mathcal{T}_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \ : \ \#\mathcal{T} \leq N\} \cup \{\mathcal{T}_0\}$
- $r > 0$ *arbitrary, marking parameter suff. small*

$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^r \eta_\ell \simeq \sup_{N>0} \Big(N^r \min_{\mathcal{T}_{\text{opt}} \in \mathcal{T}_N} \eta_{\text{opt}}\Big)$



$\Longrightarrow$ if problem allows rate N^{-r}, AFEM reproduces rate

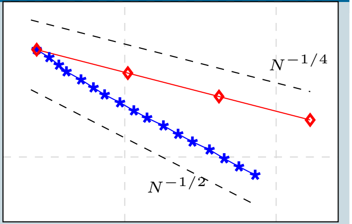
Main Theorem on h-Adaptive Algorithms

Theorem

$\Longrightarrow$ $\exists q \in (0,1) C > 0 : \forall \ell, n \geq 0 \quad \eta_{\ell+n} \leq C\, q^n\, \eta_\ell$

- $\mathcal{T}_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \ : \ \#\mathcal{T} \leq N\} \cup \{\mathcal{T}_0\}$
- $r > 0$ *arbitrary, marking parameter suff. small*

$\Longrightarrow$ $\sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^r \eta_\ell \simeq \sup_{N>0} \left(N^r \min_{\mathcal{T}_{\text{opt}} \in \mathcal{T}_N} \eta_{\text{opt}}\right)$



$\Longrightarrow$ if problem allows rate N^{-r}, AFEM reproduces rate

How to Show Optimality?

- abstract framework - *axioms of adaptivity*
- 4 properties (A1)–(A4) of estimator η_ℓ guarantee
  - (linear) convergence
  - optimal algebraic convergence behavior
- (A1): Stability, (A2): Reduction
- (A3): Discrete Reliability, (A4): Quasi-orthogonality
- model problem, discretization enter only through proof of axioms

How to Show Optimality?

- abstract framework - *axioms of adaptivity*
- 4 properties (A1)–(A4) of estimator η_ℓ guarantee
  - (linear) convergence
  - optimal algebraic convergence behavior
- (A1): Stability, (A2): Reduction
- (A3): Discrete Reliability, (A4): Quasi-orthogonality
- model problem, discretization enter only through proof of axioms

How to Show Optimality?

- abstract framework - *axioms of adaptivity*
- 4 properties (A1)–(A4) of estimator η_ℓ guarantee
  - (linear) convergence
  - optimal algebraic convergence behavior
- (A1): Stability, (A2): Reduction
- (A3): Discrete Reliability, (A4): Quasi-orthogonality
- model problem, discretization enter only through proof of axioms

How to Show Optimality?

- abstract framework - *axioms of adaptivity*
- 4 properties (A1)–(A4) of estimator η_ℓ guarantee
  - (linear) convergence
  - optimal algebraic convergence behavior
- (A1): Stability, (A2): Reduction
- (A3): Discrete Reliability, (A4): Quasi-orthogonality
- model problem, discretization enter only through proof of axioms

How to Show Optimality?

- abstract framework - *axioms of adaptivity*
- 4 properties (A1)–(A4) of estimator η_ℓ guarantee
  - (linear) convergence
  - optimal algebraic convergence behavior
- (A1): Stability, (A2): Reduction
- (A3): Discrete Reliability, (A4): Quasi-orthogonality
- model problem, discretization enter only through proof of axioms

How to Show Optimality?

- abstract framework - *axioms of adaptivity*
- 4 properties (A1)–(A4) of estimator η_ℓ guarantee
  - (linear) convergence
  - optimal algebraic convergence behavior
- (A1): Stability, (A2): Reduction
- (A3): Discrete Reliability, (A4): Quasi-orthogonality
- model problem, discretization enter only through proof of axioms

Properties of Estimator - (A1)

(A1) Stability

- $\forall \mathcal{T}_\ell \in \texttt{refine}(\mathcal{T}_0), \mathcal{U}_\ell \subset \mathcal{T}_\ell$
- $v_\ell, w_\ell \in \mathcal{P}_0^1(\mathcal{T}_\ell)$

$$\left| \Big(\sum_{T \in \mathcal{U}_\ell} \eta_\ell(T, v_\ell)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{U}_\ell} \eta_\ell(T, w_\ell)^2 \Big)^{1/2} \right| \leq C \left\| v_\ell - w_\ell \right\|_{\widetilde{H}^s(\Omega)}$$

- triangle inequality

$$\left| \left\| \widetilde{h}_\ell^s((-\Delta)^s v_\ell - f) \right\|_{L^2(\Omega)} - \left\| \widetilde{h}_\ell^s((-\Delta)^s (w_\ell) - f) \right\|_{L^2(\Omega)} \right| \leq \left\| \widetilde{h}_\ell^s (-\Delta)^s (v_\ell - w_\ell) \right\|_{L^2(\Omega)}$$

- inverse estimate

$$\left\| \widetilde{h}_\ell^s (-\Delta)^s (v_\ell - w_\ell) \right\|_{L^2(\Omega)} \leq C \left\| v_\ell - w_\ell \right\|_{\widetilde{H}^s(\Omega)}$$

Properties of Estimator - (A1)

(A1) Stability

- $\forall \mathcal{T}_\ell \in \texttt{refine}(\mathcal{T}_0), \mathcal{U}_\ell \subset \mathcal{T}_\ell$
- $v_\ell, w_\ell \in \mathcal{P}^1_0(\mathcal{T}_\ell)$

$$\left| \Big(\sum_{T \in \mathcal{U}_\ell} \eta_\ell(T, v_\ell)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{U}_\ell} \eta_\ell(T, w_\ell)^2 \Big)^{1/2} \right| \le C \, \| v_\ell - w_\ell \|_{\widetilde{H}^s(\Omega)}$$

- triangle inequality

$$\left| \left\| \widetilde{h}^s_\ell ((-\Delta)^s v_\ell - f) \right\|_{L^2(\Omega)} - \left\| \widetilde{h}^s_\ell ((-\Delta)^s (w_\ell) - f) \right\|_{L^2(\Omega)} \right| \le \left\| \widetilde{h}^s_\ell (-\Delta)^s (v_\ell - w_\ell) \right\|_{L^2(\Omega)}$$

- inverse estimate

$$\left\| \widetilde{h}^s_\ell (-\Delta)^s (v_\ell - w_\ell) \right\|_{L^2(\Omega)} \le C \, \| v_\ell - w_\ell \|_{\widetilde{H}^s(\Omega)}$$

Properties of Estimator - (A1)

(A1) Stability

- $\forall \mathcal{T}_\ell \in \texttt{refine}(\mathcal{T}_0), \mathcal{U}_\ell \subset \mathcal{T}_\ell$
- $v_\ell, w_\ell \in \mathcal{P}^1_0(\mathcal{T}_\ell)$

$$\left| \Big(\sum_{T \in \mathcal{U}_\ell} \eta_\ell(T, v_\ell)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{U}_\ell} \eta_\ell(T, w_\ell)^2 \Big)^{1/2} \right| \leq C \left\| v_\ell - w_\ell \right\|_{\widetilde{H}^s(\Omega)}$$

- triangle inequality

$$\left| \left\| \widetilde{h}_\ell^s((-\Delta)^s v_\ell - f) \right\|_{L^2(\Omega)} - \left\| \widetilde{h}_\ell^s((-\Delta)^s (w_\ell) - f) \right\|_{L^2(\Omega)} \right| \leq \left\| \widetilde{h}_\ell^s (-\Delta)^s (v_\ell - w_\ell) \right\|_{L^2(\Omega)}$$

- inverse estimate

$$\left\| \widetilde{h}_\ell^s (-\Delta)^s (v_\ell - w_\ell) \right\|_{L^2(\Omega)} \leq C \left\| v_\ell - w_\ell \right\|_{\widetilde{H}^s(\Omega)}$$

Crucial Step - Inverse Estimate

- local inverse estimate for non-local operator on locally refined mesh
- bound stronger norm by weaker norm (finite dimensional)

Lemma

- $v_\ell \in \mathcal{P}^1_0(\mathcal{T}_\ell)$

Then:

$$\left\| \widetilde{h}_\ell^s (-\Delta)^s v_\ell \right\|_{L^2(\Omega)} \leq C \left\| v_\ell \right\|_{\widetilde{H}^s(\Omega)}$$

Other Properties of Estimator - (A2),(A3),(A4)

follow essentially with standard techniques

- (A2) Reduction: properties of bisection

$$\sum_{T\in\mathcal{T}_{\ell+1}\setminus\mathcal{T}_\ell} \eta_{\ell+1}(T,v_\ell)^2 \le q_{\mathrm{red}} \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}} \eta_{\ell+1}(T,v_\ell)^2$$

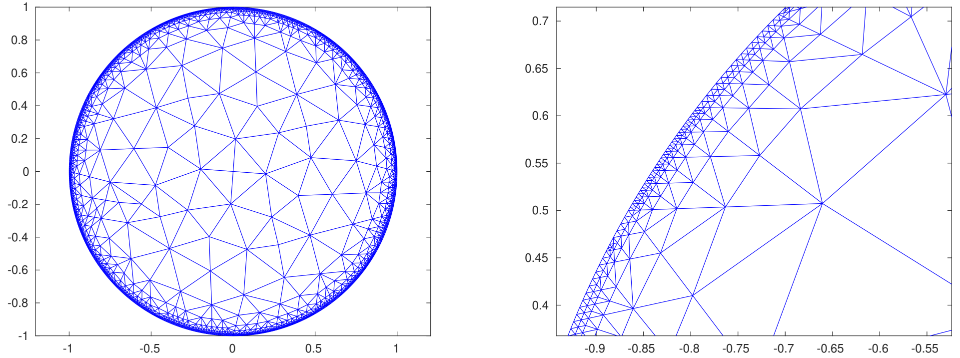
- (A3) Disc. reliability: Galerkin orthogonality, Scott-Zhang projection, scaling arg.

$$\|v_{\ell+1}-v_\ell\|^2_{\widetilde{H}^s(\Omega)} \le C_{\mathrm{rel}} \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}} \eta_\ell(T,v_\ell)^2$$

- (A4) Quasi-ortho. from ellipticity, symmetry of bilinear form

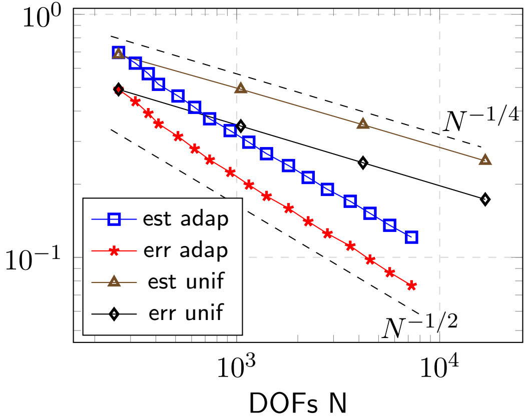
Unit Circle, f Constant

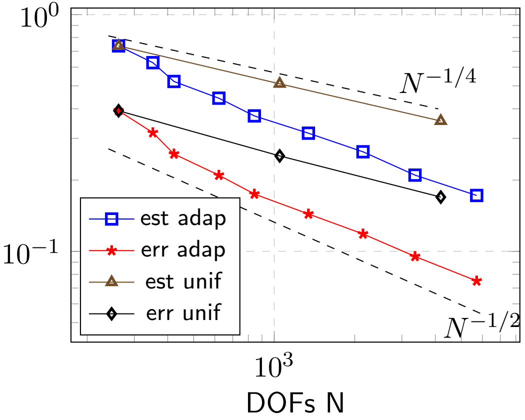
- unit circle $B_1(0) \subset \mathbb{R}^2$, f constant
- exact solution known $u(x) = \left(1 - |x|^2\right)_+^s$



Unit Circle, f Constant

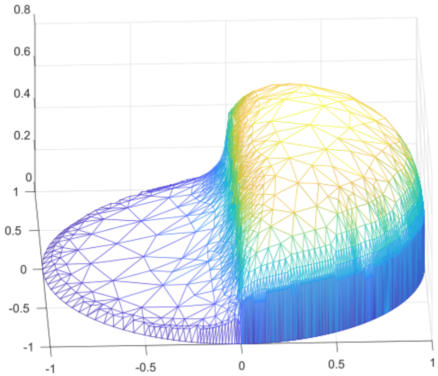
- unit circle $B_1(0) \subset \mathbb{R}^2$, f constant
- left: $s = 0.25$; right: $s = 0.75$

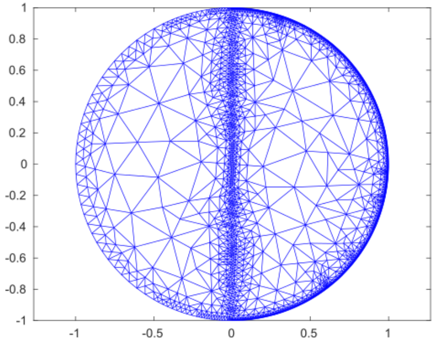




Unit Circle, f Discontinuous

- unit circle $B_1(0) \subset \mathbb{R}^2$, $f = \chi_{|\varphi|<\pi/2}$ (polar coordinates)
- exact solution known, energy computable





Unit Circle, f Discontinuous

- unit circle $B_1(0) \subset \mathbb{R}^2$, $f = \chi_{|\varphi|<\pi/2}$ (polar coordinates)
- left: $s = 0.25$; right: $s = 0.75$