FLUID RELAXATION APPROXIMATION OF THE BUSENBERG–TRAVIS CROSS-DIFFUSION SYSTEM

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Abstract. The Busenberg–Travis cross-diffusion system for segregating populations is approximated by the compressible Navier–Stokes–Korteweg equations on the torus, including a density-dependent viscosity and drag forces. The Korteweg term can be associated to the quantum Bohm potential. The singular asymptotic limit is proved rigorously using compactness and relative entropy methods. The novelty is the derivation of energy and entropy inequalities, which reduce in the asymptotic limit to the Boltzmann–Shannon and Rao entropy inequalities, thus revealing the double entropy structure of the limiting Busenberg–Travis system.

1. Introduction

The aim of this paper is to analyze a fluiddynamical approximation of the Busenberg–Travis population cross-diffusion system [7]

$$\partial_t \rho_i - \operatorname{div}\big(k_i\rho_i\nabla(\rho_1+\rho_2)\big) = 0 \quad \text{in } \mathbb{T}^d,\ t>0,\ i=1,2, \tag{1}$$

where ρ_i is the density of the ith population species, $k_i > 0$ is a diffusion coefficient, $\mathbb{T}^d$ is the multidimensional torus, and we impose the initial conditions $\rho_i(0) = \rho_i^0$ in $\mathbb{T}^d$ for $i = 1, 2$. The equations have been suggested by Busenberg and Travis [7] to describe the segregation of populations, see also [4, 21]. They have also been proposed, based on interacting particle systems, to introduce short-range repulsion in cell-cell adhesion models [12, 33].

1.1. Motivation and model setting.

Our motivation for an approximation of (1) is to recover the entropy structure of (1) from the energy and entropy of the approximating

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fluid system. Indeed, it has been shown in [28, 29] that system (1) possesses *two* entropies, the Boltzmann–Shannon entropy H_1 and the Rao entropy H_2,

$$H_1 = \sum_{i=1}^{2} \int_{\mathbb{T}^d} k_i^{-1}\rho_i(\log\rho_i - 1)dx, \quad H_2 = \frac{1}{2}\int_{\mathbb{T}^d}(\rho_1+\rho_2)^2 dx.$$

This means that formally, along solutions to (1),

$$\frac{dH_1}{dt} + \int_{\mathbb{T}^d}|\nabla(\rho_1+\rho_2)|^2 dx = 0,$$
$$\frac{dH_2}{dt} + \int_{\mathbb{T}^d}(k_1\rho_1+k_2\rho_2)|\nabla(\rho_1+\rho_2)|^2 dx = 0.$$

However, the origin of this double entropy structure remained unclear. We propose a fluiddynamical approximation that possesses the thermodynamical entropy H_1 and an energy containing H_2. Thus, the entropy structure of (1) originates from the energy and entropy of the associated fluiddynamical system.

Before we make this statement precise, we comment on the cross-diffusion system (1). The diffusion matrix associated to (1) has rank one, such that this system is of mixed hyperbolic–parabolic type. Indeed, we can reformulate (1) as a diffusion equation for the total density $\rho_1+\rho_2$ and a transport equation for one of the densities:

$$\partial_t(\rho_1+\rho_2) = \operatorname{div}\big((k_1\rho_1+k_2\rho_2)\nabla(\rho_1+\rho_2)\big), \tag{2}$$
$$\partial_t\rho_i + \operatorname{div}(\rho_i\bar{u}_i) = 0 \quad \text{in } \mathbb{T}^d,\ t>0,\ i=1,2, \tag{3}$$

where $\bar{u}_i = -k_i\nabla(\rho_1+\rho_2)$ is the velocity associated to the ith species. Observe that the equations decouple if $k := k_1 = k_2$; then $\rho_1+\rho_2$ solves a porous-medium equation, and the densities ρ_i are transported with the common velocity $\bar{u} = -k\nabla(\rho_1+\rho_2)$. We refer to [18, 19] for details on the hyperbolic–parabolic structure.

A fluiddynamical approximation of (1) takes the form

$$\begin{aligned} &\partial_t\rho_i + \operatorname{div}(\rho_i u_i) = 0, \quad i=1,2, \\ &\varepsilon\partial_t(\rho_i u_i) + \varepsilon\operatorname{div}(\rho_i u_i\otimes u_i) = \varepsilon\operatorname{div} S - k_i^{-1}\rho_i u_i - \rho_i\nabla(\rho_1+\rho_2), \end{aligned} \tag{4}$$

where u_i is the partial velocity of the ith species, $\varepsilon>0$ is a small number, S is the stress tensor, $-k_i^{-1}\rho_i u_i$ is the relaxation term, and $-\rho_i\nabla(\rho_1+\rho_2)$ is a force term. The formal limit $\varepsilon\to 0$ in (4) leads to (2)–(3). As this limit is singular, its rigorous proof is delicate.

The main difficulty comes from the force term $-\rho_i\nabla(\rho_1+\rho_2)$, since the energy of the fluiddynamical equations does not provide any gradient estimate. This issue does not occur in the relaxation-time limit of the Euler–Poisson equations, since the force reads as $-\rho_i\nabla\Phi$, and the electric potential Φ solves the Poisson equation, thus providing sufficient regularity to apply the div–curl lemma [27]. The lack of a gradient bound can be overcome by allowing for a Korteweg term in (4), leading to Euler–Korteweg equations [32] or Navier–Stokes–Korteweg equations [8]. More precisely, we add to the right-hand side of (4) the expression

$$K = \varepsilon\rho_i\nabla\bigg(\kappa(\rho_i)\Delta\rho_i + \frac{1}{2}\kappa'(\rho_i)|\nabla\rho_i|^2\bigg),$$

where $\kappa(\rho_i)$ is the capillarity coefficient. In this paper, we choose $\kappa(\rho_i) = 1/(2\rho_i)$, which leads to

$$K = \varepsilon\rho_i\nabla\bigg(\frac{\Delta\sqrt{\rho_i}}{\sqrt{\rho_i}}\bigg).$$

The expression $\Delta\sqrt{\rho_i}/\sqrt{\rho_i}$ is known in quantum mechanics as the Bohm potential, and equations (4) become the quantum Navier–Stokes equations studied in, e.g., [3, 24, 31, 35]. Other choices for $\kappa(\rho_i)$ are discussed in Remark 4. As in [24, 35], we use the density-dependent stress tensor $S = \rho_i\nabla u_i$. This dependence is needed in the derivation of the entropy inequality.

A second difficulty is due to the fact that we control the kinetic energy $\rho_i|u_i|^2$ in $L^1(\mathbb{T}^d)$ only. This is not sufficient to prevent concentration phenomena in the convective term $\rho_i u_i\otimes u_i$. A way out is the introduction of additional drag forces like in [6, Sec. 9] and used in the context of the quantum Navier–Stokes equations in [35].

Summarizing, we perform the asymptotic limit $\varepsilon\to 0$ in the compressible Navier–Stokes equations with Korteweg regularization and drag forces:

$$\partial_t\rho_i + \operatorname{div}(\rho_i u_i) = 0 \quad \text{in } \mathbb{T}^d,\ t>0,\ i=1,2, \tag{5}$$

$$\begin{aligned}\varepsilon\partial_t(\rho_i u_i) + \varepsilon\operatorname{div}(\rho_i u_i\otimes u_i) &= \varepsilon\rho_i\nabla\bigg(\frac{\Delta\sqrt{\rho_i}}{\sqrt{\rho_i}}\bigg) + \varepsilon\operatorname{div}(\rho_i\nabla u_i) \\ &\quad - \varepsilon u_i - \varepsilon\rho_i u_i|u_i|^2 - k_i^{-1}\rho_i u_i - \rho_i\nabla(\rho_1+\rho_2),\end{aligned} \tag{6}$$

subject to the initial conditions

$$\rho_i(0) = \rho_i^0, \quad \rho_i(0)u_i(0) = \rho_i^0 u_i^0 \quad \text{in } \mathbb{T}^d,\ i=1,2. \tag{7}$$

In the following, we set $\rho = (\rho_1,\rho_2)$ and $u = (u_1,u_2)$.

1.2. **Key ideas.** A priori estimates are derived by estimating the energy

$$E(\rho,u) = \int_{\mathbb{T}^d}\bigg(\frac{1}{2}(\rho_1+\rho_2)^2 + \frac{\varepsilon}{2}\sum_{i=1}^2\rho_i|u_i|^2 + \varepsilon\sum_{i=1}^2|\nabla\sqrt{\rho_i}|^2\bigg)dx, \tag{8}$$

which is the sum of the potential, kinetic, and Korteweg energies. A formal computation, made rigorous on the approximate level in Section 2.2, shows that

$$\frac{dE}{dt} + \sum_{i=1}^2\int_{\mathbb{T}^d}\big(k_i^{-1}\rho_i|u_i|^2 + \varepsilon\rho_i|\nabla u_i|^2 + \varepsilon|u_i|^2 + \varepsilon\rho_i|u_i|^4\big)dx = 0.$$

Unfortunately, this equality does not provide any gradient bound for the densities independent of ε. Our main idea is to obtain such a bound from the entropy

$$H(\rho) = \sum_{i=1}^{2} k_i^{-1} \int_{\mathbb{T}^d} \rho_i(\log \rho_i - 1)dx. \tag{9}$$

A formal computation (made rigorous for the approximate solutions in Section 2.3) shows that

$$\frac{dH}{dt} + \int_{\mathbb{T}^d} |\nabla(\rho_1 + \rho_2)|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{T}^d} \rho_i |D^2 \log \rho_i|^2 dx \le R,$$

where the remainder R depends on the unknowns ρ_i, u_i and their derivatives (like $\sqrt{\rho_i}\nabla u_i$), but it is independent of ε. The last integral on the left-hand side gives (ε-dependent) estimates in $H^2(\mathbb{T}^d)$ and $W^{1,4}(\mathbb{T}^d)$ from the inequality

$$\int_{\mathbb{T}^d} \rho_i |D^2 \log \rho_i|^2 dx \ge c(d) \int_{\mathbb{T}^d} \big(|\Delta\sqrt{\rho_i}|^2 + |\nabla \sqrt[4]{\rho_i}|^4\big) dx \tag{10}$$

for some $c(d) > 0$, which holds for sufficiently smooth functions ρ_i; see [26, Lemma 2.2] and [24, Appendix] (or [35, Lemma 2.1]) for a proof. Then the remainder R can be controlled by the bounds coming from the energy and (10). Since the limiting system (1) may possess discontinuous solutions [5], we cannot expect gradient bounds for the individual densities ρ_i but only for the sum $\rho_1 + \rho_2$. Thus, we cannot expect better estimates.

Denoting by $(\rho^\varepsilon, u^\varepsilon)$ a weak solution to (5)–(7), the energy and entropy estimates together with the Aubin–Lions lemma yield strong convergence of the sum $\bar\rho^\varepsilon := \rho_1^\varepsilon + \rho_2^\varepsilon$, but we have only weak convergence for ρ_i^ε and $\nabla\bar\rho^\varepsilon$. Thus, the limit in the product $\rho_i^\varepsilon \nabla\bar\rho^\varepsilon$ cannot be easily identified.

We show two results. First, we prove that the strong limit $\bar\rho$ of $\bar\rho^\varepsilon$ solves (2) with a "defect",

$$\partial_t\bar\rho - \operatorname{div}\big((k_1\rho_1 + k_2\rho_2)\nabla\bar\rho\big) = (k_2 - k_1)(k_2^{-1}J_2 + \rho_2\nabla\bar\rho)$$

where ρ_i is the weak limit of ρ_i^ε and J_2 is the weak limit of $\rho_2^\varepsilon u_2^\varepsilon$ (see Theorem 2). The right-hand side vanishes if $k_1 = k_2$ or if we can identify J_2 with $-k_2\rho_2\nabla\bar\rho$. Unfortunately, we have not been able to prove this identification for general values of k_1, k_2. Indeed, neither the div–curl lemma nor Feireisl's viscous flux approach can be applied because of the lack of suitable gradient bounds uniform in ε.

Second, we consider the case $k_1 = k_2 = 1$. Then $\bar\rho$ solves the quadratic porous-medium equation (see (2)), but we still have no information about the evolution of ρ_i. To derive the dynamics, we apply the relative entropy method. The idea is to compare a solution $(\rho^\varepsilon, u^\varepsilon)$ to (5)–(6) with a solution $(\bar\rho, \bar u)$ to the limit system (2)–(3) with $k_1 = k_2 = 1$ and $\bar u = -\nabla\bar\rho$. The relative entropy (more precisely: relative energy) is defined by

$$E_R(\rho^\varepsilon, u^\varepsilon|\bar\rho, \bar u) = \int_{\mathbb{T}^d} \left\{ \frac{1}{2}(\bar\rho^\varepsilon - \bar\rho)^2 + \varepsilon \sum_{i=1}^{2} \left(\frac{\rho_i^\varepsilon}{2}|u_i^\varepsilon - \bar u|^2 + |\nabla(\rho_i^\varepsilon)^{1/2}|^2 \right) \right\} dx.$$

A computation, made rigorous in Section 4, shows that

$$\frac{dE_R}{dt}(\rho^\varepsilon, u^\varepsilon|\bar\rho,\bar u) + \sum_{i=1}^2\int_{\mathbb{T}^d}\rho_i^\varepsilon|u_i^\varepsilon - \bar u|^2dx \le C\sqrt[4]{\varepsilon}.$$

Here, we need the assumption $k_1 = k_2$. We infer that $(\rho_i^\varepsilon)^{1/2}(u_i^\varepsilon - \bar u) \to 0$ strongly in $L^2(0,T;L^2(\mathbb{T}^d))$, which implies that $\rho_i^\varepsilon u_i^\varepsilon \rightharpoonup \rho_i\bar u$ weakly in $L^2(0,T;L^{4/3}(\mathbb{T}^d))$. This allows us to identify J_i with $\rho_i\bar u = -\rho_i\nabla\bar\rho$, and ρ_i solves the transport equation (3).

1.3. **State of the art.** There are many results in the literature on the relaxation limit in hyperbolic systems. General results can be found, for instance, in [17]. The relaxation limit in Euler–Poisson systems, leading to the drift-diffusion equations, was proved in [27], exploiting the regularizing effect of the Poisson equation. Relaxation-time limits were also performed in Euler–Maxwell [34] and quantum hydrodynamic equations [25]. Using compactness methods, relaxation limits in the compressible Navier–Stokes–Poisson equations [30] and in the quantum Navier–Stokes equations [3] were proved. Note that our limit is more delicate since the regularizing terms vanish in the limit.

The relative entropy method was first used by Dafermos [15] and Di Perna [16]. It was extended later by Lattanzio and Tzavaras [32] to compare the solution to the frictional Euler equations with the solution to the porous-medium equation. This technique was also applied in the analysis of the high-friction regime of Euler–Korteweg equations [23], for more general aggregation-diffusion equations [13], compressible Navier–Stokes–Korteweg systems [8], and Euler–Riesz models [1, 2].

The Busenberg–Travis system (1) can be derived from interacting particle systems in the mean-field limit, even for an arbitrary number of species [14] and for nonlinear pressures [10]. In the general case, the limiting system reads as

$$\partial_t u_i = \operatorname{div}(u_i\nabla p_i(u)), \quad p_i(u) = \sum_{j=1}^n a_{ij}u_j \quad \text{in } \mathbb{T}^d,\ i = 1,\ldots,n, \tag{11}$$

where $a_{ij} \ge 0$ are some numbers. If the matrix (a_{ij}) is positive definite, a global existence analysis can be found in [28]. If the matrix (a_{ij}) is of rank one, i.e. $a_{ij} = k_i$ like in our situation, the existence of global classical solutions to (1) (under the positivity assumption $\sum_{i=1}^n \rho_i^0 \ge c > 0$) was proved in [19]. The positivity ensures the regularity of the total density. If the positivity assumption is relaxed to nonnegativity, the existence of global measure-valued solutions can be shown [22]. Steady states may be discontinuous [11], and there is some numerical evidence [5] that this may be also true for transient solutions. The lack of regularity motivated the authors of [9] to work in the one-dimensional setting with solutions of bounded variation.

The limit in (5)–(6) seems to be new, and we believe that it contributes to the understanding of the entropy structure of the Busenberg–Travis cross-diffusion system and possibly of related models.

1.4. **Main results.** We first define our notion of weak solution. This is necessary since the Korteweg term in (6) needs special care.

Definition 1. *We call (ρ, u) with $\rho = (\rho_1, \rho_2)$ and $u = (u_1, u_2)$ a* weak solution *to* (5)–(7) *on* $(0, T)$ *if, for* $i = 1, 2$, $\rho_i \ge 0$ *in* $\mathbb{T}^d \times (0, T)$,

$$\rho_i \in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad \sqrt{\rho_i}u_i \in L^\infty(0,T;L^2(\mathbb{T}^d)),$$
$$\sqrt{\rho_i} \in L^2(0,T;H^2(\mathbb{T}^d)), \quad \rho_1+\rho_2 \in L^2(0,T;H^1(\mathbb{T}^d)),$$
$$\sqrt{\rho_i}|\nabla u_i| \in L^2(0,T;L^2(\mathbb{T}^d)), \quad \sqrt[4]{\rho_i}|u_i| \in L^4(0,T;L^4(\mathbb{T}^d)),$$

for all $\phi \in C_0^\infty(\mathbb{T}^d \times [0,T))$ *and* $\psi \in C_0^\infty(\mathbb{T}^d \times [0,T);\mathbb{R}^d)$,

$$0 = -\int_0^T\int_{\mathbb{T}^d}\rho_i\partial_t\phi dxdt - \int_{\mathbb{T}^d}\rho_i^0\phi(0)dx - \int_0^T\int_{\mathbb{T}^d}\rho_i u_i\cdot\nabla\phi dxdt, \tag{12}$$

$$\begin{aligned} 0 = &-\varepsilon\int_0^T\int_{\mathbb{T}^d}\rho_i u_i\cdot\partial_t\psi dxdt - \varepsilon\int_{\mathbb{T}^d}\rho_i^0 u_i^0\cdot\psi(0)dx - \varepsilon\int_0^T\int_{\mathbb{T}^d}\rho_i(u_i\otimes u_i):\nabla\psi dxdt \\ &+ \varepsilon\int_0^T\int_{\mathbb{T}^d}\rho_i\nabla u_i:\nabla\psi dxdt + \varepsilon\int_0^T\int_{\mathbb{T}^d}\Delta\sqrt{\rho_i}\big(2\nabla\sqrt{\rho_i}\cdot\psi + \sqrt{\rho_i}\operatorname{div}\psi\big)dxdt \\ &+ \varepsilon\int_0^T\int_{\mathbb{T}^d}(u_i+\rho_i|u_i|^2u_i)\cdot\psi dxdt + \int_0^T\int_{\mathbb{T}^d}\big(k_i^{-1}\rho_i u_i + \rho_i\nabla(\rho_1+\rho_2)\big)\cdot\psi dxdt, \end{aligned} \tag{13}$$

and the initial conditions (7) *hold in the sense of distributions.*

We impose the following assumptions:

- (A1) Parameter: $d = 1, 2, 3$, $T > 0$, and $k_1 > 0$, $k_2 > 0$.
- (A2) Initial data: $\rho_i^0 \in L^2(\mathbb{T}^d)$, $\sqrt{\rho_i^0}|u_i^0| \in L^2(\mathbb{T}^d)$, $\sqrt{\rho_i^0} \in H^1(\mathbb{T}^d)$, $\log\rho_i^0 \in L^1(\mathbb{T}^d)$ for $i = 1, 2$.

The assumption of at most $d = 3$ space dimensions is due to Sobolev embeddings (we need $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ in Lemma 10). Our analysis strongly depends on inequality (10). To avoid boundary integrals, we consider equations (5)–(6) on the torus. The regularity of the initial data is needed to obtain a finite initial energy and entropy. We may allow for reaction terms in (5) if the reactions depend nonlinearly on the total density $\rho_1 + \rho_2$ only (since we have strong convergence only for the sum). We discuss further generalizations of the nonlinearities in Remark 4.

Our first main result reads as follows.

Theorem 1 (Existence of solutions)**.** *Let Assumptions (A1)–(A2) hold. Then there exists a weak solution* (ρ, u) *to* (5)–(7) *with* $\rho = (\rho_1, \rho_2)$ *and* $u = (u_1, u_2)$ *such that*

$$\begin{aligned} E(\rho(t),u(t)) &+ \sum_{i=1}^2\int_0^t\int_{\mathbb{T}^d}k_i^{-1}\rho_i|u_i|^2dxds \\ &+ \varepsilon\sum_{i=1}^2\int_0^t\int_{\mathbb{T}^d}\big(\rho_i|\nabla u_i|^2 + |u_i|^2 + \rho_i|u_i|^4\big)dxds \le E(\rho^0,u^0), \end{aligned} \tag{14}$$

$$H(\rho(t)) + C_1(d)\varepsilon \int_0^t \int_{\mathbb{T}^d} \big(|\Delta\sqrt{\rho_i}|^2 + |\nabla\sqrt[4]{\rho_i}|^4\big)dxds \le C_2(\rho^0, u^0), \tag{15}$$

where $C_1(d) > 0$ is a constant only depending on the space dimension d, $C_2(\rho^0, u^0) > 0$ depends on the initial data (but not on ε), and we recall definitions (8) *and* (9) *of E and H. Moreover, we have the regularity*

$$\partial_t\rho_i \in L^2(0,T;L^2(\mathbb{T}^d)), \quad \partial_t(\rho_i u_i) \in L^{4/3}(0,T;H^s(\mathbb{T}^d)'), \quad s > d/2+1.$$

Therefore, the initial condition for ρ_i holds a.e. in $\mathbb{T}^d$, and the initial condition for $\rho_i u_i$ holds in $H^s(\mathbb{T}^d)'$.

The proof of the existence of solutions is based on an approximate scheme. We add the parabolic regularization $\delta\Delta\rho_i$ to the mass balance equation (5) and the regularization $\delta\varepsilon\,\mathrm{div}(u_i \otimes \nabla\rho_i)$ to the momentum balance equation (similarly as in the existence analysis for the compressible Navier–Stokes equations; see [20, Sec. 7.2]). The latter term is needed to compensate some contributions coming from the parabolic regularization when deriving the energy inequality. The local existence of solutions is shown by the Faedo–Galerkin method. The solution can be extended to a global one thanks to the energy estimate. The entropy production in (15) is obtained by applying inequality (10).

If $k_1 = k_2$, a computation similar to the proof of Lemma 8 shows that the entropy inequality (15) can be improved by replacing $C_2(\rho^0, u^0)$ by $H(\rho^0) + C\sqrt[4]{\varepsilon}$, where $C > 0$ is independent of ε.

Theorem 2 (Limit $\varepsilon \to 0$)**.** *Let $(\rho^\varepsilon, u^\varepsilon)$ be a weak solution to* (5)–(7) *as constructed in Theorem 1. Then there exists a subsequence (not relabeled) such that for $i = 1, 2$,*

$$\begin{aligned} \rho_i^\varepsilon &\rightharpoonup \rho_i \quad \textit{weakly* in } L^\infty(0,T;L^2(\mathbb{T}^d)), \\ \rho_1^\varepsilon + \rho_2^\varepsilon &\to \rho_1 + \rho_2 \quad \textit{strongly in } L^2(0,T;L^2(\mathbb{T}^d)), \\ \rho_i^\varepsilon u_i^\varepsilon &\rightharpoonup J_i \quad \textit{weakly in } L^2(0,T;L^{4/3}(\mathbb{T}^d)), \end{aligned}$$

and $\bar\rho := \rho_1 + \rho_2$ solves

$$\partial_t\bar\rho - \mathrm{div}\,\big((k_1\rho_1 + k_2\rho_2)\nabla\bar\rho)\big) = -(k_2 - k_1)\,\mathrm{div}(k_2^{-1}J_2 + \rho_2\nabla\bar\rho) \quad \textit{in } \mathbb{T}^d,\ t > 0,$$

with the initial condition $\bar\rho(0) = \rho_1^0 + \rho_2^0$ in the sense of $W^{1,4}(\mathbb{T}^d)'$.

The proof is based on the energy and entropy estimates proved in Theorem 1 and on compactness arguments. If $k_1 = k_2$, we can prove a stronger result, using the relative entropy method. Indeed, as explained in Section 1.2, we are able to prove that not only $\rho_i^\varepsilon u_i^\varepsilon \rightharpoonup J_i$ weakly in $L^2(0,T;L^{4/3}(\mathbb{T}^d))$ (which follows from the energy inequality) but also $\rho_i^\varepsilon u_i^\varepsilon \rightharpoonup \rho_i\bar u$ weakly in $L^1(0,T;L^1(\mathbb{T}^d))$ (which follows from the relative entropy inequality). This allows us to identify $J_i = \rho_i\bar u = -\rho_i\nabla\bar\rho$.

Theorem 3 (Limit $\varepsilon \to 0$, $k_1 = k_2$)**.** *Let $k_1 = k_2 = 1$, let $(\rho^\varepsilon, u^\varepsilon)$ be a weak solution to* (5)–(7) *as constructed in Theorem 1, and let $(\bar\rho, \bar u)$ be the unique smooth solution to*

$$\partial_t\bar\rho + \mathrm{div}(\bar\rho\bar u) = 0, \quad \bar u = -\nabla\bar\rho \quad \textit{in } \mathbb{T}^d,\ t > 0,$$

with initial conditions $\bar\rho(0)=\rho_1^0+\rho_2^0$ *(this requires smooth positive initial data). Then, as* $\varepsilon\to 0$,

$$\begin{aligned}\rho_1^\varepsilon+\rho_2^\varepsilon\to\bar\rho &\quad \textit{strongly in } L^2(0,T;L^2(\mathbb{T}^d)),\\ \rho_i^\varepsilon u_i^\varepsilon\rightharpoonup\rho_i\bar u &\quad \textit{weakly in } L^2(0,T;L^{4/3}(\mathbb{T}^d)).\end{aligned}$$

and the $L^2(\mathbb{T}^d)$*-weak limit* ρ_i *of* (ρ_i^ε) *solves the transport equation*

$$\partial_t\rho_i-\operatorname{div}(\rho_i\nabla(\rho_1+\rho_2))=0 \quad \textit{in } \mathbb{T}^d,\ t>0,\ i=1,2.$$

Remark 4 (Generalizations)**.** Our results can be extended in various directions. First, all results are valid for more than two species, and it is sufficient to replace the sum over $i=1,2$ by $i=1,\ldots,n$, where $n\in\mathbb{N}$ is arbitrary. Second, one may try more general Korteweg functions $\kappa(\rho_i)$. A simple choice is $\kappa(\rho_i)=1$, giving the higher-order regularization $K=\varepsilon\rho_i\nabla\Delta\rho_i$, which equals the flux of the thin-film equation. The Korteweg energy density becomes $\varepsilon|\nabla\rho_i|^2$ and the entropy production simplifies to $\varepsilon\int_{\mathbb{T}^d}(\Delta\rho_i)^2dx$, thus providing $H^2(\mathbb{T}^d)$ bounds for ρ_i. Since we do not need inequalities like (10) in this case, we may allow for bounded domains instead of the torus. However, one needs to check whether this regularization is sufficient to pass to the limit $\varepsilon\to 0$ in the regularizing terms, as such a choice does not provide gradient bounds for $\sqrt{\rho_i}$ or $\sqrt[4]{\rho_i}$. We leave this question to future works. Third, we can derive the generalized Busenberg–Travis system (11) if the matrix (a_{ij}) is symmetric and positive definite. This system is fully parabolic, which simplifies the asymptotic analysis $\varepsilon\to 0$. Indeed, we set in (6) $k_i=1$ and replace the term $-\rho_i\nabla(\rho_1+\rho_2)$ by $\rho_i\nabla p_i(u)$, where $p_i(u)$ is defined in (11). Then, using the test function $\nabla\log\rho_i$ in the weak formulation of (6) and summing over $i=1,\ldots,n$ to derive the entropy inequality, we find that

$$-\sum_{i=1}^n\int_{\mathbb{T}^d}\rho_i\nabla p_i(u)\cdot\nabla\log\rho_i dx=-\sum_{i,j=1}^n\int_{\mathbb{T}^d}a_{ij}\nabla\rho_i\cdot\nabla\rho_j dx\le-\alpha\sum_{i=1}^n\int_{\mathbb{T}^d}|\nabla\rho_i|^2dx,$$

where $\alpha>0$ is the smallest eigenvalue of (a_{ij}). This provides uniform gradient bounds for each ρ_i, and the Aubin–Lions compactness lemma implies the strong convergence of the approximating sequence of ρ_i, thus allowing us to perform the limit $\varepsilon\to 0$. □

The paper is organized as follows. We prove the existence of global weak solutions (Theorem 1) in Section 2. The limit $\varepsilon\to 0$ for general k_1, $k_2>0$ (Theorem 2) is shown in Section 3, while Section 4 is devoted to the proof of Theorem 3 in the special case $k_1=k_2$.

2. Proof of Theorem 1: existence of solutions

We show first the existence of solutions locally in time and then derive uniform estimates from the energy (8) and entropy (9), which allows us to extend the local solutions globally.

2.1. **Local existence of solutions.** The local-in-time existence of solutions can be proven by the Faedo–Galerkin method; see [20, Chap. 7] for the compressible Navier–Stokes equations and [24] for the quantum Navier–Stokes equations. Since the proof is very similar to these works, we only sketch it.

We regularize the initial data by taking $(\rho^0, u^0) \in C^\infty(\mathbb{T}^d;\mathbb{R}^4)$ such that $\rho_i^0 \geq c > 0$ for some $c > 0$ ($i = 1, 2$). This is possible by using some mollifier with parameter $\delta > 0$, proving the result for this initial datum and then passing to the limit $\delta \to 0$. Let $T > 0$, let (e_k) be an orthonormal basis of $L^2(\mathbb{T}^d)$ which is also an orthogonal basis of $H^1(\mathbb{T}^d)$, and set $X_N = \operatorname{span}\{e_1, \ldots, e_N\}$. Let the velocity $u = (u_1, u_2) \in C^0([0,T]; X_N^2)$ be given and solve the approximate equations

$$\partial_t \rho_i^N + \operatorname{div}(\rho_i^N u_i) = \delta\Delta\rho_i^N, \quad \rho_i^N(0) = \rho_i^0 \quad \text{in } \mathbb{T}^d \times (0,T),\ i = 1, 2. \tag{16}$$

The maximum principle provides the lower and upper bounds $0 < r_i \leq \rho_i^N \leq R_i$ in $\mathbb{T}^d \times (0,T)$, $i = 1, 2$, where r_i and R_i depend on δ and the $L^\infty(\mathbb{T}^d)$ norm of $\operatorname{div} u_i$, and r_i additionally depends on the lower bound $c > 0$ of the initial data. We introduce the operator $S : C^0([0,T]; X_N^2) \to C^0([0,T]; C^3(\mathbb{T}^d;\mathbb{R}^2))$ by $S(u) = \rho^N = (\rho_1^N, \rho_2^N)$. This operator is Lipschitz continuous.

Next, we solve the momentum equation on the space X_N^2. We are looking for a solution $u^N = (u_1^N, u_2^N) \in C^0([0,T]; X_N^2)$ such that for any $\psi \in C^1([0,T]; X_N^2)$ with $\psi(T) = 0$ and $i = 1, 2$,

$$\begin{aligned}
-\varepsilon\int_{\mathbb{T}^d}\rho_i^0 u_i^0\cdot\psi(0)dx &= \varepsilon\int_0^T\int_{\mathbb{T}^d}\bigg\{\rho_i^N u_i^N\cdot\partial_t\psi + \rho_i^N(u_i^N\otimes u_i^N):\nabla\psi \\
&+ \rho_i^N\nabla\bigg(\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}\bigg)\cdot\psi - \rho_i^N\nabla u_i^N:\nabla\psi - u_i^N\cdot\psi - \rho_i^N|u_i^N|^2u_i^N\cdot\psi\bigg\}dxdt \\
&- \int_0^T\int_{\mathbb{T}^d}\bigg(\frac{1}{k_i}\rho_i^N u_i^N + \rho_i^N\nabla(\rho_1^N+\rho_2^N)\bigg)\cdot\psi dxdt - \delta\varepsilon\int_0^T\int_{\mathbb{T}^d}(u_i^N\otimes\nabla\rho_i^N):\nabla\psi dxdt.
\end{aligned}\tag{17}$$

As already mentioned in the introduction, the additional term $\delta\varepsilon\operatorname{div}(u_i^N\otimes\nabla\rho_i^N)$ is introduced to deal with the term $\delta\Delta\rho_i^N$ in (16) when deriving the energy estimates; see the proof of Lemma 6. To solve problem (17), we introduce the operator family

$$\mathcal{M}[\eta] : X_N \to X_N', \quad \langle\mathcal{M}[\eta]u, w\rangle = \int_{\mathbb{T}^d}\eta u\cdot w dx,$$

where $\eta \in L^1(\mathbb{T}^d)$ satisfies $\eta \geq r := \min\{r_1, r_2\} > 0$ and $u, w \in X_N$. The operator $\mathcal{M}$ is invertible and $\mathcal{M}^{-1}$ is Lipschitz continuous as a function from $L^1(\mathbb{T}^d)$ to the space of bounded linear mappings $X_N' \to X_N$.

We can rephrase the integral equation (17) as an ordinary differential equation on X_N,

$$\frac{d}{dt}\big(\mathcal{M}[\rho_i^N]u_i^N\big) = \mathcal{N}[u, u^N], \quad \mathcal{M}[\rho_i^0]u_i^N(0) = \mathcal{M}[\rho_i^0]u_i^0, \tag{18}$$

where $\rho^N = S(u)$ and

$$\begin{aligned}
\mathcal{N}[u, u^N] = &-\operatorname{div}(\rho_i^N u\otimes u_i^N) + \operatorname{div}(\rho_i^N\nabla u_i^N) + \rho_i^N\nabla\bigg(\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}\bigg) \\
&- u_i^N - \rho_i^N|u_i|^2u_i^N - \varepsilon^{-1}\big(k_i^{-1}\rho_i^N u_i^N + \rho_i^N\nabla(\rho_1^N+\rho_2^N)\big) + \delta\operatorname{div}(u_i^N\otimes\nabla\rho_i^N).
\end{aligned}$$

By standard theory for systems of ordinary differential equations, for given $u \in C^0([0,T];X_N^2)$, there exists a unique solution $u^N \in C^1([0,T];X_N^2)$. We are looking for a fixed point $u = u^N$ of (18), and the fixed-point equation can be written in integrated form as

$$u_i^N(t) = \mathcal{M}^{-1}[S(u^N)_i(t)]\bigg(\mathcal{M}[\rho_i^0]u_i^0 + \int_0^t \mathcal{N}[u^N,u^N]ds\bigg) \quad \text{in } X_N$$

and using the Lipschitz continuity of S, $\mathcal{M}^{-1}$, and $\mathcal{N}$, we can apply Banach's fixed-point theorem on a short time interval $[0,T^*]$ for some $0 < T^* \le T$ in the space $C^0([0,T^*];X_N^2)$.

Before, we proceed with the uniform estimates, we recall the following classical lemma, which is used several times in this work.

Lemma 5. *Let v be a weak solution to $\partial_t v + \operatorname{div} F = g$ in $\mathbb{T}^d$, $t > 0$, for some integrable functions F and g in the sense of*

$$0 = -\int_0^T\int_{\mathbb{T}^d} v\partial_t\chi dxdt - \int_{\mathbb{T}^d} v(0)\chi(0)dx - \int_0^T\int_{\mathbb{T}^d}(F\cdot\nabla\chi + g\chi)dxdt$$

for functions $\chi \in C_0^\infty(\mathbb{T}^d\times[0,T))$. Then, for any $t\in[0,T]$ and $\phi \in C^\infty(\mathbb{T}^d\times[0,T])$,

$$0 = -\int_0^t\int_{\mathbb{T}^d} v\partial_s\phi dxds + \int_{\mathbb{T}^d} v\phi\Big|_{s=0}^{s=t}dx - \int_0^t\int_{\mathbb{T}^d}(F\cdot\nabla\phi + g\phi)dxds.$$

2.2. **Approximate energy inequality.** To prove the global existence of solutions, it is sufficient to show that the sequence $(u^N(t))$ is bounded in X_N^2 for $t\in[0,T^*]$ uniformly in T^*.

Lemma 6 (Energy inequality)**.** *Let (ρ^N,u^N) be the local solution to (16)–(17) constructed in Section 2.1. Then*

$$\begin{aligned}\frac{dE}{dt}(\rho^N,u^N) &+ \sum_{i=1}^2 k_i^{-1}\int_{\mathbb{T}^d}\rho_i^N|u_i^N|^2dx + \varepsilon\sum_{i=1}^2\int_{\mathbb{T}^d}\big(|u_i^N|^2 + \rho_i^N|\nabla u_i^N|^2 + \rho_i^N|u_i^N|^4\big)dx \\ &+ \delta\int_{\mathbb{T}^d}|\nabla(\rho_1^N+\rho_2^N)|^2dx + \frac{\delta\varepsilon}{2}\sum_{i=1}^2\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dx = 0,\end{aligned}$$

recalling Definition (8) of $E(\rho^N,u^N)$.

Since the lower bound $\rho_i^N \ge r_i > 0$ yields a uniform estimate for (u_i^N) in $L^2(\mathbb{T}^d)$ uniformly in time, thanks to the definition of the energy (8), we obtain the desired estimate for u^N.

Proof. We use the test function $\phi = \varepsilon|u_i^N|^2/2 - \varepsilon\Delta(\rho_i^N)^{1/2}/(\rho_i^N)^{1/2} + (\rho_1^N+\rho_2^N)$ in the weak formulation of (16) and the test function $\psi = u_i^N$ in the time-differentiated form of (17) (see Lemma 5) and add both equations. Then, using the identity $\partial_t(\rho_i^N u_i^N) + \operatorname{div}(\rho_i^N u_i^N\otimes u_i^N) = \rho_i^N(\partial_t u_i^N + u_i^N\cdot\nabla u_i^N) + \delta\Delta\rho_i^N u_i^N$ and proceeding as in the proof of [24, Lemma 3.1], some terms cancel, and we end up with

$$\varepsilon\frac{d}{dt}\int_{\mathbb{T}^d}\big(\rho_i^N|u_i^N|^2 + |\nabla(\rho_i^N)^{1/2}|^2\big)dx + \int_{\mathbb{T}^d}\partial_t\rho_i^N(\rho_1^N+\rho_2^N)dx$$

$$= -\int_{\mathbb{T}^d}\big(k_i^{-1}\rho_i^N|u_i^N|^2 + \varepsilon(|u_i^N|^2 + \rho_i^N|\nabla u_i^N|^2 + \rho_i^N|u_i^N|^4)\big)dx$$
$$- \delta\varepsilon\int_{\mathbb{T}^d}\Delta\rho_i^N\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}dx - \delta\int_{\mathbb{T}^d}\nabla\rho_i^N\cdot\nabla(\rho_1^N+\rho_2^N)dx.$$

Adding these equations for $i = 1, 2$ and using

$$\int_{\mathbb{T}^d}\Delta\rho_i^N\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}dx = \frac{1}{2}\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dx \tag{19}$$

(see [24, (3.7)]), we find that

$$\frac{d}{dt}\int_{\mathbb{T}^d}\bigg(\varepsilon\sum_{i=1}^2\big(\rho_i^N|u_i^N|^2 + |\nabla(\rho_i^N)^{1/2}|^2\big) + \frac{1}{2}(\rho_1+\rho_2)^2\bigg)dx$$
$$+ \sum_{i=1}^2\int_{\mathbb{T}^d}\big(k_i^{-1}\rho_i^N|u_i^N|^2 + \varepsilon(|u_i^N|^2 + \rho_i^N|\nabla u_i^N|^2 + \rho_i^N|u_i^N|^4)\big)dx$$
$$+ \delta\int_{\mathbb{T}^d}|\nabla(\rho_1^N+\rho_2^N)|^2dx + \frac{\delta\varepsilon}{2}\sum_{i=1}^2\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dx = 0.$$

This finishes the proof. □

The energy inequality and inequality (10) imply the following bounds.

Corollary 7 (Uniform estimates I)**.** *Let (ρ^N, u^N) be the local solution to* (16)–(17) *constructed in Section 2.1. Then there exists $C > 0$ independent of (δ, ε, N) such that for $i = 1, 2$,*

$$\|\rho_i^N\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))} + \|(\rho_i^N)^{1/2}u_i^N\|_{L^2(0,\infty;L^2(\mathbb{T}^d))} + \sqrt{\varepsilon}\|\nabla(\rho_i^N)^{1/2}\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))} \le C,$$
$$\sqrt{\varepsilon}\|(\rho_i^N)^{1/2}u_i^N\|_{L^\infty(0,\infty;L^2(\mathbb{T}^d))} + \sqrt{\varepsilon}\|(\rho_i^N)^{1/2}\nabla u_i^N\|_{L^2(0,\infty;L^2(\mathbb{T}^d))} \le C,$$
$$\sqrt{\varepsilon}\|u_i^N\|_{L^2(0,\infty;L^2(\mathbb{T}^d))} + \sqrt[4]{\varepsilon}\|(\rho_i^N)^{1/4}u_i^N\|_{L^4(0,\infty;L^4(\mathbb{T}^d))} \le C,$$
$$\sqrt{\delta\varepsilon}\|(\rho_i^N)^{1/2}D^2\log\rho_i^N\|_{L^2(0,\infty;L^2(\mathbb{T}^d))} + \sqrt[4]{\delta\varepsilon}\|\nabla(\rho_i^N)^{1/4}\|_{L^4(0,\infty;L^4(\mathbb{T}^d))} \le C.$$

2.3. **Approximate entropy inequality.** Further uniform estimates are derived from the entropy inequality. Here, we work directly with the weak formulation of (16) and with the weak formulation (17).

Lemma 8 (Entropy inequality)**.** *Let (ρ^N, u^N) be the local solution to* (16)–(17) *constructed in Section 2.1. Then*

$$H(\rho^N(t)) + \int_0^t\int_{\mathbb{T}^d}|\nabla(\rho_1^N+\rho_2^N)|^2dxds + \frac{\varepsilon}{8}\sum_{i=1}^2\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds$$
$$+ \sum_{i=1}^2\frac{4\delta}{k_i}\int_0^t\int_{\mathbb{T}^d}|\nabla(\rho_i^N)^{1/2}|^2dxds + \delta\varepsilon\sum_{i=1}^2\int_0^t\int_{\mathbb{T}^d}|\nabla\log\rho_i^N|^2dxds \le C,$$

and the constant $C > 0$ only depends on the initial data.

Proof. Recalling that ρ_i^N is smooth and positive, we compute

$$H(\rho^N(t)) - H(\rho^N(0)) = \sum_{i=1}^2 \frac{1}{k_i}\int_0^t\int_{\mathbb{T}^d} \partial_s\rho_i^N \log\rho_i^N dxds \tag{20}$$
$$= \sum_{i=1}^2 \frac{1}{k_i}\int_0^t\int_{\mathbb{T}^d} \big(\rho_i^N u_i^N - \delta\nabla\rho_i^N\big)\cdot\nabla\log\rho_i^N dxds$$
$$= \sum_{i=1}^2 \frac{1}{k_i}\int_0^t\int_{\mathbb{T}^d} u_i^N\cdot\nabla\rho_i^N dxds - \sum_{i=1}^2 \frac{4\delta}{k_i}\int_0^t\int_{\mathbb{T}^d} |\nabla(\rho_i^N)^{1/2}|^2 dxds.$$

To estimate the first term on the right-hand side, we use the test function $\nabla\log\rho_i^N$ in (17):

$$\frac{1}{k_i}\int_0^t\int_{\mathbb{T}^d} u_i^N\cdot\nabla\rho_i^N dxds = \frac{1}{k_i}\int_0^t\int_{\mathbb{T}^d}(\rho_i^N u_i^N)\cdot\nabla\log\rho_i^N dxds = I_1 + \cdots + I_7, \tag{21}$$
$$I_1 = -\varepsilon\int_0^t\int_{\mathbb{T}^d}\big(\partial_s(\rho_i^N u_i^N) + \operatorname{div}(\rho_i^N u_i^N\otimes u_i^N)\big)\cdot\nabla\log\rho_i^N dxds,$$
$$I_2 = \varepsilon\int_0^t\int_{\mathbb{T}^d}\rho_i^N\nabla\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}\cdot\nabla\log\rho_i^N dxds,$$
$$I_3 = \varepsilon\int_0^t\int_{\mathbb{T}^d}\operatorname{div}(\rho_i^N\nabla u_i^N)\cdot\nabla\log\rho_i^N dxds,$$
$$I_4 = -\varepsilon\int_0^t\int_{\mathbb{T}^d} u_i^N\cdot\nabla\log\rho_i^N dxds,$$
$$I_5 = -\varepsilon\int_0^t\int_{\mathbb{T}^d}\rho_i^N|u_i^N|^2 u_i^N\cdot\nabla\log\rho_i^N dxds,$$
$$I_6 = -\int_0^t\int_{\mathbb{T}^d}\rho_i^N\nabla(\rho_1^N+\rho_2^N)\cdot\nabla\log\rho_i^N dxds,$$
$$I_7 = \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\operatorname{div}(u_i^N\otimes\nabla\rho_i^N)\cdot\nabla\log\rho_i^N dxds.$$

Step 1: Estimation of I_1, I_2, and I_7: We start with I_2. We infer from identity (19) that

$$I_2 = -\varepsilon\int_0^t\int_{\mathbb{T}^d}\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}\Delta\rho_i^N dxds = -\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2 dxds,$$

and this expression will be used to absorb some integrals coming from the other terms. It follows from (16) that

$$I_1 = -\varepsilon\int_0^t\int_{\mathbb{T}^d}\big(\rho_i^N(\partial_s u_i^N + u_i^N\cdot\nabla u_i^N) + \delta\Delta\rho_i^N u_i^N\big)\cdot\nabla\log\rho_i^N dxds \tag{22}$$
$$= -\varepsilon\int_0^t\int_{\mathbb{T}^d}\big(\partial_s u_i^N\cdot\nabla\rho_i^N + u_i^N\cdot\nabla u_i^N\cdot\nabla\rho_i^N + \delta\Delta\rho_i^N u_i^N\cdot\nabla\log\rho_i^N\big)dxds$$
$$=: I_{11} + I_{12} + I_{13}.$$

The term I_{11} is rewritten according to

$$\begin{aligned} I_{11} &= -\varepsilon\int_0^t\int_{\mathbb{T}^d}\big(\partial_s(u_i^N\cdot\nabla\rho_i^N) - u_i^N\cdot\nabla\partial_s\rho_i^N\big)dxds \\ &= -\varepsilon\int_0^t\frac{d}{ds}\int_{\mathbb{T}^d}u_i^N\cdot\nabla\rho_i^N dxds - \varepsilon\int_0^t\int_{\mathbb{T}^d}\operatorname{div}u_i^N\big(-\operatorname{div}(\rho_i^N u_i^N) + \delta\Delta\rho_i^N\big)dxds \\ &= -\varepsilon\int_{\mathbb{T}^d}u_i^N(t)\cdot\nabla\rho_i^N(t)dx + \varepsilon\int_{\mathbb{T}^d}u_i^N(0)\cdot\nabla\rho_i^N(0)dx \\ &\quad + \varepsilon\int_0^t\int_{\mathbb{T}^d}(u_i^N\cdot\nabla\rho_i^N)\operatorname{div}u_i^N dxds + \varepsilon\int_0^t\int_{\mathbb{T}^d}\rho_i^N(\operatorname{div}u_i^N)^2dxds \\ &\quad - \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\Delta\rho_i^N\operatorname{div}u_i^N dxds =: I_{111} + \cdots + I_{115}. \end{aligned}$$

Corollary 7 shows that, for $0 < t < T$,

$$\begin{aligned} I_{111} + I_{112} &= -\varepsilon\int_{\mathbb{T}^d}u_i^N(t)\cdot\nabla\rho_i^N(t)dx + \varepsilon\int_{\mathbb{T}^d}u_i^N(0)\cdot\nabla\rho_i^N(0)dx \\ &\le 2\big(\sqrt{\varepsilon}\|(\rho_i^N)^{1/2}u_i^N\|_{L^\infty(0,T;L^2(\Omega))}\big)\big(\sqrt{\varepsilon}\|\nabla(\rho_i^N)^{1/2}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}\big) \\ &\quad + 2\varepsilon\|(\rho_i^0)^{1/2}u_i^0\|_{L^2(\mathbb{T}^d)}\|\nabla(\rho_i^0)^{1/2}\|_{L^2(\mathbb{T}^d)} \le C, \\ I_{114} &\le \varepsilon\|(\rho_i^N)^{1/2}\nabla u_i^N\|^2_{L^2(0,T;L^2(\mathbb{T}^d))} \le C. \end{aligned}$$

Furthermore, using Hölder's inequality, Corollary 7, and inequality (10),

$$\begin{aligned} I_{113} &\le 4\big(\sqrt[4]{\varepsilon}\|(\rho_i^N)^{1/4}u_i^N\|_{L^4(0,T;L^4(\mathbb{T}^d))}\big)\big(\sqrt{\varepsilon}\|(\rho_i^N)^{1/2}\nabla u_i^N\|_{L^2(0,T;L^2(\mathbb{T}^d))}\big) \\ &\quad \times \big(\sqrt[4]{\varepsilon}\|\nabla(\rho_i^N)^{1/4}\|_{L^4(0,T;L^4(\mathbb{T}^d))}\big) \\ &\le C + \frac{\varepsilon}{16c(d)}\int_0^t\int_{\mathbb{T}^d}|\nabla(\rho_i^N)^{1/4}|^4dxds \\ &\le C + \frac{\varepsilon}{16}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds, \end{aligned}$$

where in the last but one step we applied Young's inequality. For the remaining term I_{115}, we first reformulate it, and then use similar arguments as for I_{113}:

$$\begin{aligned} I_{115} &= -\delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\operatorname{div}(\rho_i^N\nabla\log\rho_i^N)\operatorname{div}u_i^N dxds \\ &= -\delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\big(\rho_i^N\Delta\log\rho_i^N + \nabla\rho_i^N\cdot\nabla\log\rho_i^N\big)\operatorname{div}u_i^N dxds \\ &= -\delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\big((\rho_i^N)^{1/2}\Delta\log\rho_i^N + 16|\nabla(\rho_i^N)^{1/4}|^2\big)(\rho_i^N)^{1/2}\operatorname{div}u_i^N dxds \\ &\le \delta\sqrt{\varepsilon}\big(\|(\rho_i^N)^{1/2}\Delta\log\rho_i^N\|_{L^2(0,T;L^2(\mathbb{T}^d))} + 16\|\nabla(\rho_i^N)^{1/4}\|^2_{L^4(0,T;L^4(\mathbb{T}^d))}\big) \\ &\quad \times \sqrt{\varepsilon}\|(\rho_i^N)^{1/2}\nabla u_i^N\|_{L^2(0,T;L^2(\mathbb{T}^d))} \end{aligned}$$

$$\leq C + \frac{\varepsilon}{16}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds.$$

In a similar way, the bounds in Corollary 7 yield

$$\begin{aligned} I_{12} &\leq 4\big(\sqrt[4]{\varepsilon}\|(\rho_i^N)^{1/4}u_i^N\|_{L^4(0,T;L^4(\mathbb{T}^d))}\big)\big(\sqrt{\varepsilon}\|(\rho_i^N)^{1/2}\nabla u_i^N\|_{L^2(0,T;L^2(\mathbb{T}^d))}\big) \\ &\quad\times\big(\sqrt[4]{\varepsilon}\|\nabla(\rho_i^N)^{1/4}\|_{L^4(0,T;L^4(\mathbb{T}^d))}\big) \\ &\leq C + \frac{\varepsilon}{16}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds. \end{aligned}$$

We conclude from (22) that

$$I_1 \leq C + \frac{3\varepsilon}{16}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds - \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\Delta\rho_i^N u_i^N\cdot\nabla\log\rho_i^N dxds,$$

and the last term cancels with a part of I_7, since

$$\begin{aligned} I_7 &= \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\big(\nabla\log\rho_i^N\cdot\nabla u_i^N\cdot\nabla\rho_i^N + \Delta\rho_i^N u_i^N\cdot\nabla\log\rho_i^N\big)dxds \\ &\leq \sqrt{\delta}\big(4\sqrt[4]{\delta\varepsilon}\|\nabla(\rho_i^N)^{1/4}\|_{L^4(0,T;L^4(\mathbb{T}^d))}\big)^2\big(\sqrt{\varepsilon}\|(\rho_i^N)^{1/2}\nabla u_i^N\|_{L^2(0,T;L^2(\mathbb{T}^d))}\big) \\ &\quad + \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\Delta\rho_i^N u_i^N\cdot\nabla\log\rho_i dxds \\ &\leq C + \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}\Delta\rho_i^N u_i^N\cdot\nabla\log\rho_i dxds. \end{aligned}$$

This shows that

$$I_1 + I_2 + I_7 \leq C - \frac{5\varepsilon}{16}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds. \tag{23}$$

Step 2: Estimation of $I_3,\ldots,I_6$*:* We continue with the estimate of I_3:

$$\begin{aligned} I_3 &= -\varepsilon\int_0^t\int_{\mathbb{T}^d}\rho_i^N\nabla u_i^N : D^2\log\rho_i^N dxds \\ &\leq \frac{\varepsilon}{16}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds + 4\varepsilon\int_0^t\int_{\mathbb{T}^d}\rho_i^N|\nabla u_i^N|^2dxds \\ &\leq \frac{\varepsilon}{16}\int_0^t\int_{\mathbb{T}^d}\rho_i^N|D^2\log\rho_i^N|^2dxds + C. \end{aligned}$$

By the approximative mass balance equation (16), we have

$$\begin{aligned} I_4 &= -\varepsilon\int_0^t\int_{\mathbb{T}^d}\frac{u_i^N\cdot\nabla\rho_i^N}{\rho_i^N}dxds = \varepsilon\int_0^t\int_{\mathbb{T}^d}\frac{1}{\rho_i^N}\big(\partial_s\rho_i^N + \rho_i^N\operatorname{div}u_i^N - \delta\Delta\rho_i^N\big)dxds \\ &= \varepsilon\int_0^t\int_{\mathbb{T}^d}\partial_s\log\rho_i^N dxds + \varepsilon\int_0^t\int_{\mathbb{T}^d}\operatorname{div}u_i^N dxds - \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}|\nabla\log\rho_i^N|^2dxds \\ &= \varepsilon\int_{\mathbb{T}^d}\log\rho_i^N(t)dx - \varepsilon\int_{\mathbb{T}^d}\log\rho_i^N(0)dx - \delta\varepsilon\int_0^t\int_{\mathbb{T}^d}|\nabla\log\rho_i^N|^2dxds \end{aligned}$$

$$\leq \varepsilon \int_{\mathbb{T}^d} (\rho_i^N(t) - 1)dx - \varepsilon \int_{\mathbb{T}^d} \log \rho_i^0 dx - \delta\varepsilon \int_{\mathbb{T}^d} |\nabla \log \rho_i^N|^2 dxds.$$

Because of the $L^2(\mathbb{T}^d)$ bound for ρ_i^N in Corollary 7, the first term on the right-hand side is bounded uniformly in (N, δ, ε). The same holds true for the second term on the right-hand side, since $\log \rho_i^0$ is assumed to be integrable. We infer that

$$I_4 \leq C - \delta\varepsilon \int_{\mathbb{T}^d} |\nabla \log \rho_i^N|^2 dxds.$$

Furthermore, using Hölder's inequality, Young's inequality, and then inequality (10),

$$\begin{aligned} I_5 &\leq 4\big(\sqrt[4]{\varepsilon}\|(\rho_i^N)^{1/4} u_i^N\|_{L^4(0,\infty;L^4(\mathbb{T}^d))}\big)^3 \big(\sqrt[4]{\varepsilon}\|\nabla(\rho_i^N)^{1/4}\|_{L^4(0,\infty;L^4(\mathbb{T}^d))}\big) \\ &\leq C + \frac{\varepsilon}{16} \int_0^t \int_{\mathbb{T}^d} \rho_i^N |D^2 \log \rho_i^N|^2 dxds. \end{aligned}$$

Finally, the term I_6 is rewritten as

$$I_6 = -\int_0^t \int_{\mathbb{T}^d} \nabla \rho_i^N \cdot \nabla(\rho_1^N + \rho_2^N) dxds,$$

and it becomes nonpositive when added for $i = 1, 2$. We conclude that

$$\begin{aligned} I_3 + \cdots + I_6 \leq C &+ \frac{\varepsilon}{8} \int_0^t \int_{\mathbb{T}^d} \rho_i^N |D^2 \log \rho_i^N|^2 dxds - \delta\varepsilon \int_{\mathbb{T}^d} |\nabla \log \rho_i^N|^2 dxds \\ &- \int_0^t \int_{\mathbb{T}^d} \nabla \rho_i^N \cdot \nabla(\rho_1^N + \rho_2^N) dxds. \end{aligned} \tag{24}$$

Step 3: End of the proof: We insert (23) and (24) into (21) and sum over $i = 1, 2$ to conclude from (20) the desired entropy inequality. □

The entropy inequality allows us to improve some bounds from Corollary 7.

Corollary 9 (Uniform estimates II). *Let (ρ^N, u^N) be the local solution to (16)–(17) constructed in Section 2.1. Then there exists $C > 0$ independent of (δ, ε, N) such that*

$$\begin{aligned} \|\nabla(\rho_1^N + \rho_2^N)\|_{L^2(0,\infty;L^2(\mathbb{T}^d))} &\leq C, \\ \sqrt{\varepsilon}\|(\rho_i^N)^{1/2}\|_{L^2(0,\infty;H^2(\mathbb{T}^d))} + \sqrt[4]{\varepsilon}\|\nabla(\rho_i^N)^{1/4}\|_{L^4(0,\infty;L^4(\mathbb{T}^d))} &\leq C. \end{aligned}$$

2.4. **Further uniform estimates.** We derive some spatial and time regularity bounds for ρ_i^N and $\rho_i^N u_i^N$ uniform in N, which are needed for the limit $N \to \infty$.

Lemma 10 (Spatial regularity). *For any $T > 0$, there exists a constant $C(\varepsilon) > 0$ independent of N and δ such that*

$$\begin{aligned} \|\rho_i^N\|_{L^\infty(0,T;L^3(\mathbb{T}^d))} + \|(\rho_i^N)^{1/2}\|_{L^4(0,T;W^{1,3}(\mathbb{T}^d))} &\leq C(\varepsilon), \\ \|\rho_i^N\|_{L^2(0,T;W^{2,3/2}(\mathbb{T}^d))} + \|\rho_i^N u_i^N\|_{L^2(0,T;W^{1,3/2}(\mathbb{T}^d))} &\leq C(\varepsilon). \end{aligned}$$

Proof. The first bound for ρ_i^N is an immediate consequence of Corollary 7 and the Sobolev embedding $H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)$, yielding a uniform bound for $(\rho_i^N)^{1/2}$ in $L^\infty(0,T;L^6(\mathbb{T}^d))$ (uniform in N and δ). It follows from the embedding $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ (here we use the condition $d \le 3$) and Corollary 9 that $(\rho_i^N)^{1/2}$ is uniformly bounded in $L^2(0,T;L^\infty(\mathbb{T}^d))$. Then, together with the uniform bound for $(\rho_i^N)^{1/2}u_i^N$ in $L^\infty(0,T;L^2(\mathbb{T}^d))$ from Corollary 7, we obtain a uniform estimate for $\rho_i^N u_i^N = (\rho_i^N)^{1/2} \cdot (\rho_i^N)^{1/2}u_i^N$ in $L^2(0,T;L^2(\mathbb{T}^d))$. Furthermore, by Corollary 9, $(\nabla(\rho_i^N)^{1/2})$ is bounded in $L^2(0,T;L^6(\mathbb{T}^d))$. Then

$$\nabla(\rho_i^N u_i^N) = 2\nabla(\rho_i^N)^{1/2} \cdot (\rho_i^N)^{1/2}u_i^N + 2(\rho_i^N)^{1/2} \cdot (\rho_i^N)^{1/2}\nabla u_i^N$$

is uniformly bounded in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$. This yields a uniform bound for $\rho_i^N u_i^N$ in $L^2(0,T;W^{1,3/2}(\mathbb{T}^d))$.

Next, we apply the Gagliardo–Nirenberg inequality with $\theta = 1/2$:

$$\begin{aligned}\|\nabla(\rho_i^N)^{1/2}\|^4_{L^4(0,T;L^3(\mathbb{T}^d))} &\le C\int_0^T \|(\rho_i^N)^{1/2}\|^{4\theta}_{H^2(\mathbb{T}^d)}\|(\rho_i^N)^{1/2}\|^{4(1-\theta)}_{L^6(\mathbb{T}^d)}dt \\ &\le C\|(\rho_i^N)^{1/2}\|^2_{L^\infty(0,T;L^6(\mathbb{T}^d))}\int_0^T \|(\rho_i^N)^{1/2}\|^2_{H^2(\mathbb{T}^d)}dt \le C,\end{aligned}$$

showing that $(\rho_i^N)^{1/2}$ is uniformly bounded in $L^4(0,T;W^{1,3}(\mathbb{T}^d))$. Because of inequality (10), the uniform bound for $(\rho_i^N)^{1/2}D^2\log\rho_i^N$ in $L^2(0,T;L^2(\mathbb{T}^d))$ implies a uniform bound for $D^2(\rho_i^N)^{1/2}$ in $L^2(0,T;\ L^2(\mathbb{T}^d))$. Thus,

$$D^2\rho_i^N = 2(\rho_i^N)^{1/2}D^2(\rho_i^N)^{1/2} + 2\nabla(\rho_i^N)^{1/2} \otimes \nabla(\rho_i^N)^{1/2}$$

is uniformly bounded in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$. Finally, the bound for $(\rho_i^N)^{1/4}$ in $L^\infty(0,T;$ $L^{12}(\mathbb{T}^d))$ and for $\nabla(\rho_i^N)^{1/4}$ in $L^4(0,T;L^4(\mathbb{T}^d))$ yield an estimate for $\nabla(\rho_i^N)^{1/2} = 2(\rho_i^N)^{1/4}$ $\nabla(\rho_i^N)^{1/4}$ in $L^4(0,T;L^3(\mathbb{T}^d))$, finishing the proof. □

Lemma 11 (Time regularity)**.** *For any $T > 0$, there exists a constant $C(\varepsilon) > 0$ independent of N and δ such that for $s > d/2 + 1$,*

$$\|\partial_t\rho_i^N\|_{L^2(0,T;L^{3/2}(\mathbb{T}^d))} + \|\partial_t(\rho_i^N u_i^N)\|_{L^{4/3}(0,T;H^s(\mathbb{T}^d)')} + \|\partial_t(\rho_i^N)^{1/2}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le C.$$

Proof. We deduce from Lemma 10 that

$$\partial_t\rho_i^N = -\operatorname{div}(\rho_i^N u_i^N) + \delta\Delta\rho_i^N$$

is uniformly bounded in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$. The estimate on $\partial_t(\rho_i^N u_i^N)$ follows from the following bounds, which are consequences of Corollaries 7 and 9, as well as from the spatial regularity of Lemma 10:

- The sequence $(\rho_i^N u_i^N \otimes u_i^N)$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}^d))$. Hence, $\operatorname{div}(\rho_i^N u_i^N \otimes u_i^N)$ is bounded in $L^\infty(0,T;H^s(\mathbb{T}^d)')$, since $H^s(\mathbb{T}^d) \hookrightarrow W^{1,\infty}(\mathbb{T}^d)$ for $s > d/2 + 1$.
- We know that $\rho_i^N\nabla u_i^N = (\rho_i^N)^{1/2} \cdot (\rho_i^N)^{1/2}\nabla u_i^N$ is bounded in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$. Thus, $\operatorname{div}(\rho_i^N\nabla u_i^N)$ is bounded in $L^2(0,T;W^{1,3}(\mathbb{T}^d)') \hookrightarrow L^2(0,T;H^s(\mathbb{T}^d)')$.

- Let $\psi \in L^4(0,T;W^{1,3}(\mathbb{T}^d;\mathbb{R}^d))$. Then, by integration by parts,
$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\rho_i^N\nabla\bigg(\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}\bigg)\cdot\psi dxds \\ &= -\int_0^T\int_{\mathbb{T}^d}\Delta(\rho_i^N)^{1/2}\big(2\nabla(\rho_i^N)^{1/2}\cdot\psi + (\rho_i^N)^{1/2}\operatorname{div}\psi\big)dxds \\ &\le 2\|\Delta(\rho_i^N)^{1/2}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\|\nabla(\rho_i^N)^{1/2}\|_{L^4(0,T;L^3(\mathbb{T}^d))}\|\psi\|_{L^4(0,T;L^6(\mathbb{T}^d))} \\ &\quad + \|\Delta(\rho_i^N)^{1/2}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\|(\rho_i^N)^{1/2}\|_{L^\infty(0,T;L^6(\mathbb{T}^d))}\|\psi\|_{L^2(0,T;W^{1,3}(\mathbb{T}^d))} \\ &\le C\|\psi\|_{L^4(0,T;W^{1,3}(\mathbb{T}^d))}.\end{aligned}$$
This proves that $\rho_i^N\nabla(\Delta(\rho_i^N)^{1/2}/(\rho_i^N)^{1/2})$ is uniformly bounded in $L^{4/3}(0,T;W^{1,3}(\mathbb{T}^d)')$ $\hookrightarrow L^{4/3}(0,T;H^s(\mathbb{T}^d)')$.
- The sequence $\rho_i^N|u_i^N|^2u_i^N = (\rho_i^N)^{1/2}u_i^N\cdot(\rho_i^N)^{1/2}|u_i^N|^2$ is bounded in $L^2(0,T;L^1(\mathbb{T}^d))$ $\hookrightarrow L^2(0,T;H^s(\mathbb{T}^d)')$, since $(\rho_i^N)^{1/2}u_i^N$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$ and $(\rho_i^N)^{1/2}$ $|u_i^N|^2$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$ by Corollary 7.
- By Lemma 10, $\rho_i^Nu_i^N$ is uniformly bounded in $L^2(0,T;W^{1,3/2}(\mathbb{T}^d))\hookrightarrow L^2(0,T;$ $L^{3/2}(\mathbb{T}^d))$.
- In view of the $L^\infty(0,T;L^3(\mathbb{T}^d))$ bound of ρ_i^N from Lemma 10 and the $L^2(0,T;L^2(\mathbb{T}^d))$ bound of $\nabla(\rho_1^N+\rho_2^N)$ from Corollary 9, $\rho_i^N\nabla(\rho_1^N+\rho_2^N)$ is bounded in $L^2(0,T;L^{6/5}(\mathbb{T}^d))$ $\hookrightarrow L^2(0,T;H^s(\mathbb{T}^d)')$.
- We deduce from the $L^2(0,T;L^2(\mathbb{T}^d))$ bound of $(\rho_i^N)^{1/2}u_i^N$ (Corollary 7) and the $L^4(0,T;L^3(\mathbb{T}^d))$ bound of $\nabla(\rho_i^N)^{1/2}$ that $u_i^N\otimes\nabla\rho_i^N = 2(\rho_i^N)^{1/2}u_i^N\otimes\nabla(\rho_i^N)^{1/2}$ is bounded in $L^{4/3}(0,T;L^{6/5}(\mathbb{T}^d))$. Hence, the sequence $\operatorname{div}(u_i^N\otimes\nabla\rho_i^N)$ is bounded in $L^{4/3}(0,T;W^{1,6}(\mathbb{T}^d)')\hookrightarrow L^{4/3}(0,T;H^s(\mathbb{T}^d)')$.

We conclude that
$$\begin{aligned}\partial_t(\rho_i^Nu_i^N) &= -\operatorname{div}(\rho_i^Nu_i^N\otimes u_i^N) + \rho_i^N\nabla\bigg(\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}\bigg) + \operatorname{div}(\rho_i^N\nabla u_i^N) - u_i^N \\ &\quad - \rho_i^Nu_i^N|u_i^N|^2 - \varepsilon^{-1}k_i^{-1}\rho_i^Nu_i^N - \varepsilon^{-1}\rho_i^N\nabla(\rho_1+\rho_2^N) + \delta\operatorname{div}(u_i^N\otimes\nabla\rho_i^N)\end{aligned}$$
is uniformly bounded in $L^{4/3}(0,T;H^s(\mathbb{T}^d)')$. Finally, the sequence
$$\begin{aligned}\partial_t(\rho_i^N)^{1/2} &= -\frac{1}{2}(\rho_i^N)^{1/2}\operatorname{div}u_i^N - 2\nabla(\rho_i^N)^{1/4}\cdot((\rho_i^N)^{1/4}u_i^N) \\ &\quad + \delta\Delta(\rho_i^N)^{1/2} + 4\delta|\nabla(\rho_i^N)^{1/4}|^2\end{aligned}$$
is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$. □

2.5. **Limit $N\to\infty$.** The spatial and time regularity for $(\rho_i^N)^{1/2}$, ρ_i^N, and $\rho_i^Nu_i^N$ allow us to apply the Aubin–Lions compactness lemma to conclude the existence of a subsequence (not relabeled) such that, as $N\to\infty$,
$$\begin{aligned}(\rho_i^N)^{1/2} &\to \sqrt{\rho_i} \quad \text{strongly in } L^2(0,T;H^1(\mathbb{T}^d)), \\ \rho_i^N &\to \rho_i \quad \text{strongly in } L^2(0,T;L^2(\mathbb{T}^d)),\end{aligned}$$

$$\rho_i^N u_i^N \to J_i \quad \text{strongly in } L^2(0,T;L^2(\mathbb{T}^d)).$$

Furthermore, we have the weak convergences (up to subsequences)

$$\begin{aligned}(\rho_i^N)^{1/2} &\rightharpoonup \sqrt{\rho_i} \quad \text{weakly in } L^2(0,T;H^2(\mathbb{T}^d)), \\ u_i^N &\rightharpoonup u_i \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T}^d)).\end{aligned}$$

It follows that $\rho_i^N u_i^N \rightharpoonup \rho_i u_i$ weakly in $L^1(0,T;L^1(\mathbb{T}^d))$, showing that $J_i = \rho_i u_i$. Moreover, by Lemma 10, $\delta\nabla\rho_i^N = 2\delta(\rho_i^N)^{1/2}\nabla(\rho_i^N)^{1/2} \to 0$ strongly in $L^4(0,T;L^2(\mathbb{T}^d))$.

With these convergences, we can pass to the limit $N\to\infty$ and $\delta\to 0$ in (16), formulated for $\phi \in C_0^\infty(\mathbb{T}^d\times[0,T))$ as

$$\begin{aligned}0 = &-\int_0^T\int_{\mathbb{T}^d}\rho_i^N\partial_t\phi dxdt - \int_{\mathbb{T}^d}\rho_i^0\phi(0)dx - \int_0^T\int_{\mathbb{T}^d}\rho_i^N u_i^N\cdot\nabla\phi dxdt \\ &+ \delta\int_0^T\int_{\mathbb{T}^d}\nabla\rho_i^N\cdot\nabla\phi dxdt,\end{aligned}$$

leading to

$$0 = -\int_0^T\int_{\mathbb{T}^d}\rho_i\partial_t\phi dxdt - \int_{\mathbb{T}^d}\rho_i^0\phi(0)dx - \int_0^T\int_{\mathbb{T}^d}\rho_i u_i\cdot\nabla\phi dxdt.$$

The limit in the momentum balance equation (17) is more involved. The strong convergence of $\rho_i^N u_i^N$ and the weak convergence of u_i^N in $L^2(0,T;L^2(\mathbb{T}^d))$ lead to

$$\rho_i^N u_i^N\otimes u_i^N \rightharpoonup \rho_i u_i\otimes u_i \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

Similarly, $(\rho_i^N)^{1/2}u_i^N \rightharpoonup \sqrt{\rho_i}u_i$ weakly in $L^2(0,T;L^2(\mathbb{T}^d))$, taking into account Corollary 7. Then, together with the strong convergences of $\nabla(\rho_i^N)^{1/2}$ and $\rho_i^N u_i^N$ in $L^2(0,T;L^2(\mathbb{T}^d))$, we find that for $\psi\in C_0^\infty(\mathbb{T}^d\times(0,T);\mathbb{R}^{d\times d}))$, integrating by parts,

$$\begin{aligned}-\int_0^T\int_{\mathbb{T}^d}\rho_i^N\nabla u_i^N : \psi dxdt &= \int_0^T\int_{\mathbb{T}^d}u_i^N\cdot\operatorname{div}(\rho_i^N\psi)dxdt \\ &= \int_0^T\int_{\mathbb{T}^d}\big(2(\rho_i^N)^{1/2}u_i^N\cdot\psi\cdot\nabla(\rho_i^N)^{1/2} + \rho_i^N u_i^N\cdot\operatorname{div}\psi\big)dxdt \\ &\to \int_0^T\int_{\mathbb{T}^d}\big(2\sqrt{\rho_i}u_i\cdot\psi\cdot\nabla\sqrt{\rho_i} + \rho_i u_i\cdot\operatorname{div}\psi\big)dxdt \\ &= -\int_0^T\langle\nabla u_i,\rho_i\psi\rangle_{H^1(\mathbb{T}^d)',H^1(\mathbb{T}^d)}dt = -\int_0^T\langle\rho_i\nabla u_i,\psi\rangle_{\mathcal{D}'(\mathbb{T}^d),\mathcal{D}(\mathbb{T}^d)}dt.\end{aligned} \tag{25}$$

Thus, $\operatorname{div}(\rho_i^N\nabla u_i^N)\to\operatorname{div}(\rho_i\nabla u_i)$ in the sense of distributions. In fact, because of the uniform bounds of $(\rho_i^N)^{1/2}$ in $L^\infty(0,T;H^1(\mathbb{T}^d))\hookrightarrow L^\infty(0,T;L^6(\mathbb{T}^d))$ and of $(\rho_i^N)^{1/2}\nabla u_i^N$ in $L^2(0,T;L^2(\mathbb{T}^d))$, this convergence also holds in the weak topology of $L^2(0,T;L^{3/2}(\mathbb{T}^d))$. The same bounds imply that

$$u_i^N\otimes\nabla\rho_i^N = 2(\rho_i^N)^{1/2}u_i^N\otimes\nabla(\rho_i^N)^{1/2} \rightharpoonup 2\sqrt{\rho_i}u_i\otimes\nabla\sqrt{\rho_i} = u_i\otimes\nabla\rho_i$$

weakly in $L^1(0,T;L^1(\mathbb{T}^d))$ and hence $\operatorname{div}(u_i^N\otimes\nabla\rho_i^N)\rightharpoonup\operatorname{div}(u_i\otimes\nabla\rho_i)$ weakly in $L^1(0,T;H^s(\mathbb{T}^d)')$.

Next, since $\Delta(\rho_i^N)^{1/2}\rightharpoonup\Delta\sqrt{\rho_i}$ weakly in $L^2(0,T;L^2(\mathbb{T}^d))$ and $\nabla(\rho_i^N)^{1/2}\to\nabla\sqrt{\rho_i}$ strongly in $L^2(0,T;L^2(\mathbb{T}^d))$, for $\psi\in C_0^\infty(\mathbb{T}^d\times(0,T);\mathbb{R}^d)$,

$$
\begin{aligned}
\int_0^T\int_{\mathbb{T}^d}\rho_i^N\nabla\bigg(\frac{\Delta(\rho_i^N)^{1/2}}{(\rho_i^N)^{1/2}}\bigg)\cdot\psi dxdt \\
&= -\int_0^T\int_{\mathbb{T}^d}\Delta(\rho_i^N)^{1/2}\big(2\nabla(\rho_i^N)^{1/2}\cdot\phi+(\rho_i^N)^{1/2}\operatorname{div}\psi\big)dxdt \\
&\to -\int_0^T\int_{\mathbb{T}^d}\Delta\sqrt{\rho_i}\big(2\nabla\sqrt{\rho_i}\cdot\psi+\sqrt{\rho_i}\operatorname{div}\psi\big)dxdt.
\end{aligned}
$$

The convergence $\rho_i^N|u_i^N|^2u_i^N\to\rho_i|u_i|^2u_i$ strongly in $L^1(0,T;L^1(\mathbb{T}^d))$ has been proved in [35, Lemma 2.3]. For completeness, we recall the proof. The strong convergences of (ρ_i^N) and $(\rho_i^N u_i^N)$ imply, up to subsequence, that $\rho_i^N\to\rho_i$ and $\rho_i^N u_i^N\to\rho_i u_i$ a.e. Hence, for a.e. (x,t), $u_i^N=(\rho_i^N u_i^N)/\rho_i^N\to u_i$ whenever $\rho_i^N(x,t)\neq 0$. For a.e. (x,t) for which $\rho_i^N(x,t)=0$, we have

$$
g_i^N := \rho_i^N|u_i^N|^2u_i^N 1_{\{|u_i^N|\le M\}}\le\rho_I^N M^3=0
$$

for any $M>0$. Consequently, $g_i^N\to\rho_i|u_i|^2u_i1_{\{|u_i|\le M\}}$ a.e. As the sequence (ρ_i^N) is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$, g_i^N is bounded in the same space. Then dominated convergence implies that

$$
g_i^N\to\rho_i|u_i|^2u_i1_{\{|u_i|\le M\}}\quad\text{strongly in } L^1(0,T;L^1(\mathbb{T}^d)). \tag{26}
$$

Now, for any $M>0$,

$$
\begin{aligned}
\int_0^T\int_{\mathbb{T}^d}\big|\rho_i^N|u_i^N|^2u_i^N-\rho|u_i|^2u_i\big|dxdt \\
&\le\int_0^T\int_{\mathbb{T}^d}\big|\rho_i^N|u_i^N|^2u_i^N1_{\{|u_i^N|\le M\}}-\rho_i|u_i|^2u_i1_{\{|u_i|\le M\}}\big|dxdt \\
&\quad+\int_0^T\int_{\mathbb{T}^d}\big(\rho_i^N|u_i^N|^3 1_{\{|u_i^N|>M\}}+\rho_i|u_i|^3 1_{\{|u_i|>M\}}\big)dxdt \\
&\le\int_0^T\int_{\mathbb{T}^d}\big|g_i^N-\rho_i|u_i|^2u_i1_{\{|u_i|\le M\}}\big|dxdt \\
&\quad+\frac{1}{M}\int_0^T\int_{\mathbb{T}^d}\big(\rho_i^N|u_i^N|^4+\rho_i|u_i|^4\big)dxdt,
\end{aligned}
$$

observing that $\rho_i|u_i|^4$ is an element of $L^1(0,T;L^1(\mathbb{T}^d))$. The convergence (26) shows that

$$
\lim_{N\to\infty}\int_0^T\int_{\mathbb{T}^d}\big|\rho_i^N|u_i^N|^2u_i^N-\rho|u_i|^2u_i\big|dxdt\le\frac{C}{M}
$$

for some $C>0$ and for all $M>0$. The limit $M\to\infty$ finishes the proof of the strong convergence of $\rho_i^N|u_i^N|^2u_i^N$.

The weak convergence of $\nabla(\rho_1^N+\rho_2^N)$ and the strong convergence of ρ_i^N in $L^2(0,T;L^2(\mathbb{T}^d))$ imply that

$$\rho_i^N\nabla(\rho_1^N+\rho_2^N) \rightharpoonup \rho_i\nabla(\rho_1+\rho_2) \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

We treat the δ-regularized terms:

$$\begin{aligned}\left|\delta\int_0^T\int_{\mathbb{T}^d}\nabla\rho_i^N\cdot\nabla\phi dxt\right| &\le \delta\|\rho_i^N\|_{L^1(0,T;L^1(\mathbb{T}^d))}\|\Delta\phi\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\to 0,\\ \left|\delta\varepsilon\int_0^T\int_{\mathbb{T}^d}(u_i^N\otimes\nabla\rho_i^N):\nabla\phi dxdt\right| &\le 2\delta\varepsilon\|(\rho_i^N)^{1/2}u_i^N\|_{L^2(0,T;L^2(\mathbb{T}^d))}\\ &\quad\times\|\nabla(\rho_i^N)^{1/2}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\|\nabla\phi\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\to 0\end{aligned}$$

as $N\to\infty$ and $\delta\to 0$. These convergences are sufficient to pass to the limit $N\to\infty$ and $\delta\to 0$ in (17).

It remains to perform the limit in the approximate energy inequality (see Lemma 6). This follows from the weak lower semicontinuity of the norms if we show that

$$\begin{aligned}(\rho_i^N)^{1/2}\nabla u_i^N &\rightharpoonup \sqrt{\rho_i}\nabla u_i \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T}^d)),\\ (\rho_i^N)^{1/4}u_i^N &\rightharpoonup \sqrt[4]{\rho_i}u_i \quad \text{weakly in } L^4(0,T;L^4(\mathbb{T}^d)).\end{aligned}$$

In fact, we deduce from the bound on $(\rho_i^N)^{1/4}u_i^N$ that $(\rho_i^N)^{1/4}u_i^N \rightharpoonup y_i$ weakly in $L^4(0,T;$ $L^4(\mathbb{T}^d))$ for some y_i. The strong convergence of $(\rho_i^N)^{1/4}$ in $L^4(0,T;L^4(\mathbb{T}^d))$ and the weak convergence of u_i^N in $L^2(0,T;L^2(\mathbb{T}^d))$ imply that $(\rho_i^N)^{1/4}u_i^N \rightharpoonup \sqrt[4]{\rho_i}u_i$ weakly in $L^{4/3}(0,T;$ $L^{4/3}(\mathbb{T}^d))$ and consequently $y_i=\sqrt[4]{\rho_i}u_i$. The remaining limit has been shown in (25).

3. Proof of Theorem 2

We want to pass to the limit $\varepsilon\to 0$ in (5)–(6). Let $(\rho_i^\varepsilon,u_i^\varepsilon)_{i=1,2}$ be a weak solution to (5)–(6) constructed in Theorem 1. We set $\rho^\varepsilon=(\rho_1^\varepsilon,\rho_2^\varepsilon)$, $u^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon)$, and $\bar\rho^\varepsilon=\rho_1^\varepsilon+\rho_2^\varepsilon$. The energy and entropy estimates from Corollaries 7 and 9 yield in the limit $N\to\infty$ and $\delta\to 0$ the existence of a constant $C>0$ independent of ε such that for $i=1,2$,

$$\begin{aligned}\|\rho_i^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}+\|\bar\rho^\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^d))}+\|(\rho_i^\varepsilon)^{1/2}u_i^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^d))} &\le C,\\ \sqrt{\varepsilon}\|(\rho_i^\varepsilon)^{1/2}\|_{L^2(0,T;H^2(\mathbb{T}^d))}+\sqrt[4]{\varepsilon}\|(\rho_i^\varepsilon)^{1/4}\|_{L^4(0,T;W^{1,4}(\mathbb{T}^d))} &\le C,\\ \sqrt{\varepsilon}\|u_i^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^d))}+\sqrt[4]{\varepsilon}\|(\rho_i^\varepsilon)^{1/4}u_i^\varepsilon\|_{L^4(0,T;L^4(\mathbb{T}^d))} &\le C.\end{aligned} \tag{27}$$

This shows that $\rho_i^\varepsilon u_i^\varepsilon=(\rho_i^\varepsilon)^{1/2}\cdot(\rho_i^\varepsilon)^{1/2}u_i^\varepsilon$ is uniformly bounded in $L^2(0,T;L^{4/3}(\mathbb{T}^d))$. Therefore, there exist subsequences (not relabeled) such that, as $\varepsilon\to 0$,

$$\begin{aligned}\rho_i^\varepsilon &\rightharpoonup \rho_i \quad \text{weakly* in } L^\infty(0,T;L^2(\mathbb{T}^d)),\\ \rho_i^\varepsilon u_i^\varepsilon &\rightharpoonup J_i \quad \text{weakly in } L^2(0,T;L^{4/3}(\mathbb{T}^d)),\ i=1,2.\end{aligned}$$

In particular, $\partial_t\bar\rho^\varepsilon = -\operatorname{div}(\rho_1^\varepsilon u_1^\varepsilon + \rho_2^\varepsilon u_2^\varepsilon)$ is uniformly bounded in $L^2(0,T;W^{1,4}(\mathbb{T}^d)')$. Thanks to the $L^2(0,T;H^1(\mathbb{T}^d))$ bound for $\bar\rho^\varepsilon$, we can apply the Aubin–Lions compactness lemma to conclude that, up to a subsequence,

$$\bar\rho^\varepsilon \to \bar\rho \quad \text{strongly in } L^2(0,T;L^p(\mathbb{T}^d)),\ p<6.$$

Then the limit in the sum of (5) over $i=1,2$, namely $\partial_t\bar\rho^\varepsilon + \operatorname{div}(\rho_1^\varepsilon u_1^\varepsilon + \rho_2^\varepsilon u_2^\varepsilon) = 0$, shows that $\bar\rho$ solves

$$\partial_t\bar\rho + \operatorname{div}(J_1+J_2) = 0. \tag{28}$$

The limit in (6) is more involved, and we perform the limit only in the sum over $i=1,2$. We treat the various expressions term by term. First, the sequence

$$\rho_i^\varepsilon u_i^\varepsilon \otimes u_i^\varepsilon = \big((\rho_i^\varepsilon)^{1/2}u_i^\varepsilon\big)\otimes\big((\rho_i^\varepsilon)^{1/2}u_i^\varepsilon\big)$$

is bounded in $L^1(0,T;L^1(\mathbb{T}^d))$, which implies that

$$\varepsilon\operatorname{div}(\rho_i^\varepsilon u_i^\varepsilon\otimes u_i^\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^1(0,T;H^s(\mathbb{T}^d)').$$

Furthermore, again by (27),

$$\sqrt{\varepsilon}\rho_i^\varepsilon\nabla u_i^\varepsilon = \sqrt{\varepsilon}\nabla(\rho_i^\varepsilon u_i^\varepsilon) - 2\sqrt{\varepsilon}((\rho_i^\varepsilon)^{1/2}u_i^\varepsilon)\otimes\nabla(\rho_i^\varepsilon)^{1/2}$$

is uniformly bounded in $L^2(0,T;W^{1,4}(\mathbb{T}^d)') + L^1(0,T;L^1(\mathbb{T}^d))$, and thus

$$\varepsilon\operatorname{div}(\rho_i^\varepsilon\nabla u_i^\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^1(0,T;W^{2,4}(\mathbb{T}^d)').$$

The Korteweg regularization in its weak formulation is estimated as follows:

$$\begin{aligned}
&\left|\varepsilon\int_0^T\int_{\mathbb{T}^d}\Delta(\rho_i^\varepsilon)^{1/2}\big(4(\rho_i^\varepsilon)^{1/4}\nabla(\rho_i^\varepsilon)^{1/4}\cdot\psi + (\rho_i^\varepsilon)^{1/2}\operatorname{div}\psi\big)dxdt\right| \\
&\quad\le \sqrt[4]{\varepsilon}\cdot\sqrt{\varepsilon}\|\Delta(\rho_i^\varepsilon)^{1/2}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\big(4\sqrt[4]{\varepsilon}\|\nabla(\rho_i^\varepsilon)^{1/4}\|_{L^4(0,T;L^4(\mathbb{T}^d))} \\
&\qquad\times\|(\rho_i^\varepsilon)^{1/4}\|_{L^\infty(0,T;L^8(\mathbb{T}^d))}\|\psi\|_{L^4(0,T;L^8(\mathbb{T}^d))} \\
&\qquad+\sqrt[4]{\varepsilon}\|(\rho_i^\varepsilon)^{1/2}\|_{L^\infty(0,T;L^4(\mathbb{T}^d))}\|\operatorname{div}\psi\|_{L^2(0,T;L^4(\mathbb{T}^d))}\big) \le \sqrt[4]{\varepsilon}C \to 0,
\end{aligned}$$

where $\psi\in L^4(0,T;W^{1,4}(\mathbb{T}^d;\mathbb{R}^d))$. The drag forces also vanish in the limit since $\varepsilon u_i^\varepsilon\to 0$ strongly in $L^2(0,T;L^2(\mathbb{T}^d))$ and

$$\varepsilon\rho_i^\varepsilon|u_i^\varepsilon|^2u_i^\varepsilon = \sqrt[4]{\varepsilon}(\rho_i^\varepsilon)^{1/4}\big(\sqrt[4]{\varepsilon}(\rho_i^\varepsilon)^{1/4}|u_i^\varepsilon|\big)^2\big(\sqrt[4]{\varepsilon}(\rho_i^\varepsilon)^{1/4}u_i^\varepsilon\big) = O(\sqrt[4]{\varepsilon})\to 0$$

strongly in $L^{4/3}(0,T;L^{8/7}(\mathbb{T}^d))$. Finally,

$$\sum_{i=1}^2\rho_i^\varepsilon\nabla\bar\rho^\varepsilon = \bar\rho^\varepsilon\nabla\bar\rho^\varepsilon \rightharpoonup \bar\rho\nabla\bar\rho \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

Hence, passing to the limit $\varepsilon\to 0$ in sum of the weak formulation (13) over $i=1,2$, we find that

$$k_1^{-1}J_1 + k_2^{-1}J_2 = -\bar\rho\nabla\bar\rho,$$

or $J_1 = -k_1\bar\rho\nabla\bar\rho - (k_1/k_2)J_2$. Together with (28), $\bar\rho$ solves

$$\partial_t\bar\rho = k_1\operatorname{div}(\bar\rho\nabla\bar\rho) - \left(1-\frac{k_1}{k_2}\right)\operatorname{div} J_2 = \operatorname{div}\left(k_1\bar\rho\nabla\bar\rho - \left(1-\frac{k_1}{k_2}\right)J_2\right).$$

Remark 12. If we can identify $J_2 = -k_2\rho_2\nabla\bar\rho$, we obtain

$$k_1\bar\rho\nabla\bar\rho + \left(1-\frac{k_1}{k_2}\right)k_2\rho_2\nabla\bar\rho = (k_1\rho_1 + k_2\rho_2)\nabla\bar\rho,$$

such that the limit equation becomes

$$\partial_t\bar\rho + \operatorname{div}\big((k_1\rho_1+k_2\rho_2)\nabla\bar\rho\big) = 0.$$

4. Proof of Theorem 3: relative entropy inequality

The solution $(\bar\rho,\bar u)$ with $\bar u = -\nabla\bar\rho$ is shown to solve fluid-type equations for which we derive an associated energy equality. Then we derive an inequality for the difference of the mass and momentum balance equations for $(\rho^\varepsilon,u^\varepsilon)$ and $(\bar\rho,\bar u)$. These results allow us to prove the relative entropy inequality and to finish the proof of Theorem 3.

4.1. **Energy equality for the limit system.** We analyze the limit system for $\bar\rho = \rho_1+\rho_2$ and $\bar u = -\nabla\bar\rho$. The mass balance equation reads as

$$\partial_t\bar\rho + \operatorname{div}(\bar\rho\bar u) = 0, \tag{29}$$

with initial condition $\bar\rho(0) = \bar\rho^0 := \rho_1^0 + \rho_2^0$. Inserting (29), the analog of the momentum balance equation for $\bar u$ becomes

$$\partial_t(\bar\rho\bar u) + \operatorname{div}(\bar\rho\bar u\otimes\bar u) = \bar\rho(\partial_t\bar u + (\bar u\cdot\nabla)\bar u) = \bar\rho\bar e, \tag{30}$$

where $\bar e := \partial_t\bar u + (\bar u\cdot\nabla)\bar u$ is the material derivative of $\bar u$. The initial condition is $\bar\rho(0)\bar u(0) = -\bar\rho^0\nabla\bar\rho^0$. According to Lemma 5, the weak formulations of (29)–(30) can be written as

$$0 = -\int_0^t\int_{\mathbb{T}^d}\bar\rho\partial_s\phi dxds + \int_{\mathbb{T}^d}\bar\rho\phi\Big|_{s=0}^{s=t}dx + \int_0^t\int_{\mathbb{T}^d}\bar\rho\nabla\bar\rho\cdot\nabla\phi dxds, \tag{31}$$

$$\begin{aligned}0 = &-\varepsilon\int_0^t\int_{\mathbb{T}^d}\bar\rho\bar u\cdot\partial_s\psi dxds + \varepsilon\int_{\mathbb{T}^d}\bar\rho\bar u\cdot\psi\Big|_{s=0}^{s=t}dx \\ &-\varepsilon\int_0^t\int_{\mathbb{T}^d}\bar\rho(\bar u\otimes\bar u):\nabla\psi dxds - \varepsilon\int_0^t\int_{\mathbb{T}^d}\bar\rho\bar e\cdot\psi dxds\end{aligned} \tag{32}$$

for test functions $\phi\in C_0^\infty(\mathbb{T}^d\times[0,t])$ and $\psi\in C_0^\infty(\mathbb{T}^d\times[0,t];\mathbb{R}^d)$.

We derive the energy equality associated to (29)–(30). For this, we use the test function $\phi = \bar\rho - \varepsilon|\bar u|^2/2$ in (31) and recall that $\nabla\bar\rho = -\bar u$:

$$\begin{aligned}0 = &-\int_0^t\int_{\mathbb{T}^d}\bar\rho\partial_s\left(\bar\rho - \frac{\varepsilon}{2}|\bar u|^2\right)dxds + \int_{\mathbb{T}^d}\bar\rho(t)\left(\bar\rho(t) - \frac{\varepsilon}{2}|\bar u(t)|^2\right)dx \\ &-\int_{\mathbb{T}^d}\bar\rho^0\left(\bar\rho^0 - \frac{\varepsilon}{2}|\bar u^0|^2\right)dx + \int_0^t\int_{\mathbb{T}^d}\bar\rho|\bar u|^2dxds + \varepsilon\int_0^t\int_{\mathbb{T}^d}\bar\rho\bar u\cdot\nabla\bar u\cdot\bar u dxds.\end{aligned}$$

Furthermore, with the test function $\psi = \bar{u}$ in (32), we find that

$$0 = -\varepsilon \int_0^t \int_{\mathbb{T}^d} \bar{\rho}\bar{u} \cdot \partial_s \bar{u} dx ds + \varepsilon \int_{\mathbb{T}^d} \bar{\rho}(t)|\bar{u}(t)|^2 dx - \varepsilon \int_{\mathbb{T}^d} \bar{\rho}^0 |\bar{u}^0|^2 dx$$
$$- \varepsilon \int_0^t \int_{\mathbb{T}^d} \bar{\rho}\bar{u} \cdot \nabla \bar{u} \cdot \bar{u} dx ds - \varepsilon \int_0^t \int_{\mathbb{T}^d} \bar{\rho}\bar{e} \cdot \bar{u} dx ds.$$

Adding the last two equations, some terms cancel, and observing that

$$\int_0^t \int_{\mathbb{T}^d} \bar{\rho} \partial_s \bar{\rho} dx ds = \frac{1}{2} \int_{\mathbb{T}^d} \bar{\rho}^2 \Big|_{s=0}^{s=t} dx$$

leads to the energy equality

$$E_0(\bar{\rho}(t), \bar{u}(t)) - E_0(\bar{\rho}^0, \bar{u}^0) + \int_0^t \int_{\mathbb{T}^d} \bar{\rho}|\bar{u}|^2 dx ds = \varepsilon \int_0^t \int_{\mathbb{T}^d} \bar{\rho}\bar{e} \cdot \bar{u} dx ds, \tag{33}$$

where the energy associated to (29)–(30) is defined by

$$E_0(\bar{\rho}, \bar{u}) = \int_{\mathbb{T}^d} \left(\frac{\bar{\rho}^2}{2} + \frac{\varepsilon}{2} \bar{\rho}|\bar{u}|^2 \right) dx.$$

4.2. **Difference of mass balance equations.** We subtract the energy equality (33) from the energy inequality (14) (recall that $k_1 = k_2 = 1$):

$$\begin{aligned} &E(\rho^\varepsilon(t), u^\varepsilon(t) - E_0(\bar{\rho}(t), \bar{u}(t)) + \int_0^t \int_{\mathbb{T}^d} \left(\sum_{i=1}^2 \rho_i^\varepsilon |u_i^\varepsilon|^2 - \bar{\rho}|\bar{u}|^2 \right) dx ds \\ &\quad + \varepsilon \sum_{i=1}^2 \int_0^t \int_{\mathbb{T}^d} \left(\rho_i^\varepsilon |\nabla u_i^\varepsilon|^2 + |u_i^\varepsilon|^2 + \rho_i^\varepsilon |u_i^\varepsilon|^4 \right) dx ds + \varepsilon \int_0^t \int_{\mathbb{T}^d} \bar{\rho}\bar{e} \cdot \bar{u} dx ds \\ &\le E(\rho^0, u^0) - E_0(\bar{\rho}^0, \bar{u}^0). \end{aligned} \tag{34}$$

We use the test function $\phi = \bar{\rho} - \varepsilon|\bar{u}|^2/2$ in the sum of the mass balance equations (12) over $i = 1, 2$ (recalling that $\bar{\rho}^\varepsilon = \rho_1^\varepsilon + \rho_2^\varepsilon$) and subtract the weak formulation (31) with the same test function (observing that the terms at $t = 0$ cancel):

$$\begin{aligned} 0 = &- \int_0^t \int_{\mathbb{T}^d} (\bar{\rho}^\varepsilon - \bar{\rho}) \partial_s \left(\bar{\rho} - \frac{\varepsilon}{2} |\bar{u}|^2 \right) dx ds + \int_{\mathbb{T}^d} (\bar{\rho}^\varepsilon - \bar{\rho})(t) \left(\bar{\rho}(t) - \frac{\varepsilon}{2} |\bar{u}(t)|^2 \right) dx \\ &- \int_0^t \int_{\mathbb{T}^d} \left(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar{\rho}\bar{u} \right) \cdot \nabla \left(\bar{\rho} - \frac{\varepsilon}{2} |\bar{u}|^2 \right) dx ds. \end{aligned} \tag{35}$$

4.3. **Difference of momentum balance equations.** We add the weak formulation (13) with test function $\psi = \bar{u}_i$ for $i = 1, 2$ and subtract the weak formulation (32) for the limit system with the same test function (recalling that $(\bar{\rho}, \bar{u})$ is assumed to be smooth):

$$0 = -\varepsilon \int_0^t \int_{\mathbb{T}^d} \left(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar{\rho}\bar{u} \right) \cdot \partial_s \bar{u} dx ds + \varepsilon \int_{\mathbb{T}^d} \left(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar{\rho}\bar{u} \right) \cdot \bar{u} \Big|_{s=0}^{s=t} dx \tag{36}$$

$$
\begin{aligned}
&- \varepsilon \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon \otimes u_i^\varepsilon - \bar\rho \bar u \otimes \bar u \bigg) : \nabla \bar u dxds \\
&+ \varepsilon \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 \big\{ \rho_i^\varepsilon \nabla u_i^\varepsilon : \nabla \bar u + \Delta (\rho_i^\varepsilon)^{1/2} \big(2\nabla (\rho_i^\varepsilon)^{1/2} \cdot \bar u + (\rho_i^\varepsilon)^{1/2} \operatorname{div} \bar u \big) \big\} dxds \\
&+ \varepsilon \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 \big(u_i^\varepsilon \cdot \bar u + \rho_i^\varepsilon |u_i^\varepsilon|^2 u_i^\varepsilon \cdot \bar u \big) dxds + \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar\rho \bar u \bigg) \cdot \bar u dxds \\
&+ \int_0^t \int_{\mathbb{T}^d} \big(\bar\rho^\varepsilon \nabla \bar\rho^\varepsilon - \bar\rho \nabla \bar\rho \big) \cdot \bar u dxds + \varepsilon \int_0^t \int_{\mathbb{T}^d} \bar\rho \bar e \cdot \bar u dxds.
\end{aligned}
$$

4.4. **Relative energy inequality.** We recall the definition of the relative energy:

$$
E_R(\rho^\varepsilon, u^\varepsilon | \bar\rho, \bar u) = \int_{\mathbb{T}^d} \bigg\{ \frac{1}{2} (\bar\rho^\varepsilon - \bar\rho)^2 + \varepsilon \sum_{i=1}^2 \bigg(\frac{\rho_i^\varepsilon}{2} |u_i^\varepsilon - \bar u|^2 + |\nabla (\rho_i^\varepsilon)^{1/2}|^2 \bigg) \bigg\} dx,
$$

where $\bar\rho^\varepsilon := \rho_1^\varepsilon + \rho_2^\varepsilon$. The main task is to derive the relative entropy inequality.

Lemma 13 (Relative energy inequality)**.** *There exists a constant $C > 0$ independent of ε such that*

$$
\begin{aligned}
&E_R(\rho^\varepsilon(t), u^\varepsilon(t) | \bar\rho(t), \bar u(t)) + \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 \rho_i^\varepsilon |u_i^\varepsilon - \bar u|^2 dxds \\
&\quad + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 \big(\rho_i^\varepsilon |\nabla (u_i^\varepsilon - \bar u)|^2 + |u_i^\varepsilon - \bar u|^2 + \rho_i^\varepsilon |u_i^\varepsilon|^2 |u_i^\varepsilon - \bar u|^2 \big) dxds \\
&\le C \sqrt[4]{\varepsilon} + C \int_0^t E_R(\rho^\varepsilon, u^\varepsilon | \bar\rho, \bar u)(s) ds.
\end{aligned}
$$

In particular, for $i = 1, 2$, as $\varepsilon \to 0$,

$$
(\rho_i^\varepsilon)^{1/2} (u_i^\varepsilon - \bar u) \to 0 \quad \textit{strongly in } L^2(0, T; L^2(\mathbb{T}^d)).
$$

Proof. We subtract (35) and (36) from (34). An elementary computation shows that

$$
\int_{\mathbb{T}^d} \bigg(\frac{1}{2} (\bar\rho^\varepsilon - \bar\rho)^2 + \frac{\varepsilon}{2} \sum_{i=1}^2 \rho_i^\varepsilon |u_i^\varepsilon - \bar u|^2 + \varepsilon \sum_{i=1}^2 |\nabla (\rho_i^\varepsilon)^{1/2}|^2 \bigg) \bigg|_{s=0}^{s=t} dx = J_1 + \cdots + J_{11}, \tag{37}
$$

where

$$
\begin{aligned}
J_1 &= - \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 \rho_i^\varepsilon |u_i^\varepsilon - \bar u|^2 dxds, \\
J_2 &= - \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar\rho^\varepsilon \bar u \bigg) \cdot \bar u dxds,
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= -\int_0^t \int_{\mathbb{T}^d} (\bar\rho^\varepsilon - \bar\rho)\partial_s\Big(\bar\rho - \frac{\varepsilon}{2}|\bar u|^2\Big)dxds, \\
J_4 &= -\varepsilon \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar\rho\bar u\bigg)\cdot \partial_s \bar u dxds, \\
J_5 &= -\int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar\rho\bar u\bigg)\cdot \nabla\Big(\bar\rho - \frac{\varepsilon}{2}|\bar u|^2\Big)dxds, \\
J_6 &= -\varepsilon \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon \otimes u_i^\varepsilon - \bar\rho\bar u \otimes \bar u\bigg) : \nabla \bar u dxds, \\
J_7 &= \int_0^t \int_{\mathbb{T}^d} (\bar\rho^\varepsilon \nabla \bar\rho^\varepsilon - \bar\rho\nabla\bar\rho)\cdot \bar u dxds, \\
J_8 &= -\varepsilon \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 \rho_i^\varepsilon \nabla u_i^\varepsilon : \nabla(u_i^\varepsilon - \bar u)dxds, \\
J_9 &= -\varepsilon \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 u_i^\varepsilon \cdot (u_i^\varepsilon - \bar u)dxds, \\
J_{10} &= -\varepsilon \int_0^t \int_{\mathbb{T}^d} \sum_{i=1}^2 \rho_i^\varepsilon |u_i^\varepsilon|^2 u_i^\varepsilon \cdot (u_i^\varepsilon - \bar u)dxds, \\
J_{11} &= \varepsilon \int_0^t \int_{\mathbb{T}^d} \Delta(\rho_i^\varepsilon)^{1/2}\big(2\nabla(\rho_i^\varepsilon)^{1/2}\cdot \bar u + (\rho_i^\varepsilon)^{1/2}\operatorname{div}\bar u\big)dxds.
\end{aligned}
$$

The expression on the left-hand side of (37) at $t = 0$ is of order ε, because of Assumption (A2) on the initial data and $(\rho^\varepsilon - \bar\rho)(0) = 0$. We wish to estimate $J_1, \ldots, J_{11}$.

We split the sum $J_3 + \cdots + J_6$ into two parts, $J_3 + \cdots + J_6 = K_1 + K_2$, where

$$
\begin{aligned}
K_1 = \frac{\varepsilon}{2}\int_0^t \int_{\mathbb{T}^d} &(\bar\rho^\varepsilon - \bar\rho)\partial_s|\bar u|^2 dxds - \varepsilon \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar\rho\bar u\bigg)\cdot \partial_s \bar u dxds \\
&+ \frac{\varepsilon}{2}\int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar\rho\bar u\bigg)\nabla|\bar u|^2 dxds \\
&- \varepsilon \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon \otimes u_i^\varepsilon - \bar\rho\bar u \otimes \bar u\bigg) : \nabla\bar u dxds \\
&=: K_{11} + \cdots K_{14}, \\
K_2 = -\int_0^t \int_{\mathbb{T}^d} &(\bar\rho^\varepsilon - \bar\rho)\partial_s \bar\rho dxds - \int_0^t \int_{\mathbb{T}^d} \bigg(\sum_{i=1}^2 \rho_i^\varepsilon u_i^\varepsilon - \bar\rho\bar u\bigg)\cdot \nabla\bar\rho dxds.
\end{aligned}
$$

Some terms cancel in $K_{11}+K_{12}$, and we end up with

$$K_{11}+K_{12}=-\varepsilon\int_0^t\int_{\mathbb{T}^d}\bigg(\sum_{i=1}^2\rho_i^\varepsilon u_i^\varepsilon-\bar\rho^\varepsilon\bar u\bigg)\cdot\partial_s\bar u dxds.$$

Also in the sum $K_{13}+K_{14}$, some terms cancel:

$$\begin{aligned}K_{13}+K_{14}&=-\varepsilon\int_0^t\int_{\mathbb{T}^d}\bigg(\sum_{i=1}^2\rho_i^\varepsilon u_i^\varepsilon-\bar\rho^\varepsilon\bar u\bigg)\cdot\nabla\bar u\cdot\bar u dxds\\&\quad-\varepsilon\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\rho_i^\varepsilon(u_i^\varepsilon-\bar u)\otimes(u_i^\varepsilon-\bar u):\nabla\bar u dxds.\end{aligned}$$

Recalling that $\bar e=\partial_t\bar u+\bar u\cdot\nabla\bar u$, we obtain

$$\begin{aligned}K_1&=-\varepsilon\int_0^t\int_{\mathbb{T}^d}\bigg(\sum_{i=1}^2\rho_i^\varepsilon u_i^\varepsilon-\bar\rho^\varepsilon\bar u\bigg)\cdot\bar e dxds\\&\quad-\varepsilon\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\rho_i^\varepsilon(u_i^\varepsilon-\bar u)\otimes(u_i^\varepsilon-\bar u):\nabla\bar u dxds.\end{aligned}$$

Since $\bar e$ and $\bar u$ are assumed to be smooth, we can estimate K_1 according to

$$\begin{aligned}K_1&\le\varepsilon C(\bar e)\|\rho_1^\varepsilon u_1^\varepsilon+\rho_2^\varepsilon u_2^\varepsilon-\bar\rho^\varepsilon\bar u\|_{L^2(0,T;L^1(\mathbb{T}^d))}\\&\quad+\varepsilon C(\nabla\bar u)\int_0^t\int_{\mathbb{T}^d}\rho_i^\varepsilon|u_i^\varepsilon-\bar u|^2dxds\le C\varepsilon,\end{aligned}$$

where the last step follows from estimates (27).

For K_2, we insert $\partial_t\bar\rho=-\operatorname{div}(\bar\rho\bar u)$, use $\nabla\bar\rho=-\bar u$, and integrate by parts:

$$K_2=-\int_0^t\int_{\mathbb{T}^d}\nabla(\bar\rho^\varepsilon-\bar\rho)\cdot(\bar\rho\bar u)dxds+\int_0^t\int_{\mathbb{T}^d}\bigg(\sum_{i=1}^2\rho_i^\varepsilon u_i^\varepsilon-\bar\rho\bar u\bigg)\cdot\bar u dxds.$$

We add to this expression

$$J_2+J_7=-\int_0^t\int_{\mathbb{T}^d}\bigg(\sum_{i=1}^2\rho_i^\varepsilon u_i^\varepsilon-\bar\rho^\varepsilon\bar u\bigg)\cdot\bar u dxds+\int_0^t\int_{\mathbb{T}^d}(\bar\rho^\varepsilon\nabla\bar\rho^\varepsilon-\bar\rho\nabla\bar\rho)\cdot\bar u dxds,$$

which yields, using again $\bar u=-\nabla\bar\rho$ and integration by parts,

$$\begin{aligned}K_2+J_2+J_7&=-\int_0^t\int_{\mathbb{T}^d}\big(\bar\rho(\nabla\bar\rho^\varepsilon-\nabla\bar\rho)+(\bar\rho\bar u-\bar\rho^\varepsilon\bar u)-(\bar\rho^\varepsilon\nabla\bar\rho^\varepsilon-\bar\rho\nabla\bar\rho)\big)\cdot\bar u dxds\\&=\int_0^t\int_{\mathbb{T}^d}(\bar\rho^\varepsilon-\bar\rho)\nabla(\bar\rho^\varepsilon-\bar\rho)\cdot\bar u dxds=-\frac12\int_0^t\int_{\mathbb{T}^d}(\bar\rho^\varepsilon-\bar\rho)^2\operatorname{div}\bar u dxds\\&\le C(\bar u)\int_0^t\int_{\mathbb{T}^d}(\bar\rho^\varepsilon-\bar\rho)^2dxds.\end{aligned}$$

This shows that

$$J_2+\cdots+J_7 = K_1+K_2+J_2+J_7 \le C\varepsilon + C\int_0^t\int_{\mathbb{T}^d}(\bar\rho^\varepsilon-\bar\rho)^2dxds.$$

Next, we estimate J_8, using the Young and Cauchy–Schwarz inequalities:

$$\begin{aligned} J_8 &= -\varepsilon\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\big(\rho_i^\varepsilon|\nabla(u_i^\varepsilon-\bar u)|^2+\rho_i^\varepsilon\nabla\bar u:\nabla(u_i^\varepsilon-\bar u)\big)dxds \\ &\le -\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\rho_i^\varepsilon|\nabla(u_i^\varepsilon-\bar u)|^2dxds+\varepsilon\sum_{i=1}^2\|\rho_i^\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}^d))}\|\nabla\bar u\|^2_{L^2(0,T;L^\infty(\mathbb{T}^d))} \\ &\le -\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\rho_i^\varepsilon|\nabla(u_i^\varepsilon-\bar u)|^2dxds+C\varepsilon, \end{aligned}$$

since the total mass of ρ_i^ε is uniformly bounded. We estimate J_9 and J_{10} in a similar way, leading to

$$J_9+J_{10}\le -\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\big(|u_i^\varepsilon-\bar u|^2+\rho_i^\varepsilon|u_i^\varepsilon|^2|u_i^\varepsilon-\bar u|^2\big)dxds+C\varepsilon.$$

By the uniform estimates (27),

$$\begin{aligned} J_{11} &\le \sqrt[4]{\varepsilon}\sum_{i=1}^2\sqrt{\varepsilon}\|\Delta(\rho_i^\varepsilon)^{1/2}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\big(4\sqrt[4]{\varepsilon}\|\nabla(\rho_i^\varepsilon)^{1/4}\|_{L^4(0,T;L^4(\mathbb{T}^d))} \\ &\quad\times\|(\rho_i^\varepsilon)^{1/4}\|_{L^4(0,T;L^4(\mathbb{T}^d))}\|\bar u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \\ &\quad+\|(\rho_i^\varepsilon)^{1/2}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}\|\operatorname{div}\bar u\|_{L^2(0,T;L^\infty(\mathbb{T}^d))}\big)\le C(\bar u)\sqrt[4]{\varepsilon}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &J_8+\cdots+J_{11} \\ &\quad\le C\sqrt[4]{\varepsilon}-\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\big(\rho_i^\varepsilon|\nabla(u_i^\varepsilon-\bar u)|^2+|u_i^\varepsilon-\bar u|^2+\rho_i^\varepsilon|u_i^\varepsilon|^2|u_i^\varepsilon-\bar u|^2\big)dxds. \end{aligned}$$

We summarize the previous estimates to infer from (37) that

$$\begin{aligned} &\int_{\mathbb{T}^d}\bigg(\frac12(\rho^\varepsilon-\bar\rho)^2+\frac{\varepsilon}{2}\sum_{i=1}^2\rho_i^\varepsilon|u_i^\varepsilon-\bar u|^2+\varepsilon\sum_{i=1}^2|\nabla(\rho_i^\varepsilon)^{1/2}|^2\bigg)(t)dx \\ &\qquad+\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\rho_i^\varepsilon|u_i^\varepsilon-\bar u|^2dxds \\ &\qquad+\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^d}\sum_{i=1}^2\big(\rho_i^\varepsilon|\nabla(u_i^\varepsilon-\bar u)|^2+|u_i^\varepsilon-\bar u|^2+\rho_i^\varepsilon|u_i^\varepsilon|^2|u_i^\varepsilon-\bar u|^2\big)dxds \end{aligned}$$

$$\leq C\sqrt[4]{\varepsilon} + C\int_0^t\int_{\mathbb{T}^d}(\bar\rho^\varepsilon - \bar\rho)^2 dxds.$$

Gronwall's lemma concludes the proof of the lemma. □

4.5. **Proof of Theorem 3.** It follows from Lemma 13 that

$$\|\rho_i^\varepsilon(u_i^\varepsilon - \bar u)\|_{L^1(0,T;L^1(\mathbb{T}^d))} \leq \|(\rho_i^\varepsilon)^{1/2}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\|(\rho_i^\varepsilon)^{1/2}(u_i^\varepsilon - \bar u)\|_{L^2(0,T;L^2(\mathbb{T}^d))} \to 0$$

as $\varepsilon \to 0$. Together with the convergence $\rho_i^\varepsilon \rightharpoonup \rho_i$ weakly* in $L^\infty(0,T;L^2(\mathbb{T}^d))$, this shows that

$$\rho_i^\varepsilon u_i^\varepsilon = \rho_i^\varepsilon(u_i^\varepsilon - \bar u) + \rho_i^\varepsilon \bar u \rightharpoonup \rho_i \bar u \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

By definition, $\bar u = -\nabla\bar\rho$. Since $\rho_i^\varepsilon u_i^\varepsilon \rightharpoonup J_i$ weakly in $L^2(0,T;L^{4/3}(\mathbb{T}^d))$, we infer that $J_i = -\rho_i\nabla\bar\rho$. In particular, ρ_i solves the transport equation $\partial_t\rho_i = -\operatorname{div} J_i = \operatorname{div}(\rho_i\nabla\bar\rho)$, while $\bar\rho$ is the solution to the porous-medium equation $\partial_t\bar\rho = \operatorname{div}(\bar\rho\nabla\bar\rho)$ with $\bar\rho(0) = \rho_1^0 + \rho_2^0$ in $\mathbb{T}^d$.

Data availability

There is no data associated to this work.

Conflict of interest

The authors have no conflict of interest to disclose.

References

[1] N. J. Alves, J. A. Carrillo, and Y.-P. Choi, Weak-strong uniqueness and high-friction limit for Euler-Riesz systems, *Commun. Math. Anal. Appl.* 3 (2024), 266–286.
[2] N. J. Alves, L. Grafakos, and A. E. Tzavaras, A bilinear fractional integral operator for Euler-Riesz systems, preprint ArXiv 2409.18309, (2024).
[3] P. Antonelli, G. C. Carnevale, C. Lattanzio, and S. Spirito. Relaxation limit from the quantum Navier–Stokes equations to the quantum drift-diffusion equations. *J. Nonlin. Sci.* 31 (2021), no. 71.
[4] M. Bertsch, M. E. Gurtin, D. Hilhorst, and L. A. Peletier, On interacting populations that disperse to avoid crowding: preservation of segregation, *J. Math. Biol.* 23 (1985), 1–13.
[5] M. Bertsch, R. Dal Passo, and M. Mimura. A free boundary problem arising in a simplified tumour growth model of contact inhibition. *Interfaces Free Bound.* 12 (2010), 235–250.
[6] D. Bresch and B. Desjardins. On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids. *J. Math. Pures Appl.* 87 (2007), 57–90.
[7] S. Busenberg and C. Travis. Epidemic models with spatial spread due to population migration. *J. Math. Biol.* 16 (1983), 181–198.
[8] G. C. Carnevale and C. Lattanzio. High friction limits of Euler–Navier–Stokes–Korteweg equations for multicomponent models. *Commun. Math. Sci.* 21 (2023), 1815–1863.
[9] J. A. Carrillo, S. Fagioli, F. Santambrogio, and M. Schmidtchen. Splitting schemes and segregation in reaction cross-diffusion systems. *SIAM J. Math. Anal.* 50 (2018), 5695–5718.
[10] J. A. Carrillo and S. Guo. Interacting particle approximation of cross-diffusion systems. Submitted for publication, 2024. arXiv: 2402.05094.
[11] J. A. Carrillo, Y. Huang, and M. Schmidtchen. Zoology of a nonlocal cross-diffusion model for two species. *SIAM J. Appl. Math.* 78 (2018), 1078–1104.

[12] J. A. Carrillo, H. Murakawa, M. Sato, H. Togashi, and O. Trush. A population dynamics model of cell-cell adhesion incorporating population pressure and density saturation. *J. Theor. Biol.* 474 (2019), 14–24.

[13] J. A. Carrillo, Y. Peng, and A. Wróblewska-Kamińska, Relative entropy method for the relaxation limit of hydrodynamic models, *Netw. Heterog. Media* 15 (2020), 369–387.

[14] L. Chen, E. Daus, and A. Jüngel. Rigorous mean-field limit and cross-diffusion. *Z. Angew. Math. Phys.* 70 (2019), no. 122, 21 pages.

[15] C. Dafermos. The second law of thermodynamics and stability. *Arch. Ration. Mech. Anal.* 70 (1979), 167–179.

[16] R. DiPerna. Uniqueness of solutions to hyperbolic conservation laws. *Indiana Univ. Math. J.* 28 (1979), 137–188.

[17] D. Donatelli and P. Marcati. Convergence of singular limits for multi-D semilinear hyperbolic systems to parabolic systems. *Trans. Amer. Math. Soc.* 356 (2004), 2093–2121.

[18] P.-E. Druet, K. Hopf, and A. Jüngel. Hyperbolic-parabolic normal form and local classical solutions for cross-diffusion systems with incomplete diffusion. *Commun. Partial Differ. Eqs.* 48 (2023), 863–894.

[19] P.-E. Druet and A. Jüngel. Analysis of cross-diffusion systems for fluid mixtures driven by a pressure gradient. *SIAM J. Math. Anal.* 52 (2020), 2179–2197.

[20] E. Feireisl. *Dynamics of Viscous Compressible Fluids*. Oxford University Press, Oxford, 2004.

[21] M. Gurtin and A. Pipkin. A note on interacting populations that disperse to avoid crowding. *Quart. Appl. Math.* 42 (1984), 87–94.

[22] K. Hopf and A. Jüngel. Convergence of a finite volume scheme and dissipative measure-valued–strong stability for a hyperbolic–parabolic cross-diffusion system. Submitted for publication, 2023. arXiv:2304.00787.

[23] X. Huo, A. Jüngel, and A. Tzavaras. High-friction limits of Euler flows for multicomponent systems. *Nonlinearity* 32 (2019), 2875–2913.

[24] A. Jüngel. Global weak solutions to compressible Navier–Stokes equations for quantum fluids. *SIAM J. Math. Anal.* 42 (2010), 1025–1045.

[25] A. Jüngel, H.-L. Li, and A. Matsumura. The relaxation-time limit in the quantum hydrodynamic equations for semiconductors. *J. Differ. Eqs.* 225 (2006), 440–464.

[26] A. Jüngel and D. Matthes. The Derrida–Lebowith–Speer–Spohn equation. *SIAM J. Math. Anal.* 39 (2008), 1996–2015.

[27] A. Jüngel and Y.-J. Peng. A hierarchy of hydrodynamic models for plasmas: zero-relaxation-time limits. *Commun. Partial Differ. Eqs.* 24 (1999), 1007–1033.

[28] A. Jüngel, S. Portisch, and A. Zurek. Nonlocal cross-diffusion systems for multi-species populations and networks. *Nonlin. Anal.* 219 (2022), no. 112800, 26 pages.

[29] A. Jüngel and A. Zurek. A finite-volume scheme for a cross-diffusion model arising from interacting many-particle population systems. In: R. Klöfkorn, E. Keilegavlen, F. Radu, and J. Fuhrmann (eds.), *Finite Volumes for Complex Applications IX – Methods, Theoretical Aspects, Examples*. Springer Proceedings in Mathematics & Statistics, vol. 323. Springer, Cham, 2020, pp. 223-231.

[30] T. Kobayashi and T. Ogawa. Fluid mechanical approximation to the degenerated drift-diffusion system from the compressible Navier–Stokes–Poisson system. *Indiana Univ. Math. J.* 62 (2013), 1021–1054.

[31] I. Lacroix-Violet and A. Vasseur. Global weak solutions to the compressible quantum Navier–Stokes equation and its semi-classical limit. *J. Math. Pures Appl.* 114 (2017), 191–210.

[32] C. Lattanzio and A. Tzavaras. Relative entropy in diffusive relaxation. *SIAM J. Math. Anal.* 45 (2013), 1563–1584.

[33] H. Murakawa and H. Togashi. Continuous models for cell–cell adhesion. *J. Theor. Biol.* 374 (2015), 1–12.

[34] Y.-J. Peng, S. Wang, and Q. Gu. Relaxation limit and global existence of smooth solutions of compressible Euler–Maxwell equations. *SIAM J. Math. Anal.* 43 (2011), 944–970.

[35] A. Vasseur and C. Yu. Global weak solutions to the compressible quantum Navier–Stokes equations with damping. *SIAM J. Math. Anal.* 48 (2016) 1489–1511.

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