# An introduction to center manifold theory

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TECHNISCHE UNIVERSITÄT WIEN

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# Local stability analysis of an equilibrium

- Consider an autonomous ODE

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

with $\mathbf{x} \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth and $f(0) = 0$.

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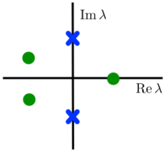
- $\text{Re}(\lambda) \neq 0 \implies$ hyperbolic.

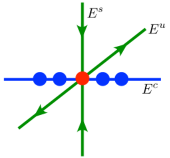
# Solutions of the linearized System $\dot{\mathbf{y}} = L\mathbf{y}$

- Phase space decomposed into invariant eigenspaces:

$$\mathbb{R}^n = E^s \oplus E^c \oplus E^u$$

$$\implies \mathbf{y}(t) = \mathbf{y}^s(t) + \mathbf{y}^c(t) + \mathbf{y}^u(t).$$



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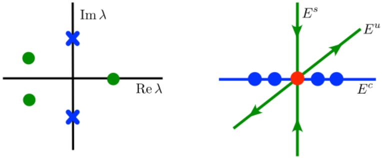


- There are $K, \alpha > 0$ and $m \in \mathbb{N}$ such that
  - $||\mathbf{y}^s(t)|| \leq Ke^{-\alpha t}||\mathbf{y}^s(0)||$ for $t \geq 0$,
  - $||\mathbf{y}^c(t)|| \leq K|t|^m||\mathbf{y}^c(0)||$ for $t \in \mathbb{R}$,
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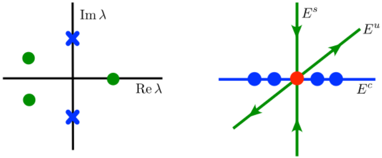


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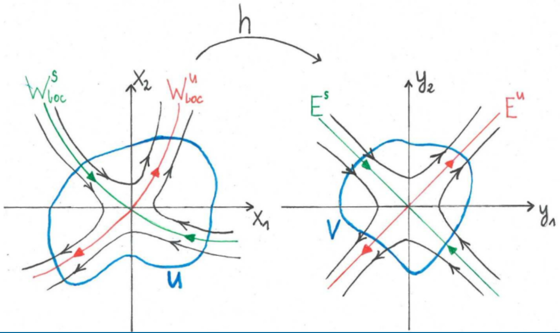
- Dynamics of $\dot{\mathbf{x}} = f(\mathbf{x})$ close to $\bar{\mathbf{x}}$ from its linearization $\dot{\mathbf{y}} = Df(\bar{\mathbf{x}})\mathbf{y} = L\mathbf{y}$?

# Linearization suffices for a hyperbolic equilibrium

**Hartman-Grobman (1960)**

If $\bar{\mathbf{x}}$ is hyperbolic, i.e, $E^c = \{0\} \implies \exists$ neigbourhoods $\bar{\mathbf{x}} \in U$ and $0 \in V$ and a homeomorphism $h : U \to V$ such that the two flows induced by $\dot{\mathbf{x}} = f(\mathbf{x})$ and $\dot{\mathbf{y}} = Df(\hat{\mathbf{x}})\mathbf{y} = A\mathbf{y}$ are topologically conjugated.
In particular $h(\mathbf{x}(t)) = e^{Lt} h(\mathbf{x}(0))$.



# Center manifolds at non-hyperbolic equilibria

In a suitable basis the systems take the form

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \quad f, g \in C^r, \ r \geq 1 \\ \dot{y} &= By + g(x,y), \quad (x,y) \in E^c \times (E^s \times E^u) \end{aligned} \tag{1}$$

**Center manifold**

A locally invariant manifold of (1)

$$W^c = \{(x,y) \in E^c \times (E^s \times E^u) : y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0\}$$

for some $\delta > 0$ and $h \in C^r$ is called a center manifold.

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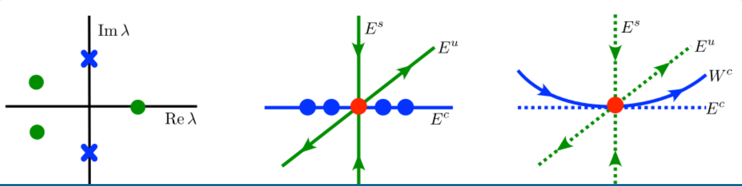


# Existence of a center manifold

**(Existence)**

There exists a $n_c$-dimensional center manifold $W^c$ for system (1) given as a graph $y = h(x)$.

**(Smoothness)**

Nonlinearities $f \in C^r$ and $g \in C^r \implies h \in C^r$ for $1 \leq r < \infty$.
Vector field analytic: $W^c$ is not necessarily analytic.

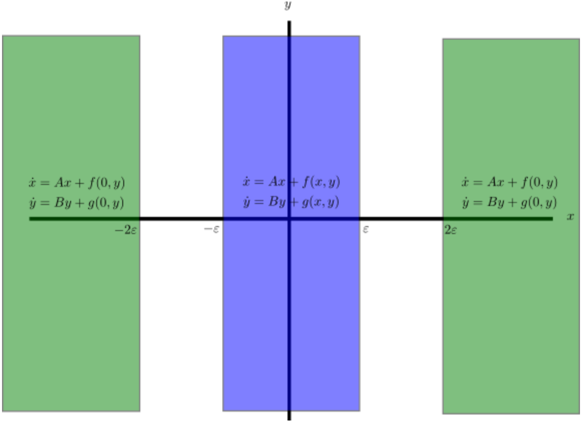
**(Non-Uniqueness)**

$W^c$ in general not unique.

# Sketch of proof for $E^u = \{0\}$

- Localize the analysis with a smooth cut-off $\Psi : \mathbb{R}^{n_c} \to [0,1]$, $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$ such that

$$F(x,y) := f(x\Psi(\frac{x}{\varepsilon}), y), \quad G(x,y) := g(x\Psi(\frac{x}{\varepsilon}), y), \quad \varepsilon > 0.$$



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- Define for $L$, $M > 0$ the Banach space

$$\mathcal{B}_L^M := \{h : E^c \to E^s, Lip(h) \leq L, ||h(x)|| \leq M \, \forall x \in E^c, h(0) = 0\}$$

equipped with $||\cdot||_\infty$.

# Use variation of constants to define an operator $T$.

- For $h \in \mathcal{B}_L^M$ and $x_0 \in E^c$, define $x(t, x_0, h)$ as solution of

$$\dot{x} = Ax + F(x, h(x)),\ x(0, x_0, h) = x_0$$

- Variation of the constant formula for $\dot{y} = By + G(x, y)$:

$$\begin{aligned} & y(s) = C(s)e^{Bs} \\ & \implies \dot{C}(s) = e^{-Bs}G(x(s, x_0, h), h(x(s, x_0, h))) \\ & \implies C(t) - \lim_{s \to -\infty} C(s) = \int_{-\infty}^{t} e^{-Bs}G(x(s, x_0, h), h(x(s, x_0, h)))\, ds \end{aligned}$$

- Note: $C(0) = y(0) = h(x_0)$.
- Want: $\lim_{s \to -\infty} C(s) = 0$ to sort out purely exponentially growing solutions for $s \to -\infty$.

# Show that $T$ is a contraction

- Graph of $h$ is a center manifold if

$$h(x_0) = \int_{-\infty}^{0} e^{-Bs} G(x(s, x_0, h), h(x(s, x_0, h)))\, ds.$$

- Define an operator $T : \mathcal{B}_L^M \to \mathcal{B}_L^M$

$$Th(x_0) := \int_{-\infty}^{0} e^{-Bs} G(x(s, x_0, h), h(x(s, x_0, h)))\, ds.$$

- Show with estimates on $A$, $B$, $F$, and $G$ that $T$ is a contraction for suitable $L$, $M$ and $\varepsilon$ small enough.

# Non-uniqueness of $W^c$

- Consider the system

$$\dot{x} = x^2$$
$$\dot{y} = -y$$

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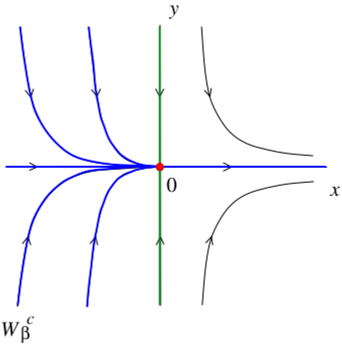
- For all $\beta \in \mathbb{R}$ the graph of

$$h(x) = \begin{cases} \beta e^{\frac{1}{x}}, x < 0 \\ 0, \quad x \geq 0 \end{cases}$$

is a center manifold $W^c_\beta$.



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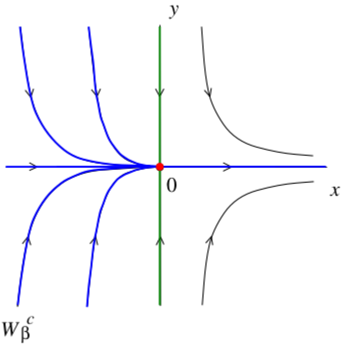


- Non-uniqueness in general since the constructed manifolds depend on cut-off.

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- Center manifolds are exponentially close.
- $n_u = 0$: Center manifold determines stability.
- Especially useful if $\dim(W^c)$ small.

# Computation of a center manifold

- Invariance of $y = h(x)$ gives

$$\dot{y} = Dh(x)\dot{x} \iff Bx + g(x, h(x)) = Dh(x)[Ax + f(x, h(x))]$$

$$\iff \mathcal{N}(h(x)) := Bx + g(x, h(x)) - Dh(x)[Ax + f(x, h(x))] = 0$$

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- Taylor series can usually be computed.

# Rough eigenspace reduction fails

- Consider the system

$$\dot{x} = -xy - x^6$$
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- $E^s = \{x = 0\}$, $E^c = \{y = 0\}$
- Rough eigenspace approximation: $\dot{x} = -x^6$ suggests unstable.

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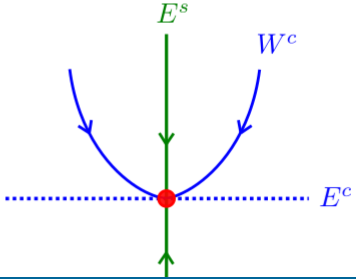
  with equilibrium $(0, 0)$ and eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$.
- Ansatz for $W^c$: $h(x) = ax^2 + bx^3 + \mathcal{O}(x^4)$
- Invariance equation $\mathcal{N}(h(x)) = 0$: $h(x) = x^2 + \mathcal{O}(x^4)$.
- Reduction on $W^c$: $\dot{x} = -x^3 + \mathcal{O}(x^5) \implies$ stable.



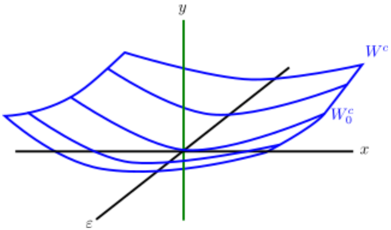
# Extention to parameter dependent systems

Consider a system $\dot{\mathbf{x}} = f(\mathbf{x}, \varepsilon)$ depending on parameters $\varepsilon \in \mathbb{R}^{n_p}$.

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \varepsilon) \\ \dot{\varepsilon} &= 0 \\ \dot{y} &= By + g(x, y, \varepsilon), \quad (x, y) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_s + n_u} \end{aligned} \tag{2}$$

with $\sigma(A) \subset i\mathbb{R}$ and $\sigma(B) \cap i\mathbb{R} = \emptyset$.

- $n_c + n_p$ center directions $\implies$ $W^c$ is a graph $y = h(x, \varepsilon)$.
- $W^c$ exists for $|\varepsilon|$ small enough $\implies$ all bifurcating solutions contained in $W^c$.



# Andronov-Hopf bifurcation

Consider a planar system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \varepsilon), \quad \mathbf{x} \in \mathbb{R}^2, \varepsilon \in \mathbb{R}.$$

- Assume $\bar{\mathbf{x}} = 0$ is equilibrium for $\varepsilon = 0$ with eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$.

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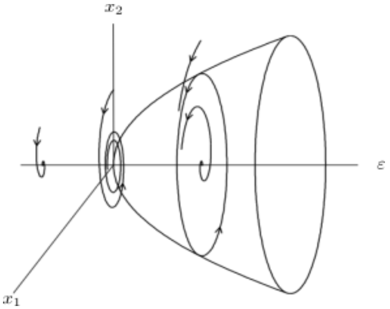
- Assume $\bar{\mathbf{x}} = 0$ is equilibrium for $\varepsilon = 0$ with eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$.
- Generic dependence on $\varepsilon$ and first Lyapunov coefficient $l_1(0) \neq 0$ $\Longrightarrow$ limit cycle bifurcates from $\bar{\mathbf{x}} = 0$.



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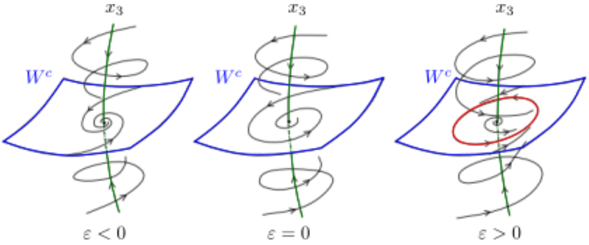
- $\exists$ 2−d attracting center manifold $W^c \implies$ under generic conditions in reduced system: Andronov-Hopf bifurcation on $W^c$.



# Towards center manifolds for infinite-dim systems

- Let $\mathcal{Z} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{X}$ continuously embedded Banach spaces.
- Consider a dynamical system on $\mathcal{X}$ of the form

$$\frac{d\mathbf{u}}{dt} = \mathbf{L}u + \mathbf{N}(u)$$

  with a closed linear operator **L** and **N** nonlinear.
- Spectrum $\sigma$ of **L**

$$\sigma = \sigma_s \cup \sigma_c \cup \sigma_u$$

  with $\sigma_s = \{\lambda \in \sigma : \mathrm{Re}(\lambda) < 0\}$, $\sigma_c = \{\lambda \in \sigma : \mathrm{Re}(\lambda) = 0\}$, $\sigma_u = \{\lambda \in \sigma : \mathrm{Re}(\lambda) > 0\}$.
- A center manifold exists under the following three assumptions A1)-A3).

# Linearization and spectral gap

$$\frac{d\mathbf{u}}{dt} = \mathbf{L}u + \mathbf{N}(u), \quad \sigma = \sigma_s \cup \sigma_c \cup \sigma_u$$

A1)

- $\mathbf{L} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$
- For some $r \geq 2$ $\exists$ neighbourhood $\mathcal{V} \subset \mathcal{Z}$ of 0:

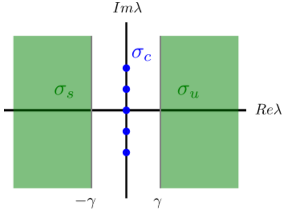
$$\mathbf{N} \in C^r(\mathcal{V}, \mathcal{Y}), \quad \mathbf{N}(0) = D\mathbf{N}(0) = 0.$$

A2)

- $\exists \gamma > 0$ :

$$\inf_{\lambda \in \sigma_u} (\operatorname{Re} \lambda) > \gamma, \quad \sup_{\lambda \in \sigma_s} (\operatorname{Re} \lambda) < -\gamma$$

- $\sigma_c$ only contains finitely many eigenvalues with finite multiplicity $\Longrightarrow$ $E^c$ finite dimensional.



# Solution of the linear problem

$$\frac{d\mathbf{u}}{dt} = \mathbf{L}u + \mathbf{N}(u), \quad \sigma = \sigma_s \cup \sigma_c \cup \sigma_u$$

- For $\nu > 0$ consider Banach space of exponentially growing functions

$$\mathcal{C}_\nu(\mathbb{R}, \mathcal{X}) = \{u \in C^0(\mathbb{R}, \mathcal{X}) : ||u||_{\mathcal{C}_\nu} = \sup_{t \in \mathbb{R}}(e^{-\nu|t|}||u(t)||_{\mathcal{X}}) < \infty\}.$$

A3)

- For any $\nu \in (0, \gamma]$ and $f \in \mathcal{C}_\nu(\mathbb{R}, \mathcal{Y}_h)$ the linear problem

$$\frac{du_h}{dt} = \mathbf{L_h} u_h + f(t)$$

corresponding to $\sigma_s$ and $\sigma_u$ has a unique solution $u_h = \mathbf{K}_h f \in \mathcal{C}_\nu(\mathbb{R}, \mathcal{Z}_h)$.

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$$\frac{du_h}{dt} = \mathbf{L_h}u_h + f(t)$$

corresponding to $\sigma_s$ and $\sigma_u$ has a unique solution $u_h = \mathbf{K}_h f \in \mathcal{C}_\nu(\mathbb{R}, \mathcal{Z}_h)$.

- A1) and A2) relatively easy to check, compared with A3).
- For semilinear PDEs A3) can be replaced with estimates on the resolvent $(L - \lambda I)^{-1}$.

# Center manifold theorem for PDEs

$$\frac{d\mathbf{u}}{dt} = \mathbf{L}u + \mathbf{N}(u) \tag{3}$$

**Theorem**

Assume A1), A2) and A3) hold. Then there exists a map $H \in C^r(E^c, E^s \times E^u)$ with $H(0) = DH(0) = 0$ and a neighbourhood $U$ of the origin such that the manifold

$$W^c = \{u_0 + H(u_0) : u_0 \in E^c\}$$

is locally invariant and contains the set of bounded solutions of (3).

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- Reduction to a finite-dim system.
- Dynamics on $W^c$ is an ODE.

# References


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