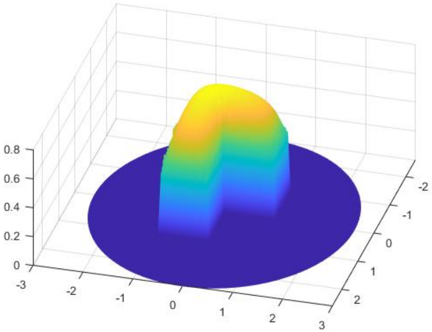
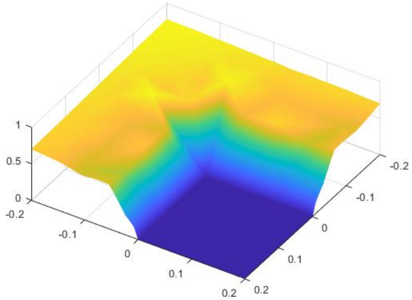
Mesh design principles

Solution with $f = \text{const}$, $s = 0.25$

- Ω Lipschitz polygon





- vertex and edge singularities!

Edge, Vertex, and Edge-Vertex Neighborhoods

- for vertices $\mathbf{v}$ of Ω: vertex weight $r_{\mathbf{v}}(x) := \operatorname{dist}(x, \mathbf{v})$
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- in $\omega_{\mathbf{ve}}$ and $\omega_{\mathbf{e}}$ fitted coordinates: $(x_{\parallel}, x_{\perp})$ and corr. differentiation as $D_{x_{\parallel}}$, $D_{x_{\perp}}$

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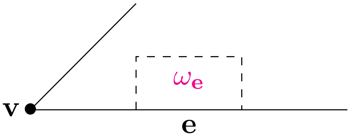

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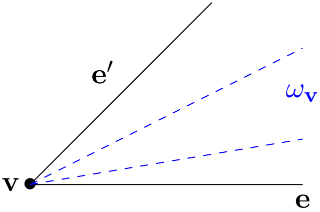

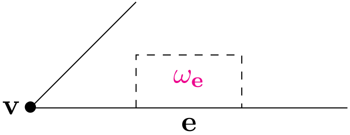

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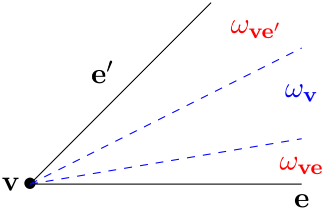

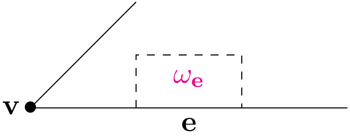

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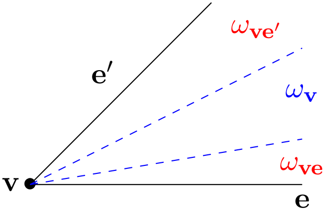

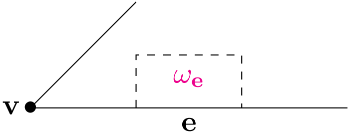

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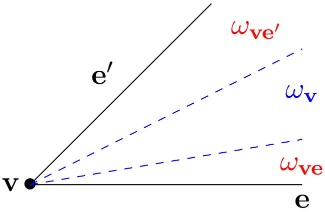

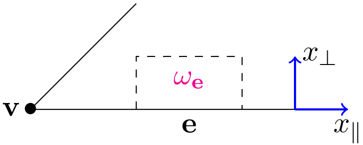

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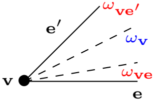
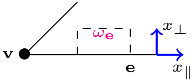
Weighted Analytic Regularity





Theorem (F., Marcati, Melenk, Schwab '21)

$\Omega \subset \mathbb{R}^2$ *polygon,* f *analytic on* $\overline{\Omega}$, u *solution,* ε *arbitrary. Then,* $\forall p \in \mathbb{N}$:

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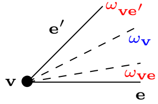

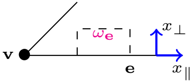

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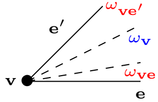

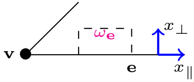

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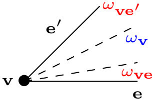

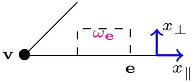

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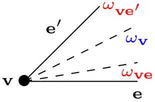

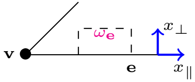

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Remark:

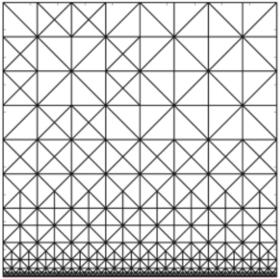
- the presence of the "mitigating" terms $-1/2 - s$ is essential for approximation theory

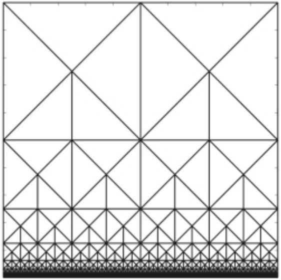
Graded meshes, grading factor $\mu > 1$

- shape-regular mesh
- $h_T \sim h\,\mathrm{dist}(T,\partial\Omega)^{1-1/\mu}$ if $\overline{T} \cap \partial\Omega = \emptyset$
- $h_T \sim h^\mu$ if $\overline{T} \cap \partial\Omega \neq \emptyset$

Convergence behaviour

$$\inf_{v \in \mathcal{P}_0^1(\mathcal{T})} \|u - v\|_{\widetilde{H}^s(\Omega)} \leq C h^{2-s}$$

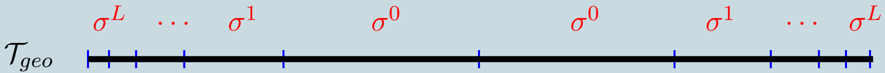




Anisotropic meshes

Geometric grading (non-shape regular)

1D mesh $\mathcal{T}_{geo}$ (L layers of refinement, $\sigma \in (0,1)$)



2d observation: regularity different in different directions

- meshes should be (anisotropically) geometrically graded towards edges
- meshes should be (isotropically) geometrically graded towards corners

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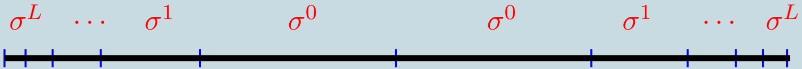

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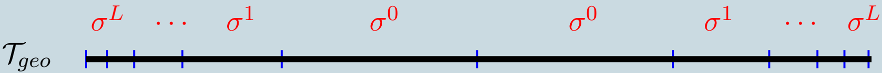

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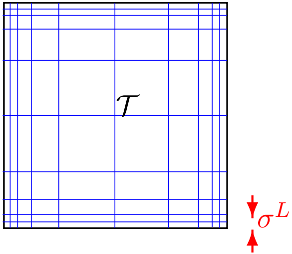
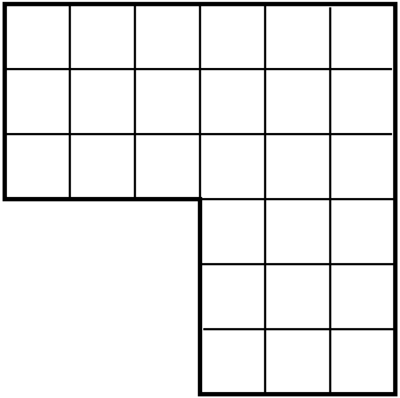


Mesh Construction via Mesh Patches



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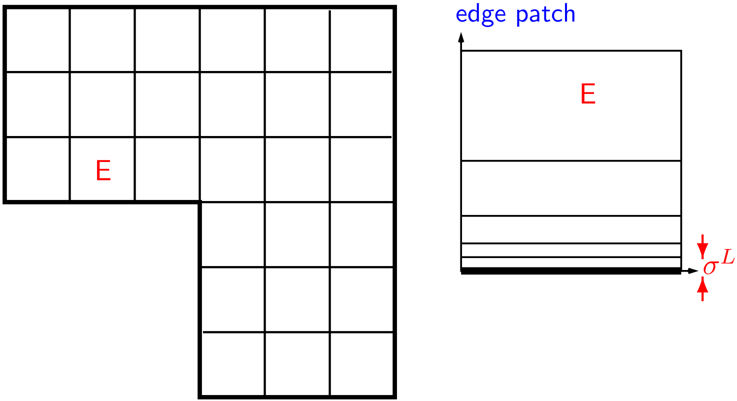

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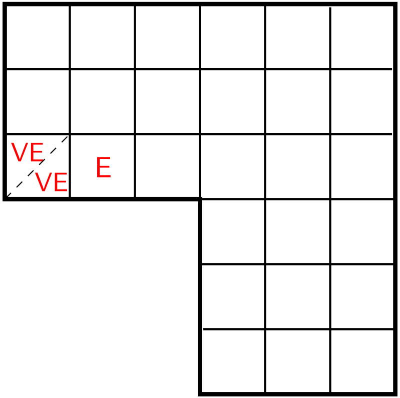
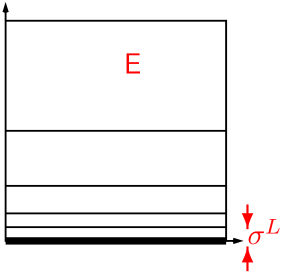
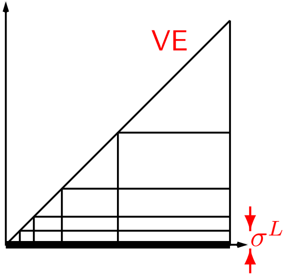


edge patch



vertex-edge patch



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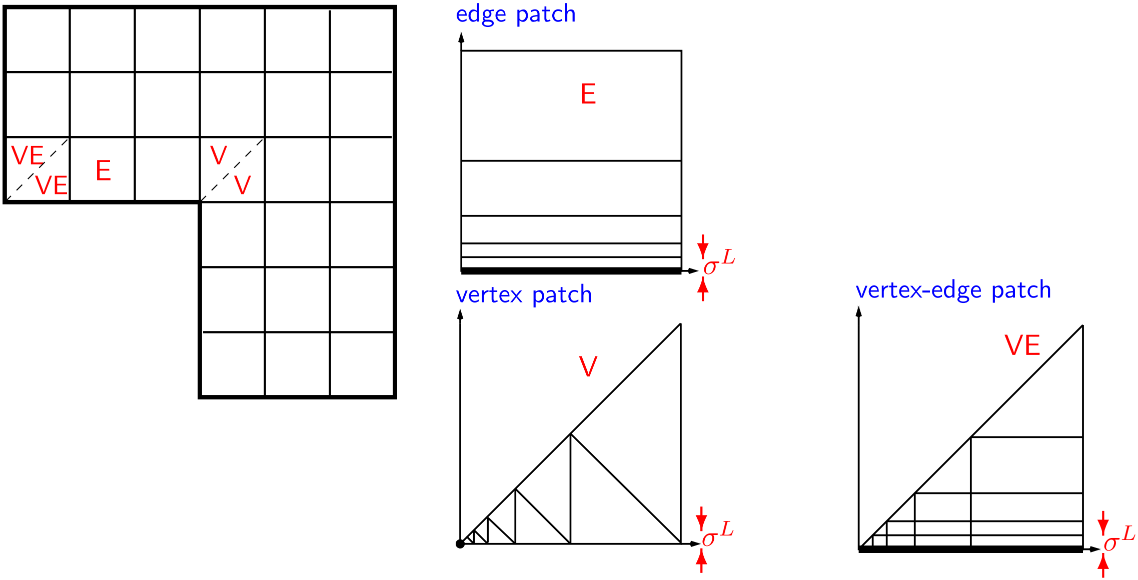

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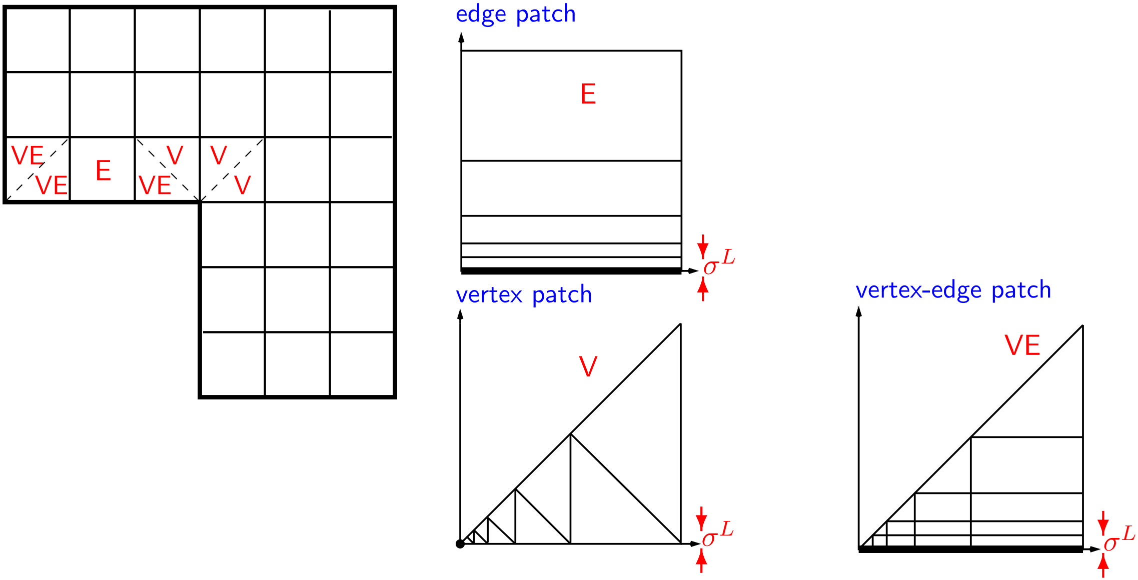

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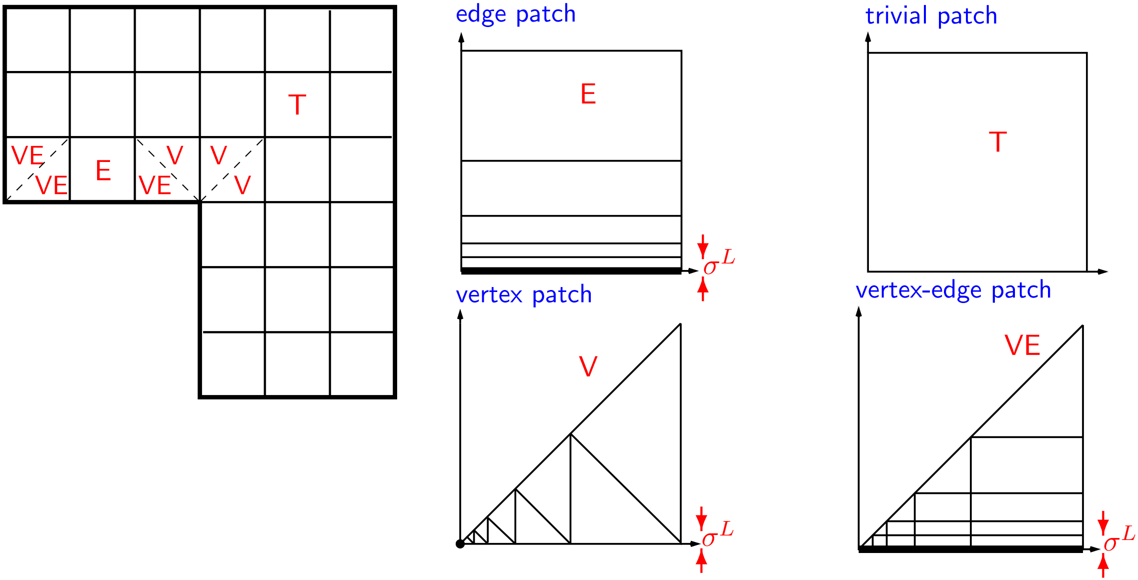

1D Numerics

Convergence behavior

$$\|u - u_N\|_{\widetilde{H}^s(\Omega)} \leq C \exp(-b\sqrt[4]{N})$$