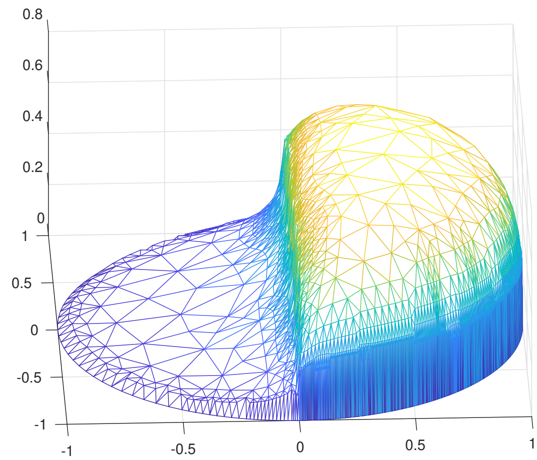
Non-local Operators

(winter term 2024/25)



Lecture Notes

Markus Faustmann, Jens Markus Melenk
Institute of Analysis und Scientific Computing
TU Wien

Contents

Chapter 1

Introduction

1.1 Non-local Operators

In this lecture, we study the analytical behavior of certain non-local operators and present some numerical methods for these.

We call an operator $\mathcal{A} : X \to Y$ acting between two function spaces X, Y of functions $u : \mathbb{R}^d \to \mathbb{R}$ **local**, if for all $x \in \mathbb{R}^d$ the value $(\mathcal{A}u)(x)$ (if point-evaluation is possible) only depends on the values of $u|_{B_\varepsilon(x)}$ for all $\varepsilon > 0$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^d \ : \ |x - y| < \varepsilon\}$ denotes the open ball of radius ε around $x \in \mathbb{R}^d$.

Classical examples of local operators are, e.g., differential operators such as the Laplacian

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(x)$$

on the function space $C^2(\mathbb{R}^d)$, since computing derivatives at a point x only needs the function values in a neighborhood of x.

Conversely, if an operator is not local, we call it a **non-local operator**.

Non-local operators appear oftentimes in physics by modeling of non-local effects such as gravity or quantum entanglement. As a simple mathematical example, we consider the integral operator

$$\mathcal{A}u(x) := \int_0^1 (x - y)u(y)\, dy \qquad u \in L^2(0, 1).$$

From the definition, one directly sees, that the computation of $\mathcal{A}u(x)$ needs all values $u(y)$ for $y \in (0, 1)$. This also leads to the effect, that, even if u is locally supported, e.g., u is the characteristic function of a sub-interval $u = \chi_{(1/4,1/2)}$, we compute that $\operatorname{supp} \mathcal{A}u := \overline{\{x \ : \ \mathcal{A}u(x) \neq 0\}} = [0, 1]$, i.e., $\mathcal{A}u$ does have global support.
This observation is particularly important for numerics, since the discretization of local operators (e.g. by finite element methods for the Laplacian) usually leads to sparse linear systems of equations, which can be solved efficiently. In contrast, non-local operators lead to fully-populated matrices,

and computations with these can be too expensive. Therefore, an additional challenge for non-local operators is to derive efficient numerical methods that circumvent the problem of fully populated systems.

In this lecture, we are primarily concerned with non-local operators of convolution type with singular kernel

$$\mathcal{A}u(x) := \int k(x-y)u(y)\, dy,$$

where the kernel function k is singular for $x = y$. This also includes some definitions of so called fractional differential operators like

$$(-\Delta)^s \qquad \text{for } s \in (0,1),$$

i.e., differential operators with non-integer order of differentiation. A formal definition of those is not straight forward and other definitions, which are not based on singular integrals exist, see Chapter 2.

In the following, we briefly motivate where such classes of operators appear.

1.2 Recasting a PDE as an Integral Equation

Consider the Poisson equation on an unbounded exterior domain $\mathbb{R}^d \backslash \overline{\Omega}$, where Ω is a bounded domain

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \mathbb{R}^d \backslash \Omega, \\ u|_{\partial\Omega} &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

where f, g are given data.
We refer to the PDE lecture, for the fact that there exists a fundamental solution (or Green's function) $G(x,y) = G(x-y)$ satsifying $-\Delta G(x,y) = \delta_y$ (delta distribution). Assuming sufficient regularity, we can use Green's second formula with u and $v(y) = G(x-y)$ to obtain

$$(-\Delta u, v)_{L^2(\mathbb{R}^d \backslash \overline{\Omega})} + (\partial_n u, v)_{L^2(\partial\Omega)} = (u, -\Delta v)_{L^2(\mathbb{R}^d \backslash \overline{\Omega})} + (u, \partial_n v)_{L^2(\partial\Omega)}.$$

Using the definition of the fundamental solution and the PDE, we (formally!) obtain the representation formula

$$u(x) = (f, G(x,\cdot))_{L^2(\mathbb{R}^d \backslash \overline{\Omega})} + (\partial_n u, G(x,\cdot))_{L^2(\partial\Omega)} - (g, \partial_n G(x,\cdot))_{L^2(\partial\Omega)}.$$

This can now be written in terms of the non-local operators (of convolution type) $\widetilde{N}$ (Newton potential), $\widetilde{V}$ (single-layer potential), $\widetilde{K}$ (double-layer potential) as

$$u(x) = \widetilde{N} f(x) + \widetilde{V}\phi(x) - \widetilde{K} g(x) \quad \text{for } x \in \Omega, \tag{1.1}$$

where $\phi := \partial u / \partial n$ and

$$\widetilde{N} f(x) = \int_\Omega G(x,y) f(y) dy, \qquad \widetilde{V}\phi(x) := \int_{\partial\Omega} G(x,y)\phi(y) ds_y$$

$$\widetilde{K} g(x) := \int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y) g(y) ds_y \qquad x \in \Omega.$$

In the representation formula (1.1), there are still two unknowns, u in the exterior domain and the normal trace ϕ on $\partial\Omega$. However, taking the traces on $\partial\Omega$ (i.e. considering the limit process $\Omega \ni \widetilde{x} \to x \in \partial\Omega$), one obtains (the appearing operators are just the traces of the potentials defined above)

$$g(x) = N_0 f(x) + V\phi(x) - (K - 1/2)g(x) \quad \text{for } x \in \partial\Omega$$

or rewritten as

$$V\phi = -N_0 f + (K + 1/2)g \quad \text{on } \partial\Omega.$$

Now, the only unknown is the normal trace ϕ, which solves a non-local integral equation. For solvability of such an equation, we refer to standard textbooks on integral equations (in fact, the operator V is elliptic on appropriate Sobolev spaces).

While the derived formulation seems to be more complicated than the PDE, which we started with, the integral equation has a very destinct advantage in that it is now posed only on a bounded set $\partial\Omega$ instead of an unbounded domain $\mathbb{R}^d \backslash \overline{\Omega}$. Thus, ideas similar to the standard finite element method can be employed for discretization, which is directly not possible for the PDE on the unbounded domain.

Example. A classical example for application of the above reformulation is acoustic scattering or radar. Hereby, a so called incident wave, here $u^{\text{inc}}(x) = e^{ikx}$, is sent to an object and the goal is to compute a so called scattered wave u^s by an object Ω. This is inherently a problem posed on Ω^c (unbounded!). Note that the scattered wave contains information about Ω, which can e.g. be used to reconstruct the object (as done in radar).

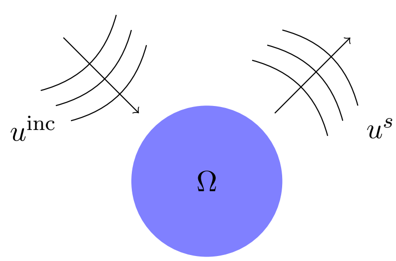


Figure 1.1: Scattering by an object Ω.

In this example, the total acoustic field is given as $u = u^{\text{inc}} + u^s$ and satsifies the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^d \backslash \Omega$$

together with a so called Sommerfeld radiation condition "boundary condition at infinity" given for $r = |x|$ by

$$\partial_r u^s - iku^s = o(r^{-1}).$$

Then, u can be written using the ideas from above as

$$u(x) = u^{\text{inc}} - \int_{\partial\Omega} G(x-y)\partial_n u(y)dy \qquad G(x-y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

and taking the normal derivative on both sides gives a boundary integral operator formulation. Discretization with a Galerkin method ("boundary element method" - BEM) gives the following simulation for the total field (c.f. [BS21]).

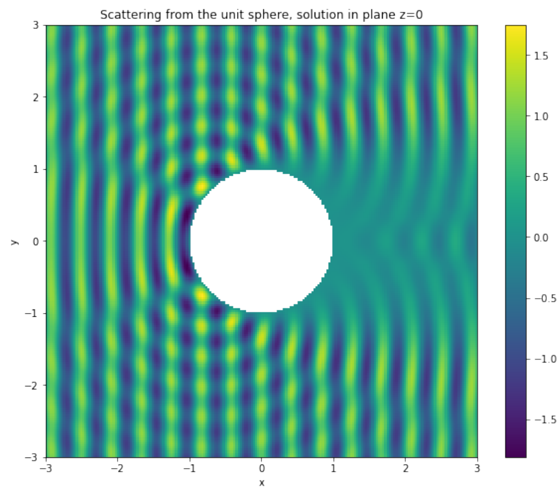


Figure 1.2: Scattering at unit sphere ($d = 3$), solution plot in the plane $z = 0$, figure take from [BS21].

1.3 Fractional Operators

Recently (since ∼2000), more complex physical and biological models started to take non-local diffusive effects into account. A simple model for such operators is the fractional Laplacian

$$(-\Delta)^s \qquad s \in (0,1),$$

where the choice of s gives an additional model-parameter for more precise modeling.
As such, further applications in finance, image processing and materials science followed.
Mathematically, these operators are also of interest as some crucial differences to the Laplacian appear. As already mentioned, fractional differential operators are non-local, which makes their analysis and numerical approximation challenging. In fact, even the precise mathematical definition of these operators is not straight forward. On the whole space $\mathbb{R}^d$ there are multiple definitions, which turn out to be equivalent. A formally easy way to define the fractional Laplacian is to use the Fourier transformation to write

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\zeta|^{2s}\mathcal{F}u),$$

which, however, is unpractical for numerical methods. A definition, which is more in line with the singular integral operators of the previous subsection is given by the so called **integral fractional Laplacian** defined pointwise as the Cauchy principal value

$$(-\Delta)^s u(x) := C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy.$$

Restricting oneself from the full-space to a subdomain Ω makes matters more complicated, as different definitions turn out to be not equivalent any more. In this lecture, we discuss two different definitions, the integral fractional Laplacian, which is the formula above applied to functions vanishing outside of the domain, and the so called **spectral fractional Laplacian**, which uses the eigenvalues λ_k and eigenfunctions φ_k of $(-\Delta)$ to define

$$(-\Delta)^s_\sigma u := \sum_{k=1}^{\infty} \lambda_k^s u_k \varphi_k, \qquad u_k := \int_\Omega u \varphi_k dx.$$

There are some other definitions, like the regional fractional Laplacian, which we do not discuss in detail in the following, and given your application in mind you need to choose the appropriate definition accordingly.

Example. An example for a simulation of a physical process involving fractional derivatives is given by phase separation by means of the fractional Cahn-Hilliard equation

$$\partial_t u + (-\Delta)^\alpha_\sigma(-\Delta u + f(u)) = 0.$$

The following simulations from [AM17] show the effect of including non-integer powers of the Laplacian in the equation as an effect called "coarsening" of the structures is way less prominent for smaller values of α.

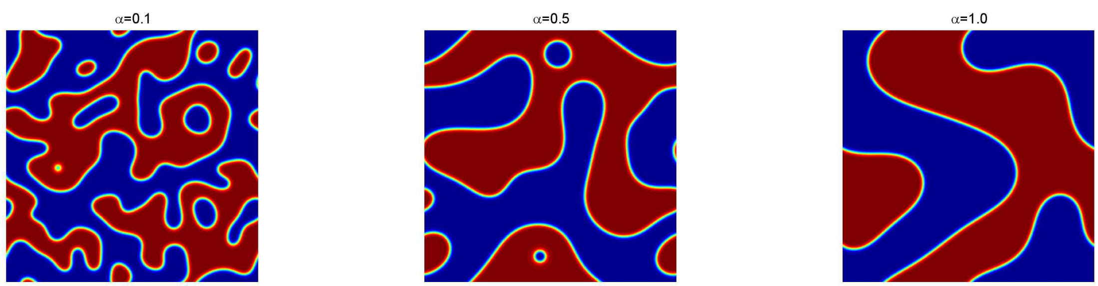


Figure 1.3: Phase separation by means of a (non-local) Cahn-Hilliard equation (random initial data), figures from [AM17].

A probabilistic motivation for fractional PDEs involving the *integral* fractional Laplacian will be given in the next section.

1.4 Further examples of non-local operators

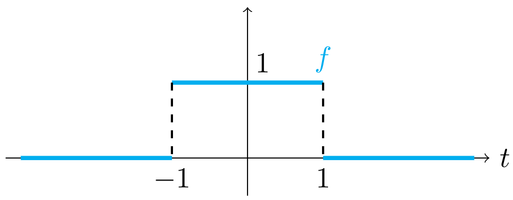
Integral transformations

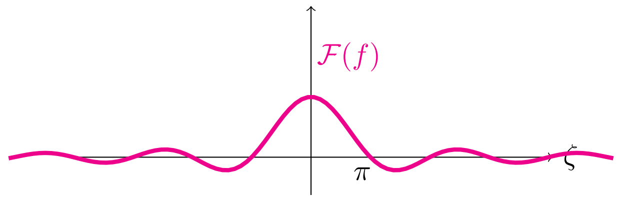
The Fourier transform

$$\mathcal{F}(f)(\zeta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\zeta t} dt$$

is non-local as, for example, the step function $f(t) = \chi_{[-1,1]}$ is transformed to

$$\mathcal{F}(\chi_{[-1,1]})(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\zeta t} dt = \frac{2}{\sqrt{2\pi}\zeta} \sin(\zeta).$$





In the same way, the Laplace transformation

$$\mathcal{L}(f)(s) := \int_0^\infty f(t)e^{-st}\,dt$$

is non-local.

1.5 Outlook and Literature

In the following, we briefly sketch the content of the lecture notes and mention important works used to write this lecture notes. In this version of the lecture notes, we take fractional operators as model problem. Many techniques employed can also be used for other non-local operators such as the integral operators appearing in the reformulation of the Poisson equation as integral equation. Details for this specific model problem can be found in text books on the boundary element method, such as [Ste08a, SS11] or in a previous version of the lecture (held in winter term 19/20) with lecture notes downloadable at

Numerical methods for fractional operators are studied quite recently and as a consequence few textbooks suitable for lectures are available, and the course is mainly designed using research articles. As such, this lecture notes can also be seen as a collection of readable literature for an introduction to the topic and further studies.

Chapter 2: Definitions of the Fractional Laplacian

Different definitions – and their equivalences – can be found in the rather technical paper of Kwasnicki [Kwa17]. Among them, we mention the ground breaking work of Caffarelli and Silvestre, [CS07], where the fractional Laplacian was identified as a Dirichlet-to-Neumann operator for a degenerated elliptic PDE. A good overview article is [LPG$^+$20].

- We start with a probabilistic motivation by looking at a random walk approach with arbitrary long jumps. It turns out that taking the limit of the discrete random-walk gives a fractional heat equation $u_t = (-\Delta)^s u$, with the integral fractional Laplacian, whereas a fixed length random-walk gives the classical heat equation.
- We formally introduce different definitions of the fractional Laplacian: the integral definition, the Fourier definition, the definition via a heat-semigroup as well as the famous Caffarelli-Silvestre extension problem, which gives a PDE-approach to fractional diffusion. On the full-space $\mathbb{R}^d$, we prove that all definitions are equivalent.
- Having understood the definitions in $\mathbb{R}^d$, we turn to the case of a bounded subdomain $\Omega \subset \mathbb{R}^d$. We start by introduction Sobolev spaces of non-integer order and their properties, further

information on that can be found in the book of McLean, [McL00]. Afterwards, we state the integral and spectral definition and discuss that they are indeed different operators. Moreover, we formulate the Caffarelli–Silvestre extension for both operators, which is very useful for analysis and numerical methods. Good references for that are [CS07, Sti10, ST10].

Chapter 3: Numerical Approximation

Here, we mainly focus on the integral fractional Laplacian. A good overview article on numerical methods for fractional PDEs is by Nochetto et al., [BBN+18], which collects results of Acosta and Borthagaray, [AB17] for the integral fractional Laplacian, as well as Nochetto, Otarola, Salgado, [NOS15] for the spectral fractional Laplacian. The more recent overview article [BLN22] gives more focus on regularity and mesh design for finite element methods.

- We start by deriving a weak formulation for the equation $(-\Delta)^s u = f$ using the integral fractional Laplacian, which leads to the non-local bilinear form

$$a(u,v) \simeq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}} dy\, dx.$$

  Hereby, analyzing a Galerkin approximation has two main difficulties: the non-locality of the bilinear form, i.e., plugging in two functions with disjoint support gives in general a non-zero contribution, as well as the non-locality of the energy norm in a fractional Sobolev space. The non-locality of the norm does not allow for elementwise a-priori estimates. However, we mention a remedy for this problem by proving a localizable upper bound for the energy norm.

- With a Ceá-type best approximation estimate, the a-priori analysis of a finite element approximation then comes down to a question of providing a quasi-interpolation operator with the right approximation properties – here the Scott-Zhang projection – as well as regularity of the solution. Assuming sufficient regularity, we prove convergence of the FEM approximation.

- Regularity for the integral fractional Laplacian is rather delicate, and it turns out that solutions tend to be not more regular than $H^{1/2+s-\varepsilon}(\Omega)$ even if the geometry and right-hand side are smooth. However, the singularities in the derivatives at the boundary $\partial\Omega$ can be captured in certain weighted Sobolev spaces, which is the main observation of [FMMS22]. Assuming analyticity of the data, one even obtains analytic regularity of the solutions in weighted spaces.

- The discussion on regularity gives an idea on how to construct suitable meshes in order to obtain better convergence rates for fractional PDEs. Finite regularity results together with graded meshes, [BLN22], can be used to improve convergence rates to $h^{1-\varepsilon}$. Employing geometric meshes, the weighted analytic regularity can be used to obtain exponential convergence, which can be found in [BFM+23, FMMS23].

- Finally, we comment on adaptive methods for optimal mesh design, [AG17, FMP21].

Chapter 4: Numerical quadrature

The calculation of the stifness matrix for the integral fractional Laplacian comes down to evaluation of certain singular integrals. While FEM error analysis oftentimes assumes that the bilinear forms

can be evaluated exactly, this is in practice not always possible. A way to analyze this so called variational crime is by means of the Strang lemma. In order to employ this, a separate analysis of the quality of the used quadrature formula has to be made. Here, we consider a weighted Gaussian quadrature formula, so called Gauss-Jacobi quadrature, which fits the given problem class very well.
Using the transformation theorem, certain variable substitutions can be made such that the singularities of the integrad can be mapped to the coordinate axis [CvPS15] and afterwards the weights in the Gaussian quadrature takes care of the singular behavior. For this ongoing research, we refer to [BFM24].

Chapter 5: Data-sparse approximation

A main problem for numerical methods for non-local operators is that usually discretizations lead to dense stiffness matrices with $\mathcal{O}(N^2)$ entries. Thus, computations such as solution of the corresponding linear system tend to be very expensive. A way to circumvent this is to use approximations to the system matrices in data-sparse formats to reduce the complexity to $\mathcal{O}(\mathcal{N} \log \mathcal{N})$.
Here, we only consider the matrix compression format of hierarchical matrices ($\mathcal{H}$-matrices), which approximates matrix blocks by low-rank approximations, and give a very brief introduction in the topic. More details can be found in the book of Hackbusch, [Hac15], or a special lecture on this topic with lecture notes downloadable at https://www.tuwien.at/mg/asc/faustmann/lehre/skripten.

Chapter 6: FEM for the Extension Problem - Spectral Fractional Laplacian

For the spectral definition of the fractional Laplacian, an overview on numerical methods is given in [BBN$^+$18]. We also mention [NOS15] for the PDE approach using the Cafarelli-Silvestre extension and [BMN$^+$19] for more advanced numerical methods and their analysis.

- Here, the PDE interpretation using the Caffarelli–Silvestre extension problem is the main tool used, and the PDE (in $d+1$-dimensions) is approximated using FEM. Again, questions of approximation properties and regularity are discussed, where especially the regularity in the extended dimension is crucial.

- Finally, we discuss sparse grids as a method to improve the complexity of the approximation with respect to the problem size.

Chapter 7: Dunford–Taylor Approach

For this approach using the Dunford-Taylor calculus (see e.g. the book of Yoshida, [Yos80]), we refer to the article of Bonito and Pasciak, [BP15] and the follow up work [BLP19]. Results on the used sinc-quadrature can be found, e.g., in [Ste93, LB92].

- Using Cauchy's integral formula gives yet another definition of the fractional Laplacian with the so-called Dunford–Taylor calculus. In fact, one can deform Cauchy's formula to the real-axis to obtain an operator-valued integral over $\mathbb{R}^+$, the so called Balakrishnan formula.

- With the Balakrishnan formula at hand, one can derive a numerical method using FEM to approximate the shifted Laplacian in the integrand and so-called sinc-quadrature to approximate the integral, which converges exponentially. Balancing the FEM error and the quadrature error gives convergence rates up to h^{2-s} for highly regular solutions.

- The Balakrishnan formula only holds for the spectral integral Laplacian. However, one can derive a similar formula for the bilinear form corresponding to the integral fractional Laplacian using the Fourier transform.

Chapter 2

Definitions of the Fractional Laplacian

2.1 The Integral Fractional Laplacian

2.1.1 Probabilistic Motivation

As a motivation for the fractional Laplacian, we start with a random walk approach with arbitrarily long jumps. Let P be a probability measure on $\mathbb{N}$ given by

$$P(I) := c(s) \sum_{j \in I} \frac{1}{j^{1+2s}} \qquad \text{for } I \subset \mathbb{N},$$

where $c(s) := \left(\sum_{j=1}^{\infty} \frac{1}{j^{1+2s}} \right)^{-1}$ is a normalization constant (depending on s) such that $P(\mathbb{N}) = 1$.

We study the motion of a particle described as follows: We assume that the particle does discrete jumps and denote by h the minimal possible jump-width in space and by τ the step size in time and link them by $\tau = h^{2s}$.

By $u(x,t)$ we denote the probability of a particle being at the place x at time t . The particle moves in a way that for each timestep τ, it chooses a random direction $\nu \in \partial B_1(0) \subset \mathbb{R}^d$ according to a uniform distribution as well as a $j \in \mathbb{N}$ according to the probability distribution P and it makes a step in space in the direction $jh\nu$.

Example. In one space dimension $d = 1$ and allowing only forward/backward jumps of length h both with probability 1/2 gives a classical random walk. Then, we have

$$u(x, t + \tau) = \frac{1}{2} u(x + h, t) + \frac{1}{2} u(x - h, t)$$

and assuming $2\tau = h^2$ yields

$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{u(x + h, t) - 2u(x, t) + u(x - h, t)}{h^2}.$$

Taking the limit $h, \tau \to 0$ gives the heat equation

$$u_t = \Delta u. \qquad \blacksquare$$

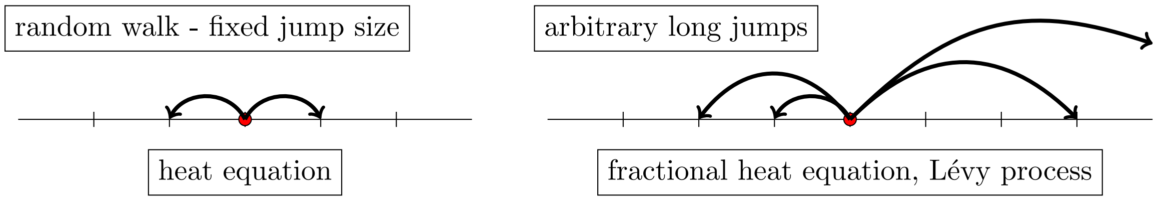


Figure 2.1: Random walk with fixed or arbitrary jump sizes.

In contrast to the previous example, we allow jumps of arbitrary length (although long jumps have small probability).

Example. The value $u(x,t+\tau)$ is given by the sum of all probabilities of the particle being at time t at location $x+jh\nu$ multiplied with the probability of jumping from there to x, i.e.,

$$u(x,t+\tau) = \frac{c(s)}{|\partial B_1(0)|}\int_{\partial B_1(0)} \sum_{j\in\mathbb{N}} \frac{u(x+jh\nu,t)}{j^{1+2s}} ds_\nu.$$

The change of variables $\nu \to -\nu$ gives

$$u(x,t+\tau) = \frac{c(s)}{|\partial B_1(0)|}\int_{\partial B_1(0)} \sum_{j\in\mathbb{N}} \frac{u(x-jh\nu,t)}{j^{1+2s}} ds_\nu,$$

so that

$$u(x,t+\tau) = \frac{1}{2}\frac{c(s)}{|\partial B_1(0)|}\int_{\partial B_1(0)} \sum_{j\in\mathbb{N}} \frac{u(x+jh\nu,t)+u(x-jh\nu,t)}{j^{1+2s}} ds_\nu. \tag{2.1}$$

Next, we obtain

$$u(x,t+\tau)-u(x,t) = \frac{c(s)}{|\partial B_1(0)|}\int_{\partial B_1(0)} \sum_{j\in\mathbb{N}} \frac{u(x+jh\nu,t)+u(x-jh\nu,t)-2u(x,t)}{2j^{1+2s}} ds_\nu.$$

Using $\tau = h^{2s}$, this leads to

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{c(s)}{2|\partial B_1(0)|}\int_{\partial B_1(0)} \sum_{j\in\mathbb{N}} h\frac{u(x+jh\nu,t)+u(x-jh\nu,t)-2u(x,t)}{(jh)^{1+2s}} ds_\nu.$$

We recognize the sum as a Riemann sum for an integral. Taking (formally) the limit $h\to 0$ (which also implies $\tau \to 0$) and using polar coordinates, we arrive at

$$\begin{aligned}
u_t(x,t) &= \frac{c(s)}{2|\partial B_1(0)|}\int_{\partial B_1(0)}\int_0^\infty \frac{u(x+z,t)+u(x-z,t)-2u(x,t)}{|z|^{1+2s}} dz ds_\nu \\
&= \frac{c(s)}{2|\partial B_1(0)|}\int_{\mathbb{R}^d} \frac{u(x+y,t)+u(x-y,t)-2u(x,t)}{|y|^{d+2s}} dy \\
&=: C(d,s)A^s u(x,t).
\end{aligned}$$

We will later on identify the operator A^s as a s-th (non-integer!) power of the Laplacian, called the fractional Laplacian. Therefore, the limit of a random walk with arbitrary long jumps leads to the so-called fractional heat equation. The constant $C(d, s)$, which will be specified in Def. 2.1, is simply introduced for consistency with other, equivalent, definitions of the fractional Laplacian. ■

Example (payoff model). Suppose one moves in a random walk in Ω as described in the preceeding example. Assume the walk is stopped if one jumps to a point $y \in \mathbb{R}^d \setminus \Omega$, where one gets a payoff $u(y)$. What is the expected payoff $u(x)$ if one starts at $x \in \Omega$?
The expected payoff at $x \in \Omega$ is the average of the expected payoffs of all points x' that can be reached from x by a jump of length $jh\nu$ weighted by the probability of the jump. Hence, we expect

$$u(x) = \sum_j \frac{c(s)}{|\partial B_1(0)|} \int_{\partial B_1(0)} \frac{u(x + jh\nu)}{j^{1+2s}} \, ds_\nu.$$

Noting $\sum_j c(s) j^{-(1+2s)} = 1$, this leads to

$$0 = \sum_j \frac{c(s)}{|\partial B_1(0)|} \int_{\partial B_1(0)} \frac{u(x + jh\nu) - u(x)}{j^{1+2s}} \, ds_\nu.$$

Dividing by h^{2s} and recognizing the sum as a Riemann sum for an integral, we obtain (formally) in the limit $h \to 0$

$$\begin{aligned} 0 &= \frac{c(s)}{|\partial B_1(0)|} \int_{\partial B_1(0)} \int_{r=0}^{\infty} \frac{u(x + r\nu) - u(x)}{r^{1+2s}} \, dr \, ds_\nu \\ &= \frac{c(s)}{|\partial B_1(0)|} \int_{z \in \mathbb{R}^d} \frac{u(x + z) - u(x)}{|z|^{d+2s}} \, dz =: (-\Delta)^s u. \end{aligned} \tag{2.2}$$

We conclude (formally) that the sought function u satisfies a fractional PDE

$$\begin{aligned} (-\Delta)^s u &= 0 \quad \text{in } \Omega, \\ u &= u_0 \quad \text{in } \mathbb{R}^d \setminus \Omega. \end{aligned}$$

■

We note that in the previous example, we could have done the "symmetrization" step as in (2.1). The definition of $(-\Delta)^s$ from (2.2) is the more common one in literature and made mathematically precise in Def. 2.1. Lemma 2.2 below shows that the operators A^s and $(-\Delta)^s$ in the previous examples are indeed equivalent.

2.1.2 The Integral Fractional Laplacian

The previous example motivates the definition of the fractional Laplacian as a non-local singular integral operator.
Initially, we define it on **Schwartz functions** in the space

$$\mathcal{S} := \Big\{ u \in C^\infty(\mathbb{R}^d) \ : \ \sup_{x \in \mathbb{R}^d} |(1 + |x|)^\alpha \partial^\beta u(x)| < \infty \quad \forall \alpha \in \mathbb{N}_0, \beta \in \mathbb{N}_0^d \Big\}.$$

Definition 2.1. *Let $s \in (0,1)$ and $u \in \mathcal{S}$. Then, the pointwise defined operator*

$$(-\Delta)^s u(x) := C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy \tag{2.3}$$
$$= C(d,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy$$

is called **integral fractional Laplacian**. *Here, P.V. denotes the Cauchy principal value and the appearing normalization constant is given by $C(d,s) := \frac{2^{2s} s\Gamma(s+d/2)}{\pi^{d/2}\Gamma(1-s)}$ with the Gamma-function $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$ for $z > 0$.*

Remark. The operator $(-\Delta)^s$ is well-defined pointwise for functions in $\mathcal{S}$. Let $\varepsilon > 0$ be fixed and choose an arbitrary $R > \varepsilon$. Then,

$$\int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy = \int_{B_R(x) \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy + \int_{B_R(x)^c} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy. \tag{2.4}$$

For the first integral, we use Taylor expansion $u(x)-u(y) = (x-y)\cdot\nabla u(x) + R(x,y)$, where the remainder satisfies $|R(x,y)| \leq \frac{1}{2}\|D^2u\|_{L^\infty(\mathbb{R}^d)}|x-y|^2$, and polar coordinates to obtain

$$\int_{B_R(x) \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy = \int_{B_R(x) \setminus B_\varepsilon(x)} \frac{(x-y)\cdot\nabla u(x)}{|x-y|^{d+2s}} dy + \int_{B_R(x) \setminus B_\varepsilon(x)} \frac{R(x,y)}{|x-y|^{d+2s}} dy. \tag{2.5}$$

Here, the first integral in (2.5) vanishes by a symmetry argument since

$$\int_{B_R(x) \setminus B_\varepsilon(x)} \frac{(x-y)\cdot\nabla u(x)}{|x-y|^{d+2s}}\, dy = \int_{r=\varepsilon}^R \int_{\nu \in S^{d-1}} \frac{r\nu \cdot \nabla u(x)}{r^{d+2s}} r^{d-1}\, d\nu dr$$
$$= \nabla u(x) \cdot \int_{r=\varepsilon}^R r^{-2s} \int_{\nu \in S^{d-1}} \nu d\nu dr = 0.$$

For the second integral in (2.5), we estimate

$$\int_{B_R(x) \setminus B_\varepsilon(x)} \left| \frac{R(x,y)}{|x-y|^{d+2s}} \right| dy \leq \int_{r=\varepsilon}^R \int_{\nu \in S^{d-1}} \frac{r^2\|D^2u\|_{L^\infty(\mathbb{R}^d)}}{2r^{d+2s}} r^{d-1}\, d\nu dr \lesssim \|D^2u\|_{L^\infty(\mathbb{R}^d)} \int_{r=\varepsilon}^R r^{1-2s}\, dr$$
$$= C(R^{2-2s} - \varepsilon^{2-2s}) \lesssim C \quad \forall \varepsilon > 0.$$

Hence, the limit $\varepsilon \to 0$ exists by Lebesgue Dominated Convergence. Finally, the last term in (2.4) can be simply bounded using polar coordinates by

$$\int_{B_R(x)^c} \left| \frac{u(x)-u(y)}{|x-y|^{d+2s}} \right| dy \lesssim 2\|u\|_{L^\infty(\mathbb{R}^d)} \int_{B_R(x)^c} \frac{1}{|x-y|^{d+2s}} dy \lesssim \int_{r=R}^\infty \frac{1}{r^{1+2s}} dr \simeq R^{-2s} < \infty.$$

Therefore, we have shown that the pointwise definition is well-defined. ■

Remark. For $s \in (0,1/2)$, the Cauchy principal value in the integral in the definition of the fractional Laplacian is not necessary, since it exists as an improper integral. To see this, we again employ the splitting of (2.4) (with $\varepsilon = 0$) and only investigate the first term (the second term follows directly as in the previous remark). Using the Taylor expansion $u(x) - u(y) = (x-y) \cdot \nabla u(\zeta)$ and Polar coordinates, we obtain

$$\int_{B_R(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy \leq \|u\|_{C^1(\mathbb{R}^d)} \int_{B_R(x)} \frac{1}{|x-y|^{d+2s-1}} dy \simeq \int_{r=0}^{R} \frac{1}{r^{2s}} dr < \infty,$$

since $s < 1/2$ implies $1 - 2s > 0$ and, consequently, the last integral exists. ■

The definition using the Cauchy principal value is oftentimes not handy for computations. The following lemma gives an equivalent representation of the integral fractional Laplacian $(-\Delta)^s$ without the need of the Cauchy principal value (even for $s \geq 1/2$) by using a weighted difference quotient of second order (see also the motivating example in the beginning of the section).

Lemma 2.2. *Let $s \in (0,1)$, $u \in \mathcal{S}$ and $(-\Delta)^s u$ given by Definition 2.1. Then, for $x \in \mathbb{R}^d$, we have*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+2s}} dy.$$

Proof. Using the transformations $z = y - x$ as well as $\widehat{z} = -z$, we obtain

$$\begin{aligned} (-\Delta)^s u(x) &= -C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x+z) - u(x)}{|z|^{d+2s}} dz \\ &= -C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x-\widehat{z}) - u(x)}{|\widehat{z}|^{d+2s}} d\widehat{z}. \end{aligned}$$

Relabeling $\widehat{z}$ to z and adding both equations leads to

$$2(-\Delta)^s u(x) = -C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x+z) - 2u(x) + u(x-z)}{|z|^{d+2s}} dz.$$

It remains to show that the integral indeed exists as an improper integral. We use the same arguments as in the previous remarks, i.e., we split $\mathbb{R}^d = B_R(0) \cup B_R(0)^c$ and use Taylor expansion. Here, the second order difference quotient has the Taylor expansion

$$\begin{aligned} u(x+z) - 2u(x) + u(x-z) &= u(x) + z \cdot \nabla u(x) + \frac{1}{2} z^T D^2 u(\zeta_1) z - 2u(x) \\ &\quad + u(x) - z \cdot \nabla u(x) + \frac{1}{2} z^T D^2 u(\zeta_2) z \\ &= \mathcal{O}(|z|^2) \end{aligned}$$

with suitable points $\zeta_1, \zeta_2 \in \mathbb{R}^d$. Therefore, we obtain with Polar coordinates

$$\int_{B_R(0)} \frac{u(x+z) - 2u(x) + u(x-z)}{|z|^{d+2s}} dz \lesssim \|u\|_{C^2(\mathbb{R}^d)} \int_{B_R(0)} \frac{1}{|z|^{d+2s-2}} dz \lesssim \int_{r=0}^{R} r^{1-2s} dr < \infty.$$

Since

$$\int_{B_R(0)^c} \frac{u(x+z) - 2u(x) + u(x-z)}{|z|^{d+2s}} dz \lesssim 4\|u\|_{L^\infty(\mathbb{R}^d)} \int_{B_R(0)^c} \frac{1}{|z|^{d+2s}} dz \lesssim \int_{r=R}^{\infty} r^{-1-2s} dr < \infty,$$

this gives the existence of the integral. ■

2.2 The Fourier Definition

The Fourier transformation $\mathcal{F}$ is given by

$$f \mapsto \mathcal{F}(f)(\zeta) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\zeta \cdot z} f(z)\, dx$$

for sufficiently smooth functions f. The inverse Fourier transform is given by

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\zeta \cdot x} f(\zeta)\, d\zeta.$$

Definition 2.3. **Sobolev spaces** $H^t(\mathbb{R}^d)$ *for* $t \in \mathbb{R}$ *on* $\mathbb{R}^d$ *can be defined using the Fourier transformation* $\mathcal{F}$ *as*

$$H^t(\mathbb{R}^d) = \{u \in \mathcal{S}' \ : \ \int_{\mathbb{R}^d} (1 + |\zeta|^2)^t |\mathcal{F}u(\zeta)|^2 \, d\zeta < \infty\}.$$

Fourier transformation turns differentiation into multiplication in the Fourier image. Hence, a possible way of defining fractional powers of differential operators would be to do the multiplication with $|\zeta|^{2s}$ in the Fourier space (which is well-defined for $u \in \mathcal{S}$) and then transform back, i.e., define a s-th power of $(-\Delta)$ as

$$\mathcal{F}^{-1}(|\zeta|^{2s} \mathcal{F}u(\zeta)).$$

In general, for an operator $\mathcal{A}$, a function $S_{\mathcal{F}}(\zeta) : \mathbb{R}^d \to \mathbb{R}$ is called **Fourier symbol** (or simply: symbol) of $\mathcal{A}$ if

$$\mathcal{F}(\mathcal{A}u)(\zeta) = S_{\mathcal{F}}(\zeta) \mathcal{F}u(\zeta).$$

The following theorem shows that the Fourier symbol of the integral fractional Laplacian $(-\Delta)^s$ from Def-2.1 is indeed $|\zeta|^{2s}$, i.e., the integral and the Fourier definition are equivalent.

Theorem 2.4. *Let* $s \in (0, 1)$ *and* $(-\Delta)^s$ *be the integral fractional Laplacian. Then, for* $u \in \mathcal{S}$, *we have*

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\zeta|^{2s} \mathcal{F}u).$$

The proof of this theorem needs the following technical lemma, which provides a different representation of the constant $C(d, s)$.

Lemma 2.5. *The constant* $C(d, s)$ *from Definition 2.1 satisfies*

$$C(d, s) = \left(\int_{\mathbb{R}^d} \frac{1 - \cos(z_1)}{|z|^{d+2s}} dz \right)^{-1}.$$

Proof. See [BV16, Lem. 3.1.3.]. ■

Proof of Theorem 2.4. We use Lemma 2.2 to write for $x \in \mathbb{R}^d$

$$(-\Delta)^s u(x) = -\frac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+2s}} dy.$$

The proof consists of three steps.
1. step. We show that the integrand is in $L^1(\mathbb{R}^{2d})$: With the same arguments as in the proof of Lemma 2.2 (Taylor-expansion), we estimate

$$\begin{aligned}\frac{|u(x+y) - 2u(x) + u(x-y)|}{|y|^{d+2s}} \leq \chi_{B_1(0)}(y)|y|^{2-d-2s}\|D^2 u\|_{L^\infty(B_1(x))} \\ + \chi_{B_1(0)^c}(y)|y|^{-d-2s}|u(x+y) - 2u(x) + u(x-y)|.\end{aligned} \tag{2.6}$$

The first term (2.6) in $L^1(\mathbb{R}^{2d})$ since $u \in \mathcal{S}$ so that $\|D^2 u\|_{L^\infty(B_1(x))} \leq C(1+|x|)^{-(d+1)}$. The second term is seen to be in $L^1(\mathbb{R}^{2d})$ by Fubini since

$$\begin{aligned}&\int_{y\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} \chi_{B_1(0)^c}(y)|y|^{-d-2s}|u(x+y) - 2u(x) + u(x-y)|\, dx\, dy \\ &\quad \leq 4 \int_{y\in\mathbb{R}^d} \chi_{B_1(0)^c}(y)|y|^{-d-2s}\, dy \|u\|_{L^1(\mathbb{R}^d)} \leq C\|u\|_{L^1(\mathbb{R}^d)} < \infty.\end{aligned}$$

2. step. We use $\mathcal{F}(u(x+y))(\zeta) = e^{i\zeta\cdot y}\mathcal{F}(u(x))$ and interchange the integral and the Fourier transform to obtain

$$\begin{aligned}\mathcal{F}((-\Delta)^s u)(\zeta) &= -\frac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{\mathcal{F}(u(x+y) - 2u(x) + u(x-y))}{|y|^{d+2s}} dy \\ &= -\frac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{e^{i\zeta\cdot y} + e^{-i\zeta\cdot y} - 2}{|y|^{d+2s}} dy\ (\mathcal{F}u)(\zeta) \\ &= C(d,s) \int_{\mathbb{R}^d} \frac{1 - \cos(\zeta\cdot y)}{|y|^{d+2s}} dy\ (\mathcal{F}u)(\zeta).\end{aligned}$$

3. step. We show that the integral satisfies

$$C(d,s)I(\zeta) := C(d,s) \int_{\mathbb{R}^d} \frac{1 - \cos(\zeta\cdot y)}{|y|^{d+2s}} dy = |\zeta|^{2s}, \tag{2.7}$$

which directly gives the Fourier symbol of $(-\Delta)^s$ and, therefore, proves the theorem.
For $\zeta = (\zeta_1, \ldots, \zeta_d)$ with $|\zeta|$ small, we estimate

$$\frac{1 - \cos(\zeta_1)}{|\zeta|^{d+2s}} \leq \frac{|\zeta_1|^2}{|\zeta|^{d+2s}} \leq \frac{1}{|\zeta|^{d+2s-2}}.$$

This shows that the integral

$$\int_{\mathbb{R}^d} \frac{1 - \cos(\zeta_1)}{|\zeta|^{d+2s}} d\zeta$$

is indeed finite. We show that $I(\zeta) = I(|\zeta|e_1)$, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$.
For $d = 1$ this is obvious since $I(\zeta) = I(-\zeta)$ due to the same property for the cosine.
For $d \geq 2$, we take a rotation R such that $R(|\zeta|e_1) = \zeta$. Noting that $\det R = 1$, we use the transformation $y \mapsto R^T y$ to write

$$\begin{aligned} I(\zeta) &= \int_{\mathbb{R}^d} \frac{1 - \cos(R(|\zeta|e_1) \cdot y)}{|y|^{d+2s}} dy = \int_{\mathbb{R}^d} \frac{1 - \cos(|\zeta|e_1 \cdot R^T y)}{|R^T y|^{d+2s}} dy \\ &= \int_{\mathbb{R}^d} \frac{1 - \cos(|\zeta|e_1 \cdot y)}{|y|^{d+2s}} dy = I(|\zeta|e_1). \end{aligned}$$

With this equality, we may use the transformation $z = |\zeta|y$ and compute

$$\begin{aligned} I(\zeta) = I(|\zeta|e_1) &= \int_{\mathbb{R}^d} \frac{1 - \cos(|\zeta|y_1)}{|y|^{d+2s}} dy = \frac{1}{|\zeta|^d} \int_{\mathbb{R}^d} \frac{1 - \cos(z_1)}{|z/|\zeta||^{d+2s}} dz \\ &= C(d,s)^{-1} |\zeta|^{2s}, \end{aligned}$$

where the last step follows from Lemma 2.5. This concludes the proof of the theorem. ∎

Remark. From the previous theorem we can deduce for $u \in \mathcal{S}$

$$\begin{aligned} \lim_{s \to 0^+} (-\Delta)^s u &= u, \\ \lim_{s \to 1^-} (-\Delta)^s u &= -\Delta u \end{aligned}$$

in $H^t(\mathbb{R}^d)$ for any $t \in \mathbb{R}$.

In literature, a very common and more direct way to define non-integer order Sobolev spaces (which also easily generalizes to domains $\Omega \subset \mathbb{R}^d$) is by means of the so called **Aronstein-Slobodeckij (semi-)norm**

$$\begin{aligned} |u|^2_{H^s(\mathbb{R}^d)} &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \, dx \, dy \\ \|u\|^2_{H^s(\mathbb{R}^d)} &:= |u|^2_{H^s(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

The following theorem shows equivalence of this norm to the norm defined via Fourier transformation and that the Aronstein-Slobodeckij semi-norm can be expressed in terms of the fractional Laplacian.

Theorem 2.6. *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^d)$. Then,*

$$|u|^2_{H^s(\mathbb{R}^d)} = \frac{2}{C(d,s)} \| |\zeta|^s \mathcal{F} u \|^2_{L^2(\mathbb{R}^d)} = \frac{2}{C(d,s)} \| (-\Delta)^{s/2} u \|^2_{L^2(\mathbb{R}^d)}.$$

Proof. Using the transformation $z = x - y$, we obtain

$$\begin{aligned} |u|^2_{H^s(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \frac{u(z+y) - u(y)}{|z|^{d/2+s}} \right|^2 dy dz \\ &= \int_{\mathbb{R}^d} \left\| \frac{u(z+\cdot) - u(\cdot)}{|z|^{d/2+s}} \right\|^2_{L^2(\mathbb{R}^d)} dz. \end{aligned}$$

Now, Plancherel's formula gives

$$\begin{aligned}\int_{\mathbb{R}^d} \left\| \frac{u(z+\cdot)-u(\cdot)}{|z|^{d/2+s}} \right\|^2_{L^2(\mathbb{R}^d)} dz &= \int_{\mathbb{R}^d} \left\| \mathcal{F}\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{d/2+s}} \right) \right\|^2_{L^2(\mathbb{R}^d)} dz \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|e^{i\zeta\cdot z}-1|^2}{|z|^{d+2s}} |\mathcal{F}u(\zeta)|^2 d\zeta dz \\ &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1-\cos(\zeta\cdot z)}{|z|^{d+2s}} |\mathcal{F}u(\zeta)|^2 dz d\zeta.\end{aligned}$$

With (2.7) the latter is equal to

$$\begin{aligned}|u|^2_{H^s(\mathbb{R}^d)} &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1-\cos(\zeta\cdot z)}{|z|^{d+2s}} |\mathcal{F}u(\zeta)|^2 dz d\zeta = 2C(d,s)^{-1}\int_{\mathbb{R}^d} |\zeta|^{2s}|\mathcal{F}u(\zeta)|^2 d\zeta \\ &= 2C(d,s)^{-1}\||\zeta|^s\mathcal{F}u\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

We stress that this argument shows the equivalence of the Sobolev-norms defined via Slobodeckij-seminorm and via Fourier transformation. For $u \in \mathcal{S}$, Theorem 2.4 gives with Plancherel's formula

$$\begin{aligned}|u|^2_{H^s(\mathbb{R}^d)} &= 2C(d,s)^{-1}\||\zeta|^s\mathcal{F}u\|^2_{L^2(\mathbb{R}^d)} = 2C(d,s)^{-1}\|\mathcal{F}((-\Delta)^{s/2}u)\|^2_{L^2(\mathbb{R}^d)} \\ &= 2C(d,s)^{-1}\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Since $\mathcal{S}$ is dense in $H^s(\mathbb{R}^d)$, we can extend this to the case of $u \in H^s(\mathbb{R}^d)$. ∎

Remark 2.7 *Since Plancherel's theorem has the form* $\langle f, g\rangle_{L^2(\mathbb{R}^d)} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2(\mathbb{R}^d)}$, *inspection of the proof of Theorem 2.6 shows that for* u, $v \in H^s(\mathbb{R}^d)$, *we have obtained*

$$\frac{C(d,s)}{2}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}} = \int_{\mathbb{R}^d} \left((-\Delta)^{s/2}u\right)\left((-\Delta)^{s/2}v\right).$$

As a corollary of the previous theorems, we obtain that the fractional Laplacian is an operator of order $2s$.

Corollary 2.8.

(i) $(-\Delta)^s$ *is an operator of order* $2s$, *i.e.,* $(-\Delta)^s : H^\ell(\mathbb{R}^d) \to H^{\ell-2s}(\mathbb{R}^d)$ *is bounded for arbitrary* $\ell \in \mathbb{R}$.

(ii) If $u \in L^2(\mathbb{R}^d)$ *solves* $(-\Delta)^s u = f$ *in* $\mathbb{R}^d$ *with* $f \in H^\ell(\mathbb{R}^d)$ *for some* $\ell \geq 0$, *then we have* $u \in H^{\ell+2s}(\mathbb{R}^d)$.

Proof. (i) follows directly from Theorem 2.4 and the definition of Sobolev spaces in terms of the Fourier transformation.
The *shift theorem* (ii) is seen by a bootstrapping argument. The Fourier transforms satisfy

$$(1+|\zeta|^{2s})\mathcal{F}(u) = \mathcal{F}(f) + \mathcal{F}(u) = \mathcal{F}(f+u). \tag{2.8}$$

By assumption, $f+u \in H^{\min(\ell,0)}(\mathbb{R}^d)$. Then (2.8) implies $u \in H^{\min(\ell,0)+2s}(\mathbb{R}^d) = H^{\min(\ell+2s,2s)}(\mathbb{R}^d)$. Thus, the right-hand side of (2.8) is actually in $H^{\min(\ell,\ell+2s,2s)}(\mathbb{R}^d)$ so that by (2.8), we have $u \in H^{\min(\ell,2s)+2s}(\mathbb{R}^d) = H^{\min(\ell+2s,4s)}(\mathbb{R}^d)$. One can repeat these arguments and obtains in the k-th step $u \in H^{\min(\ell+2s,2sk)}(\mathbb{R}^d)$. Since k is arbitrary, $u \in H^{\ell+2s}(\mathbb{R}^d)$. ■

Remark. We will see later that statement (ii) does not transfer to the case of bounded domains $\Omega \subset \mathbb{R}^d$ due to the effect of boundary singularities. The shift theorem is only valid in the limited range $\ell < 1/2$.

2.3 Definition via Heat Semigroup

In this subsection, we give yet another equivalent definition of the fractional Laplacian, this time by using the semigroup of the heat equation.

As a motivation, we look at the Gamma-function, use integration by parts and the scaling $\tau = \lambda t$ for arbitrary $\lambda > 0$. Then,

$$\begin{aligned}
-s\Gamma(-s) = \Gamma(1-s) &= \int_0^\infty \tau^{-s} e^{-\tau}\, d\tau = -\int_0^\infty \tau^{-s} \frac{d}{d\tau}(e^{-\tau}-1)\, d\tau \\
&= -s \int_0^\infty \tau^{-s-1}(e^{-\tau}-1)\, d\tau \\
&= -s\lambda^{-s} \int_0^\infty t^{-s-1}(e^{-\lambda t}-1)\, dt. \qquad (2.9)
\end{aligned}$$

Solving for λ^s gives the expression

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(e^{-\lambda t}-1)\, dt. \qquad (2.10)$$

Using this approach for the fractional power, we formally replace $\lambda = -\Delta$ to obtain a possible definition of $(-\Delta)^s$

$$(-\Delta)^s = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(e^{t\Delta}-I)\, dt.$$

Here, $e^{t\Delta}$ is the heat semigroup since the function $U(x,t) := e^{t\Delta}u(x)$ solves the PDE

$$\begin{aligned}
\partial_t U = \frac{\partial}{\partial t}(e^{t\Delta}u(x)) &= \Delta(e^{t\Delta}u(x)) = \Delta U, \\
U(x,0) &= u(x),
\end{aligned}$$

i.e., the heat equation on $\mathbb{R}^d \times \mathbb{R}^+$ with initial data u.

The following theorem shows that this formal approach can indeed be justified.

Theorem 2.9. *Let $s \in (0,1)$, $u \in \mathcal{S}$ and $U(x,t)$ solve*

$$\begin{aligned} \partial_t U &= \Delta U \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ U(x,0) &= u(x) \qquad x \in \mathbb{R}^d. \end{aligned}$$

Then, the integral fractional Laplacian $(-\Delta)^s$ satisfies

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt.$$

Proof. We refer to the PDE-lecture (e.g., [Eva98, Thm. 1, p. 47]) for the fact that

$$U(x,t) = \int_{\mathbb{R}^d} G(x-y,t)u(y)dy = \int_{\mathbb{R}^d} G(y,t)u(x-y)dy$$

with the heat kernel (Green's function) $G(x,t) = (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}$ satisfying $\int_{\mathbb{R}^d} G(y,t)dy = 1$. Since G is normalized, we may write

$$\int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt = \int_0^\infty \int_{\mathbb{R}^d} t^{-s-1} G(y,t)(u(x-y) - u(x))\, dy\, dt.$$

Now, we use the substitution $\tau := \frac{|y|^2}{4t}$ noting that $dt = -\frac{|y|^2}{4\tau^2} d\tau$ and obtain

$$\begin{aligned} &\int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt \\ &\qquad = \int_0^\infty \int_{\mathbb{R}^d} |y|^{-2s-2}(4\tau)^{s+1}\tau^{d/2}(\pi|y|^2)^{-d/2} e^{-\tau}(u(x-y) - u(x))dy \left(\frac{|y|^2}{4\tau^2} d\tau\right) \\ &\qquad = 4^s \pi^{-d/2} \int_0^\infty \tau^{d/2+s-1} e^{-\tau} d\tau \frac{1}{2} \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+2s}} dy. \end{aligned}$$

where we "symmetrized" the term $u(x-y) - u(x)$. Lemma 2.2 gives that the integral in y is equal to $-C(d,s)^{-1}(-\Delta)^s u(x)$ and the integral in τ is just $\Gamma(\frac{d}{2}+s)$ by definition of the Gamma-function. Putting everything together, we obtain

$$\frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt = -\frac{2^{2s}\Gamma(\frac{d}{2}+s)}{\pi^{d/2}\Gamma(-s)C(d,s)} (-\Delta)^s u(x).$$

By the choice of $C(d,s)$ in Definition 2.1 (noting $-s\Gamma(-s) = \Gamma(1-s)$) the constant in front of $(-\Delta)^s u(x)$ is equal to one, which finishes the proof. ∎

2.4 The Caffarelli–Silvestre Extension

In this section, we discuss yet another approach to interpret the fractional Laplacian. This time, we study a PDE-approach, where the fractional Laplacian is given as a Dirichlet-to-Neumann operator of a degenerated elliptic PDE in one additional space dimension. In literature, this is often called Caffarelli–Silvestre extension, see, e.g., [CS07].

The Caffarelli–Silvestre extension problem reads as follows: For a given function $u : \mathbb{R}^d \to \mathbb{R}$ and $\alpha \in (-1, 1)$ (we will use $\alpha = 1 - 2s$), we seek a function $\mathcal{U} = \mathcal{U}(x, y) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\operatorname{div}(y^\alpha \nabla \mathcal{U}(x, y)) = 0 \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \tag{2.11a}$$

$$\mathcal{U}(x, 0) = u(x) \quad \text{on } \mathbb{R}^d. \tag{2.11b}$$

Equation (2.11a) can equivalently be written as

$$\Delta_x \mathcal{U} + \frac{\alpha}{y} \partial_y \mathcal{U} + \partial_y^2 \mathcal{U} = 0. \tag{2.12}$$

The extension problem (2.11) is not fully determined yet.[1] We seek $\mathcal{U}$ as

$$\mathcal{U} := \operatorname{argmin} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^+} y^\alpha |\nabla \mathcal{U}(x, y)|^2 \, dx \, dy \, : \, \mathcal{U}(\cdot, 0) = u \right\}. \tag{2.13}$$

This minimization problem is well-posed (in an appropriate function space, see Section 2.5.5) and has a unique solution. The system (2.11) is then the Euler-Lagrange equation for this minimization problem.

Remark. For $\alpha = 0$, the extension $\mathcal{U}$ is a harmonic function in $\mathbb{R}^d \times \mathbb{R}^+$ with prescribed boundary value. The map $u \mapsto \mathcal{U}$ is well-studied and has the form $\mathcal{U}(x, y) = \int_{x \in \mathbb{R}^d} P(x - z, y) u(z) \, dz$ with the **Poisson kernel** P. The corresponding "Poisson kernel" for $\alpha \neq 0$ is derived in [CS07] (see also the discussions in Remarks 2.11 and 2.13) and used below.

The following theorem links the solution of (2.11) to the integral fractional Laplacian.

Theorem 2.10. *Let $s \in (0, 1)$, $u \in \mathcal{S}$, $\alpha = 1 - 2s$ and $\mathcal{U}$ a solution of (2.11) with boundary data u. Then, we have*

$$\lim_{y \to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -d_s (-\Delta)^s u,$$

with the constant $d_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

Proof. There exists an integral representation of the solution $\mathcal{U}$ of (2.11) in terms of a "Poisson kernel". We refer to [CS07] for the fact that

$$\mathcal{U}(x, y) = C_1 \int_{\mathbb{R}^d} \frac{y^{2s}}{(|x - z|^2 + |y|^2)^{(d+2s)/2}} u(z) dz \tag{2.14}$$

solves (2.11) with an appropriate normalization constant C_1. We start by noting that

$$C_1 \int_{z \in \mathbb{R}^d} \frac{y^{2s}}{(|x - z|^2 + y^2)^{(d+2s)/2}} \, dz = 1 \qquad \forall x \in \mathbb{R}^d, y > 0.$$

[1] Example for $\alpha = 0$: the function $\mathcal{U}(x, y) := \sin(kx) \sinh(ky)$ for $k > 0$ satisfies $\Delta \mathcal{U} = 0$ and $\mathcal{U}(\cdot, 0) = 0$.

This follows from the fact that the minimum norm solution of (2.13) for $u \equiv 1$ is $\mathcal{U} \equiv 1$ and observing that (2.14) gives a representation of $\mathcal{U}$.
Hence,

$$\mathcal{U}(x,0) = u(x) = u(x) \cdot 1 = u(x) C_1 \int_{z\in\mathbb{R}^d} \frac{y^{2s}}{(|x-z|^2+y^2)^{(d+2s)/2}}\, dz.$$

Using the integral representation of $\mathcal{U}$, we compute

$$\begin{aligned}
\lim_{y\to 0^+} \frac{\mathcal{U}(x,y)-\mathcal{U}(x,0)}{y^{1-\alpha}} &= \lim_{y\to 0^+} \frac{C_1}{y^{1-\alpha}} \int_{\mathbb{R}^d} \frac{y^{2s}}{(|x-z|^2+|y|^2)^{(d+2s)/2}} (u(z)-u(x)) dz \\
&= \lim_{y\to 0^+} C_1 \int_{z\in\mathbb{R}^d} \frac{u(x+z)-u(x)}{(|z|^2+|y|^2)^{(d+2s)/2}}\, dz \\
&= \lim_{y\to 0^+} \frac{C_1}{2} \int_{z\in\mathbb{R}^d} \frac{u(x+z)+u(x-z)-2u(x)}{(|z|^2+|y|^2)^{(d+2s)/2}}\, dz
\end{aligned}$$

where we used the same argument as in the proof of Lemma 2.2 to "symmetrize" the integrand. Since u is smooth, we can use Lebesgue dominated convergence to pull the limit $y \to 0$ inside the integral and we arrive at

$$\lim_{y\to 0^+} \frac{\mathcal{U}(x,y)-\mathcal{U}(x,0)}{y^{1-\alpha}} = \frac{C_1}{2} \int_{z\in\mathbb{R}^d} \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{d+2s}}\, dz = -\frac{C_1}{C(d,s)} (-\Delta)^s u(x).$$

Using the explicit value of C_1 ultimately leads to the stated representation.
To finish the proof, we ascertain that $(1-\alpha)\lim_{y\to 0+}(g(y)-g(0))y^{-1+\alpha} = \lim_{y\to 0+} y^\alpha g'(y)$ for $g \in C^1$. This follows from de l'Hôspital's rule[2]. ■

Remark 2.11 *See [CS07] for various motivations for the Poisson kernel for the extension problem (2.11). It can be checked that (2.14) really represents the Poisson kernel in the following way:*
a) a direct calculation shows that

$$P(x,y) := \frac{y^{2s}}{(|x|^2+|y|^2)^{(d+2s)/2}}$$

satisfies the differential equation;
b) $\lim_{y\to 0} P(x,y) = c\delta_x$ *for some suitable* $c \in \mathbb{R}$. *(That is,* $\lim_{y\to 0}(P(\cdot,y) * u)(x) \to u(x)$ *for* $u \in C_0^\infty(\Omega)$.*)*
To see b), the key observation is that

$$P(x,y) = y^{-d} P(x/y, 1) \tag{2.15}$$

[2]Alternative to l'Hôspital's rule: We make the change of variables $z = y^{1-\alpha}$ and consider the function $G(z) := g(y) = g(z^{1/(1-\alpha)})$. Then

$$G'(z) = g'(z^{1/(1-\alpha)}) \frac{z^{\alpha/(1-\alpha)}}{1-\alpha} = g'(y) \frac{y^\alpha}{1-\alpha}$$

Hence,

$$\lim_{y\to 0} g'(y) y^\alpha = (1-\alpha) \lim_{z\to 0} G'(z) = (1-\alpha) \lim_{z\to 0} \frac{G(z)-G(0)}{z} = (1-\alpha) \lim_{y\to 0} \frac{g(y)-g(0)}{y^{1-\alpha}}.$$

This shows that the left limit exists if the right limit exists.

so that for $u \in C_0^\infty(\mathbb{R}^d)$ *one has*

$$\int_{z \in \mathbb{R}^d} P(x-z,y)u(z)\,dz = \int_{z \in \mathbb{R}^d} y^{-d} P((x-z)/y,1)u(z)\,dz = \int_{z' \in \mathbb{R}^d} P(z',1)u(yz')\,dz'.$$

Hence, (by Lebesgue Dominated Convergence)

$$\lim_{y \to 0} \int_{z \in \mathbb{R}^d} P(x-z,y)u(z)\,dz = u(x) \int_{z' \in \mathbb{R}^d} P(z',1)\,dz',$$

which expresses $\lim_{y \to 0} P(x,y) = c\delta_x$ *with* $c = \int_{z' \in \mathbb{R}^d} P(z',1)\,dz'$.
An alternative approach to compute the Poisson kernel P *is with Fourier techniques as discussed in Remark 2.13.*

Remark. In the following chapter, we present an alternative proof for the statement of Theorem 2.10 on bounded domains, which could also be used for the problem posed on the whole space.

Representation of $\mathcal{U}$ using Fourier techniques*

One way to derive a formula for the extension $\mathcal{U}$ (and subsequently derive the Poisson kernel) is to start from the minimization problem (2.13) and use Fourier techniques. In what follows, we will proceed somewhat formally. Details could be found, e.g., in [BV16, Sec. 3.3].

We start with a 1D minimization problem.

Lemma 2.12. *Let* $s \in (0,1)$ *and* $\alpha = 1-2s \in (-1,1)$. *Let* $z > 0$. *The minimizer* ψ *of the problem*

$$\text{minimize } J_z(v) := \int_{y=0}^{\infty} y^\alpha (z^2|v(y)|^2 + |v'(y)|^2)\,dy \quad \text{under the constraint } v(0) = 1 \tag{2.16}$$

has the form $\psi(y) = c_s (zy)^s K_s(zy)$, *where* K_s *denotes the second kind modified Bessel function and* $c_s = 2^{1-s}/\Gamma(s)$.

Proof. Given a function v with $v(0) = 1$, set $v_z(y) := v(zy)$, which also satisfies $v_z(0) = 1$. The change of variables $yz = \tilde{y}$ shows

$$J_z(v_z) = z^{2-\alpha-d} J_1(v).$$

Since this holds for all functions v, we see that if v is a minimizer of the J_1, then v_z is a minimizer of J_z. The stated formula for ψ reflects this, and we may restrict our attention to the case $z = 1$. The Euler-Lagrange equations for the minimization problem are

$$y^\alpha \psi - (y^\alpha \psi')' = 0, \qquad \psi(0) = 1.$$

This reads

$$\psi - \frac{\alpha}{y}\psi' - \psi'' = 0.$$

The two linearly independent solutions are $y^s K_s(y)$ and $y^s I_s(y)$ with $s = (1+\alpha)/2$, where I_s and K_s are the first and second kind modified Bessel functions.[3] The function I_s is exponentially growing so that the condition that $J_1(\psi)$ be finite implies that $\psi(y) = c_s y^s K_s(y)$ for some $c \in \mathbb{R}$. K_s is exponentially decaying as $y \to \infty$. To compute the coefficient c_s, we use that by [AS64, (9.6.9)]

$$\lim_{y\to 0}\left(\frac{1}{2}y\right)^s K_s(y) = \frac{1}{2}\Gamma(s),$$

so that the choice $c_s = 2^{1-s}/\Gamma(s)$ indeed leads to $\lim_{y\to 0}\psi(y) = 1$. ■

Let $\mathcal{F}$ be the Fourier transformation in the x-variable. Using Parseval, the minimization problem (2.13) takes the form

$$\int_{y=0}^{\infty} y^\alpha \int_{\zeta\in\mathbb{R}^d} |\nabla\mathcal{U}(x,y)|^2 = \int_{\zeta\in\mathbb{R}^d}\int_{y=0}^{\infty} y^\alpha\left(|\zeta|^2|\mathcal{F}\mathcal{U}(\zeta,y)|^2 + |\partial_y\mathcal{F}\mathcal{U}(\zeta,y)|^2\right)$$

We have $\mathcal{F}\mathcal{U}(\zeta,0) = \mathcal{F}u(\zeta)$, i.e., $\mathcal{F}\mathcal{U}(\zeta,0)$ is given for each ζ. Minimizing for fixed ζ, we expect from Lemma 2.12 that

$$\mathcal{F}\mathcal{U}(\zeta,y) = \mathcal{F}\mathcal{U}(\zeta,0)\psi(y|\zeta|) = \mathcal{F}u(\zeta)\psi(y|\zeta|).$$

By inverse Fourier transformation we obtain a representation of $\mathcal{U}$:

$$\mathcal{U}(x,y) = \mathcal{F}^{-1}(\mathcal{F}u(\zeta)\psi(|\zeta|y)).$$

With $\widetilde{P}(x) = \mathcal{F}^{-1}(\psi)$ and $\mathcal{F}^{-1}(\psi(|\zeta|y)) = y^{-d}\widetilde{P}(x/y)$ we obtain

$$(2\pi)^{d/2}\mathcal{U}(x,y) = u * (y^{-d}\widetilde{P}(\cdot/y))(x) = y^{-d}\int_{\mathbb{R}^d} u(z)\widetilde{P}((x-z)/y)\,dz. \tag{2.17}$$

Remark 2.13 *Formula (2.17) shows that the Poisson kernel $P(x,y) = y^{-d}P(x/y,1)$ (cf. (2.15)) should satisfy $P(z,1) = c\widetilde{P}(z)$ for some $c \in \mathbb{R}$. Hence, P can be computed from the (inverse) Fourier transform of ψ. Since ψ is a radial function, so is $\widetilde{P}$, i.e., $\widetilde{P}(x) = P'(|x|)$. The Fourier transform of ψ can be computed explicitly, [DLMF, (10.32.11)], and one arrives at the form (2.14).*

2.5 The Fractional Laplacian on Bounded Domains

From now on, we consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. Our goal is to (numerically) solve the equation

$$\begin{aligned}(-\Delta)^s u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Omega^c.\end{aligned} \tag{2.18}$$

[3] It is instructive to consider the special case $\alpha = 0$. Then, the linearly independent solutions are e^z and e^{-z} and the integrability condition implies $\psi(z) = c_s e^{-z}$.

2.5.1 The Sobolev spaces $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$

We recall for $s \in (0,1)$ the Sobolecki seminorm for functions defined on $\omega \subset \mathbb{R}^d$:

$$|u|^2_{H^s(\omega)} := \int_\omega \int_\omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dx\,dy. \tag{2.19}$$

In view of the fact that the solution u of interest in (2.18) is supported by $\overline{\Omega}$, we will require function spaces of functions on Ω. For $s \in (0,1)$ and functions u defined on Ω, we let $\widetilde{u}$ be its zero-extension to $\mathbb{R}^d$, i.e., $\widetilde{u}(x) = u(x)$ for $x \in \Omega$ and $\widetilde{u}(x) = 0$ for $x \in \mathbb{R}^d \backslash \Omega$, and define

$$|u|^2_{\widetilde{H}^s(\Omega)} := |\widetilde{u}|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{d+2s}}\,dx\,dy, \tag{2.20}$$

$$\|u\|^2_{\widetilde{H}^s(\Omega)} := \|u\|^2_{L^2(\Omega)} + |u|^2_{\widetilde{H}^s(\Omega)}. \tag{2.21}$$

We define $\widetilde{H}^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{\widetilde{H}^s(\Omega)}$. We have:

Lemma 2.14. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $s \in (0,1)$. Then:*

(i) For all $u \in \widetilde{H}^s(\Omega)$ there holds $\|u\|_{\widetilde{H}^s(\Omega)} \sim \|\widetilde{u}\|_{H^s(\mathbb{R}^d)}$.

(ii) $\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^d) \mid \operatorname{supp} u \subset \overline{\Omega}\}$. Furthermore, $\widetilde{H}^s(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^d)$ (and of $H^s(\Omega)$) and thus a Hilbert space.

(iii) There is $C > 0$ depending only on Ω such that for all $u \in \widetilde{H}^s(\Omega)$

$$C^{-1}\|u\|^2_{\widetilde{H}^s(\Omega)} \leq \|\operatorname{dist}(\cdot,\partial\Omega)^{-s}u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)} \leq C\|u\|^2_{\widetilde{H}^s(\Omega)}. \tag{2.22}$$

(iv) For $s \in (0,1)$ the embedding $\widetilde{H}^s(\Omega) \subset H^s(\Omega)$ is continuous. The embedding $H^s(\Omega) \subset L^2(\Omega)$ is compact.

(v) The seminorm $|\cdot|_{\widetilde{H}^s(\Omega)}$ is a norm equivalent to the full norm $\|\cdot\|_{\widetilde{H}^s(\Omega)}$ on $\widetilde{H}^s(\Omega)$.

(vi) The space $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$.

Proof. *Proof of (i):* see Theorem 2.6.
Proof of (ii): See, e.g., [McL00]. An alternative argument that $\widetilde{H}(\Omega)$ is a Hilbert space comes from the fact that it is the completion of $C_0^\infty(\Omega)$ under a norm induced by a scalar product.
Proof of (iii): We illustrate the proof for the case $d = 1$ and $\Omega = (0,1)$.

$$|u|^2_{\widetilde{H}^s(\Omega)} = \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\widetilde{u}(x)-\widetilde{u}(y)|^2}{|x-y|^{d+2s}}\,dx\,dy = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dx\,dy + 2\int_{x\in\Omega}\int_{y\in\Omega^c} \frac{|u(x)|^2}{|x-y|^{d+2s}}\,dy\,dx.$$

We have $\Omega^c = (-\infty,0) \cup (1,\infty)$, and the integral over Ω^c is decomposed further into integral over these two contribution. For example, for $x \in (0,1)$ and $y \in (-\infty,0)$, we have

$$\int_{y=-\infty}^0 \frac{1}{|x-y|^{1+2s}}\,dy = \int_{y=-\infty}^0 \frac{1}{(x-y)^{1+2s}}\,dy = \frac{1}{2s}(x-y)^{-2s}|_{y=-\infty}^{y=0} \sim x^{-2s} = \operatorname{dist}(x,0)^{-2s}.$$

An analogous computation gives $\int_{y=1}^{\infty} \frac{1}{|x-y|^{1+2s}}\,dy \sim \operatorname{dist}(x,1)^{-2s}$, and we conclude for $x \in \Omega$ that

$$\int_{y\in\Omega^c} \frac{1}{|x-y|^{d+2s}}\,dy \sim \operatorname{dist}(x,\partial\Omega)^{-2s}.$$

Hence,

$$|u|^2_{\widetilde{H}^s(\Omega)} \sim |u|^2_{H^s(\Omega)} + \|\operatorname{dist}(\cdot,\partial\Omega)^{-s}u\|^2_{L^2(\Omega)}. \tag{2.23}$$

This calculation for the case $d = 1$ generalizes to higher dimensions, see, e.g., [Gri11, Lemma 1.3.2.6].
Proof of (iv): The continuity of the embedding $\widetilde{H}^s(\Omega) \subset H^s(\Omega)$ is expressed by the (2.23). The compact embedding is shown in the literature, see, e.g., [McL00].
Proof of (v): This Poincaré-type estimate follows directly from (2.23).
Proof of (vi): [McL00]. ■

Remark (Difference between $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$). The term $\|\operatorname{dist}(\cdot,\partial\Omega)^{-s}u\|_{L^2(\Omega)}$ in the norm of $\widetilde{H}^s(\Omega)$ expresses that functions $u \in \widetilde{H}^s(\Omega)$ vanish on $\partial\Omega$ in an appropriate sense. In fact, for $s > 1/2$, functions $u \in H^s(\Omega)$ have a meaningful trace γu on $\partial\Omega$ (in $L^2(\partial\Omega)$), and the trace γu on $\partial\Omega$ of $u \in \widetilde{H}^s(\Omega)$ vanishes on $\partial\Omega$.

Remark. For $0 < s < s' < 1$ we have the (continuous) embeddings $H^{s'}(\Omega) \subset H^s(\Omega)$ and $\widetilde{H}^{s'}(\Omega) \subset \widetilde{H}^s(\Omega)$. For $s \in (0,1)$, we have the continuous embeddings $H^1(\Omega) \subset H^s(\Omega)$ and $H^1_0(\Omega) \subset \widetilde{H}^s(\Omega)$.

Remark (the spaces $H^\sigma(\Omega)$ and $\widetilde{H}^\sigma(\Omega)$ for $\sigma \geq 1$).

- Sobolev spaces $H^\sigma(\Omega)$ with $\sigma > 1$ are defined as follows: for $\sigma \notin \mathbb{N}$ one write $\sigma = \lfloor\sigma\rfloor + s$ with $s \in (0,1)$ and defines the norm by

$$|u|^2_{H^\sigma(\Omega)} := \sum_{|\alpha|=\lfloor\sigma\rfloor} |D^\alpha u|^2_{H^s(\Omega)}, \qquad \|u\|^2_{H^\sigma(\Omega)} := \|u\|^2_{H^{\lfloor\sigma\rfloor}(\Omega)} + |u|^2_{H^\sigma(\Omega)}. \tag{2.24}$$

- For $\sigma \geq 1$, the norm $\|\cdot\|_{\widetilde{H}^\sigma(\Omega)}$ is defined in terms of the zero-extension by

$$\|u\|_{\widetilde{H}^\sigma(\Omega)} := \|\widetilde{u}\|_{H^\sigma(\mathbb{R}^d)}.$$

  The space $\widetilde{H}^\sigma(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{\widetilde{H}^\sigma(\Omega)}$. Functions in $\widetilde{H}^\sigma(\Omega)$ are such that their derivatives up to order $|\alpha| < \sigma - 1/2$ vanish on $\partial\Omega$.

For completeness, we also introduce negative order spaces as dual spaces: For $\sigma > 0$ we set

$$H^{-\sigma}(\Omega) := \left(\widetilde{H}^\sigma(\Omega)\right)', \tag{2.25}$$

$$\widetilde{H}^{-\sigma}(\Omega) := \left(H^\sigma(\Omega)\right)'. \tag{2.26}$$

equipped with the usual norms

$$\|u\|_{H^{-\sigma}(\Omega)} := \sup_{v\in\widetilde{H}^\sigma(\Omega)} \frac{\langle u,v\rangle}{\|v\|_{\widetilde{H}^\sigma(\Omega)}}, \qquad \|u\|_{\widetilde{H}^{-\sigma}(\Omega)} := \sup_{v\in H^\sigma(\Omega)} \frac{\langle u,v\rangle}{\|v\|_{H^\sigma(\Omega)}}.$$

Remark ($H^\sigma(\Omega)$ vs. $\widetilde{H}^\sigma(\Omega)$ vs. $H^s_0(\Omega)$). Sometimes, the spaces $H^\sigma_0(\Omega)$, $\sigma \geq 0$, are used in the literature. They are defined as the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{H^s(\Omega)}$. In summary, one has for $\sigma \geq 0$:

$$\widetilde{H}^\sigma(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{\widetilde{H}^\sigma(\Omega)}}, \qquad H^\sigma_0(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^\sigma(\Omega)}}, \qquad H^\sigma(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{H^\sigma(\Omega)}}.$$

These spaces are related to each other in the following way:

$$\widetilde{H}^\sigma(\Omega) = H^\sigma_0(\Omega) \quad \text{(equiv. norms)} \qquad \text{if} \quad \sigma - \frac{1}{2} \notin \mathbb{N}_0, \tag{2.27}$$

$$\widetilde{H}^s(\Omega) = H^s_0(\Omega) = H^s(\Omega) \quad \text{(equiv. norms)} \quad \text{for } 0 \leq s < 1/2, \tag{2.28}$$

$$H^s_0(\Omega) = H^s(\Omega) \quad \text{(equiv. norms)} \quad \text{for } 0 \leq s \leq 1/2.$$

The spaces $H_0^{1/2}(\Omega)$ and $\widetilde{H}^{1/2}(\Omega)$ do not coincide. Frequently in the literature the notation $H_{00}^{1/2}(\Omega)$ is used as an alternative notation for the space $\widetilde{H}^{1/2}(\Omega)$. A good reference for properties of this spaces is [McL00].

2.5.2 The Integral Definition

For a function $u : \Omega \to \mathbb{R}$, we denote its zero-extension by $\widetilde{u} : \mathbb{R}^d \to \mathbb{R}$. Then, we can easily extend the definition (2.3) by

$$(-\Delta)^s u(x) := (-\Delta)^s \widetilde{u}(x) = C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{\widetilde{u}(x) - \widetilde{u}(y)}{|x-y|^{d+2s}} dy.$$

Obviously, this definition is well-defined for functions $u \in C_0^\infty(\Omega)$. By density, it can be extended as a bounded linear operator $(-\Delta)^s : \widetilde{H}^s(\Omega) \to H^{-s}(\Omega) = \left(\widetilde{H}^s(\Omega)\right)'$. We will discuss the problem (2.18) in more detail in Section 3.1.

Remark. Sometimes in literature, a different definition, the so-called **regional fractional Laplacian**,

$$(-\Delta)^s_\Omega u(x) := C(d,s,\Omega)\ P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{d+2s}} dy,$$

where integration is restricted to Ω, is used. While some of the results presented in the following also hold for the regional fractional Laplacian, there are also some considerable differences, and in the following, we will not discuss the regional definition any more.

2.5.3 The Spectral Definition

We present a different notion of the fractional Laplacian in bounded domains.
The Laplacian $-\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is a symmetric, self-adjoint operator with compact inverse. As such, classical spectral theory provides eigenfunctions φ_k and eigenvalues λ_k satisfying

$$-\Delta \varphi_k = \lambda_k \varphi_k, \qquad \varphi_k|_{\partial\Omega} = 0$$

that are an orthonormal basis[4] in $L^2(\Omega)$ as well as an orthogonal basis in $H_0^1(\Omega)$ such that

$$-\Delta u = \sum_{k=1}^{\infty} \lambda_k u_k \varphi_k, \qquad \text{where} \quad u_k := \int_\Omega u\varphi_k dx.$$

Using this expansion, and the spectral theorem, a possible way to define the s-th power of the Laplacian would be to take the s-th power of the eigenvalues, i.e.,

$$(-\Delta)_\sigma^s u := \sum_{k=1}^{\infty} \lambda_k^s u_k \varphi_k, \qquad \text{where} \quad u_k := \int_\Omega u\varphi_k dx$$

for $u \in C_0^\infty(\Omega)$. This series represents an operator $C_0^\infty(\Omega) \to L^2(\Omega)$. In terms of the spaces $\widetilde{H}^s(\Omega)$ and $H^{-s}(\Omega)$, this series definition can be extended by density to the space $\widetilde{H}^s(\Omega)$ as an operator $(-\Delta)_\sigma^s : \widetilde{H}^s(\Omega) \to H^{-s}(\Omega)$. This operator is called the **spectral fractional Laplacian** $(-\Delta)_\sigma^s$.

Remark 2.15 *The operators $(-\Delta)^s$ and $(-\Delta)_\sigma^s$ on Ω are indeed inherently different. We refer to [SV14] for the following results regarding the eigenvalues and eigenfunctions:*

- *Let λ_1 denote the smallest eigenvalue. Then, the smallest eigenvalue of $(-\Delta)^s$ is a lower bound of the smallest eigenvalue of $(-\Delta)_\sigma^s$, i.e, there holds*

$$\lambda_1((-\Delta)^s) < \lambda_1((-\Delta)_\sigma^s).$$

- *The eigenfunctions of $(-\Delta)_\sigma^s$ coincide with the eigenfunctions of the Laplacian $(-\Delta)$, the smoothness does only depend on Ω. For instance, on the unit sphere the eigenfunctions are in $C^\infty(B_1(0))$.*

  However, the eigenfunctions for $(-\Delta)^s$ are only Hölder-continuous for some Hölder-exponent β, and [SV14, Pro. 2] states that $e_1 \notin W^{1,\infty}(B_1(0))$ (for $d > 2s$), where e_1 is the eigenfunction corresponding to the smallest eigenvalue $\lambda_1((-\Delta)^s)$.

The differences in the operators can also be seen in the asymptotics of the solutions of the equations $(-\Delta)^s u = f$ and $(-\Delta)_\sigma^s \widetilde{u} = f$. We refer to [Gru15, CS16], where f, Ω are assumed to be sufficiently smooth, for the asymptotic behavior for x close to $\partial\Omega$:

- $u \simeq \operatorname{dist}(x, \partial\Omega)^s + v(x)$, *with v smooth;*
- $\widetilde{u} \simeq \begin{cases} \operatorname{dist}(x, \partial\Omega)^{2s} + \widetilde{v}(x) & 0 < s < 1/2, \\ \operatorname{dist}(x, \partial\Omega) + \widetilde{v}(x) & 1/2 < s < 1, \end{cases}$ *with $\widetilde{v}$ smooth.*

[4] "elementary" proof of the orthogonalities: we have $(\nabla\varphi_k, \nabla v)_{L^2} = \lambda_k(\varphi_k, v)_{L^2}$ for all $v \in H_0^1(\Omega)$. Hence $\lambda_k(\varphi_k, \varphi_\ell)_{L^2} = (\nabla\varphi_k, \nabla\varphi_\ell)_{L^2} = (\nabla\varphi_\ell, \nabla\varphi_k)_{L^2} = \lambda_\ell(\varphi_\ell, \varphi_k)_{L^2} = \lambda_\ell(\varphi_k, \varphi_\ell)_{L^2}$. Thus, for $\lambda_k \neq \lambda_\ell$, one has $(\varphi_k, \varphi_\ell)_{L^2} = 0$ Then, $(\nabla\varphi_k, \nabla\varphi_\ell)_{L^2} = \lambda_k(\varphi_k, \varphi_\ell)_{L^2} = 0$ of $k \neq \ell$.

2.5.4 The Caffarelli-Silvestre extension for $(-\Delta)^s_\sigma$

Remark 2.16 *The extension problem (2.11) is defined for $u \in \mathcal{S}$. In particular, therefore, for $u \in C_0^\infty(\Omega) \subset \mathcal{S}$. As such, the solution $\mathcal{U}$ of the extension problem (2.11) satisfies by Theorem 2.10*

$$\lim_{y \to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -d_s(-\Delta)^s \widetilde{u} = -d_s(-\Delta)^s u.$$

There is, however, no reason to assume that this co-normal derivative coincides with $(-\Delta)^s_\sigma u$: The lifting $\mathcal{U}$ of (2.11) does not "see" the domain Ω — it is the lifting for a full-space problem and only the data u happens to be in $C_0^\infty(\Omega)$. In contrast, the domain Ω enters the definition of $(-\Delta)^s_\sigma$ in an essential way since it depends on the eigenpairs $(\varphi_k, \lambda_k)_k$, which in turn depend on Ω.

In the context of the spectral fractional Laplacian, the correct extension is the solution of the following problem:

$$\mathcal{L}\mathcal{U} := \operatorname{div}(y^\alpha \nabla \mathcal{U}(x,y)) = 0 \qquad \text{in } \Omega \times \mathbb{R}^+ \tag{2.29a}$$
$$\mathcal{U} = 0 \qquad \text{on } \partial\Omega \times (0, \infty) \tag{2.29b}$$
$$\mathcal{U}(x, 0) = u(x) \quad \text{in } \Omega. \tag{2.29c}$$

As in the case of the equations (2.11), a decay condition as $y \to \infty$ needs to be imposed to make the solution unique. That is, we seek $\mathcal{U}$ as the

$$\mathcal{U} := \operatorname{argmin}\left\{ \int_{\Omega \times \mathbb{R}^+} y^\alpha |\nabla \mathcal{U}|^2 \, dx\, dy \ : \ \mathcal{U}(\cdot, 0) = u(\cdot), \quad \mathcal{U} = 0 \quad \text{ on } \partial\Omega \times \mathbb{R}^+ \right\}. \tag{2.30}$$

The following theorem shows that the Neumann-data of the extension problem (2.29) actually gives the spectral fractional Laplacian.

Theorem 2.17. *Let $s \in (0,1)$, $\alpha = 1 - 2s$ and $\mathcal{U}$ a solution of (2.29) with boundary data $u \in C_0^\infty(\Omega)$. Then, we have*

$$\lim_{y \to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -d_s(-\Delta)^s_\sigma u, \qquad d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}.$$

Proof of Theorem 2.17 via series expansions*

We show Theorem 2.17 using expansion in eigenfunctions $(\varphi_k)_k$. A more more general technique based on the Spectral Theorem is given in Section 2.5.4. That theory generalizes to the case of unbounded Ω, e.g., $\Omega = \mathbb{R}^d$. In particular, the case of the operator $(-\Delta)^s$ could be treated in that framework. A good exposition is [Sti10].
Let $u(x) = \sum_k u_k \varphi_k(x)$. For the solution $\mathcal{U}$ of the minimization problem (2.30) we make the ansatz

$$\mathcal{U}(x, y) = \sum_k u_k \varphi_k(x) \psi_k(y)$$

for functions ψ_k to be determined. The condition $\mathcal{U}(\cdot,0) = u(\cdot)$ implies that $\psi_k(0) = 1$. By the orthogonality properties of the eigenfunctions φ_k, we have

$$\int_{\Omega\times\mathbb{R}^+} y^\alpha |\nabla\mathcal{U}|^2 = \sum_k |u_k|^2 \int_{y=0}^{\infty} y^\alpha (\lambda_k |\psi_k(y)|^2 + |\psi_k'(y)|^2)\, dy$$

Hence, the minimization problem (2.30) is reduced to a 1D minimization problem for each k. Lemma 2.12 provides the solution

$$\mathcal{U}(x,y) = \sum_k u_k \varphi_k(x)\psi_k(y), \qquad \psi_k(y) = \psi(\sqrt{\lambda_k}y), \qquad \psi(z) = c_s z^s K_s(z). \tag{2.31}$$

The cornomal derivative can be explicitly computed

$$\begin{aligned}
\lim_{z\to 0} z^\alpha \partial_z \psi_k(z) &= c_s\sqrt{\lambda_k} \lim_{z\to 0} z^{1-2s}\psi'(\sqrt{\lambda_k}z) = c_s\sqrt{\lambda_k}\lambda_k^{-1/2+s} \lim_{z\to 0} z^{1-2s}(z^s K_s(z))' \\
&\overset{\text{[AS64, (9.6.28)]}}{=} -c_s\lambda_k^s \lim_{z\to 0} z^{1-2s} z^s K_{s-1}(z) = -c_s\lambda_k^s \lim_{z\to 0} z^{1-s}K_{s-1}(z) \\
&\overset{\text{[AS64, (9.6.9)]}}{=} -c_s\lambda_k^s 2^{-s}\Gamma(1-s) = -\frac{2^{1-s}}{\Gamma(s)} 2^{-s}\Gamma(1-s)\lambda_k^s = -d_s\lambda_k^s
\end{aligned}$$

By summation over all k, the stated representation of $(-\Delta)^s_\sigma$ follows.
Instead of representing $\mathcal{U}$ by a sum, is also possible to express it as an integral using the solution operator for the heat equation. We have

Lemma 2.18. *Let $u \in C_0^\infty(\Omega)$. The solution $\mathcal{U}$ of (2.29) is given by*

$$\mathcal{U}(x,y) = \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} e^{t\Delta}((-\Delta)^s_\sigma u)(x) e^{-y^2/(4t)} \frac{dt}{t^{1-s}} \tag{2.32}$$

$$= \frac{y^{2s}}{4^s\Gamma(s)} \int_{t=0}^{\infty} e^{t\Delta}u(x)e^{-y^2/(4t)} \frac{dt}{t^{1+s}} = \frac{1}{\Gamma(s)} \int_{r=0}^{\infty} e^{y^2/(4r)\Delta}u(x)e^{-r}\frac{dr}{r^{1-s}}. \tag{2.33}$$

Proof. We recall that, since φ_k is an eigenfunction of Δ, we have

$$e^{t\Delta}\varphi_k = e^{-t\lambda_k}\varphi_k \tag{2.34}$$

We restrict our attention to the case where u is only a finite sum of Fourier terms as then summability is not an issue.
Proof of (2.32): A first approach could be to use [DLMF, Eq. (10.32.10)], which shows

$$\int_{t=0}^{\infty} e^{-t\lambda_k} e^{-y^2/(4t)} \frac{dt}{t^{1-s}} \sim \psi(\sqrt{\lambda_k}y)$$

and then compute compute (2.32) or (2.33) explicitly to arrive at (2.31). Instead, we define

$$\widetilde{\mathcal{U}}(x,y) = \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} e^{t\Delta}((-\Delta)^s_\sigma u)(x)e^{-y^2/(4t)} \frac{dt}{t^{1-s}}$$

and show that a) $\widetilde{\mathcal{U}}(x,0) = u(x)$ and b) $\widetilde{\mathcal{U}}$ satisfies

$$\Delta_x \widetilde{\mathcal{U}} + \partial_y^2 \widetilde{\mathcal{U}} + \frac{\alpha}{y}\partial_y \widetilde{\mathcal{U}} = 0. \tag{2.35}$$

Since the function $\widetilde{\mathcal{U}}$ satisfies[5] $\int_{\Omega\times\mathbb{R}^+} y^\alpha |\nabla \widetilde{\mathcal{U}}|^2 < \infty$, it will be the sought minimizer of (2.30). (The minimizer is unique!) To see that $\widetilde{\mathcal{U}}(\cdot,0) = u$, we compute

$$\begin{aligned}
\widetilde{\mathcal{U}}(x,0) &= \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} e^{t\Delta}((-\Delta)^s_\sigma u)(x) \frac{dt}{t^{1-s}} = \sum_k \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} e^{-t\lambda_k} \lambda_k^s u_k \varphi_k(x) \frac{dt}{t^{1-s}} \\
&= \sum_k \lambda_k^s u_k \varphi_k(x) \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} e^{-t\lambda_k} \frac{dt}{t^{1-s}} = \sum_k \lambda_k^s u_k \varphi_k(x) \frac{\lambda_k^{-s}}{\Gamma(s)} \int_{\tau=0}^{\infty} e^{-\tau} \frac{d\tau}{\tau^{1-s}} = \sum_k u_k \varphi_k(x).
\end{aligned}$$

To see (2.35), we use $-\Delta\varphi_k = \lambda_k \varphi_k$ and compute

$$\begin{aligned}
-\Delta\widetilde{\mathcal{U}}(x,y) &= \frac{1}{\Gamma(s)} \sum_k \lambda_k^s \lambda_k u_k \varphi_k(x) \int_{t=0}^{\infty} e^{-t\lambda_k} e^{-y^2/(4t)} \frac{dt}{t^{1-s}} \\
&= -\frac{1}{\Gamma(s)} \sum_k \lambda_k^s u_k \varphi_k(x) \int_{t=0}^{\infty} \partial_t \left(e^{-t\lambda_k}\right) e^{-y^2/(4t)} \frac{dt}{t^{1-s}} \\
&= \frac{1}{\Gamma(s)} \sum_k \lambda_k^s u_k \varphi_k(x) \int_{t=0}^{\infty} e^{-t\lambda_k} e^{-y^2/(4t)} \left(\frac{y^2}{4t^2} + \frac{s-1}{t}\right) \frac{dt}{t^{1-s}} \\
&= \frac{1-2s}{y} \partial_y \widetilde{\mathcal{U}}(x,y) + \partial_y^2 \widetilde{\mathcal{U}}(x,y).
\end{aligned}$$

This shows (2.32).
Proof of (2.33):

$$\begin{aligned}
\mathcal{U}(x,y) = \widetilde{\mathcal{U}}(x,y) &= \sum_k u_k \varphi_k(x) \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} e^{-t\lambda_k} \lambda_k^s e^{-y^2/(4t)} \frac{dt}{t^{1-s}} \\
&= \sum_k u_k \varphi_k(x) \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} e^{-t\lambda_k} (t\lambda_k)^s e^{-y^2/(4t)} \frac{dt}{t} \\
&\overset{r=y^2/(4t\lambda_k)}{=} \sum_k u_k \varphi_k(x) \frac{1}{\Gamma(s)} \int_{r=0}^{\infty} e^{-y^2/(4r)} \left(\frac{y^2}{4r}\right)^s e^{-r\lambda_k} \frac{dr}{r} \\
&= \sum_k u_k \varphi_k(x) \frac{y^{2s}}{4^s \Gamma(s)} \int_{r=0}^{\infty} e^{-y^2/(4r)} e^{-r\lambda_k} \frac{dr}{r^{1+s}} \\
&= \frac{y^{2s}}{4^s\Gamma(s)} \int_{r=0}^{\infty} e^{r\Delta} u(x) e^{-y^2/(4r)} \frac{dr}{r^{1+s}},
\end{aligned}$$

which is the first equality in (2.33). The second one is simply a change of variables. ■

[5] the key observation is that $\widetilde{\mathcal{U}}$ and $\partial_y \widetilde{\mathcal{U}}$ decay exponentially in y. For example, for $\widetilde{\mathcal{U}}$, one observes that the maximum of $t \mapsto e^{-\lambda t} e^{-y^2/(4t)}$ is $e^{-y\sqrt{\lambda}}$ attained at $\bar{t} := y/(2\sqrt{\lambda})$. This leads to $\int_{t=0}^{\infty} e^{-\lambda t} e^{-y^2/(4t)} t^{-(1-s)}\,dt = \int_{t=0}^{\bar t} \cdots + \int_{t=\bar t}^{\infty} \cdots \leq e^{-y\sqrt{\lambda}} \int_{t=0}^{\bar t} t^{-(1-s)}\,dt + \int_{t=\bar t}^{\infty} e^{-\lambda t}\,dt \leq C_s e^{-y\sqrt{\lambda}} + \lambda^{-1} e^{-\lambda \bar t} \leq C_s e^{-y\sqrt{\lambda}} + C\lambda^{-1} e^{-y\sqrt{\lambda}/2}$. Note that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$

As in the case of the integral fractional Laplacian in Theorem 2.9, the spectral fractional Laplacian $(-\Delta)^s_\sigma$ can be represented in terms of the heat semigroup:

Lemma 2.19.

$$(-\Delta)^s_\sigma = \frac{1}{\Gamma(-s)} \int_{t=0}^{\infty} \left(e^{t\Delta} - \mathrm{I}\right) \frac{dt}{t^{1+s}}$$

As the proof shows, such a representation is a fairly general property when defining fractional powers of positive definite selfadjoint operators.
Proof. From (2.9) we have

$$\int_{t=0}^{\infty} (e^{-t\lambda_k} - 1) \frac{dt}{t^{1+s}} = \lambda_k^s \Gamma(-s).$$

We proceed again formally and assume that $u = \sum_k u_k \varphi_k$ for a finite sum. Then,

$$\begin{aligned}(-\Delta)^s_\sigma u(x) = \sum_k \lambda_k^s u_k \varphi_k(x) &= \frac{1}{\Gamma(-s)} \sum_k u_k \varphi_k(x) \int_{t=0}^{\infty} (e^{-t\lambda_k} - 1) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_{t=0}^{\infty} (e^{t\Delta} - \mathrm{I}) u(x) \frac{dt}{t^{1+s}}\end{aligned}$$

which finishes the proof. ∎

Proof of Theorem 2.17 via spectral theory

For the proof of the theorem, we make use of classical spectral theory, which we briefly recall here.

The spectral theorem for (un-)bounded self adjoint operators A states the existence of a unique spectral measure $E : \mathcal{A} \to \mathcal{L}(L^2(\Omega))$, where $\mathcal{A}$ is a sigma-algebra on the spectrum $\sigma(A)$ and $\mathcal{L}(L^2(\Omega))$ are the linear operators mapping from $L^2(\Omega)$ to $L^2(\Omega)$, such that

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

Moreover, for $f, g \in L^2(\Omega)$, the spectral measure E weakly defines a complex measure $E_{f,g}$ with total variation bounded by $\|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}$ in terms of

$$E_{f,g}(T) := (E(T)f, g) \quad \forall T \in \mathcal{A}.$$

Using this notation as well as the functional calculus, that comes with this setting, we may write for the spectral fractional Laplacian

$$(-\Delta)^s_\sigma = \int_0^\infty \lambda^s dE(\lambda), \tag{2.36}$$

as the spectrum of the Laplacian is contained in $\mathbb{R}^+$.
Moreover, the heat-semigroup has the representation

$$e^{t\Delta} = \int_0^\infty e^{-t\lambda} dE(\lambda).$$

Using this together with formula (2.10), we can formally interchange integrals to write

$$\begin{aligned}(-\Delta)^s_\sigma &= \int_0^\infty \lambda^s dE(\lambda) = \int_0^\infty \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) t^{-1-s} dt \, dE(\lambda) \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\Delta} - I) t^{-1-s} dt.\end{aligned}$$

This formal computation can also be easily justified and we observe that the heat equation approach from Section 2.3 coincides with the spectral fractional Laplacian on bounded domains.

We are now in the position to proof the preceding theorem.

Proof of Theorem 2.17. We prove that the formula

$$\mathcal{V}(x,y) := \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta}(-\Delta)^s_\sigma u(x) \, e^{-y^2/(4t)} t^{s-1} dt \tag{2.37}$$

defines a solution to (2.29) in the weak $L^2(\Omega)$ sense, i.e., we have $(\mathcal{L}\mathcal{V}, g)_{L^2(\Omega)} = 0$ for all $g \in L^2(\Omega)$ as well as weak boundary conditions. Then, computing the Neumann-data proves the theorem.
1. step. At first, we show that there holds $\mathcal{V}(\cdot, y) \in L^2(\Omega)$ and

$$(\mathcal{V}(\cdot,y), g)_{L^2} = \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} \; e^{-y^2/(4t)} t^{s-1} dt. \tag{2.38}$$

In order to do so, let $R > 0$ and define $\mathcal{V}_R$ as

$$\mathcal{V}_R(x,y) := \frac{1}{\Gamma(s)} \int_0^R e^{t\Delta}(-\Delta)^s_\sigma u(x) \, e^{-y^2/(4t)} t^{s-1} dt.$$

As $e^{-y^2/(4t)} t^{s-1}$ is integrable and $e^{t\Delta}(-\Delta)^s_\sigma u$ is bounded, we may interchange the L^2-scalar product with the integration in t by standard rules of Bochner integration to obtain

$$\begin{aligned}(\mathcal{V}_R(\cdot,y), g)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^R \left(e^{t\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} \; e^{-y^2/(4t)} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^R \int_0^\infty e^{-t\lambda} \lambda^s dE_{u,g}(\lambda) \, e^{-y^2/(4t)} t^{s-1} dt,\end{aligned}$$

where the last equality follows from the spectral calculus. Now interchanging the integration with Fubini's theorem, we obtain together with the scaling $\tau = t\lambda$

$$\begin{aligned}(\mathcal{V}_R(\cdot,y), g)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^R e^{-t\lambda} \lambda^s e^{-y^2/(4t)} t^{s-1} dt \, dE_{u,g}(\lambda) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^{\lambda R} e^{-\tau} \tau^{s-1} e^{-y^2/(4\tau)\lambda} d\tau \, dE_{u,g}(\lambda) \\ &\leq \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty e^{-\tau} \tau^{s-1} d\tau \, dE_{u,g}(\lambda) \\ &\leq \|u\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, for all fixed $y \in \mathbb{R}^+$, we have $\mathcal{V}_R(\cdot, y) \in L^2(\Omega)$, and for any sequence $(R_j)_{j\in\mathbb{N}}$ satisfying $R_j \to \infty$ the sequence $(\mathcal{V}_{R_j}(\cdot, y))_{j\in\mathbb{N}}$ is a Cauchy sequence that converges weakly to $\mathcal{V}(\cdot, y)$, which implies $\mathcal{V}(\cdot, y) \in L^2(\Omega)$. Taking the limit $R \to \infty$ also immediately shows equation (2.38).

2. step. Similar arguments can be used to show that $\mathcal{V}(\cdot, y) \in \mathrm{dom}(-\Delta)$ by proving that the limit $\lim_{\tau\to 0^+} \left(\frac{e^{\tau\Delta}\mathcal{V}(\cdot,y)-\mathcal{V}(\cdot,y)}{\tau}, g\right)_{L^2}$ exists for all $g \in L^2(\Omega)$. We refer to [ST10] for details.

3. step. $\mathcal{V}$ satisfies the boundary condition at $y = 0$. We use (2.38) and the transformation $\tau = t\lambda$ as in step 1 to compute with the Lebesgue dominated convergence theorem and the definition of the Γ-function

$$
\begin{aligned}
(\mathcal{V}(\cdot, y), g)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty e^{-\tau}\tau^{s-1} e^{-y^2/(4\tau)\lambda} d\tau \; dE_{u,g}(\lambda) \\
&\xrightarrow[y\to 0]{} \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty e^{-\tau}\tau^{s-1} d\tau \; dE_{u,g}(\lambda) = (u, g)_{L^2} .
\end{aligned}
$$

4. step. $\mathcal{V}$ is differentiable and satisfies equation (2.29). With the help of Lebesgue dominated convergence, we compute

$$
\lim_{h\to 0} \left(\frac{\mathcal{V}(\cdot, y+h) - \mathcal{V}(\cdot, y)}{h}, g\right)_{L^2} = \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} \; \partial_y e^{-y^2/(4t)} t^{s-1} dt.
$$

Thus, $\mathcal{V}$ is differentiable in y, and the y-derivatives are just applied to $e^{-y^2/(4t)}$. An elementary calculation gives

$$
\left(\partial^2_{y^2} + \frac{1-2s}{y}\partial_y\right) e^{-y^2/(4t)} t^{s-1} = \partial_t \left(\frac{e^{-y^2/(4t)}}{t^{1-s}}\right) .
$$

This identity and integration by parts prove

$$
\begin{aligned}
\left(\left(\partial^2_{y^2} + \frac{1-2s}{y}\partial_y\right)\mathcal{V}(\cdot, y), g\right)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} \; \partial_t \left(e^{-y^2/(4t)} t^{s-1}\right) dt \\
&= -\frac{1}{\Gamma(s)} \int_0^\infty \partial_t \left(e^{t\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} \; e^{-y^2/(4t)} t^{s-1} dt \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty \lambda e^{-t\lambda} \lambda^s dE_{u,g}(\lambda) \; e^{-y^2/(4t)} t^{s-1} dt \\
&= ((-\Delta)\mathcal{V}(\cdot, y), g)_{L^2}
\end{aligned}
$$

and $\mathcal{V}$, indeed, solves the differential equation (2.29).

5. step. Finally, we compute the Neumann data. With (2.38) and the substitution $\tau = y^2/(4t)$, we obtain

$$
\begin{aligned}
\left(\frac{\mathcal{V}(\cdot, y) - \mathcal{V}(\cdot, 0)}{y^{2s}}, g\right)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} \; \frac{(e^{-y^2/(4t)} - 1)}{y^{2s}} t^{s-1} dt \\
&= \frac{1}{4^s\Gamma(s)} \int_0^\infty \left(e^{\frac{y^2}{4\tau}\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} \; (e^{-\tau} - 1)\tau^{s+1} d\tau.
\end{aligned}
$$

Now, dominated convergence implies

$$\lim_{y\to 0}\left(e^{\frac{y^2}{4\tau}\Delta}(-\Delta)^s_\sigma u, g\right)_{L^2} = ((-\Delta)^s_\sigma u, g)_{L^2}\,,$$

and taking the limit $y \to 0$ in the previous equation gives

$$\lim_{y\to 0}\left(\frac{\mathcal{V}(\cdot,y)-\mathcal{V}(\cdot,0)}{y^{2s}}, g\right)_{L^2} = \frac{\Gamma(-s)}{4^s\Gamma(s)}\,((-\Delta)^s_\sigma u, g)_{L^2}\,,$$

which concludes the proof. ■

2.5.5 Function spaces for the Caffarelli-Silvestre extension

A suitable function space for the extension problem is given by weighted Sobolev spaces in the extended variable.
For a subset $D \subset \mathbb{R}^d \times \mathbb{R}^+$, we define the space $L^2(D; y^\alpha)$ as $L^2(D)$-space with measure $y^\alpha d\lambda(x, y)$ and norm

$$\|v\|^2_{L^2(D;y^\alpha)} = \int_D |v|^2 y^\alpha dxdy < \infty.$$

Imposing this weighted integrability also for the gradient, we define the weighted Sobolev space $H^1(D; y^\alpha)$ as

$$H^1(D; y^\alpha) := \{w \in L^2(D; y^\alpha) \,:\, |\nabla w| \in L^2(D; y^\alpha)\}$$

with norm

$$\|w\|^2_{H^1(D;y^\alpha)} := \|w\|^2_{L^2(D;y^\alpha)} + \|\nabla w\|^2_{L^2(D;y^\alpha)}.$$

These spaces satisfy:

- For $\alpha = 0$, we have the standard $H^1(D)$-space.
- $H^1(D; y^\alpha)$ is a Hilbert space.
- $C^\infty(\overline{D}) \cap H^1(D; y^\alpha)$ is dense in $H^1(D; y^\alpha)$.
- Let Ω be bounded and $\mathcal{Y} > 0$, then

$$\begin{aligned} &H^1(\Omega \times (0, \mathcal{Y})) \hookrightarrow H^1(\Omega \times (0, \mathcal{Y}); y^\alpha) \qquad &&\alpha \in (0, 1)\\ &H^1(\Omega \times (0, \mathcal{Y})) \hookleftarrow H^1(\Omega \times (0, \mathcal{Y}); y^\alpha) \qquad &&\alpha \in (-1, 0)\end{aligned}$$

For the full-space extension problem (or the extension problem for the integral fractional Laplacian), the appropriate function space for the minimization problem (2.13) and in turn its Euler-Lagrange equations (2.11) is the so called Beppo-Levi space

$$\mathrm{BL}^1_\alpha := \{U \in L^2_{loc}(\mathbb{R}^d \times \mathbb{R}^+) \,:\, \nabla U \in L^2(\mathbb{R}^d \times \mathbb{R}^+; y^\alpha)\}.$$

Lemma 2.20 ([KM19]). *Let $\alpha \in (-1,1)$. The trace operator* tr *at $y = 0$ can be uniquely extended (from $C^\infty(\mathbb{R}^d \times \overline{\mathbb{R}^+}) \cap H^1(\mathbb{R}^d \times \mathbb{R}^+; y^\alpha)$) to a bounded linear operator mapping $H^1_\alpha(\mathbb{R}^d \times \mathbb{R}^+) \to H^s(\mathbb{R}^d)$ with the trace inequalities*

$$\begin{aligned}
&\|\operatorname{tr} V\|_{L^2(\Omega)} \le C\left(\|V\|_{L^2(\Omega\times\mathbb{R}^+;y^\alpha)} + \|V\|^{(1-\alpha)/2}_{L^2(\Omega\times\mathbb{R}^+;y^\alpha)} \|\partial_y V\|^{(1+\alpha)/2}_{L^2(\Omega\times\mathbb{R}^+;y^\alpha)}\right)\\
&|\operatorname{tr} V|_{H^s(\mathbb{R}^d)} \le C\|\nabla V\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+;y^\alpha)}
\end{aligned}$$

for all measurable subsets $\Omega \subset \mathbb{R}^d$.

Proof. We only show the first estimate, the second estimate follows from transformation into polar coordinates, the fundamental theorem of calculus and a weighted Hardy inequality, see [KM19]. By density of smooth functions as stated above, we may assume that V is sufficiently small.
Step 1: Assume that $\operatorname{supp} V \subset \mathbb{R}^d \times [0, \mathcal{Y}]$ for some $\mathcal{Y} > 1$. Then, $\|\partial_y V(x,\cdot)\|_{L^2(\mathbb{R}^+;y^\alpha)} = 0$ implies $V(x,\cdot) \equiv 0$. We thus may assume $\|\partial_y V(x,\cdot)\|_{L^2(\mathbb{R}^+;y^\alpha)} \neq 0$ in the following. Hölder's inequality gives for any $z > 0$

$$|V(x,0)| \le |V(x,z)| + |\int_0^z \partial_y V(x,y)dy| \le C\left(|V(x,z)| + z^{(1-\alpha)/2}\|\partial_y V(x,\cdot)\|_{L^2(\mathbb{R}^+;y^\alpha)}\right).$$

A scaling argument (transforming $(z,2z)$ to $(0,1)$), a 1d trace inequality on the interval $(0,1)$ and transforming back show

$$|V(x,z)|^2 \le C\Big(z^{-1}\int_z^{2z} |V(x,y)|^2 dy + z\int_z^{2z} |\partial_y V(x,y)|^2 dy\Big).$$

For $y \in (z,2z)$, there holds $1 \lesssim z^{-\alpha}y^\alpha$, which gives

$$|V(x,z)|^2 \le C\Big(z^{-1-\alpha}\|V(x,\cdot)\|^2_{L^2(\mathbb{R}^+;y^\alpha)} + z^{1-\alpha}\|\partial_y V(x,\cdot)\|^2_{L^2(\mathbb{R}^+;y^\alpha)}\Big).$$

Now, we may select $z = \|V(x,\cdot)\|_{L^2(\mathbb{R}^+;y^\alpha)}\|\partial_y V(x,\cdot)\|^{-1}_{L^2(\mathbb{R}^+;y^\alpha)}$ and obtain

$$|V(x,0)|^2 \le C\|V(x,\cdot)\|^{1-\alpha}_{L^2(\mathbb{R}^+;y^\alpha)}\|\partial_y V(x,\cdot)\|^{1+\alpha}_{L^2(\mathbb{R}^+;y^\alpha)}. \tag{2.39}$$

Integration over Ω shows the claimed estimate.
Step 2: Take a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset [0,2]$, $\chi \equiv 1$ on $[0,1]$, $\|\chi\|_{L^\infty(\mathbb{R}^+)} + \|\chi'\|_{L^\infty(\mathbb{R}^+)} \le C$. Then, estimate (2.39) can be applied to the function χV and gives

$$\begin{aligned}
|V(x,0)|^2 = |\chi V(x,0)|^2 &\le C\|\chi V(x,\cdot)\|^{1-\alpha}_{L^2(\mathbb{R}^+;y^\alpha)}\|\partial_y(\chi V(x,\cdot))\|^{1+\alpha}_{L^2(\mathbb{R}^+;y^\alpha)}\\
&\le C\Big(\|V(x,\cdot)\|^2_{L^2(\mathbb{R}^+;y^\alpha)} + \|V(x,\cdot)\|^{1-\alpha}_{L^2(\mathbb{R}^+;y^\alpha)}\|\partial_y V(x,\cdot)\|^{1+\alpha}_{L^2(\mathbb{R}^+;y^\alpha)}\Big).
\end{aligned} \tag{2.40}$$

Integration over Ω gives the result. ∎

For elements $U \in \mathrm{BL}^1_\alpha$, one can also extend the trace operator at $y = 0$ by density arguments.

Chapter 3

Numerical Approximation

3.1 The Integral Fractional Laplacian - Weak Formulation

In this section $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. We consider the integral fractional Laplacian $(-\Delta)^s$ and study the model problem

$$(-\Delta)^s u = f \quad \text{in } \Omega, \tag{3.1a}$$
$$u = 0 \quad \text{on } \Omega^c \tag{3.1b}$$

with a given right-hand side $f \in H^{-s}(\Omega)$.

3.1.1 Variational chacterization of the integral Fractional Laplacian

Lemma 3.1. *For u, $v \in \mathcal{S}$ there holds*

$$\int_{\mathbb{R}^d} (-\Delta)^s uv \, dx = \frac{C(d,s)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}} \, dy \, dx. \tag{3.2}$$

Furthermore,

$$\int_{\mathbb{R}^d} v(x) P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} \, dy \, dx = -\int_{\mathbb{R}^d} P.V. \int_{\mathbb{R}^d} v(y) \frac{u(x)-u(y)}{|x-y|^{d+2s}} \, dy \, dx. \tag{3.3}$$

Proof. We present two approaches for (3.2). Given $h \in \mathbb{R}^d$, we define the difference operator $(D_h u)(x) := u(x+h) - u(x)$. A change of variables argument shows

$$\int_{\mathbb{R}^d} (D_h f)(x) g(x) \, dx = -\int_{\mathbb{R}^d} f(x) D_{-h} g(x) \, dx.$$

We note that $(D_h D_{-h} f)(x) = f(x+h) - 2f(x) + f(x-h)$. Recalling Lemma 2.2 and noting as in the proof of Lemma 2.2 that Fubini's theorem is applicable gives

$$\begin{aligned}
\int_{\mathbb{R}^d} (-\Delta)^s uv &= -\frac{C(d,s)}{2} \int_{x\in\mathbb{R}^d} \int_{z\in\mathbb{R}^d} v(x) \frac{(D_{-z} D_z u)(x)}{|z|^{d+2s}}\, dz\, dx \\
&= -\frac{C(d,s)}{2} \int_{z\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} v(x) \frac{(D_{-z} D_z u)(x)}{|z|^{d+2s}}\, dz\, dx \\
&= \frac{C(d,s)}{2} \int_{z\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} D_z v(x) \frac{(D_z u)(x)}{|z|^{d+2s}}\, dz\, dx \\
&= \frac{C(d,s)}{2} \int_{z\in\mathbb{R}^d} \int_{x\in\mathbb{R}^d} \frac{(v(x+z) - v(x))(u(x+z) - u(x))}{|z|^{d+2s}}\, dz\, dx
\end{aligned}$$

which is (3.2).
An alternative proof is given by Fourier techniques. Using Remark 2.7 we get

$$\begin{aligned}
\langle (-\Delta)^s u, v\rangle_{L^2(\mathbb{R}^d)} &\overset{\text{Parseval}}{=} \langle \mathcal{F}(-\Delta)^s u, \mathcal{F}v\rangle_{L^2(\mathbb{R}^d)} = \langle |\zeta|^{2s} \mathcal{F}u, \mathcal{F}v\rangle_{L^2(\mathbb{R}^d)} = \langle |\zeta|^s \mathcal{F}u, |\zeta|^s \mathcal{F}v\rangle_{L^2(\mathbb{R}^d)} \\
&= \langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v\rangle_{L^2(\mathbb{R}^d)} \\
&\overset{\text{Rem. 2.7}}{=} \frac{C(d,s)}{2} \int_{x\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{d+2s}}.
\end{aligned}$$

Proof of (3.3): We employ (3.2), which gives

$$\int_x P.V. \int_y v(x) \frac{u(x) - u(y)}{|x-y|^{d+2s}} = \int_x P.V. \int_y (v(x) - v(y)) \frac{u(x) - u(y)}{|x-y|^{d+2s}} + \int_x P.V. \int_y v(y) \frac{u(x) - u(y)}{|x-y|^{d+2s}}.$$

In the first integral, the integrand is integrable so that the P.V. operation is not required. Recognizing this integral as (up to the factor $C(d,s)/2$) the right-hand side of (3.2), we get

$$\int_x P.V. \int_y v(x) \frac{u(x) - u(y)}{|x-y|^{d+2s}} = 2 \int_x P.V. \int_y v(x) \frac{u(x) - u(y)}{|x-y|^{d+2s}} + \int_x P.V. \int_y v(y) \frac{u(x) - u(y)}{|x-y|^{d+2s}}.$$

Simplifying shows the result. ∎

3.1.2 Weak formulation

A weak formulation for (3.1) is obtained by multiplying the equation (3.1a) with a test function $v \in C_0^\infty(\Omega)$ and integrating over $\mathbb{R}^d$ to get in view of Lemma 3.1

$$\int_{\mathbb{R}^d} (-\Delta)^s u \cdot v dx = \frac{C(d,s)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{d+2s}}\, dy\, dx = \int_{\mathbb{R}^d} f(y) v(y)\, dy.$$

Lemma 3.2. *The bilinear form*

$$a(u,v) := \frac{C(d,s)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}\, dx\, dy \tag{3.4}$$

is a symmetric, continuous bilinear form on $\widetilde{H}^s(\Omega) \times \widetilde{H}^s(\Omega)$. *Furthermore, it is coercive, i.e., there is* $C_{\text{coer}} > 0$ *(depending only on* Ω*) such that*

$$C_{\text{coer}} \|u\|^2_{\widetilde{H}^s(\Omega)} \leq a(u,u) \qquad \forall u \in \widetilde{H}^s(\Omega).$$

Proof. Follows directly from Lemma 2.14. Note that $a(\cdot,\cdot)$ is—up to the scaling factor $C(d,s)/2$—the inner product that induces the seminorm $|\cdot|_{\widetilde{H}^s(\Omega)}$. By Lemma 2.14, this seminorm is a norm equivalent to the full norm $\|\cdot\|_{\widetilde{H}^s(\Omega)}$. ■

Hence, with the bilinear form $a(\cdot,\cdot)$ of (3.4), we obtain the weak formulation

$$a(u,v) = \int_{\mathbb{R}^d} f(y)v(y)\, dy \qquad \forall v \in C_0^\infty(\Omega). \tag{3.5}$$

For $f \in L^2(\Omega)$, the right-hand side $v \mapsto \int_{\mathbb{R}^d} f(y)v(y)\, dy$ extends to a linear functional on $\widetilde{H}^s(\Omega)$. We will therefore *define* the weak solution of (3.1) as the solution of the problem:

$$\text{Find } u \in \widetilde{H}^s(\Omega) \text{ such that} \qquad a(u,v) = \int_\Omega f v dx \quad \forall v \in \widetilde{H}^s(\Omega). \tag{3.6}$$

As it is written, (3.6) assumes that f is a function. However, the weak formulation (3.6) makes sense if f is a linear functional on $\widetilde{H}^s(\Omega)$, i.e., if $f \in H^{-s}(\Omega) = \left(\widetilde{H}^s(\Omega)\right)'$.
By Lemma 3.2 we can apply the Lax-Milgram theorem and obtain unique solvability of (3.6):

Theorem 3.3. *There exists a unique weak solution* $u \in \widetilde{H}^s(\Omega)$ *of* (3.6), *which also satisfies*

$$\|u\|_{\widetilde{H}^s(\Omega)} \lesssim \|f\|_{H^{-s}(\Omega)}.$$

3.1.3 Galerkin method

The Galerkin method for (3.6) is as follows: Let $X_h \subset \widetilde{H}^s(\Omega)$ be a closed subspace, e.g., a finite-dimensional subspace. Then the Galerkin approximation u_h to the solution u of (3.6) is given by

$$\text{Find } u_h \in X_h \text{ such that} \qquad a(u_h,v) = \int_\Omega f v\, dx \qquad \forall v \in X_h. \tag{3.7}$$

The classical Céa-Lemma gives existence, uniqueness, and quasi-optimality:

Lemma 3.4 (Céa-Lemma). *Let Ω be a bounded Lipschitz domain. There exists $C > 0$ depending only on Ω and $s \in (0,1)$ such that, for any closed $X_h \subset \widetilde{H}^s(\Omega)$, problem (3.7) has a unique solution u_h, which satisfies*

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq C \inf_{v_h \in X_h} \|u - v_h\|_{\widetilde{H}^s(\Omega)}. \tag{3.8}$$

Proof. Existence and uniqueness of u_h follows from the fact that the closed subspace X_h is again a Hilbert space (cf. Lemma 2.14, (ii)) so that the Lax-Milgram lemma is applicable to (3.7).
By subtracting (3.6) and (3.7), we obtain the *Galerkin orthogonality*

$$a(u - u_h, v_h) = 0 \qquad \forall v_h \in X_h. \tag{3.9}$$

Hence, we get with the coercivity of $a(\cdot,\cdot)$ and arbitrary $v_h \in X_h$

$$\begin{aligned} C_{\mathrm{coer}}\|u-u_h\|^2_{\widetilde{H}^s(\Omega)} &\overset{\text{coercivity}}{\leq} a(u-u_h, u-u_h) \overset{(3.9)}{=} a(u-u_h, u) \overset{(3.9)}{=} a(u-u_h, u-v_h) \\ &\overset{\text{continuity}}{\leq} C\|u-u_h\|_{\widetilde{H}^s(\Omega)}\|u-v_h\|_{\widetilde{H}^s(\Omega)}. \end{aligned}$$

Since v_h is arbitrary, we arrive at

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq C \inf_{v_h \in X_h} \|u - v_h\|_{\widetilde{H}^s(\Omega)},$$

which finishes the proof. ■

The discrete solution u_h in (3.7) can be computed as the solution of a linear system of equations, if X_h is finite-dimensional: let $(\varphi_i)_{i=1}^N$ be a basis of X_h. Making the ansatz $u_h = \sum_j \mathbf{u}_j \varphi_j$ for a vector $\mathbf{u} \in \mathbb{R}^N$, problem (3.7) is equivalent to:

$$\text{Find } \mathbf{u} \in \mathbb{R}^N \text{ s.t.} \quad \sum_j \mathbf{u}_j a(\varphi_j, \varphi_i) = \int_\Omega f\varphi_i, \qquad i = 1,\ldots,N. \tag{3.10}$$

With the *stiffness matrix* $\mathbf{A} \in \mathbb{R}^{N\times N}$ and the *load vector* $\mathbf{l} \in \mathbb{R}^N$ given by

$$\mathbf{A}_{ij} = a(\varphi_j, \varphi_i), \qquad \mathbf{l}_i = \int_\Omega f\varphi_i \tag{3.11}$$

(3.10) is equivalent to solving the linear system $\mathbf{Au} = \mathbf{l}$.
The procedure so far is very much remiscent of the classical FEM for elliptic problem such as the Poisson problem. However, there are differences discussed in the following remark.

Remark 3.5 • *The bilinear form $a(\cdot,\cdot)$ is non-local, i.e., for non-trivial functions φ_i, $\varphi_j \geq 0$ satisfying* $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$, *we get*

$$\begin{aligned} a(\varphi_i,\varphi_j) &= \frac{C(d,s)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\varphi_i(x)-\varphi_i(y))(\varphi_j(x)-\varphi_j(y))}{|x-y|^{d+2s}}dy\,dx \\ &= -\frac{C(d,s)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\varphi_i(x)\varphi_j(y)+\varphi_i(y)\varphi_j(x))}{|x-y|^{d+2s}}dy\,dx \\ &= -C(d,s)\int_{\operatorname{supp}\varphi_i}\int_{\operatorname{supp}\varphi_j} \frac{\varphi_i(x)\varphi_j(y)}{|x-y|^{d+2s}}dy\,dx < 0. \end{aligned} \tag{3.12}$$

Thinking of φ_i as local basis functions, this computation shows that the stiffness matrices will be densely populated in contrast to the classical FEM case.

- *The energy norm ($\widetilde{H}^s(\Omega)$-norm) is non-local, and the seminorm part is not additive. Indeed, let $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$. Then,*

$$|v|^2_{H^s(\Omega)} = |v|^2_{H^s(\Omega_1)} + |v|^2_{H^s(\Omega_2)} + 2\int_{\Omega_1}\int_{\Omega_2} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dy\, dx,$$

and the last term cannot be bounded by the first two terms on the right-hand side.

This leads to problems in the finite element analysis as error estimates are usually derived elementwise. However, Proposition 3.8 gives a way to circumvent this by deriving a localized upper bound for the seminorm.

3.1.4 Classical piecewise polynomial approximation

Meshes and discrete spaces

Let $\mathcal{T}$ be a regular mesh on $\Omega \subset \mathbb{R}^d$. That is:

Definition 3.6 (regular mesh).

1. *(covering) The sets $T \in \mathcal{T}$ are open, pairwise disjoint, and $\cup_{T\in\mathcal{T}}\overline{T} = \overline{\Omega}$.*
2. *(affine element maps) Each T is the image under an affine transformation $F_T : \widehat{T} \to T$ of the reference d-simplex $\widehat{T}$.*
3. *(regular mesh) Two elements T, $T' \in \mathcal{T}$ satisfy the following: either T and T' coincide or the intersection $\overline{T} \cap \overline{T'}$ is a j-simplex for some $j \in \{0, \dots, d-1\}$. For example, for $d = 2$, elements T, T' can only share a vertex or a full edge.*

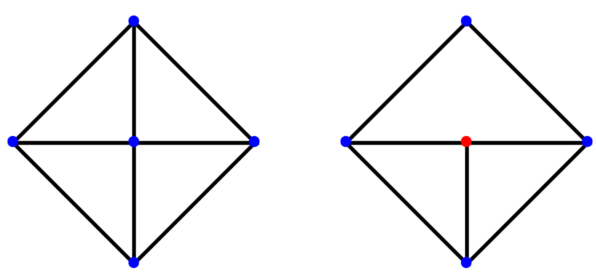


Figure 3.1: A regular mesh (left) and a mesh violating the notion of regular meshes (right).

We denote by $h_T := \operatorname{diam} T$ the size of the element T.
We will additionally often require the notion of **shape-regularity**. For spatial dimension $d \geq 2$, a mesh $\mathcal{T}$ is called σ-shape-regular (or just shape-regular), if

$$\max_{T\in\mathcal{T}} \frac{h_T^d}{|T|} \leq \sigma. \tag{3.13}$$

For $d = 1$, we call a mesh shape-regular, if

$$\frac{h_T}{h_{T'}} \leq \sigma \quad \text{ if } \overline{T} \cap \overline{T'} \neq \emptyset. \tag{3.14}$$

We call a mesh $\mathcal{T}$ **quasi-uniform**, if it is shape-regular and additionally[1]

$$\min_{T\in\mathcal{T}} h_T \sim \max_{T\in\mathcal{T}} h_T =: h. \tag{3.15}$$

The simplest spaces of piecewise polynomials on a regular mesh $\mathcal{T}$ are the space of piecewise constant functions $\mathcal{P}^0(\mathcal{T})$, the space $\mathcal{P}^1(\mathcal{T})$ of piecewise linears, and the space $\mathcal{P}^1_0(\mathcal{T})$ of piecewise linears that vanish on $\partial\Omega$ given by

$$\mathcal{P}^0(\mathcal{T}) := \{v \in L^2(\Omega) \colon v|_T \in P_0(T) \quad \forall T \in \mathcal{T}\}, \tag{3.16}$$
$$\mathcal{P}^p(\mathcal{T}) := \{v \in C(\overline{\Omega}) \colon v|_T \in P_p(T) \quad \forall T \in \mathcal{T}\}, \qquad p \geq 1 \tag{3.17}$$
$$\mathcal{P}^p_0(\mathcal{T}) := \{v \in C(\overline{\Omega}) \colon v|_T \in P_p(T) \quad \forall T \in \mathcal{T}, \quad v|_{\partial\Omega} = 0\}, \quad p \geq 1. \tag{3.18}$$

Here, the space $P_p(T)$ is the space of polyomials of (at most) degree $p \in \mathbb{N}_0$. We have

$$\mathcal{P}^0(\mathcal{T}) \subset \widetilde{H}^s(\Omega) \quad \text{if } s < 1/2, \tag{3.19}$$
$$\mathcal{P}^1_0(\mathcal{T}) \subset \widetilde{H}^s(\Omega) \quad \text{for } s \in (0,1). \tag{3.20}$$

Exercise 3.7 *Any course on FEM shows $\mathcal{P}^1_0(\mathcal{T}) \subset H^1_0(\Omega) \subset \widetilde{H}^s(\Omega)$ so that (3.20) holds. Show (3.19) for the case $d = 1$. Show that $\mathcal{P}^0 \not\subset \widetilde{H}^s(\Omega)$ for $s \geq 1/2$.*

For each element $T \in \mathcal{T}$ we define the **element patch** ω_T by

$$\omega_T := \text{interior} \bigcup \{\overline{T'} \ : \ \overline{T'} \cap \overline{T} \neq \emptyset\}.$$

Approximation from $\mathcal{P}^1_0(\mathcal{T})$

Before we can prove *a priori* estimates, we need to discuss how to deal with the fact that the norm $\|\cdot\|_{\widetilde{H}^s(\Omega)}$, in which we measure the error, is nonlocal. This issue was already raised in Remark 3.5. This norm can in fact be localized to some extent as the following Theorem 3.8, due to [Fae00, Fae02], shows:

Theorem 3.8. *Let $s \in (0,1)$ and $u \in H^s(\Omega)$. Then,*

$$|u|^2_{H^s(\Omega)} \leq \sum_{T\in\mathcal{T}} \int_T \int_{\omega_T} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy\, dx + \frac{C}{s h_T^{2s}} \|u\|^2_{L^2(T)},$$

where the constant $C > 0$ depends only on d and the shape-regularity constant σ.

[1] Since any *fixed* mesh is – technically speaking – quasiuniform, the notion of quasi-uniform mesh is actually only meaningful when considering a family of meshes $\mathcal{T}_h$; then quasi-uniformity requires the shape-regularity parameter σ and the implied constants in (3.15) not to depend on h.

Proof. Let $T \in \mathcal{T}$ and define $\omega_T^c := \Omega \backslash \omega_T$. We write

$$|u|_{H^s(\Omega)}^2 \leq \sum_{T \in \mathcal{T}} \int_T \int_{\omega_T} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dy\, dx + \int_T \int_{\omega_T^c} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dy\, dx.$$

The first term has the right form already, and it remains to estimate the second one. Using the triangle inequality and Fubini's theorem, we get

$$\begin{aligned}\int_T \int_{\omega_T^c} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dy\, dx &\leq 2 \int_T |u(x)|^2 \int_{\omega_T^c} |x-y|^{-d-2s} dy\, dx + 2 \int_{\omega_T^c} |u(y)|^2 \int_T |x-y|^{-d-2s} dx\, dy \\ &=: 2 I_{T,1} + 2 I_{T,2}.\end{aligned}$$

In the following, we show that $\sum_{T \in \mathcal{T}} I_{T,1} = \sum_{T \in \mathcal{T}} I_{T,2}$. With the characteristic function $\chi_{\omega_T^c}$ of the set ω_T^c, we write

$$\begin{aligned}\sum_{T \in \mathcal{T}} I_{T,2} &= \sum_{T \in \mathcal{T}} \int_\Omega \chi_{\omega_T^c}(y) |u(y)|^2 \int_T |x-y|^{-d-2s} dx\, dy \\ &= \int_\Omega |u(y)|^2 \underbrace{\sum_{T \in \mathcal{T}} \chi_{\omega_T^c}(y) \int_T |x-y|^{-d-2s} dx}_{=: f(y)}\, dy.\end{aligned}$$

Let $T' \in \mathcal{T}$ be the element with $y \in T'$. (Since we are integrating in y, we may ignore the mesh skeleton, which is a (Lebesgue) nullset.) We note that $\chi_{\omega_T^c}(y) \neq 0$ if and only if $T \subset \omega_{T'}^c$. Hence,

$$f(y) = \sum_{T \in \mathcal{T}} \chi_{\omega_T^c}(y) \int_T |x-y|^{-d-2s} dx = \sum_{T \in \mathcal{T}\colon T \subset \omega_{T'}^c} \int_T |x-y|^{-d-2s}\, dx = \int_{\omega_{T'}^c} |x-y|^{-d-2s} dx.$$

This finally leads to

$$\sum_{T \in \mathcal{T}} I_{T,2} = \sum_{T' \in \mathcal{T}} \int_{T'} |u(y)|^2 \int_{\omega_{T'}^c} |x-y|^{-d-2s} dx\, dy = \sum_{T' \in \mathcal{T}} I_{T',1}.$$

Using this equality gives the estimate

$$|u|_{H^s(\Omega)}^2 \leq \sum_{T \in \mathcal{T}} \int_T \int_{\omega_T} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dy\, dx + 4 \sum_{T \in \mathcal{T}} I_{T,1},$$

and we have to estimate $I_{T,1}$. Due to the assumption of shape-regularity of $\mathcal{T}$, there is a constant $c > 0$ depending only on the σ-shape regularity constant such that $\omega_T^c \subseteq \Omega \backslash B_{ch_T}(x)$. This allows one to estimate using polar coordinates

$$\begin{aligned}I_{T,1} &= \int_T |u(x)|^2 \int_{\omega_T^c} |x-y|^{-d-2s} dy\, dx \\ &\lesssim \int_T |u(x)|^2 \int_{ch_T}^\infty r^{-1-2s} dr\, dx \lesssim \frac{1}{sh_T^{2s}} \|u\|_{L^2(T)}^2,\end{aligned}$$

which finishes the proof. ■

With Theorem 3.8 in hand, we can formulate an approximation result assuming regularity of the solution. We emphasize that the regularity assumption $u \in H^2(\Omega)$ is not realistic for this problem class. Nevertheless, the proof shows how approximation results in nonlocal norms such as the $\|\cdot\|_{\widetilde{H}^s(\Omega)}$-norm can be obtained.

The proof of the approximation properties follows the proof of the "classical" FEM. As it is done there, we will work with scaling arguments and norm equivalences on finite-dimensional spaces. In order for these tools to work, we require that there be a finite number of patch types. That is, scaling a patch ω_T to size 1 leads to a patch $\widehat{\omega}_T$ that is a from a fixed, finite set of reference configurations. This can be achieved by constructing the mesh in a certain way. In 1D, e.g., by only bisecting elements (and enforcing shape regularity in the sense that neighboring elements have comparable size) and in dimension $d \geq 2$ by generating the mesh with Newest Vertex Bisection (NVB), see, e.g., [KPP13] for the case $d = 2$ and to [Ste08b] for the case $d \geq 3$.

Lemma 3.9. *Let $d \in \{1, 2, 3\}$. Let the exact solution $u \in \widetilde{H}^s(\Omega)$ of (3.6) satisfy the regularity assumption $u \in H^2(\Omega)$. Let the mesh $\mathcal{T}$ be σ-shape regular, generated by NVB and be quasi-uniform with maximal mesh width h. Then, there is $C > 0$ depending only on Ω, d, s such that the piecewise linear interpolant $Iu \in \mathcal{P}^1_0(\mathcal{T})$ satisfies*

$$\|u - Iu\|_{\widetilde{H}^s(\Omega)} \leq Ch^{2-s}\|u\|_{H^2(\Omega)}.$$

Proof. For simplicity of exposition, we restrict our attention to the case $d = 1$.
Let $T \in \mathcal{T}$ and ω_T be its patch. Let $F_{\omega_T} : \widehat{\omega}_T \to \omega_T$ be the affine transformation that realizes the scaling of ω_T to size 1. Throughout, we indicate by the notation $\widehat{\cdot}$ the pull-back to the reference patch, e.g., $\widehat{v} := v \circ F_{\omega_T}$. We note that by shape-regularity, we have $\operatorname{diam}\omega_T \sim h_T$ with implied constants depending only on the σ-shape-regularity.
Scaling arguments show the familiar equivalences

$$|v|^2_{H^j(\omega_T)} \sim h_T^{d-2j}|\widehat{v}|^2_{H^j(\widehat{\omega}_T)}, \qquad j \in \mathbb{N}_0.$$

For fractional Sobolev norms, a similar equivalence follows from the definition of the seminorm $|\cdot|_{H^s(\omega_T)}$:

$$\begin{aligned}|v|^2_{H^s(\omega_T)} &= \int_{x\in\omega_T}\int_{y\in\omega_T} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2s}}\,dy\,dx = \frac{|\omega_T|^2}{|\widehat{\omega}_T|^2}\int_{x\in\widehat{\omega}_T}\int_{y\in\widehat{\omega}_T} \frac{|\widehat{v}(x)-\widehat{v}(y)|^2}{|F_{\omega_T}(x)-F_{\omega_T}(y)|^{d+2s}}\,dy\,dx \\ &\sim \frac{h_T^{2d}}{h_T^{d+2s}}\int_{x\in\widehat{\omega}_T}\int_{y\in\widehat{\omega}_T} \frac{|\widehat{v}(x)-\widehat{v}(y)|^2}{|x-y|^{d+2s}}\,dy\,dx \sim h_T^{d-2s}|\widehat{v}|^2_{H^s(\widehat{\omega}_T)}.\end{aligned}$$

Step 1 *(error estimate in $H^s(\omega_T)$-seminorm):* We denote by $\widehat{I}$ the piecewise linear interpolation operator on $\widehat{\omega}_T$. Since I and $\widehat{I}$ reproduce piecewise linear functions, we get for any piecewise linear function $\widehat{v}$ on $\widehat{\omega}_T$ that $\widehat{I}\widehat{v} = \widehat{v}$. Hence, we get by affine scaling

$$
\begin{aligned}
|u - Iu|_{H^s(\omega_T)} &\sim h_T^{d/2-s} |\widehat{u} - \widehat{I}\widehat{u}|_{H^s(\omega_T)} = h_T^{d/2-s} |\widehat{u} - \widehat{v} - \widehat{I}(\widehat{u} - \widehat{v})|_{H^s(\widehat{\omega}_T)} \\
&\le h_T^{d/2-s} \left[|\widehat{u} - \widehat{v}|_{H^s(\widehat{\omega}_T)} + |\widehat{I}(\widehat{u} - \widehat{v})|_{H^s(\widehat{\omega}_T)} \right] \\
&\overset{\text{norm equiv. on finite-dim. spaces}}{\le} C h_T^{d/2-s} \left[|\widehat{u} - \widehat{v}|_{H^s(\widehat{\omega}_T)} + |\widehat{I}(\widehat{u} - \widehat{v})|_{C(\overline{\widehat{\omega}}_T)} \right] \\
&\overset{\text{cont. of } \widehat{I}}{\le} C h_T^{d/2-s} \left[|\widehat{u} - \widehat{v}|_{H^s(\widehat{\omega}_T)} + |\widehat{u} - \widehat{v}|_{C(\overline{\widehat{\omega}}_T)} \right] \\
&\overset{\text{embedding } H^2 \subset C}{\le} C h_T^{d/2-s} \|\widehat{u} - \widehat{v}\|_{H^2(\widehat{\omega}_T)}.
\end{aligned}
$$

Since $\widehat{v}$ is arbitrary, we get with the Deny-Lions Lemma (i.e., compactness of H^2 in H^1; employed elementwise for all scaled elements in the scaled patch $\widehat{\omega}_T$)[2]

$$
|u - Iu|_{H^s(\omega_T)} \le C h_T^{d/2-s} \inf_{\widehat{v}} \|\widehat{u} - \widehat{v}\|_{H^2(\widehat{\omega}_T)} \le C h_T^{d/2-s} |\widehat{u}|_{H^2(\widehat{\omega}_T)} \sim h_T^{d/2-s} h_T^{-d/2+2} |u|_{H^2(\omega_T)}. \tag{3.21}
$$

Step 2 (*error estimate in* $L^2(\omega_T)$*):* By similar arguments, one obtains (see also FEM literature)

$$
\|u - Iu\|_{L^2(\omega_T)} \le C h_T^2 |u|_{H^2(\omega_T)}. \tag{3.22}
$$

Step 3 *(error estimate in weighted* $L^2(\omega_T)$ *with* T *touching* $\partial\Omega$*):* Let T be an element adjacent to the boundary. To fix notation, let the element map $F_T : (0,1) \to T$ be such that 0 is mapped to $\partial\Omega$, i.e., $F_T(0) \in \partial\Omega$. Then, scaling gives

$$
\| \operatorname{dist}(\cdot, \partial\Omega)^{-s} (u - Iu)\|_{L^2(T)} \sim h_T^{d/2-s} \| |x|^{-s} (\widehat{u} - \widehat{I}\widehat{u})\|_{L^2(\widehat{T})}.
$$

Since $(\widehat{u} - \widehat{I}\widehat{u})(0) = 0$, we can write $(\widehat{u} - \widehat{I}\widehat{u})(x) = \int_0^x (\widehat{u} - \widehat{I}\widehat{u})'(t)\,dt$ and estimate with a Cauchy-Schwarz inequality

$$
|(\widehat{u} - \widehat{I}\widehat{u})(x)| \le C x^{1/2} \|(\widehat{u} - \widehat{I}\widehat{u})'\|_{L^2(\widehat{T})}.
$$

We conclude

$$
\begin{aligned}
\| \operatorname{dist}(\cdot, \partial\Omega)^{-s} (u - Iu)\|_{L^2(T)} &\sim h_T^{d/2-s} \| |x|^{-s} (\widehat{u} - \widehat{I}\widehat{u})\|_{L^2(\widehat{T})} \lesssim h_T^{d/2-s} \|x^{1/2-s}\|_{L^2(\widehat{T})} \|(\widehat{u} - \widehat{I}\widehat{u})'\|_{L^2(\widehat{T})} \\
&\lesssim h_T^{d/2-s} \|(\widehat{u} - \widehat{I}\widehat{u})'\|_{L^2(\widehat{T})}.
\end{aligned}
$$

We may now proceed as in the other cases of scaling arguments to arrive at

$$
\| \operatorname{dist}(\cdot, \partial\Omega)^{-s} (u - Iu)\|_{L^2(T)} \lesssim h_T^{2-s} |u|_{H^2(T)}. \tag{3.23}
$$

Step 4 *(error estimate in weighted* $L^2(\omega_T)$ *with* T *not touching* $\partial\Omega$*):* By shape-regularity of the mesh $\mathcal{T}$, we have $\operatorname{dist}(T, \partial\Omega) \gtrsim h_T$. Hence, we get from (3.22)

$$
\| \operatorname{dist}(\cdot, \partial\Omega)^{-s} (u - Iu)\|_{L^2(T)} \lesssim h_T^{-s} \|u - Iu\|_{L^2(T)} \overset{(3.22)}{\lesssim} h_T^{2-s} |u|_{H^2(T)}.
$$

[2] in 1D, one can alternatively simply take $\widetilde{v} = \widehat{u}(x_0) + (x - x_0)\widehat{u}'(x_0)$ as the Taylor polynomial at a point x_0 and use $u(x) - \widehat{v} = \int_{t=x_0}^x (x - t)\widehat{u}''(t)\,dt$.

Step 5 *(conclusion of the proof):* Since the patches ω_T, $T \in \mathcal{T}$, have a finite overlap property (in fact, any element T' is in not more than 3 patches), we can estimate with Theorem 3.8

$$\begin{aligned}\|u - Iu\|^2_{\widetilde{H}^s(\Omega)} &\lesssim \sum_{T\in\mathcal{T}} |u - Iu|^2_{H^s(\omega_T)} + h_T^{-2s}\|u - Iu\|^2_{L^2(T)} + \|\operatorname{dist}(\cdot, \partial\Omega)^{-s}(u - Iu)\|^2_{L^2(T)} \\ &\lesssim \sum_{T\in\mathcal{T}} h_T^{2(2-s)}|u|^2_{H^2(\omega_T)} \lesssim h^{2(2-s)}|u|^2_{H^2(\Omega)},\end{aligned}$$

which finishes the proof. ■

By combining the approximation result from Lemma 3.9 with the best approximation result Lemma 3.4 we obtain:

Corollary 3.10. *Let $u \in H^2(\Omega) \cap \widetilde{H}^s(\Omega)$ solve (3.1). Let $u_h \in \mathcal{P}^1_0(\mathcal{T})$ be the Galerkin approximation to u given by (3.7) with $X_h = P^1_0(\mathcal{T})$. Then*

$$\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq Ch^{2-s}\|u\|_{H^2(\Omega)}.$$

3.1.5 Approximation of low-regularity functions with Clément/Scott-Zhang interpolation

Lemma 3.9 assumed significant regularity and could then work with the nodal interpolation operator. If the solution has less regularity, then a pointwise evalution of the solution u may not be possible and the construction of an approximation has to be done in a different way. The standard technique in FEM in this situation is to use a Clément or Scott-Zhang type operator. Both the Clément interpolant and the Scott-Zhang interpolant rely on local averaging of the function to be approximated.

Clément interpolation

The Clément interpolation operator $I^{Cl} : L^2(\Omega) \to \mathcal{P}^1(\mathcal{T})$, introduced in [Clé75], has the form

$$I^{Cl}u = \sum_{z\in\mathcal{N}(\mathcal{T})} \varphi_z \frac{1}{|\mathcal{A}(z)|} \int_{\mathcal{A}(z)} u\, dx.$$

Here, $\mathcal{N}(\mathcal{T})$ is the collection of all vertices of the triangulation $\mathcal{T}$ and the elements of $\mathcal{N}(\mathcal{T})$ are called *nodes*. The function φ_z is the standard hat function associated with the node $z \in \mathcal{N}(\mathcal{T})$. For each node $z \in \mathcal{N}(\mathcal{T})$, we select (arbitrarily) an element $T_z \in \mathcal{T}$ such that z is a vertex of T and we then set $\mathcal{A}(z) := T_z$.

Exercise 3.11 *Let the mesh $\mathcal{T}$ be regular and shape-regular. Show that I^{Cl} is locally stable in L^2, i.e., there is $C > 0$ depending only on Ω and the shape-regularity constant σ such that*

$$\|I^{Cl}u\|_{L^2(T)} \leq C\|u\|_{L^2(\omega_T)} \qquad \forall T \in \mathcal{T}.$$

Conclude that $\|I^{Cl}u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}$.

The Clément interpolation operator is in fact stable in other Sobolev norms. Notably, one has $\|I^{Cl}u\|_{H^1(T)} \leq C\|u\|_{H^1(\omega_T)}$.

Scott-Zhang interpolation

The Scott-Zhang projection I^{SZ}, introduced in [SZ90], is a refinement of the Clément interpolation operator. Its main advantage over the Clément operator is that a) it is projection and b) (homogeneous) Dirichlet boundary conditions can easily be satisfied.
We define the set of $(d-1)$-faces (e.g., edges in 2D) $\mathcal{E}(\mathcal{T})$ as all faces of elements in $\mathcal{T}$.
Let $\mathcal{A}(z)$ be a so called averaging set, which is for a fixed node $z \in \mathcal{N}(\mathcal{T})$ either an edge $E_z \in \mathcal{E}(\mathcal{T})$ with $z \in \overline{E_z}$ or an element $T_z \in \mathcal{T}$ with $z \in \overline{T_z}$.
Moreover, for all $z \in \mathcal{N}(\mathcal{T})$, there exists a local dual basis function, i.e., $\psi_z \in \mathcal{P}^1(\mathcal{A}(z))$ such that

$$\int_{\mathcal{A}(z)} \psi_z \varphi_{z'} = \delta_{zz'} \qquad \forall z' \in \mathcal{N}(\mathcal{T}),$$

where $\varphi_{z'}$ again denotes the usual hat function associated with the node $z' \in \mathcal{N}(\mathcal{T})$. With these notations, we may define the Scott-Zhang projection.

Definition 3.12. *Let $z \in \mathcal{N}(\mathcal{T})$. Choose $\mathcal{A}(z)$ either as*

- *$\mathcal{A}(z) = E_z$ with $E_z \subseteq \partial\Omega$, for $z \in \partial\Omega$;*
- *$\mathcal{A}(z) = T_z$ with $z \in \overline{T_z}$ for $z \in \Omega$.*

Then, the Scott-Zhang projection I^{SZ} is defined as

$$I^{SZ}v := \sum_{z \in \mathcal{N}(\mathcal{T})} \varphi_z \int_{\mathcal{A}(z)} \psi_z v \, dx.$$

We stress that the choice of averaging set is not unique, as there might be multiple elements with $z \in \overline{T_z}$.

The Scott-Zhang projection is well-defined for $s > 1/2$ and maps $H^s(\Omega) \to \mathcal{P}^1(\mathcal{T})$. Moreover, as the averaging set is chosen as boundary edge, for nodes at the boundary, it also satisfies I^{SZ} : $\widetilde{H}^s(\Omega) \to \mathcal{P}^1_0(\mathcal{T})$, i.e., homogeneous Dirichlet boundary conditions are preserved. The Scott-Zhang operator $I^{Sz} : H^s(\Omega) \to \mathcal{P}^1(\mathcal{T})$ is stable for $s > 1/2$, i.e., $\|I^{SZ}v\|_{H^s(\Omega)} \le C\|v\|_{H^s(\Omega)}$ for some $C > 0$ depending only on s, Ω, and the shape-regularity constant σ. I^{SZ} is a projection on $\mathcal{P}^1(\mathcal{T})$, i.e., $I^{SZ}v = v$ for all $v \in \mathcal{P}^1(\mathcal{T})$.

Remark. For $s \le 1/2$ the Scott-Zhang operator needs to be modified. There are two options:

1. For nodes $z \in \partial\Omega$, one selects as the averaging set also an element rather than an edge. This then leads to an operator $H^s(\Omega) \to \mathcal{P}^1(\mathcal{T})$. Note that the boundary conditions are not satisfied. This operator is then stable mapping $H^s(\Omega) \to \mathcal{P}^1(\mathcal{T})$ and a projection.
2. The values at nodes $z \in \partial\Omega$ is simply set to zero, i.e., the sum extends only over interior nodes. In that case, one obtains an operator that actually maps $\widetilde{H}^s(\Omega) \to \mathcal{P}^1_0(\mathcal{T})$. This operator is stable $\widetilde{H}^s(\Omega) \to \mathcal{P}^1_0(\mathcal{T})$ and a projection on $\mathcal{P}^1_0(\mathcal{T})$.

The Scott-Zhang operator has local approximation properties in H^s, which is stated in the following proposition, [Cia13].

Proposition 3.13. *Let $T \in \mathcal{T}$, $\max\{1/2, s\} < \ell \leq 2$, $v \in H^\ell(\Omega)$ and I^{SZ} be the Scott-Zhang projection of Definition 3.12. Then,*

$$\int_T \int_{\omega_T} \frac{|(v - I^{SZ}v)(x) - (v - I^{SZ}v)(y)|^2}{|x-y|^{d+2s}} dydx \leq C h_T^{2(\ell - s)} |v|^2_{H^\ell(\omega_T)},$$

where the constant $C > 0$ depends only on Ω, d, s, ℓ, and the shape-regularity of $\mathcal{T}$ and blows up for $s \to 1$.

Proposition 3.13 immediately gives approximation results, if the solution u has merely $H^{s+t}(\Omega)$-regularity for some $0 < t \leq 2 - s$. Asserting regularity of the solution u will be the topic of the next section.

3.2 The Integral Fractional Laplacian - Regularity

Regularity assertions for the solution are the key to obtaining rates of convergence of Galerkin method.
A first question is: If the right-hand side f of the equation satisfies $f \in H^r(\Omega)$ for some $r > -s$, for which $\ell \geq s$ can we expect that $u \in H^\ell(\Omega)$?

Comparing to the Laplace equation, the answer of this problem is more involved, and a field of active current research, in particular on polygonal Lipschitz domains, we especially mention the overview articles [BBN+18, BLN22] as well as [BN23a, FMMS22, BN23b]. For smooth domains or Hölder regularity, we refer to [Gru15, ROS14].

3.2.1 Regularity in Sobolev spaces

We start with an example highlighting the difference to the integer order Laplacian.

Example 3.14. *Let $\Omega = B_1(0)$, $f \equiv 2^{2s}\Gamma(1+s)^2$. Then, the exact solution of (3.1) is given by*

$$u(x) = (1 - |x|^2)^s_+, \qquad \text{where } g_+ := \max\{g, 0\}.$$

(See, e.g., [BV16, Sec. 2.6] and Figure 3.2.) One can check that $u \in \widetilde{H}^{s+1/2-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, but $u \notin H^{s+1/2}(\Omega)$.
Therefore, even for smooth Ω, f, one cannot expect more regularity than $s + 1/2 - \varepsilon$, which is vastly different from the case of the integer Laplacian, where the same problem setting would give smooth solutions.

Remark. One way to see that one should expect limited regularity is by considering the special case $s = 1/2$. Then, the solution of the Caffarelli-Silvestre extension problem is harmonic in $\mathbb{R}^d \times \mathbb{R}^+$. The

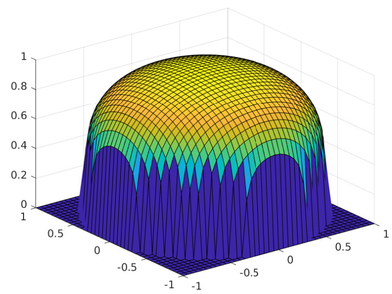


Figure 3.2: solution of $(-\Delta)^s u = 1$; $u = 0$ on Ω^c on the unit circle.

boundary conditions at $y = 0$ are of mixed type: Neumann boundary conditions at $\Omega \times \{0\}$ and Dirichlet boundary conditions at $\Omega^c \times \{0\}$. One expects a singularity at $\partial\Omega$.

Example 3.14 shows that a "shift theorem" in Sobolev spaces may only hold in limited range. The following result shows that expected shift by $2s$ holds until essentially the regularity $H^{s+1/2}$ is reached:

Theorem 3.15 ([BN23a]). *Let Ω be a bounded Lipschitz domain. Then, for $r \in (-s, 0]$ set $\gamma := \min\{s + r, 1/2\}$. Then for every $\varepsilon > 0$ and $f \in H^r(\Omega)$, the solution u of (3.1) satisfies*

$$\|u\|_{H^{s+\gamma-\varepsilon}(\Omega)} \leq C\|f\|_{H^r(\Omega)}.$$

Example 3.14 shows that one should expect the leading order behavior of the solution near the boundary to be $u \sim \operatorname{dist}(\cdot, \partial\Omega)^s$ (at least for smooth $\partial\Omega$). Example 3.14 also suggests that the loss of regularity is limited to the vicinity of $\partial\Omega$ if f is smooth. This is indeed the case:

Lemma 3.16. *(Interior regularity, [FMP19, Lemma 4.4]) Let $\omega \subset\subset \Omega$ be a domain. Let $f \in C^\infty(\overline{\omega})$. Then the solution u of (3.1) satisfies $u \in C^\infty(\overline{\omega})$.*

Proof. The proof uses the classical difference method of Nirenberg. For the Laplacian, it can be found in most PDE textbooks, e.g., [Eva98]. We proceed similarly using the extension problem as it is a PDE and afterwards employ the trace theorem to obtain the result for u.
As mentioned in Section 2.5.5, an appropriate function space for the Caffarelli-Silvestre extension is the space BL^1_α. Here, we additionally require the exterior boundary condition, i.e. we take the space

$$\mathrm{BL}^1_{\alpha,0} = \{U \in L^2_{loc}(\mathbb{R}^d \times \mathbb{R}^+) \,:\, \nabla U \in L^2(\mathbb{R}^d \times \mathbb{R}^+; y^\alpha), \operatorname{tr} U|_{\Omega^c} = 0\}.$$

Then, there exists an unique solution $U \in \mathrm{BL}^1_{\alpha,0}$ for the weak form of the extension problem given by

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} y^\alpha \nabla U \cdot \nabla V \; dydx = -\frac{1}{d_s} \int_{\mathbb{R}^d} (-\Delta)^s u \operatorname{tr} V \, dx = -\frac{1}{d_s} \int_\Omega f \operatorname{tr} V dx \qquad \forall V \in \mathrm{BL}^1_{\alpha,0}.$$

We now choose a suitable test-function. For that, select a cut-off function $\chi \in C_0^\infty(\Omega)$ with $\chi \equiv 1$ in a neighborhood of $\overline{\omega}$. We will view χ as defined on $\mathbb{R}^d \times \mathbb{R}^+$ with no dependence on y. We consider the difference operators

$$D_h U(x,y) := \frac{U(x+he_j,y)-U(x,y)}{h}$$

for some unit vector e_j. For h sufficiently small, one considers the test function $V := D_{-h}(\chi^2 D_h U)$. For $h \neq 0$ this is an admissible test function as V satisfies the correct boundary conditions. Inserting this test function into the variational formulation gives

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} y^\alpha \nabla U \cdot \nabla (D_{-h}(\chi^2 D_h U))\, dydx = \int_\Omega f \operatorname{tr} D_{-h}(\chi^2 D_h U) dx.$$

Moving D_{-h} to the other terms by a change of variables argument, we rewrite this as

$$\begin{aligned}\int_\Omega f \operatorname{tr} D_{-h}(\chi^2 D_h U) &= -\int_{\mathbb{R}^d\times\mathbb{R}^+} y^\alpha \nabla D_h U \cdot \nabla(\chi^2 D_h U)\\ &= -\int_{\mathbb{R}^d\times\mathbb{R}^+} y^\alpha \chi^2 \nabla D_h U \cdot \nabla D_h U - 2\int_{\mathbb{R}^d\times\mathbb{R}^+} y^\alpha \chi \nabla D_h U \cdot (\nabla\chi) D_h U.\end{aligned}$$

Thus, rearranging (also moving D_{-h} onto f) and Cauchy-Schwarz give

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^+} y^\alpha \chi^2 |\nabla D_h U|^2 &= \int_\Omega D_h f \operatorname{tr}(\chi^2 D_h U) - 2\int_{\mathbb{R}^d\times\mathbb{R}^+} y^\alpha \chi \nabla D_h U \cdot (\nabla\chi) D_h U\\ &\leq C\|\chi D_h f\|_{L^2(\Omega)} \|\operatorname{tr}(\chi D_h U)\|_{L^2(\Omega)}\\ &\quad + 2\|y^{\alpha/2}\chi\nabla D_h U\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \|y^{\alpha/2}(\nabla\chi) D_h U\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}.\end{aligned} \tag{3.24}$$

Now, the trace estimate from Lemma 2.20 gives

$$\begin{aligned}\|\operatorname{tr}(\chi D_h U)\|_{L^2(\Omega)} &\leq C\|y^{\alpha/2}\nabla(\chi D_h U)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}\\ &\leq C\|y^{\alpha/2}(\nabla\chi) D_h U\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} + C\|y^{\alpha/2}\chi\nabla(D_h U)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}\end{aligned} \tag{3.25}$$

Inserting this in (3.24), we can employ Young's inequality to move the term $\|y^{\alpha/2}\chi\nabla D_h U\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}$ on the right-hand side to the left-hand side and get

$$\|y^{\alpha/2}\chi\nabla D_h U\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \leq C\|\chi D_h f\|^2_{L^2(\Omega)} + C\|y^{\alpha/2}(\nabla\chi) D_h U\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}. \tag{3.26}$$

Furthermore, the L^2_α-norm of the difference quotient D_h can be bounded uniformly by the L^2_α-norm of the derivative ∂_{x_j} uniformly in h (see [Eva98, Sec. 6.3] for details), which leads to

$$\|y^{\alpha/2}\chi\nabla D_h U\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \leq C\|\chi \partial_{x_j} f\|^2_{L^2(\Omega)} + C\|y^{\alpha/2}(\nabla\chi) \partial_{x_j} U\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}$$

for some $C > 0$ independent of h. Now taking the limit $h \to 0$ (see [Eva98, Thm. 3, Sec. 5.8.2] for details) gives

$$\|y^{\alpha/2}\chi\nabla \partial_{x_j} U\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \leq C\|\chi \partial_{x_j} f\|^2_{L^2(\Omega)} + C\|y^{\alpha/2}(\nabla\chi) \partial_{x_j} U\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \leq C \tag{3.27}$$

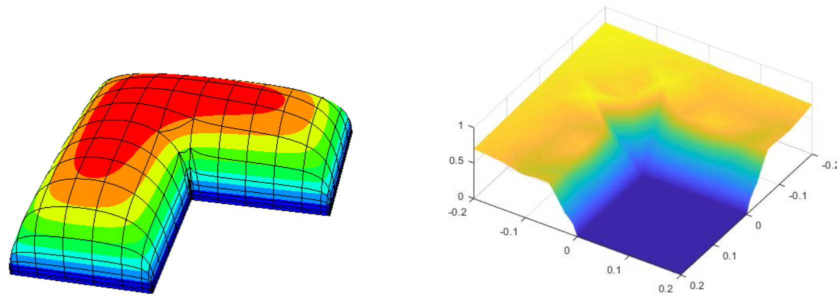


Figure 3.3: Solution of $(-\Delta)^s u = 1$; $u = 0$ on Ω^c on the L-shaped domain and zoom to the reentrant corner.

as $f \in H^1(\Omega)$ and $U \in \mathrm{BL}^1_\alpha$. From this and the trace theorem (3.25) one can infer

$$\|\partial_{x_j} u\|_{L^2(\omega)} < \infty.$$

That is, u is locally in H^1 and not only in H^s. One can bootstrap this argument to see that u is actually in $C^\infty(\omega)$ since f is smooth. ∎

An illustration of the fact that the solution is smooth inside Ω is given in Fig. 3.3 for smooth right-hand side f: away from $\partial\Omega$ the solution u is seen to be smooth; one also discerns that one approaches $\partial\Omega$, the regularity is lost. This can be conveniently captured with weighted spaces as discussed next.

3.2.2 Regularity in weighted Sobolev spaces

Lemma 3.16 shows that the loss of regularity (for smooth f) happens near the boundary $\partial\Omega$. This suggests that the solution is in a weighted space. For that, we introduce the notation

$$r_{\partial\Omega}(x) := \operatorname{dist}(x, \partial\Omega)$$

as well as $\Omega^+ := \Omega \times (0, \infty) \subset \mathbb{R}^d \times \mathbb{R}^+$.

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^d$ be a polygonal Lipschitz domain. Let $f \in H^2(\Omega)$ and u be the solution of (3.1). Then, for $t > 1/2$ there holds*

$$\|r_{\partial\Omega}^{t-s} \nabla u\|_{L^2(\Omega)} \le C \|f\|_{H^1(\Omega)}, \tag{3.28}$$
$$\|r_{\partial\Omega}^{t-s+1} D^2 u\|_{L^2(\Omega)} \le C \|f\|_{H^2(\Omega)}. \tag{3.29}$$

Proof. We only illustrate the result for $d = 1$. For the more involved cases $d = 2, 3$ the stated estimates follow from the more general estimates discussed below.
Step 1: Trace estimate. By the trace estimate of Lemma 2.20, we may show appropriate regularity results for solutions to the Caffarelli-Silvestre extension problem. In particular, we use

(2.40) with $V = \partial_x U$ to obtain

$$|\partial_x U(x,0)|^2 \lesssim \|\partial_x U(x,\cdot)\|^2_{L^2(\mathbb{R}_+;y^\alpha)} + \|\partial_x U(x,\cdot)\|^{1-\alpha}_{L^2(\mathbb{R}_+;y^\alpha)} \|\partial_y \partial_x U(x,\cdot)\|^{1+\alpha}_{L^2(\mathbb{R}_+;y^\alpha)}.$$

Multiplying with $r_{\partial\Omega}^{2(t-s)}$, and using $\alpha = 1-2s$ provides

$$\begin{aligned} r_{\partial\Omega}^{2(t-s)} |\partial_x U(x,0)|^2 &\lesssim \|r_{\partial\Omega}^{t-s} \partial_x U(x,\cdot)\|^2_{L^2(\mathbb{R}_+;y^\alpha)} \\ &\quad + \|r_{\partial\Omega}^{t-1} \partial_x U(x,\cdot)\|^{1-\alpha}_{L^2(\mathbb{R}_+;y^\alpha)} \|r_{\partial\Omega}^{t} \partial_y \partial_x U(x,\cdot)\|^{1+\alpha}_{L^2(\mathbb{R}_+;y^\alpha)}. \end{aligned}$$

Integration over Ω gives in view of $u(x) = U(x,0)$ together with $s < 1$ (which induces $r_{\partial\Omega}^{-s} \le r_{\partial\Omega}^{-1}$) and $|\partial_x U| \le |\nabla U|$ as well as $|\partial_y \partial_x U| \le |\nabla \partial_x U|$ that

$$\begin{aligned} \|r_{\partial\Omega}^{t-s} \partial_x u\|^2_{L^2(\Omega)} &\lesssim \|r_{\partial\Omega}^{t-1} \partial_x U\|^2_{L^2(\Omega^+;y^\alpha)} + \|r_{\partial\Omega}^{t-1} \partial_x U\|^{1-\alpha}_{L^2(\Omega^+;y^\alpha)} \|r_{\partial\Omega}^{t} \partial_y \partial_x U\|^{1+\alpha}_{L^2(\Omega^+;y^\alpha)} \\ &\lesssim \|r_{\partial\Omega}^{t-1} \nabla U\|^2_{L^2(\Omega^+;y^\alpha)} + \|r_{\partial\Omega}^{t-1} \nabla U\|^{1-\alpha}_{L^2(\Omega^+;y^\alpha)} \|r_{\partial\Omega}^{t} \nabla \partial_x U\|^{1+\alpha}_{L^2(\Omega^+;y^\alpha)}. \end{aligned}$$

Now, this estimate suggests that we need to control weighted norms for derivatives of the solution to the extension problem.

Step 2: Additional regularity for the extension problem. For any $\widehat{t} \in [0,1/2)$, there holds

$$\int_{\mathbb{R}_+} y^\alpha \|\nabla U(\cdot,y)\|^2_{H^{\widehat{t}}(\Omega)} dy < C_{\widehat{t}} \|f\|^2_{H^1(\Omega)}. \tag{3.30}$$

We omit the technical proof that employs again difference quotient arguments, details can be found in [BFM+23].
This additional regularity can be used to obtain

$$\|r_{\partial\Omega}^{-\widehat{t}} \nabla U\|^2_{L^2(\Omega^+;y^\alpha)} \le C_{\mathrm{reg}} C_{\widehat{t}} \|f\|^2_{H^1(\Omega)}, \tag{3.31}$$

which directly follows from the embedding result $\|r_{\partial\Omega}^{-\widehat{t}} v\|_{L^2(\Omega)} \le C_{\mathrm{reg}} \|v\|_{H^{\widehat{t}}(\Omega)}$ for $\widehat{t} \in [0,1/2)$ and any $v \in H^{\widehat{t}}(\Omega)$ ([Gri11, Thm. 1.4.4.3]) employed with $v = \nabla U(\cdot,y)$ and multiplying by y^α, and integrating in y.

Step 3: Exploiting interior regularity to control $\|r_{\partial\Omega}^{t} \nabla \partial_x U\|_{L^2(\Omega^+;y^\alpha)}$.
In order to obtain an estimate for the L^2-norm over $\Omega = (a,b)$, we dyadically cover $\Omega \subset \bigcup_{i\in\mathbb{N}} B_{cr_i}(x_i)$ with intervals $B_{cr_i}(x_i) \subset \Omega$, where the points x_i run through the set $\{a + (b-a)2^{-j} : j \in \mathbb{N}\} \cup \{b - (b-a)2^{-j} : j \in \mathbb{N}\}$, $c \in (1/2,1)$ is fixed and $r_i = \operatorname{dist}(x_i, \partial\Omega)$. A geometric series argument gives $\sum_i r_i^\varepsilon < \infty$ for any chosen $\varepsilon > 0$.
Using a cut-off function χ with $\chi \equiv 1$ on $B_{cr_i}(x_i)$, $\operatorname{supp}\chi \subset B_{r_i}(x_i)$ and $\|\chi\|_{L^\infty} + \|\chi'\|_{L^\infty} \le \frac{C}{r_i}$, (3.27) gives

$$\|\nabla \partial_x U\|_{L^2(B^+_{cr_i};y^\alpha)} = \|\chi \nabla \partial_x U\|_{L^2(B^+_{cr_i};y^\alpha)} \lesssim \|f\|_{H^1(\Omega)} + r_i^{-1} \|\partial_x U\|_{L^2(B^+_{r_i};y^\alpha)} \tag{3.32}$$

$$\overset{(3.31)}{\lesssim} r_i^{\widehat{t}-1} \|f\|_{H^1(\Omega)}. \tag{3.33}$$

Consequently, for given $t > 1/2$, there is $0 < \varepsilon < t - 1/2$ such that $\widehat{t} := 1 - t + \varepsilon \in [0, 1/2)$ and we obtain

$$\begin{aligned} \|r_{\partial\Omega}^{t}\nabla\partial_x U\|^2_{L^2(\Omega^+;y^\alpha)} &\lesssim \sum_{i\in\mathbb{N}} r_i^{2t}\|\nabla\partial_x U\|^2_{L^2(B_{cr_i}(x_i)^+;y^\alpha)} \\ &\lesssim \sum_{i\in\mathbb{N}} r_i^{2\varepsilon}\|f\|^2_{H^1(\Omega)} \leq C\|f\|^2_{H^1(\Omega)}. \end{aligned}$$

This shows (3.28), estimate (3.29) follows with the same technique and only uses interior regularity for higher order derivatives exploting that $\partial_x U$ solves the extension problem with data $\partial_x f$. ■

Example 3.18 *The regularity estimates in the previous Theorem can be easily verified for the solution u of Example 3.14 on $B_1(0) \subset \mathbb{R}^2$. Introducing polar coordinates (r, φ), u is a function of r only and satisfies $|\nabla u| \sim |\partial_r u| \sim (1 - r^2)^{s-1} r$. As $\operatorname{dist}(x, \partial\Omega) = 1 - |x| = 1 - r$, we have for any $t > 1/2$*

$$\|r_{\partial\Omega}^{t-s}\nabla u\|_{L^2(\Omega)} \lesssim \int_{r=0}^{1} (1-r)^{2(t-s)}(1-r^2)^{2(s-1)} r^2 dr \leq \int_{r=0}^{1} (1-r)^{2t-2}\, dr < \infty.$$

since $2t - 2 > -1$. The verification of (3.29) is left as an exercise for the reader.

The result (3.28) can be further refined. The reason is that the solution can be expected to the smooth in the tangential direction and the lack of (higher) regularity is only due to the behavior of the solution in the normal direction. We do formulate this result in a more precise way in the following.

On polygonal domains $\Omega \subset \mathbb{R}^2$, as seen in Figure 3.3, solutions to fractional PDEs typically exhibit both vertex and edge singularities. Note that this is conceptually different than, e.g., for the Poisson equation, where only vertex singularities appear in 2d! We capture the singular behaviour by introducing vertex and edge weights defined as

$$r_{\mathbf{v}}(x) := \operatorname{dist}(x, \mathbf{v}), \qquad r_{\mathbf{e}}(x) := \operatorname{dist}(x, \mathbf{e}),$$

and decompose the polygon into vertex neighborhoods $\omega_{\mathbf{v}}$ ($r_{\mathbf{v}} < \varepsilon$ and $r_{\mathbf{e}} \geq \varepsilon r_{\mathbf{v}}$), vertex-edge neighborhoods $\omega_{\mathbf{ve}}$ ($r_{\mathbf{v}} < \varepsilon$ and $r_{\mathbf{e}} < \varepsilon r_{\mathbf{v}}$), edge neighborhoods $\omega_{\mathbf{e}}$ ($r_{\mathbf{v}} \geq \varepsilon$ and $r_{\mathbf{e}} \leq \varepsilon$) and a remainder Ω_{int} (away from the boundary). See Figure 3.4 for an illustration. We also note that the neighborhoods may overlap a bit and for each vertex/edge a different ε may be chosen.
In order to exploit (interior) regularity, it is useful to employ locally fitted coordinates in the neighborhoods $\omega_{\mathbf{ve}}$, $\omega_{\mathbf{e}}$: we denote by $\mathbf{e}_\parallel$ and $\mathbf{e}_\perp$ unit vectors such that $\mathbf{e}_\parallel$ is tangential to $\mathbf{e}$ and $\mathbf{e}_\perp$ is normal to $\mathbf{e}$ (see Figure 3.5). We introduce the differential operators

$$D_{x_\parallel} v := \mathbf{e}_\parallel \cdot \nabla_x v, \qquad\qquad D_{x_\perp} v := \mathbf{e}_\perp \cdot \nabla_x v$$

corresponding to differentiation in the tangential and normal direction. Higher order tangential and normal derivatives in $\omega_{\mathbf{e}}$ or $\omega_{\mathbf{ve}}$ are inductively defined by $D^j_{x_\parallel} v := D_{x_\parallel}(D^{j-1}_{x_\parallel} v)$ and $D^j_{x_\perp} v := D_{x_\perp}(D^{j-1}_{x_\perp} v)$ for $j > 1$.

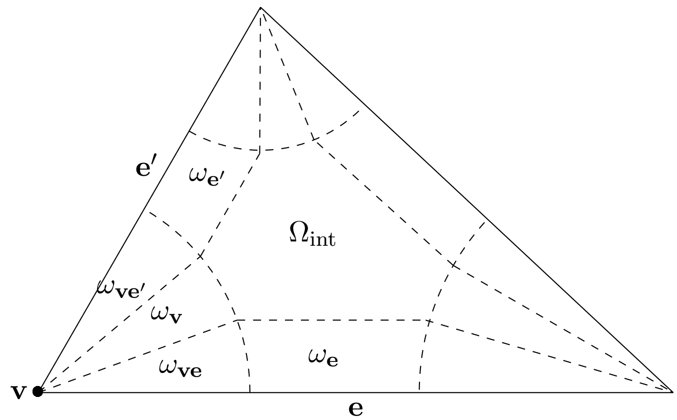


Figure 3.4: Decomposition of Ω.

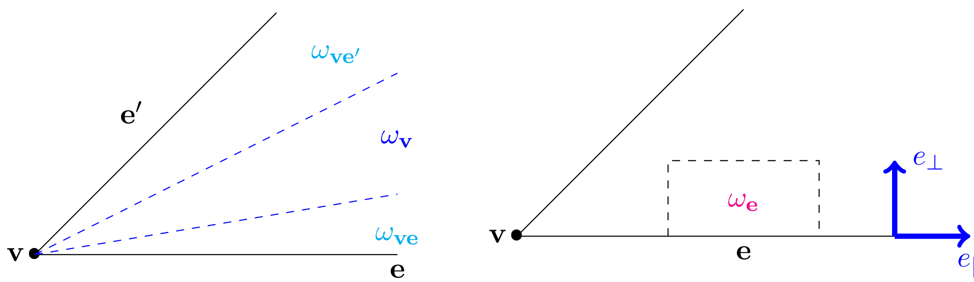


Figure 3.5: Neighborhoods and notations near vertices and edges.

The following theorem shows so called *weighted analytic regularity*, i.e., control for all (tangential and normal) derivatives in the corresponding neighborhoods of the boundary described above.

Theorem 3.19. *Let f be analytic on $\overline{\Omega}$ and u be the solution of (3.1). Then, there is $C, \gamma > 0$ such that for all $p \in \mathbb{N}$ there holds:*

1. *u is analytic on each $K \subset\subset \Omega$.*
2. *For all $\beta \in \mathbb{N}_0^2$ there holds with $|\beta| = p$*
$$\|r_{\mathbf{v}}^{p-1/2-s+\varepsilon} \partial^\beta u\|_{L^2(\omega_{\mathbf{v}})} \leq C\gamma^{p+1} p^p.$$
3. *For all $(p_\perp, p_\|) \in \mathbb{N}_0^2$ there holds with $p = p_\perp + p_\|$*
$$\|r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} D_{x_\perp}^{p_\perp} D_{x_\|}^{p_\|} u\|_{L^2(\omega_{\mathbf{e}})} \leq C\gamma^{p+1} p^p,$$
$$\|r_{\mathbf{e}}^{p_\perp-1/2-s+\varepsilon} r_{\mathbf{v}}^{p_\|+\varepsilon} D_{x_\perp}^{p_\perp} D_{x_\|}^{p_\|} u\|_{L^2(\omega_{\mathbf{ve}})} \leq C\gamma^{p+1} p^p.$$

Proof. In principle, the idea of proof is similar to the proof of Theorem 3.17: All estimates are done

for the Caffarelli-Silvestre extension problem as local PDE, additional regularity - here in tangential direction - is derived, control of higher order derivatives is achieved by employing interior regularity and finally, everything is glued together with a covering argument.
However, for each of the neighborhoods $\omega_{\mathbf{v}}, \omega_{\mathbf{e}}, \omega_{\mathbf{ve}}$ some (slight, technical) differences have to be taken into account. We refer to the paper [FMMS22], which is dedicated to the proof of the theorem. ■

3.2.3 Impact of regularity on mesh design: shape-regular meshes

Weighted regularity results such as (3.28) and in Theorem 3.19 suggest not to use quasi-uniform meshes but rather meshes that are refined towards the boundary. In that way, the convergence rate as measured in "error vs. problem size" can be improved. For more details, we refer to [BN23a, BN23b].

The typical mesh design principle that arises from (3.28) is the so-called *μ-graded mesh* (with $\mu > 1$) with the following properties:

1. $\mathcal{T}$ consists of shape-regular elements, parameter $h > 0$ given,
2. the element size h_T should satisfy $h_T \sim h \operatorname{dist}(T, \partial\Omega)^{1-1/\mu}$ if $\overline{T} \cap \partial\Omega = \emptyset$,
3. the element size h_T should satisfy[3] $h_T \sim h^\mu$ if $\overline{T} \cap \partial\Omega \neq \emptyset$.

An example of a graded mesh can be seen in Figure 3.6 (left).

Theorem 3.20. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a polygonal/polyhedral domain and the mesh $\mathcal{T}$ be μ-graded in the above sense. If the regularity assertion of Theorem 3.17 holds and if $\mu > 1$ is such that*

$$(2-s)(1-1/\mu) > 1 + t - s, \tag{3.34}$$

then one has

$$\inf_{v \in \mathcal{P}^1_0(\mathcal{T})} \|u - v\|_{\widetilde{H}^s(\Omega)} \leq C h^{2-s} \left(\|r_{\partial\Omega}^{t-s} \nabla u\|_{L^2(\Omega)} + \|r_{\partial\Omega}^{1+t-s} D^2 u\|_{L^2(\Omega)} \right) \leq C h^{2-s}.$$

For the number of degrees of freedom, we have

$$N := \dim \mathcal{P}^1_0(\mathcal{T}) \sim h^{-\mu(d-1)}.$$

Proof. We only show the complexity estimate: We decompose $\mathcal{T} = \mathcal{T}_{\partial\Omega} \cup \mathcal{T}_\Omega$ with $\mathcal{T}_{\partial\Omega} := \{T \in \mathcal{T} : \overline{T} \cap \partial\Omega \neq \emptyset\}$ and $\mathcal{T}_\Omega := \{T \in \mathcal{T} : \overline{T} \cap \partial\Omega = \emptyset\}$. As the trace mesh on $\partial\Omega$ is a quasi-uniform mesh with mesh size $h_{min} := \min_{T \in \mathcal{T}} h_T \sim h^\mu$, we have

$$\sum_{T \in \mathcal{T}_{\partial\Omega}} 1 \sim h_{min}^{-(d-1)} \sim h^{-\mu(d-1)}.$$

[3] this is natural: by shape reglarity, the elements adjacent to elements touching $\partial\Omega$ have size $O(h_{min})$ and are $O(h_{min})$ from $\partial\Omega$. Hence, these element satisfy $h_T \sim h_{min} = h h_{min}^{1-1/\mu}$, which implies $h_{min} \sim h^\mu$

For the elements in $\mathcal{T}_\Omega$, we estimate

$$\sum_{T\in\mathcal{T}_\Omega} 1 \sim \sum_{T\in\mathcal{T}_\Omega} \int_T h_T^{-d} \lesssim h^{-d} \sum_{T\in\mathcal{T}_\Omega} \int_T r_{\partial\Omega}^{-d(1-1/\mu)}$$
$$\lesssim h^{-d} \int_{r=ch_{min}}^{\mathrm{diam}(\Omega)} r^{-d(1-1/\mu)} \sim h^{-d} h_{min}^{-d+1+d/\mu} \sim h^{-\mu(d-1)}.$$

The error estimate in the $\widetilde{H}^s(\Omega)$-norm is very technical and employs the "localization" of Theorem 3.8 as illustrated in Lemma 3.9, using only the weighted regularity assumption. Hereby, scaling arguments for the weighted norms allow to recover the correct powers of h. We note that elementwise (or patchwise) approximations produce powers h_T^{2-s} by employing suitable interpolants. This motivates (3.34): condition 2. in the defintion of the graded mesh gives $h_T^{2-s} \sim h^{2-s} r_{\partial\Omega}^{(2-s)(1-1/\mu)} \leq h^{2-s} r_{\partial\Omega}^{1+t-s}$, which is the appropriate weight function for the weighted regularity. ■

We note that also exponentially graded meshes towards the boundary, motivated by choosing $\mu = \infty$ in condition 2, can be employed.

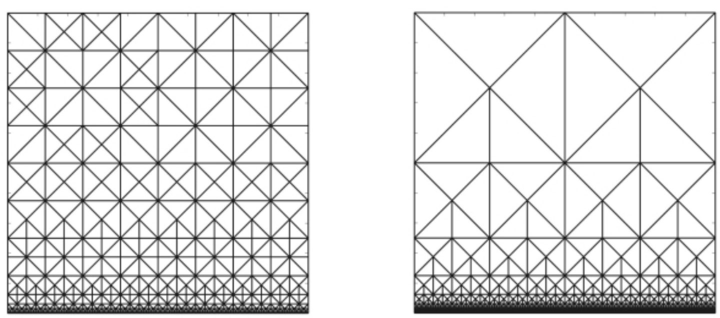


Figure 3.6: Graded, shape-regular meshes towards an edge; left: $\mu = 3$, right: exponentially graded mesh "$\mu = \infty$".

3.2.4 Impact of regularity on mesh design: anisotropic meshes

The approximation result of Theorem 3.20 with its $\mathcal{O}(h^{2-s})$ estimate is optimal in terms of h. In terms of "error vs. N", one obtains $\mathcal{O}(N^{-(2-s)/(\mu(d-1))})$. For the approximation of smooth functions from $\mathcal{P}_0^1$ on quasi-uniform meshes the the optimal rate is $\mathcal{O}(N^{-(2-s)/d})$. Since typically $\mu(d-1) > d$, the rate (as measured in "error vs. N") achieved by Theorem 3.20 is not optimal.
The regularity result of Theorem 3.19 shows that the solution has much less variation in the tangential direction than in the direction normal to $\partial\Omega$. This suggests to use **anisotropic elements**, i.e., elements where the aspect ratio may be unbounded. An archetypal mesh is depicted in Fig. 3.7. This can restore the optimal complexity.

These idea can also be employed to exponentially graded meshes towards the boundary, which, for the shape-regular case, look like meshes in Figure 3.6. We note that in 1d, the meshes can be

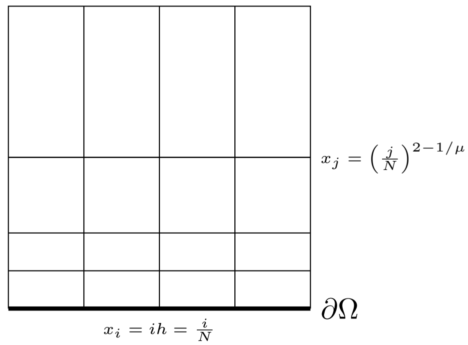


Figure 3.7: An anisotropic graded mesh given by the nodes $(x_i, x_j) = (i/N, (j/N)^{2-1/\mu})$. The 1D mesh with nodes $x_j = (j/N)^{2-1/\mu}$ satisfies $h_i \sim h x_i^{1-1/\mu}$.

explicitly defined on $(-1,1)$ by a grading factor $\sigma \in (0,1)$ and a number of refinement layers $L \in \mathbb{N}$ by the nodes

$$x_0 := -1, \quad x_i := -1 + \sigma^{L-i+1},\ i = 1, \dots, L \qquad x_{i+1} := 1 - \sigma^{i-L},\ i = L, \dots, 2L \quad x_{2L+2} := 1, \tag{3.35}$$

see Figure 3.8. The key feature of these meshes is that elements with $T \cap \partial\Omega = \emptyset$ satisfy $h_T \sim \operatorname{dist}(T, \partial\Omega)$ and elements with $T \cap \partial\Omega \neq \emptyset$ behave like $h_T \sim \sigma^L$.

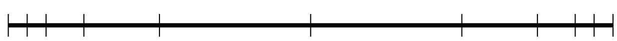


Figure 3.8: A geometric mesh in 1d.

Mesh design principles of anisotropic exponentially graded meshes for $d > 1$ are more involved. Here, we mention a construction from [FMMS23]: The polygonal geometry, e.g., the L-shaped domain, is at first decomposed into a so called *macro-triangulation*, which is a coarse mesh (possibly consisting of triangles and quadrilaterals), where each element of the macro-triangulation gets a *refinement-label*, see Figure 3.9.

The refinement labels are given according to the used refinement pattern (see Figure 3.10):

- **trivial macro elements** (interior elements) are not refined;
- **vertex macro elements** (triangular macro elements that only **share a point** with $\partial\Omega$) are geometrically refined towards the point in an *isotropic* way;
- **edge-vertex macro elements** (triangular macro elements that **share an edge** with $\partial\Omega$) are geometrically refined towards the edge in an *anisotropic* way and towards one corner in an isotropic way;
- **edge macro elements** (quadrilateral macro elements that share an edge with $\partial\Omega$) are geometrically refined towards the edge in an *anisotropic* way.

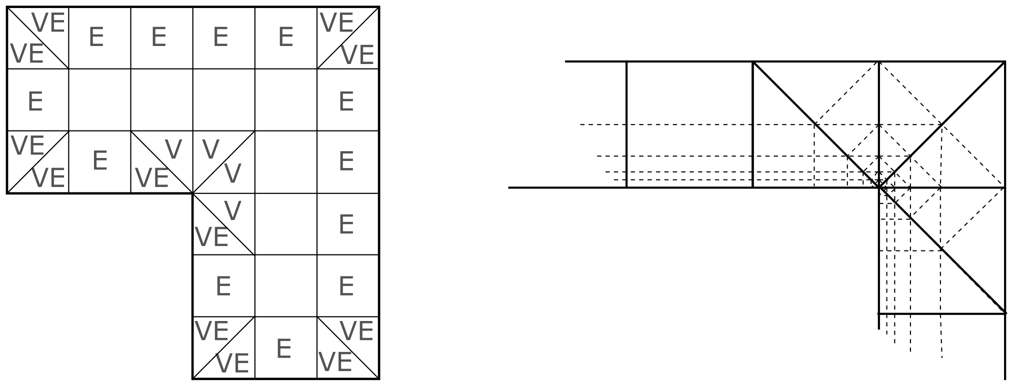


Figure 3.9: Macro triangulation of the L-shaped domain and zoom in to the reentrant corner in the refinement patterns.

Generating a mesh in this way, if each element on the boundary is refined exactly $L \in \mathbb{N}$ times (”layers of refinement”), one can glue the corresponding geometrically refined (and possibly anisotropic) macro elements together to obtain a mesh without hanging nodes, see Figure 3.9.

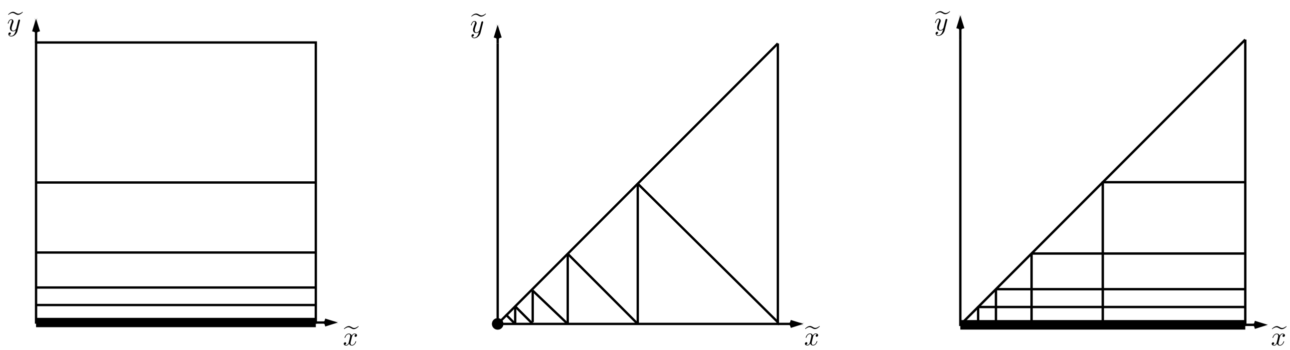


Figure 3.10: Refinement patterns for the different macro element types.

The mesh design principles of this section allow the construction of geometrically graded, regular meshes towards $\partial\Omega$ denoted by $\mathcal{T}_{\text{geo}}$. Combining such meshes with higher order finite elements, gives an hp-FEM approximation: Let $p \geq 1$ and $u_N \in \mathcal{P}_0^p(\mathcal{T}_{\text{geo}})$ solve

$$a(u_N, v_N) = \int_\Omega f v_N \, dx \qquad \forall v_N \in \mathcal{P}_0^p(\mathcal{T}_{\text{geo}}). \tag{3.36}$$

Note that the dimension $N := \dim \mathcal{P}_0^p(\mathcal{T}_{\text{geo}})$ of the FEM space behaves like $N \sim (pL)^d$.
Now, the derived weighted analytic regularity can be used to show exponential convergence of the hp-FEM approximation.

Theorem 3.21. *Let $\Omega \subset \mathbb{R}^d$ with $d = 1, 2$ be a polygonal Lipschitz domain and $\mathcal{T}_{\text{geo}}$ be a geometric mesh with L layers of refinement towards $\partial\Omega$ given by (3.35) (d=1) or the mesh construction via refinements of mesh patches (Figure 3.9, $d = 2$). Let $u_N \in \mathcal{P}_0^p(\mathcal{T}_{\text{geo}})$ solve (3.36) and $u \in \widetilde{H}^s(\Omega)$ solve (3.6).*
Then, choosing $L \sim p$ refinement layers gives for the FEM error

$$\|u - u_N\|_{\widetilde{H}^s(\Omega)} \leq C \exp(-b \sqrt[2d]{N}), \tag{3.37}$$

with $N := \dim \mathcal{P}_0^p(\mathcal{T}_{\text{geo}})$ and some constants $C, b > 0$ independent of L, p, N.

We only show the theorem for $d = 1$. For its proof, we need a few ingredients such as the following embedding result (shown via classical interpolation theory using the K-functional, see [BFM+23]).

Lemma 3.22. *Let $\beta' \in [0, 1)$, $\sigma \in (0, 1 - \beta')$. Then, there is $C_{\beta',s} > 0$ such that*

$$\|v\|_{\widetilde{H}^\sigma(\Omega)} \leq C_{\beta',s} \left[\|r_{\partial\Omega}^{\beta'} v'\|_{L^2(\Omega)} + \|r_{\partial\Omega}^{\beta'-1} v\|_{L^2(\Omega)} \right] \tag{3.38}$$

for all v such that right-hand side is finite.

By definition of the *hp*-FEM approximation and the Céa-Lemma, it suffices to construct a suitable approximation in $\mathcal{P}_0^p(\mathcal{T}_{\text{geo}})$, which is done element-by-element:

- For $T \in \mathcal{T}_{\text{geo}}$ with $\overline{T} \cap \partial\Omega \neq \emptyset$, we take $v_N|_T = I^{\text{lin}} v_N|_T$ as the linear interpolant in the endpoints of T.
- For $T \in \mathcal{T}_{\text{geo}}$ with $\overline{T} \cap \partial\Omega = \emptyset$, we take the Gauss-Lobatto interpolant of degree p on the element (see Appendix for details).

For the linear interpolant, we obtain approximation properties in weighted norms.

Lemma 3.23. *Let $\beta' \in [0, 1)$, $\varepsilon > 0$. Then there is $C_{\beta',\varepsilon} > 0$ such that the following holds: For $\widehat{v} \in C([0, 1])$ let $I\widehat{v}$ be the linear interpolant in the endpoints 0, 1. Then, provided the right-hand side is finite*

$$\|x^{\beta'}(\widehat{v} - I\widehat{v})'\|_{L^2(0,1)} + \|x^{\beta'-1}(\widehat{v} - I\widehat{v})\|_{L^2(0,1)} \leq C_{\beta',\varepsilon} \|x^{\min\{\beta'+1, 3/2-\varepsilon\}} \widehat{v}''\|_{L^2(0,1)}.$$

Proof. *Step 1:* Let $\widetilde{\pi}_1 \widehat{v} \in P_1$ be the linear interpolant of $\widehat{v}$ in the points 1/2 and 1. We now show that for any $\alpha > -1$ there holds

$$\|x^{\alpha/2}(\widehat{v} - \widetilde{\pi}_1 \widehat{v})'\|_{L^2(0,1)} \leq C_\alpha \|x^{\alpha/2+1} \widehat{v}''\|_{L^2(0,1)}. \tag{3.39}$$

This follows from

$$(\widehat{v} - \widetilde{\pi}_1 \widehat{v})'(x) = 2 \int_{1/2}^1 \widehat{v}'(x) - \widehat{v}'(t) dt = 2 \int_{1/2}^1 \int_t^x \widehat{v}''(\tau) d\tau dt,$$

which implies

$$\begin{aligned}\int_0^1 x^\alpha |(\widehat{v}-\widetilde{\pi}_1\widehat{v})'(x)|^2 dx &= 4\int_0^1 x^\alpha \left|\int_{1/2}^1\int_t^x \widehat{v}''(\tau)d\tau dt\right|^2 dx\\ &\leq 2\int_{1/2}^1\int_0^1 x^\alpha \left|\int_t^x \widehat{v}''(\tau)d\tau\right|^2 dxdt\\ &\leq 2\int_{1/2}^1\int_0^1 x^\alpha \left(\left|\int_x^1 |\widehat{v}''(\tau)|d\tau + \int_t^1 |\widehat{v}''(\tau)|d\tau\right|^2\right) dxdt\\ &\lesssim \int_{1/2}^1\int_0^1 x^\alpha \left(\left|\int_x^1 |\widehat{v}''(\tau)|d\tau\right|^2 + \left|\int_t^1 |\widehat{v}''(\tau)|d\tau\right|^2\right) dxdt.\end{aligned}$$

For the first term on the right-hand side, we employ a Hardy inequality [DL93, Chap. 2, Thm. 3.1], which gives

$$\int_0^1 x^\alpha \left|\int_x^1 |\widehat{v}''(\tau)|d\tau\right|^2 dx \lesssim \int_0^1 x^{\alpha+2}|\widehat{v}''(x)|^2 dx.$$

For the remaining term, we estimate

$$\begin{aligned}\int_{1/2}^1\int_0^1 x^\alpha \left|\int_t^1 |\widehat{v}''(\tau)|d\tau\right|^2 dxdt &\lesssim \int_{1/2}^1\int_0^1 x^\alpha \int_{1/2}^1 |\widehat{v}''(\tau)|^2 d\tau dxdt\\ &\lesssim \int_{1/2}^1 |\widehat{v}''(\tau)|^2 d\tau \lesssim \int_{1/2}^1 \tau^{\alpha+2}|\widehat{v}''(\tau)|^2 d\tau\end{aligned}$$

as $\tau^{-\alpha-2} \leq C$ on $[1/2, 1]$. In total, we arrive at the sought estimate

$$\int_0^1 x^\alpha |(\widehat{v}-\widetilde{\pi}_1\widehat{v})'(x)|^2 dx \lesssim \int_0^1 x^{\alpha+2}|\widehat{v}''(x)|^2 dx + \int_{1/2}^1 x^{\alpha+2}|\widehat{v}''(x)|^2 dx.$$

A maximum norm estimate is then obtained from

$$\begin{aligned}|(\widehat{v}-\widetilde{\pi}_1\widehat{v})(x)| &\leq \int_x^1 |(\widehat{v}-\widetilde{\pi}_1\widehat{v})'(t)|\,dt \leq \sqrt{\int_0^1 t^{-1+2\varepsilon}\,dt \int_0^1 t^{1-2\varepsilon}|(\widehat{v}-\widetilde{\pi}_1\widehat{v})'(t)|^2\,dt}\\ &\overset{(3.39)}{\leq} C_\varepsilon \|x^{3/2-\varepsilon}\widehat{v}''\|_{L^2(0,1)}.\end{aligned}$$

Step 2 ($\beta' < 1/2$): Abbreviate $e := \widehat{v} - I\widehat{v}$ and note $e(x) = \int_0^x e'(t)\,dt$. For $\beta' \in [0, 1/2)$, Hardy's inequality [DL93, Chap. 2, Thm. 3.1] is applicable and yields

$$\|x^{\beta'-1}e\|_{L^2(0,1)} \leq C\|x^{\beta'}e'\|_{L^2(0,1)}.$$

We estimate with $w = \widetilde{\pi}_1\widehat{v} \in P_1$

$$\begin{aligned}\|x^{\beta'}e'\|_{L^2(0,1)} &\leq \|x^{\beta'}(\widehat{v}-w)'\|_{L^2(0,1)} + \|x^{\beta'}(I(\widehat{v}-w))'\|_{L^2(0,1)}\\ &\overset{P_1 \text{ finite dimensional}}{\lesssim} \|x^{\beta'}(\widehat{v}-w)'\|_{L^2(0,1)} + \|\widehat{v}-w\|_{L^\infty(0,1)}\\ &\overset{\text{Step 1, } w=\widetilde{\pi}_1\widehat{v}}{\lesssim} \|x^{\min\{\beta'+1,3/2-\varepsilon\}}\widehat{v}''\|_{L^2(0,1)}.\end{aligned}$$

Step 3 ($\beta' > 1/2$): From the representation $e(x) = \int_0^x e'(t)\,dt$ we get for any $\alpha \in [0, 1/2)$ by the Cauchy-Schwarz inequality $|e(x)| \le C_\alpha x^{1/2-\alpha}\|x^\alpha e'\|_{L^2(0,1)}$. Hence, for α sufficiently close to $1/2$,

$$\|x^{\beta'-1}e\|_{L^2(0,1)} \lesssim \sqrt{\int_0^1 x^{2\beta'-2+1-2\alpha}\,dx}\|x^\alpha e'\|_{L^2(0,1)} \lesssim \|x^\alpha e'\|_{L^2(0,1)}.$$

We conclude, since $\alpha < 1/2 < \beta'$, that $\|x^{\beta'-1}e\|_{L^2(0,1)} + \|x^{\beta'}e'\|_{L^2(0,1)} \lesssim \|x^\alpha e'\|_{L^2(0,1)}$. Applying Step 2 with α taking the role of β' there, we get, by selecting α sufficiently close to $1/2$

$$\|x^\alpha e'\|_{L^2(0,1)} \lesssim \|x^{3/2-\varepsilon}\widehat{u}''\|_{L^2(0,1)}.$$

Step 4 ($\beta' = 1/2$): Given $\varepsilon > 0$, we note for $\varepsilon' > 0$ sufficiently small that $\min\{\beta'+1, 3/2-\varepsilon\} = 3/2-\varepsilon = \min\{\beta'-\varepsilon'+1, 3/2-\varepsilon\}$ so that we obtain the result by applying Step 2 to the choice $\beta' = 1/2 - \varepsilon'$. ■

Let $I = (-1, 1)$. The Gauss-Lobatto interpolant is a Lagrange interpolation in the Gauss-Lobatto points $\{x_j \,:\, j = 0, \ldots, p\}$, which are the endpoints ± 1 together with the extrema of the Legendre polynomials, i.e., the zeros of $x \mapsto (1-x^2)L_p'(x)$ with L_p being the p-th Legendre polynomial. The operator then reads as

$$I_p^{GL} : C(I) \to P_p, \quad g \mapsto \sum_{j=0}^p g(x_j) \prod_{\ell \ne j} \frac{x - x_\ell}{x_j - x_\ell}.$$

As usual for Lagrange interpolation, using the projection property of the operator I^{GL} on P_p, we obtain a quasi-best approximation

$$\|g - I_p^{GL} g\|_{L^\infty(I)} \le (1 + \Lambda_p^\infty) \inf_{v \in P_p} \|g - v\|_{L^\infty(I)},$$

where Λ_p^∞ is the so called Lebesgue constant. For the Gauss-Lobatto points, there holds the quasi-optimal upper bound

$$\Lambda_p^\infty \le C \ln(p+1).$$

The operator I_p^{GL} also is stable in H^1, see e.g. [BM97],

$$\|I_p^{GL} u\|_{H^1(I)} \le C\|u\|_{H^1(I)} \qquad \forall u \in H^1(I)$$

with a constant $C > 0$ independent of p. Moreover, we have suitable approximation properties stated in the following lemma.

Lemma 3.24. *Let $p \in \mathbb{N}$ and $f \in C^\infty(-1,1)$ satisfy*

$$\|f^{(n)}\|_{L^2(-1,1)} \le C_f \gamma^n n! \qquad \forall n \in \mathbb{N}_0,\ n \ge p+1$$

with some constants C_f, γ independent of n. Then, there exist constants $C, b > 0$, depending only on γ such that

$$\inf_{v \in P_p} \|f - v\|_{L^\infty(-1,1)} + \|f' - v'\|_{L^\infty(-1,1)} \le C C_f e^{-bp}.$$

Proof. This is a combination of Proposition A.1 and Lemma A.2 from the appendix. ∎

Proof of Theorem 3.21. The Céa-Lemma gives

$$\|u - u_N\|_{\widetilde{H}^s(\Omega)} \leq C \inf_{v_N \in \mathcal{P}_0^p(\mathcal{T}_{\text{geo}})} \|u - v_N\|_{\widetilde{H}^s(\Omega)}.$$

Selecting $\beta' := 1 - s - \varepsilon'$ for ε' sufficiently small, Lemma 3.22 shows

$$\|u - v_N\|_{\widetilde{H}^s(\Omega)} \leq C \left[\|r_{\partial\Omega}^{\beta'}(u - v_N)'\|_{L^2(\Omega)} + \|r_{\partial\Omega}^{\beta'-1}(u - v_N)\|_{L^2(\Omega)} \right],$$

which can be estimated by summing elementwise error contributions. As discribed above we distinguish between elements touching the boundary and elements away from the boundary.
Step 1: Set $\beta := 1/2 - s + \varepsilon$ with some $\varepsilon > 0$. For the approximation on the elements abutting $\partial\Omega$, we observe that, for given ε, we can select ε' sufficiently small to ensure

$$3/2 - s - \varepsilon = \beta + 1 < \min\{\beta' + 1, 3/2 - \varepsilon\} = \min\{2 - s - \varepsilon', 3/2 - \varepsilon\}. \tag{3.40}$$

In Lemma 3.23, we may therefore replace $x^{\min\{3/2-\varepsilon,\beta'+1\}}$ by $x^{\beta+1}$. Combining then Lemma 3.23 with a scaling argument, we get for T_i with $\overline{T}_i \cap \partial\Omega \neq \emptyset$

$$\|r_{\partial\Omega}^{\beta'-1}(u - v_N)\|_{L^2(T_i)} + \|r_{\partial\Omega}^{\beta'}(u - v_N)'\|_{L^2(T_i)} \leq C h_i^{1-\beta-(1-\beta')} \|r_{\partial\Omega}^{\beta+1} u''\|_{L^2(T_i)}.$$

Step 2: For the remaining elements T_i with $\overline{T}_i \cap \partial\Omega = \emptyset$, we show

$$\|r^{\beta'-1}(u - v_N)\|_{L^2(T_i)} + \|r^{\beta'}(u - v_N)'\|_{L^2(T_i)} \leq C h_i^{1-\beta-(1-\beta')} e^{-bp}$$

for some $b > 0$ independent of p. Let $T = [x_{i-1}, x_i]$. We only show the estimate for the case that $\operatorname{dist}(T, \partial\Omega) = \operatorname{dist}(\{x_{i-1}\}, \partial\Omega) =: r_i$. Denote by $\widehat{u} := u|_T \circ F_T$ the pullback of u on T to the reference interval and note that $F_T' = h_T/2$. The above estimate follows from scaling with by

$$\begin{aligned}
\|\widehat{u}^{(n)}\|_{L^2(I)} &= (h_T/2)^{n-1/2} \|u^{(n)}\|_{L^2(T)} \leq (h_T/2)^{n-1/2} r_i^{-\beta+1-n} \|r_{\partial\Omega}^{\beta-1+n} u^{(n)}\|_{L^2(T)} \\
&\leq \left(\frac{h_T}{2r_i}\right)^{n-1/2} r_i^{-\beta+1/2} \|r_{\partial\Omega}^{\beta-1+n} u^{(n)}\|_{L^2(T)} \\
&\leq C h_T^{1/2-\beta} (\gamma C_\sigma)^n n!,
\end{aligned}$$

where we used the weighted regularity estimate and that the geometric mesh satisfies $h_T = C_\sigma r_i$ with a constant depending only on the mesh grading factor σ. Consequently, Lemma 3.24 implies

$$\|\widehat{u} - I_p^{GL}\widehat{u}\|_{H^1(I)} \leq C h_T^{1/2-\beta} e^{-bp}.$$

Reversing the element transformation, we obtain a polynomial $v_T := (I_p^{GL}\widehat{u}) \circ F_T^{-1} \in P_p(T)$ satisfying $u(x_{i-1}) = v_T(x_{i-1})$ and $u(x_i) = v_T(x_i)$ and the scalings

$$\begin{aligned}
\|r_{\partial\Omega}^{\beta'-1}(u - v_T)\|_{L^2(T)} &\leq C h_T^{1/2+\beta'-1} \|\widehat{u} - I_p^{GL}\widehat{u}\|_{L^2(I)} \\
\|r_{\partial\Omega}^{\beta'}(u - v_T)'\|_{L^2(T)} &\leq C h_T^{1/2+\beta'-1} \|(\widehat{u} - I_p^{GL}\widehat{u})'\|_{L^2(I)}
\end{aligned}$$

Inserting the estimate in the H^1-norm on the reference element from above gives

$$\|r_{\partial\Omega}^{\beta'-1}(u - v_T)\|_{L^2(T)} + \|r_{\partial\Omega}^{\beta'}(u - v_T)'\|_{L^2(T)} \leq C h_T^{1/2-\beta+\beta'-1/2} e^{-bp}.$$

Step 3: Summation over all elements using step 1 and step 2. Noting that $\beta' - \beta = 1/2 + \varepsilon - \varepsilon' > 0$, we obtain from summation over all elements and using $h_i = O(\sigma^L)$ for the elements abutting on $\partial\Omega$ that

$$\|r_{\partial\Omega}^{\beta'-1}(u - v_T)\|_{L^2(T)} + \|r_{\partial\Omega}^{\beta'}(u - v_T)'\|_{L^2(T)} \lesssim \sigma^{L(\beta'-\beta)} + e^{-bp} = \sigma^{L(1/2+\varepsilon-\varepsilon')} + e^{-bp},$$

which implies the statement of the theorem by suitably adjusting ε and noting that $N \sim Lp \sim p^2$. ■

Example 3.25 *We consider $\Omega = (-1, 1)$ and $f = 1$ and employ a geometric mesh in 1d with $\sigma = 0.6$.*
Figure 3.11 shows the convergence of the energy norm error of the hp-FEM approximation plotted against the increasing number of refinement layers L. Note that $N \sim Lp \sim L^2$. The root exponential convergence of Theorem 3.21 is clearly visible.

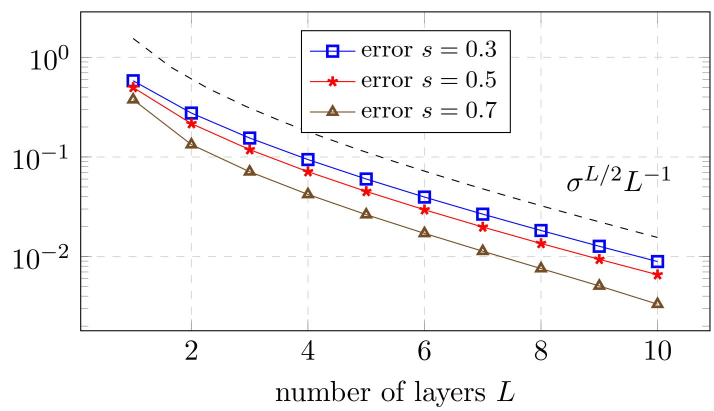
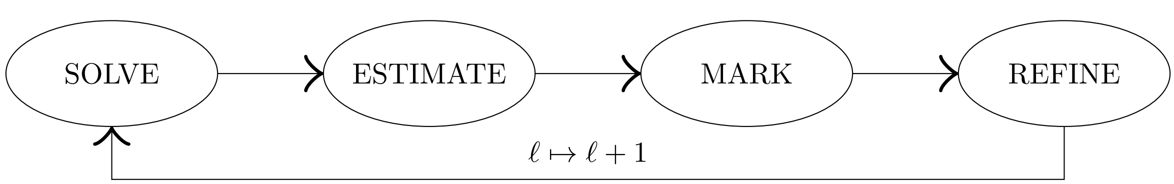


Figure 3.11: Exponential convergence of *hp*-FEM in 1d.

3.2.5 Adaptive FEM for fractional PDEs

A different take on mesh design is given by adaptive finite element methods, in the sense that meshes are not designed by hand to exploit (weighted) regularity estimates, but are done by sucessive refinement of a coarse mesh $\mathcal{T}_0$. In particular, the classical adaptive FEM loop



Algorithm 1 (Standard adaptive algorithm)

% Input: Initial triangulation $\mathcal{T}_0$, adaptivity parameter $0 < \theta \leq 1$.

for $\ell = 0, 1, 2, \ldots$ **do**

(i) Compute the solution $u_\ell \in \mathcal{P}^1_0(\mathcal{T}_\ell)$ of

$$a(u_\ell, v_\ell) = (f, v_\ell)_{L^2} \quad \forall v_\ell \in \mathcal{P}^1_0(\mathcal{T}_\ell).$$

(ii) Compute error indicators $\eta_\ell(T, u_\ell)$ for all $T \in \mathcal{T}_\ell$ (to be specified below).

(iii) Determine a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of minimal cardinality such that

$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell)^2. \tag{3.41}$$

(iv) Generate the coarsest NVB refinement $\mathcal{T}_{\ell+1} := \text{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ of $\mathcal{T}_\ell$ such that all marked elements $T \in \mathcal{T}_\ell$ have been bisected.

end for

written down in the following algorithm can be employed for non-local problems as well, [AG17], [FMP19].

The step Solve means computing the solution on the current mesh $\mathcal{T}_\ell$, which means solving the linear system given by (3.11). The steps Mark (Dörfler marking) and Refine (newest vertex bisection) are exactly the same as for standard elliptic FEM and we refer to any lecture on (A)FEM for details.

Thus, we focus in the Estimate step, which means that we have to construct an *a-posteriori* error indicator that automatically detects, where the FEM mesh should be refined.

We start by defining the piecewise constant mesh size function $h_\ell \in L^\infty(\Omega)$ by

$$h_\ell|_T := h_\ell(T) := |T|^{1/d} \quad \text{for all } T \in \mathcal{T}_\ell. \tag{3.42}$$

A natural choice would be a **residual** error estimator $\|f - (-\Delta)^s v_\ell\|_{H^{-s}(\Omega)}$. However, as one actually wants local (elementwise) defined error indicators, the non-locality of H^{-s}-norm does not allow this to be decomposed into elementwise norms. A modification is to realize the H^{-s}-norm as weighted L^2-norm

$$\|h_\ell^s(f - (-\Delta)^s v_\ell)\|^2_{L^2(\Omega)} = \sum_{T \in \mathcal{T}_\ell} \|h_\ell^s(f - (-\Delta)^s v_\ell)\|^2_{L^2(T)}. \tag{3.43}$$

While such an approach works for small s, it is not well-defined for $s \geq 3/4$: By Corollary 2.8, the application of the operator $(-\Delta)^s v$ produces an L^2-function only, if $v \in H^t$ with $t \geq 2s$. However, in the norm above the operator is applied to a function $v_\ell \in \mathcal{P}^1_0(\mathcal{T}_\ell)$, which – as a piecewise polynomial – is not smooth across element edges, i.e., derivatives have singularities at the so called mesh skeleton. In particular, one only has $v_\ell \in H^t$ for $t < 3/2$, which violates the requirement of $t \geq 2s$ for $s \geq 3/4$.

Again, this singular behaviour can be measured in certain weighted spaces. Here, for a mesh $\mathcal{T}_\ell$, we introduce the distance function to the mesh skeleton as

$$\omega_\ell(x) := \inf_{T\in\mathcal{T}_\ell} \operatorname{dist}(x,\partial T) = \inf_{T\in\mathcal{T}_\ell}\inf_{y\in\partial T}|x-y|. \tag{3.44}$$

Lemma 3.26 ([FMP19]). *Let ω_ℓ be given by (3.44). Given $0<s<1$, fix $\beta > 2s-3/2$, e.g., $\beta := s-1/2$. For any $v_\ell \in \mathcal{P}^1_0(\mathcal{T}_\ell)$, we then have $\omega_\ell^\beta(-\Delta)^s v_\ell \in L^2(\Omega)$.*

Proof. For $x\in T$, we split the fractional Laplacian into a principal value part and a smoother, integrable part

$$\begin{aligned} C(d,s)^{-1}(-\Delta)^s u_\ell(x) = \text{P.V.} &\int_{B_{\operatorname{dist}(x,\partial T)}(x)} \frac{u_\ell(x)-u_\ell(y)}{|x-y|^{d+2s}}dy \\ &+ \int_{\mathbb{R}^d\setminus B_{\operatorname{dist}(x,\partial T)}(x)} \frac{u_\ell(x)-u_\ell(y)}{|x-y|^{d+2s}}dy. \end{aligned} \tag{3.45}$$

Using polar coordinates $y = x + r\nu$, $\nu\in S^{d-1}$, where S^{d-1} is the $(d-1)$-dimensional unit sphere, and exploiting that $u_\ell|_T \in P_1(T)$, we may compute the principal value part

$$\begin{aligned} \text{P.V.}\int_{B_{\operatorname{dist}(x,\partial T)}(x)} \frac{u_\ell(x)-u_\ell(y)}{|x-y|^{d+2s}}dy &= \lim_{\varepsilon\to 0}\int_{B_{\operatorname{dist}(x,\partial T)}(x)\setminus B_\varepsilon(x)} \frac{u_\ell(x)-u_\ell(y)}{|x-y|^{d+2s}}dy \\ &= \lim_{\varepsilon\to 0}\nabla u_\ell|_T\cdot\int_{B_{\operatorname{dist}(x,\partial T)}(x)\setminus B_\varepsilon(x)} \frac{x-y}{|x-y|^{d+2s}}dy \\ &= \lim_{\varepsilon\to 0}\nabla u_\ell|_T\cdot\int_{\nu\in S^{d-1}}\int_{r=\varepsilon}^{\operatorname{dist}(x,\partial T)} r^{-2s}\nu\,dr d\nu = 0, \end{aligned} \tag{3.46}$$

where the last equality follows from interchanging the integration in ν and r. For the second part in the decomposition of $(-\Delta)^s u_\ell$ in (3.45), we abbreviate $R = \operatorname{diam}(\Omega)$, and a similar computation using the Lipschitz continuity of u_ℓ provides (note: $\|u_\ell\|_{W^{1,\infty}(\Omega)} := \|u_\ell\|_{L^\infty(\Omega)} + \|\nabla u_\ell\|_{L^\infty(\Omega)}$)

$$\begin{aligned} &\left|\int_{\mathbb{R}^d\setminus B_{\operatorname{dist}(x,\partial T)}(x)} \frac{u_\ell(x)-u_\ell(y)}{|x-y|^{d+2s}}dy\right| \\ &\quad\le \left|\int_{B_R(x)\setminus B_{\operatorname{dist}(x,\partial T)}(x)} \frac{u_\ell(x)-u_\ell(y)}{|x-y|^{d+2s}}dy\right| + \left|\int_{B_R(x)^c} \frac{u_\ell(x)-u_\ell(y)}{|x-y|^{d+2s}}dy\right| \\ &\quad\lesssim \|u_\ell\|_{W^{1,\infty}(\Omega)}\left(\int_{B_R(x)\setminus B_{\operatorname{dist}(x,\partial T)}(x)} \frac{|x-y|}{|x-y|^{d+2s}}dy + \int_{B_R(x)^c}\frac{1}{|x-y|^{d+2s}}dy\right) \\ &\quad\lesssim \|u_\ell\|_{W^{1,\infty}(\Omega)}\left(\int_{r=\operatorname{dist}(x,\partial T)}^R r^{-2s}dr + \int_{r=R}^\infty r^{-1-2s}dr\right) \\ &\quad\lesssim \|u_\ell\|_{W^{1,\infty}(\Omega)}(\operatorname{dist}(x,\partial T)^{1-2s}+1). \end{aligned} \tag{3.47}$$

Since $\operatorname{dist}(x,\partial T)^{\beta+1-2s} = (\omega_\ell|_T)^{\beta+1-2s}$ is square-integrable in view of $\beta > 2s-3/2$, Lemma 3.26 follows. ∎

By the previous lemma it is natural to just modify the weight in (3.43) as

$$\widetilde{h}_\ell^s := \begin{cases} h_\ell^s & \text{for } 0 < s \leq 1/2, \\ h_\ell^{s-\beta}\omega_\ell^\beta & \text{for } 1/2 < s < 1 \text{ and } \beta := s - 1/2 \end{cases}$$

and introduce the error estimator

$$\eta_\ell(v_\ell) = \left(\sum_{T\in\mathcal{T}_\ell} \eta_\ell(T, v_\ell)^2\right)^{1/2}, \qquad \eta_\ell(T, v_\ell) := \|\widetilde{h}_\ell^s\big(f - (-\Delta)^s v_\ell\big)\|_{L^2(T)}. \tag{3.48}$$

This error estimator indeed gives an upper bound for the error:

Theorem 3.27 ([FMP19, Thm. 2.3]). *For $0 < s < 1$ and $f \in L^2(\Omega)$ the weighted residual error estimator* (3.48) *is reliable:*

$$\|u - u_\ell\|_{\widetilde{H}^s(\Omega)} \leq C_{\mathrm{rel}}\, \eta_\ell(u_\ell). \tag{3.49}$$

Using the proposed weighted residual error estimator to steer the adaptive algorithm, then gives optimal algebraic convergence rates for the AFEM error as formulated in the following theorem.

Theorem 3.28 ([FMP19]). *If $0 < \theta \ll 1$ is sufficiently small, then the error estimator converges with the best possible algebraic rate: For each $t > 0$, there exist c_{opt}, $C_{\mathrm{opt}} > 0$ such that*

$$c_{\mathrm{opt}}\, \mathbb{A}_t(u) \leq \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^t\, \eta_\ell \leq C_{\mathrm{opt}}\, \mathbb{A}_t(u), \tag{3.50}$$

$$\mathbb{A}_t(u) := \sup_{N\in\mathbb{N}_0} (N+1)^t \min_{\mathcal{T}_{\mathrm{opt}}\in\{\mathcal{T}\in\mathrm{refine}(\mathcal{T}_0)\,:\,\#\mathcal{T}-\#\mathcal{T}_0\leq N\}} \eta_{\mathrm{opt}}, \tag{3.51}$$

where $\mathrm{refine}(\mathcal{T}_0)$ *is the (infinite) set of all NVB refinements of the initial triangulation $\mathcal{T}_0$ and η_{opt} is the weighted residual error estimator corresponding to the triangulation $\mathcal{T}_{\mathrm{opt}}$.*

Proof. The proof follows the framework *axioms of adaptivity* of [CFPP14], showing 4 properties of the error estimator: stability, reduction, discrete reliability and quasi-orthogonality. For details, we refer to [FMP19]. The key challenge hereby is stability, which follows from an inverse estimate

$$\|\widetilde{h}_\ell^s(-\Delta)^s v_\ell\|_{L^2(\Omega)} \lesssim \|v_\ell\|_{\widetilde{H}^s(\Omega)},$$

which can be shown using the interior regularity estimates from Lemma 3.16. ■

Example 3.29 *We consider the unit circle $\Omega = B_1(0)$, and choose a discontinuous right-hand side*

$$f(x,y) = \chi_{\{x>0\}}(x,y) = \begin{cases} 1 & \textit{for } x > 0 \\ 0 & \textit{otherwise} \end{cases}.$$

and solve $(-\Delta)^s u = f$ *in* Ω, $u = 0$ *on* Ω^c. *For this problem, an exact solution is known, see, e.g., [AG17] and the energy norm of the solution can be explicitly computed.*
As the right-hand side is discontinuous, we expect singularities of the solution along the line $x = 0$ *in addition to the generic singularities on the whole boundary* $\partial\Omega$ *as in Example 3.14.*
In Figure 3.12, an adaptively generated mesh and the discrete solution are plotted. As expected, the adaptive algorithm does not only refine the mesh at the boundary, but also along the discontinuity of f *at the line* $x = 0$. *However, the refinement towards the boundary tends to be stronger than that towards the discontinuity line of the right-hand side.*

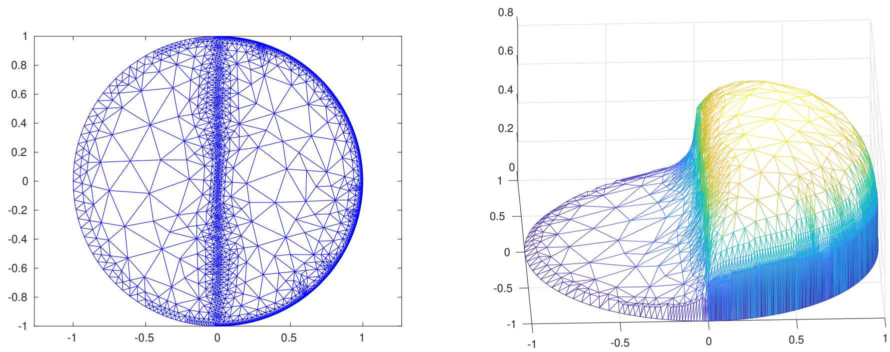


Figure 3.12: Adaptively generated mesh ($s = 0.25$) for problem with discontinuous right-hand side (left) and computed Galerkin solution (right).

Finally, in Figure 3.13, we compare convergence rates for the proposed a-posteriori error estimator and the error in the energy norm for uniform refinement and adaptive refinement done with Algorithm 1 with the parameter choice $s = 0.25$. *The convergence rate for uniform refinement is limited to* $h^{1/2} \simeq N^{-1/4}$ *by the regularity of the exact solution. The adaptive algorithm picks up a faster convergence rate* $h \simeq N^{-1/2}$ *as suggested by Theorem 3.28.*

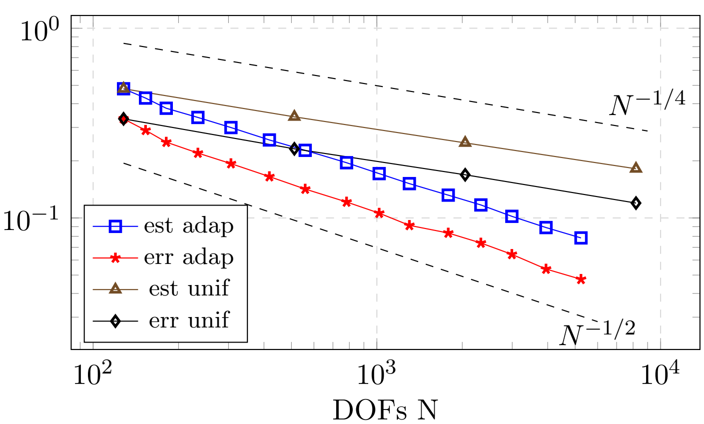


Figure 3.13: Energy norm error and error estimator with discontinuous right-hand side on the unit circle for uniform and adaptive refinement; parameter $s = 0.25$.

Chapter 4

Numerical integration

In the previous sections, we assumed that the stiffness matrices corresponding to all FEM discretizations can be assembled exactly. However, the presence of the kernel function $|x-y|^{-d-2s}$ in the double integrals makes an exact calculation unfeasible.
There are various techniques to efficiently evaluate the integrals arsing in the stiffness matrix. We discuss here the transformation technique described in detail in [CS12, CvPS15, BFM24].

The use of a quadrature incurs errors in the numerical scheme. However, this kind of error ("variational crime") can be analyzed using the Strang lemma below, see, e.g., [BS08] for a proof. The key property of the quadrature schemes that we will see is that they converge *exponentially* with the number of quadrature points. Hence, the quadrature errors can be controlled very effectively with little extra computational work.

Lemma 4.1 (First Strang Lemma). *Let $u \in \widetilde{H}^s(\Omega)$ be the solution of (3.6) and $V_h \subset \widetilde{H}^s(\Omega)$ be a FEM space. Let $\widetilde{a}_n(\cdot,\cdot)$ be a bilinear form satisfying*

$$\widetilde{\alpha}_n \|v\|^2_{\widetilde{H}^s(\Omega)} \leq \widetilde{a}_n(v,v) \text{ for all } v \in V_h \tag{4.1}$$

with some constant $\widetilde{\alpha}_n > 0$. Let $\widetilde{l}_n(\cdot)$ be a linear form on V_h and $\widetilde{u}_{h,n} \in V_h$ solve

$$\widetilde{a}_n(\widetilde{u}_{h,n}, v) = \widetilde{l}_n(v) \qquad \forall v \in V_h.$$

Then, with the continuity constant C_a of the bilinear form $a(\cdot,\cdot)$, there holds

$$\|u - \widetilde{u}_{h,n}\|_{\widetilde{H}^s(\Omega)} \leq \left(1 + \frac{C_a}{\widetilde{\alpha}_n}\right) \left(\inf_{v \in V_h} \left(\|u-v\|_{\widetilde{H}^s(\Omega)} + \sup_{w \in V_h} \frac{|a(v,w) - \widetilde{a}_n(v,w)|}{\|w\|_{\widetilde{H}^s(\Omega)}} \right) \right.$$
$$\left. + \sup_{w \in V_h} \frac{|l(w) - \widetilde{l}_n(w)|}{\|w\|_{\widetilde{H}^s(\Omega)}} \right).$$

In our case the bilinear form $\widetilde{a}_n(\cdot,\cdot)$ and linear form $\widetilde{l}_n(\cdot)$ are approximations computed with numerical quadrature. The Strang lemma then gives an error estimate that consists of a best-aprroximation error in the FEM space and two so called *consistency error* terms, which measure the quality of the approximations to the exact bilinear form $a(\cdot,\cdot)$ and linear form $l(\cdot)$.

4.1 Quadrature rules

4.1.1 1D quadrature

Goal: quadrature rules that converge exponentially (in the number of quadrature points).

On the reference interval $I := (0, 1)$, we consider a (weighted) quadrature formula

$$\int_0^1 f(x)\omega(x)\,dx \approx Q_n(f) := \sum_{i=1}^{n} w_i f(\xi_i), \tag{4.2}$$

with *quadrature points* x_i, $i = 1, \ldots, n$, and *quadrature weights* $w_i > 0$.
Here, we consider the quadrature points to be either

- the points are zeros of the Legendre polynomials L_n (for $\omega \equiv 1$), which gives classical Gaussian quadrature formula, denoted by Q_n^G.
- the zeros of orthogonal polynomials w.r.t. the weighted $L^2(I;\omega)$-inner product (for $\omega(x) = x^\alpha(1-x)^\beta$ with $\alpha, \beta > -1$), the so called Jacobi polynomials J_n, which gives so calle Gauss-Jacobi quadrature, denoted by Q_n^{GJ}.

In fact, classical Gaussian quadrature can be seen as a special case of Gauss-Jacobi quadrature with $\alpha = \beta = 0$.
The quadrature weights are determined by

$$w_i = \int_0^1 \omega(x)\ell_i(x)\,dx \qquad \ell_i(x) = \prod_{j\neq i} \frac{x - x_j}{x_i - x_j}.$$

We have:

Lemma 4.2 (Gauss-Jacobi quadrature). *The following holds:*

(i) (point distribution) The n zeros $(x_i)_{i=1}^n$ of J_n are pairwise distinct and satisfy $0 < x_i < 1$.

(ii) The weights $w_i > 0$ satisfy $\sum_{i=1}^n w_i = \int_0^1 \omega(x)dx =: C_\omega$.

(iii) The rule (4.2) is exact for polynomials of degree $2n-1$:

$$\int_0^1 \omega(x)\pi(x)\,dx = Q_n(\pi) \qquad \forall \pi \in P_{2n-1} \tag{4.3}$$

(iv) (quasi-bestapproximation)

$$\left|\int_0^1 \omega(x)f(x)\,dx - Q_n(f)\right| \leq C \inf_{v\in P_{2n-1}} \|f - v\|_{L^\infty(0,1)} \tag{4.4}$$

(v) (stability)

$$|Q_n(f)| \leq C_\omega \max_i |f(x_i)| \leq C_\omega \|f\|_{L^\infty(0,1)}. \tag{4.5}$$

Proof. For classical Gaussian quadrature, see any course on numerical analysis, the arguments for Gauss-Jacobi quadrature follow in the same way. ■

A quadrature rule for an interval (a, b) is then defined by scaling:

$$Q_{n,(a,b)}(f) := (b-a)\sum_{i=1}^{n} w_i f(a + (b-a)\xi_i) \tag{4.6}$$

with corresponding error bound

$$\left|\int_a^b \omega(x)f(x)\,dx - Q_{n,(a,b)}(f)\right| \leq C(b-a)\inf_{v\in P_{2n-1}} \|f-v\|_{L^\infty(a,b)}. \tag{4.7}$$

For classical Gaussian quadrature ($\omega \equiv 1$) and analytic integrands f, we can use Lemma 3.24 to bound the best-approximation error

$$\inf_{v\in P_n} \|f-v\|_{L^\infty(0,1)} \leq Ce^{-bn} \quad \forall n \in \mathbb{N}_0 \tag{4.8}$$

and thus obtain exponential convergence of the quadrature formula with respect to the number of quadrature points.

Besides analytic functions, an important class of integrands consists of functions that are singular at the endpoints, e.g., functions of the form $f(x) = x^\alpha(1-x)^\beta \tilde{f}(x)$ for some smooth/analytic $\tilde{f}$ and integration $\int_0^1 f(x)\,dx$. There are two common techniques to deal with such integrands:

1. Use *composite* quadrature rules like hp-quadrature: Define a geometric mesh as in Figure 3.8 with $2L+1$ elements. Use appropriately scaled Gaussian quadrature on each element, i.e.,

$$Q^{hp}_{L,n}(f) := \sum_{i=0}^{2L} Q_{n,(x_i,x_{i+1})}(f). \tag{4.9}$$

2. Gauss-Jacobi quadrature: Gaussian quadrature with weight function $\omega(x) = x^\alpha(1-x)^\beta$.

For Gauss-Jacobi quadrature the same arguments as for Gaussian quadrature can be made to obtain exponential convergence of the quadrature formula.

4.1.2 hp-quadrature*

For hp-quadrature the error also converges exponentially as stated in the following lemma.

Lemma 4.3. *Let C_f, $\gamma_f > 0$, $\beta < 1$. Let $f \in C^\infty(0,1)$ satisfy*

$$\sup_{x\in(0,1)} |x^{\beta+m} f^{(m)}(x)| \leq C_f \gamma_f^m m! \qquad \forall m \in \mathbb{N}_0. \tag{4.10}$$

Then, there are C, $b > 0$ depending only on γ_f and $\sigma \in (0,1)$ such that

$$\left| \int_0^1 f(x)\,dx - Q^{hp}_{n,L}(f) \right| \leq C \left[n^{2\beta} \sigma^{(1-\beta)L} + e^{-bn} \right]. \tag{4.11}$$

The number of quadrature points is Ln.
In particular, by selecting $L \sim n$, we get

$$\left| \int_0^1 f(x)\,dx - Q^{hp}_{n,L}(f) \right| \leq Ce^{-b'n} \tag{4.12}$$

and the number of quadrature points is $N = Ln \sim n^2$.

Proof. Since $\int_0^1 f(x)\,dx = \sum_{i=0}^L \int_{x_i}^{x_{i+1}} f(x)\,dx$, we write the error as

$$\left| \int_0^1 f(x)\,dx - Q^{hp}_{n,L}(f) \right| \leq \sum_{i=0}^{L} \left| \int_{x_i}^{x_{i+1}} f(x)\,dx - Q^G_{n,(x_i,x_{i+1})}(f) \right|$$

and estimate each term in the sum separately. We denote the element size by $h_i := (x_{i+1} - x_i)$. We start with the observation that

$$|f(x)| \leq C_f x^{-\beta}, \qquad |f^{(m)}(x)| \leq C_f x^{-m-\beta} \gamma_f^m m! \quad \forall m \in \mathbb{N}_0. \tag{4.13}$$

The element (x_0, x_1): We use the stability of Lemma 4.2, (v) as well as $\operatorname{dist}(\xi_i, -1) \geq cn^{-2}$, which holds for the Gaussian quadrature nodes by Taylor expansion of the cosine, to obtain

$$\begin{aligned}
\left| \int_{x_0}^{x_1} f(x)\,dx - Q^G_{n,(x_0,x_1)}(f) \right| &\leq \left| \int_{x_0}^{x_1} f(x)\,dx \right| + h_0 \max_{i=1,\dots,n} |f(\xi_i h_0/2)| \\
&\overset{(4.13)}{\leq} C_f \frac{1}{1-\beta} x_1^{1-\beta} + h_0 C_f C (h_0 n^{-2})^{-\beta} \\
&\leq C C_f n^{2\beta} h_0^{1-\beta} = C C_v n^{2\beta} \sigma^{L(1-\beta)}.
\end{aligned}$$

The elements (x_i, x_{i+1}), $i \geq 1$: The key observation is that

$$\begin{aligned}
&\operatorname{dist}(0, (x_i, x_{i+1})) = x_i, \qquad\qquad h_i = x_{i+1} - x_i = x_i \frac{1-\sigma}{\sigma}, \\
&\frac{h_i}{\operatorname{dist}(0,(x_i,x_{i+1}))} = \frac{h_i}{x_i} \leq \frac{1-\sigma}{\sigma}.
\end{aligned} \tag{4.14}$$

We denote by $\widehat{f}(\xi) := f(x_i + (\xi+1)h_i/2)$ the pull-back to the reference interval $(-1,1)$. We compute with the relation $x = x_i + (\xi+1)h_i/2$

$$\begin{aligned}
\left| \widehat{f}^{(m)}(\xi) \right| &= \left| \left(\frac{h_i}{2} \right)^m f^{(m)}(x) \right| \leq C_f (h_i 2^{-1} \gamma_f)^m x^{-m-\beta} m! \leq C_f (h_i 2^{-1}\gamma_f)^m x_i^{-m-\beta} m! \\
&\overset{(4.14)}{\leq} C_f x_i^{-\beta} ((1-\sigma)\sigma^{-1} 2^{-1} \gamma_f)^m m! =: C_f x_i^{-\beta} \gamma^m m!
\end{aligned}$$

Combining Lemma 4.2, (v) and Lemma 3.24 yields the existence of $b > 0$ depending only on γ_f and σ (in particular, it is independent of i!) such that

$$\left| \int_{-1}^{1} \widehat{f}(\xi)\, d\xi - Q_n^G(\widehat{f}) \right| \leq C C_f x_i^{-\beta} e^{-bn}.$$

We use this as follows:

$$\left| \int_{x_i}^{x_{i+1}} f(x)\, dx - Q_{n,(x_i,x_{i+1})}^G(f) \right| = \frac{h_i}{2} \left| \int_{-1}^{1} \widehat{f}(\xi)\, d\xi - Q_n^G(\widehat{f}) \right| \leq C C_f h_i x_i^{-\beta} e^{-bn} \leq C C_f x_i^{1-\beta} e^{-bn}.$$

Since $1 - \beta > 0$, the sum $\sum_i x_i^{1-\beta}$ is a (truncated) geometric series that is finite. We conclude

$$\sum_{i=1}^{L} \left| \int_{x_i}^{x_{i+1}} f(x)\, dx - Q_{n,(x_i,x_{i+1})}^G(f) \right| \leq C c_f e^{-bn}.$$

■

Remark. An important class of functions that is covered by Lemma 4.3 are functions of the form $x^\alpha \widetilde{f}(x)$ for function a $\widetilde{f}$ analytic on $[0,1]$, where $\alpha > -1$. One can show that there are C, $\gamma > 0$ depending only on $\widetilde{f}$ and α such that

$$|x^{n-\alpha} D_x^n (x^\alpha \widetilde{f}(x))| \leq C \gamma^n n! \qquad \forall n \in \mathbb{N}_0.$$

Remark 4.4 *The proof of Lemma 4.3 relies on two ingredients: a) the approximation on the first element (x_0, x_1) essentially relies on (x_0, x_1) being small; b) the approximation on the remaining elements (x_i, x_{i+1}) is uniformly exponential in n, which is due to the fact that the pull-backs of f to the reference element can be controlled uniformly in i. This shows that the* geometric mesh *will also deliver exponential convergence in other norms, e.g., L^2, H^1. See, e.g., the discussion in [AM18, Thm. 8 of the printed version or Thm. 3.13 of the preprint version].*

Example 4.5 *We illustrate the power of the hp-quadrature scheme in Fig. 4.1 for different integrands. Note that while quadrature for integrals of the form $\int_0^1 x^\alpha \widetilde{f}(x)\, dx$ could be handled well by suitable Gauss-Jacobi quadrature, integrals of the form $\int_0^1 \exp(\sin x^\alpha)\, dx$ are not accessible to Gauss-Jacobi quadrature.*

4.1.3 2D quadrature

While 1D quadrature rules are sufficient to approximate the linear form $l(\cdot)$, the bilinear form $a(\cdot,\cdot)$ consists a double integral and thus needs a two dimensional quadrature formula.

Here, we consider tensor product quadrature, i.e., we approximate by a 1D quadrature formula in each coordinate direction, denoted by an additional subscript $Q_{n,x}$. The tensor product rule then reads as

$$\int_0^1 \int_0^1 \Phi(x,y)\, dy\, dx \approx Q_n^{2D}(\Phi) := Q_{n,x} \circ Q_{n,y}(\Phi) = \sum_{i,j=1}^{n} \omega_i \omega_j \Phi(x_i, y_j).$$

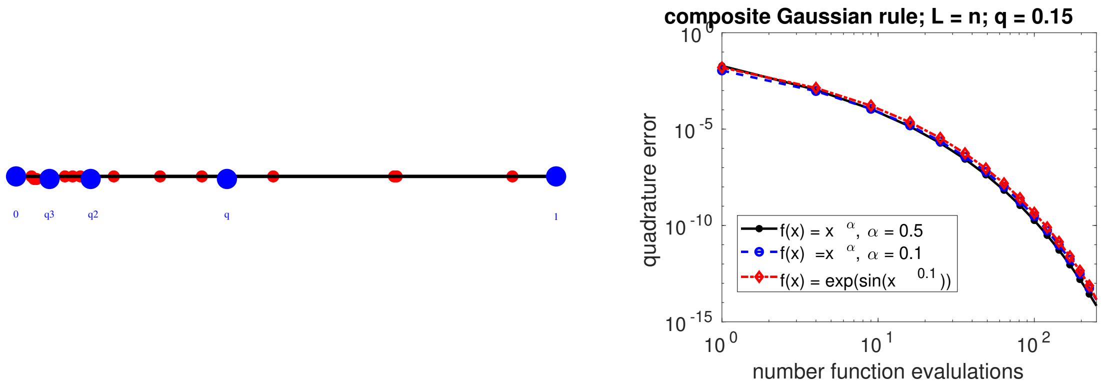


Figure 4.1: example of 1D hp-quadrature. Left: mesh (blue nodes) and quadrature points (red dots). Right: convergence behavior

The error estimate can in this case be reduced to the 1D case:

$$\begin{aligned}
&\left|\int_0^1\int_0^1 \Phi(x,y)\,dy\,dx - Q_n^{2D}(\Phi)\right| \\
&\quad\le \left|\int_0^1\int_0^1 \Phi(x,y)\,dy - Q_{n,y}(\Phi(x,\cdot))\,dx\right| + \left|\int_0^1 Q_{n,y}(\Phi(x,\cdot))\,dx - Q_{n,x}(Q_{n,y}(\Phi))\right| \\
&\quad\le \sup_{x\in(0,1)}\left|\int_0^1 \Phi(x,y)dy - Q_{n,y}(\Phi(x,\cdot))\right| + \left|Q_{n,y}\left(\int_0^1 \Phi(x,\cdot)\,dx - Q_{n,x}(\Phi)\right)\right| \\
&\quad\le \sup_{x\in(0,1)}\left|\int_0^1 \Phi(x,y)dy - Q_{n,y}(\Phi(x,\cdot))\right| + \sum_{i=1}^n \omega_i\left|\int_0^1 \Phi(x,y_i)\,dx - Q_{n,x}(\Phi(\cdot,y_i))\right| \\
&\quad\le \sup_{x\in(0,1)}\left|\int_0^1 \Phi(x,y)dy - Q_{n,y}(\Phi(x,\cdot))\right| + C_\omega \sup_{y\in(0,1)}\left|\int_0^1 \Phi(x,y)dx - Q_{n,x}(\Phi(\cdot,y))\right|.
\end{aligned}$$

The errors on the right-hand side are now essentially 1D quadrature errors and can be bounded with exponential convergent error bounds (for sufficiently nice integrands) from the one dimensional case.
With $F_{(-1,1)} : x \mapsto 2x-1$, we denote by $\widehat{\mathcal{E}}_\rho := F_{(-1,1)}^{-1}(\mathcal{E}_\rho)$ the mapped Bernstein ellipse around $(0,1)$. Then, the following error bounds hold.

Lemma 4.6. *Let $\widetilde{\rho} > 1$ and let $\Phi : \widehat{\mathcal{E}}_{\widetilde{\rho}} \times \widehat{\mathcal{E}}_{\widetilde{\rho}} \to \mathbb{C}$ be such that $\Phi(\cdot,y)$ is holomorphic on $\widehat{\mathcal{E}}_{\widetilde{\rho}}$ for all $y \in (0,1)$ and $\Phi(x,\cdot)$ is holomorphic on $\widehat{\mathcal{E}}_{\widetilde{\rho}}$ for all $x \in (0,1)$. Then, for every $1 < \rho < \widetilde{\rho}$, the quadrature error can be estimated by*

$$\left|\int_0^1\int_0^1 \Phi(x,y)\,dy\,dx - G_n^{2D}(\Phi)\right| \le C\rho^{-2n+1}\left(\sup_{y\in(0,1)}\|\Phi(\cdot,y)\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} + \sup_{x\in(0,1)}\|\Phi(x,\cdot)\|_{L^\infty(\widehat{\mathcal{E}}_\rho)}\right), \tag{4.15}$$

where the constant C is independent of n and Φ.

Now, we bring the integrals needed to assemble the stiffness matrix to the form required for the tensor product quadrature and employ suitable combinations of 1D quadrature rules.

4.2 Evaluation of the stiffness matrix

Recall that the entries of the stiffness matrix are given by

$$A_{ij} = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{d+2s}} \, dx \, dy.$$

By splitting the integrals into integrals over elements (and possibly also allow $T' = \Omega^c$), we have to evaluate integrals of the form

$$\int_{T \times T'} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{d+2s}} \, dx \, dy =: \int_{T \times T'} \frac{g(x,y)}{|x-y|^{d+2s}} \, dx \, dy.$$

We note that the functions φ_i, φ_j are smooth on T and T'. To simply further the presentation, we assume that the basis functions φ_i, φ_j are in $C(\overline{\Omega})$ (e.g., the classical FEM hat functions).
To keep the presentation of the main ideas simple, we consider the case $d = 1$ in detail, extensions for $d > 1$ need to take many more cases into account and can be treated by results of [CS12]. The elements are then intervals and the integration domains are rectangles $T \times T'$. For the evaluation of the integral, three cases that can arise:

1. T and T' are separated. Then, the integrand is smooth (analytic), and a standard (Gaussian) quadrature will perform well.
2. $T = T'$.
3. T and T' are neighboring elements that share one vertex.

In the cases 2 and 3, the integrand is singular. We now show how to transform such integrals in such a way that we have to evaluate integrals of the form

$$\int_{\xi=0}^{1} \int_{\eta=0}^{1} \widetilde{g}(\xi,\eta) \xi^{\alpha} (1-\eta)^{\beta} \, d\eta \, d\xi \tag{4.16}$$

for some smooth[1] function $\widetilde{g}$. Such integrals can very effectively be evaluated with certain types of weighted Gaussian quadrature. In fact, we use tensorproduct quadrature rules to approximate the double integral.

The case of identical elements $T = T'$

Let $T = (a,b)$, $h_T = b - a$ and denote by $F_T : (0,1) \to T,\ x \mapsto h_T x + a$. We define $\widehat{g}(x,y) := g(F_T(x), F_T(y))$ mapping $(0,1)^2 \to \mathbb{R}$ and obtain from the transformation theorem

$$\int_{T \times T} \frac{g(x,y)}{|x-y|^{1+2s}} \, dx \, dy = h_T^2 \int_{(0,1)^2} \frac{\widehat{g}(x,y)}{|h_T(x-y)|^{1+2s}} \, dx \, dy = h_T^{1-2s} \int_{(0,1)^2} \frac{\widehat{g}(x,y)}{|x-y|^{1+2s}} \, dx \, dy.$$

[1] in fact, analytic

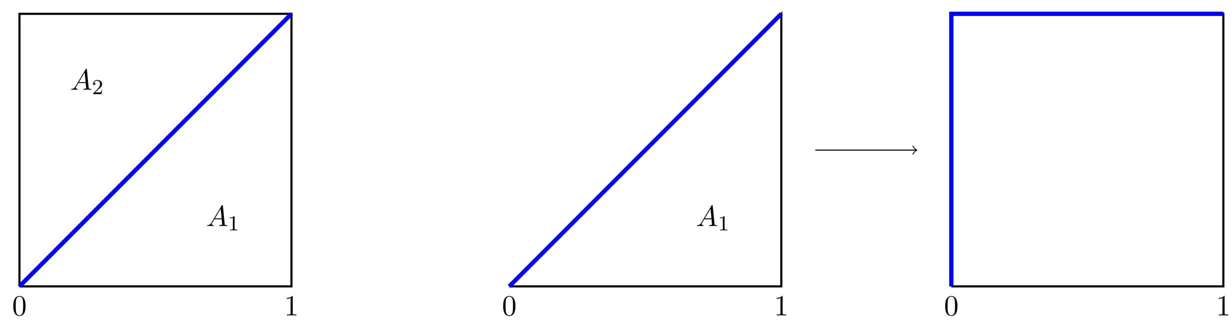


Figure 4.2: quadrature on $(0,1)^2$ for function with singularity at $x = y$: split $(0,1)^2$ into A_1 and A_2, singularities marked in blue (left); Duffy transformation (right).

The integrand is not smooth along the diagonal $x = y$. It is of the general form

$$\int_{(0,1)^2} f(x, y, x-y)\, dx\, dy$$

where the function f is smooth in the first two variables and has a singularity in the third variable, i.e., $f(x, y, z) = \widehat{g}(x, y)|z|^{\alpha}$ for $\alpha = -(1 + 2s) + 2$ and smooth $\widehat{g}$.
In order to deal with the singularity at $x = y$, we split the domain of integration $(0,1)^2$ into the triangles $A_1 := \{(x, y) \mid 0 < x < 1,\ 0 < y < x\}$ and $A_2 := \{(x, y) \mid 0 < y < 1,\ 0 < x < y\}$. As the integrand is invariant under the transformation $(x, y) \mapsto (y, x)$, we notice that both integrals are the same, i.e.,

$$\int_{(0,1)^2} f(x, y, x-y)\, dx\, dy = 2 \int_{A_1} f(x, y, x-y)\, dx\, dy.$$

For the integral over A_1, we employ the so called *Duffy transformation*, i.e., the mapping $(0,1)^2 \to A_1,\ (x, y) \mapsto (x, xy)$ (see Figure 4.2). Note that this transforms the singularity at $x = y$ to the axis parallel lines $\{(x, y) \in (0,1)^2 \ :\ x = 0\} \cup \{(x, y) \in (0,1)^2 \ :\ y = 1\}$. Again employing the transformation theorem, this gives

$$\begin{aligned}\int_{A_1} \widehat{g}(x, y)|x-y|^{-1-2s}\, dx\, dy &= \int_{(0,1)^2} \widehat{g}(x, xy)|x - xy|^{-1-2s} x\, dx\, dy \\ &= \int_{(0,1)^2} \frac{\widehat{g}(x, xy)}{x^2(1-y)^2} x^{2-2s}(1-y)^{1-2s}\, dx\, dy.\end{aligned}$$

Note that the function

$$\widetilde{g}(x, y) = \frac{\widehat{g}(x, xy)}{x^2(1-y)^2} = \frac{(\widehat{\varphi}_i(x) - \widehat{\varphi}_i(xy))(\widehat{\varphi}_j(x) - \widehat{\varphi}_j(xy))}{(x - xy)^2}$$

in the integrand is indeed smooth as it is even a polynomial by the fundamental theorem of algebra. Thus, we have an integral of the form

$$\int_{(0,1)^2} \widetilde{g}(x, y) x^{2-2s}(1-y)^{1-2s}\, dx\, dy,$$

with $\widetilde{g}$ smooth, for which Gauss-Jacobi quadrature is made for.

The case T and T' are adjacent elements

We now assume that $T = (a,b)$ and $T' = (b,c)$ share the common node b and take element transformations $F_T : (0,1) \to T,\ x \mapsto -h_T x + b$ and $F_{T'} : (0,1) \to T',\ x \mapsto h_{T'} x + b$ that both map 0 to the common point p. For simplicity of presentation, we assume $h_T = h_{T'}$ (otherwise some - rather irrelevant - quotients of $h_T/h_{T'}$ appear in the nominator). By the transformation theorem, we obtain for $\widehat{g}(x,y) = g(F_T(x), F_{T'}(y))$

$$\int_{T\times T'} \frac{g(x,y)}{|x-y|^{1+2s}}\,dx\,dy = h_T^2 \int_{(0,1)^2} \frac{\widehat{g}(x,y)}{|F_T(x)-F_{T'}(y)|^{1+2s}}\,dx\,dy = h_T^{1-2s}\int_{(0,1)^2} \frac{\widehat{g}(x,y)}{|x+y|^{1+2s}}\,dx\,dy.$$

Now, the integrand only has a singularity at $x = y = 0$. In order to deal with that, we again split the square $(0,1)^2$ into the triangles $A_1 := \{(x,y) \mid 0 < x < 1,\ 0 < y < x\}$ and $A_2 := \{(x,y) \mid 0 < y < 1,\ 0 < x < y\}$, i.e,

$$\int_{(0,1)^2} \frac{\widehat{g}(x,y)}{|x+y|^{1+2s}}\,dx\,dy = \int_{A_1} \frac{\widehat{g}(x,y)}{|x+y|^{1+2s}}\,dx\,dy + \int_{A_2} \frac{\widehat{g}(x,y)}{|x+y|^{1+2s}}\,dx\,dy.$$

Both integrals can be treated with Duffy transformations (for A_1 we take $(x,y) \mapsto (x,xy)$, for A_2 we take $(x,y) \mapsto (xy,y)$) and lead to essentially the same cases just with flipped roles of x and y. We thus only consider the integral over A_1, which after transformation leads to

$$\begin{aligned}\int_{A_1} \frac{\widehat{g}(x,y)}{|x+y|^{1+2s}}\,dx\,dy &= \int_{(0,1)^2} \frac{\widehat{g}(x,xy)}{|x+xy|^{1+2s}} x\,dx\,dy \\ &= \int_{(0,1)^2} \frac{\widehat{g}(x,xy)}{x^2(1+y)^{1+2s}} x^{2-2s}\,dx\,dy.\end{aligned}$$

Here, we note that $1/(1+y)^{1+2s}$ is smooth and again by the fundamental theorem of algebra, $\frac{\widehat{g}(x,xy)}{x^2}$ is a polynomial. Thus, we have an integral of the form

$$\int_{(0,1)^2} \widetilde{g}(x,y) x^{2-2s}\,dx\,dy,$$

with $\widetilde{g}$ smooth, for which a Gauss-Jacobi quadrature (in x-variable), and a classical Gaussian quadrature (in y-variable) is made for.

Separated elements $\overline{T} \cap \overline{T'} = \emptyset$

This time, after transformation to the reference interval, the integrand is not singular, i.e.,

$$\int_{T\times T'} \frac{g(x,y)}{|x-y|^{1+2s}}\,dx\,dy = h_T h_{T'} \int_{(0,1)^2} \frac{\widehat{g}(x,y)}{|F_T(x)-F_{T'}(y)|^{1+2s}}\,dx\,dy = h_T h_{T'} \int_{(0,1)^2} \widetilde{g}(x,y)\,dx\,dy.$$

with $\widetilde{g}$ smooth. Thus, Gaussian quadrature in both variables suffices.

Complement part ($T' = \Omega^c$)

Here, using that the basis functions φ_i, φ_j vanish outside of Ω, we consider

$$\int_T \int_{\Omega^c} \frac{g(x,y)}{|x-y|^{1+2s}}\,dy\,dx = \int_T \int_{\Omega^c} \frac{\varphi_i(x)\varphi_j(x)}{|x-y|^{1+2s}} dydx = \int_T \varphi_i(x)\varphi_j(x) \int_{\Omega^c} \frac{1}{|x-y|^{1+2s}} dydx.$$

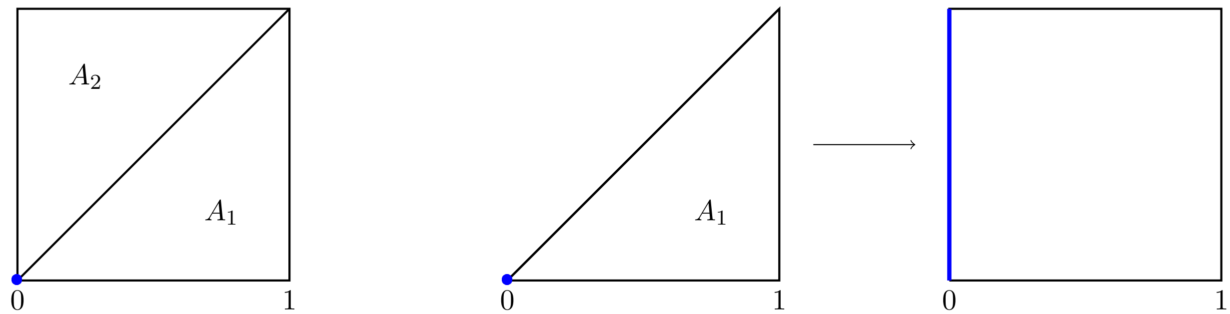


Figure 4.3: quadrature on $(0,1)^2$ for function with singularity at $(0,0)$: split $(0,1)^2$ into A_1 and A_2 (left); Duffy transformation (right), singularities marked in blue.

The inner integral can be computed explicitly and produces a smooth function (in x), if T is separated from the boundary or a function with singularity at one boundary point, if T touches the boundary. In the first case, Gaussian quadrature for x suffices, in the second case, one essentially is in the same case as in the case of adjacent elements above and one uses Gauss-Jacobi quadrature.

4.3 Consistency error estimates

By the Strang lemma, we have to provide bounds for the approximation of the bilinear form $a(\cdot,\cdot)$ denoted by $\widetilde{a}(\cdot,\cdot)$ and the linear form $l(\cdot)$ by $\widetilde{l}(\cdot)$ that are computed via the quadrature formulas in the previous subsections.

Lemma 4.7 (Consistency error for l). *Let f be analytic in $\overline{\Omega}$. Let $l(v) = \langle f, v\rangle_{L^2(\Omega)}$ be the linear form and $\widetilde{l}_n(\cdot)$ its approximation by 1D Gaussian quadrature. Then, there exists a constant $\rho > 1$ depending only on f such that*

$$|l(v) - \widetilde{l}_n(v)| \leq C_{s,f}\, p\, \rho^{p-2n+1} \|v\|_{\widetilde{H}^s(\Omega)} \qquad \text{for all } \ v \in \mathcal{P}_0^p(\mathcal{T}), \tag{4.17}$$

where $C_{s,f} > 0$ is a constant that depends only on s and f.

Proof. As f is analytic on $[0,1]$ there exists an analytic extension to a Bernstein ellipse $\widehat{\mathcal{E}}_\rho$ for some $\rho > 1$. Using Proposition A.1 with n replaced by $2n-1$ gives for a fixed element T after pull back to the reference element

$$\left| \int_0^1 \widehat{f}_T(x)\widehat{v}_T(x)dx - Q_n(\widehat{f}_T\widehat{v}_T)\right| \leq C\rho^{-2n+1}\|\widehat{f}_T\widehat{v}_T\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \leq C_{\rho,f}\,\rho^{-2n+1}\|\widehat{v}_T\|_{L^\infty(\widehat{\mathcal{E}}_\rho)}.$$

It remains to estimate the L^∞-norm of $\widehat{v}_T$. As $\widehat{v}_T \in P_p$ we may employ the With the Bernstein inequality [DL93, Chap. 4, Thm. 2.2] and a polynomial inverse inequality, we obtain

$$\|\widehat{v}_T\|_{L^\infty(\widehat{\mathcal{E}}_\rho)} \leq \rho^p \|\widehat{v}_T\|_{L^\infty(0,1)} \leq C\rho^p p \|\widehat{v}_T\|_{L^2(0,1)}.$$

Transforming back to the physical element and summing over all elements, we obtain the stated result as $\|v\|_{L^2(\Omega)} \leq \|v\|_{H^s(\Omega)}$. ∎

Lemma 4.8 (Consistency error for a). *Let $a(\cdot,\cdot)$ be the bilinear form (3.4) and $\widetilde{a}_n(\cdot,\cdot)$ be its approximation using quadrature as described in the previous subsection. Then, there exists a constant $\rho > 1$ that depends only on the shape regularity constant γ such that for all $u, v \in \mathcal{P}_0^p(\mathcal{T})$ there holds*

$$|a(u,v) - \widetilde{a}_n(u,v)| \leq C_s p^6 \rho^{2p-2n+1} \|u\|_{\widetilde{H}^s(\Omega)} \|v\|_{\widetilde{H}^s(\Omega)}, \tag{4.18}$$

where C_s is independent of p, n.

Proof. In principle, the proof follows the idea of the previous proof, i.e., estimating the quadrature error by L^∞-norms of the integrands, thereby, as double integrals are involved, using Lemma 4.6 instead of Proposition A.1. In order to do so, one has to treat all the cases of Section 4.2 differently and employ suitable inverse type estimates for each case. We refer to [BFM24] for the technical details. ■

Using a perturbation argument, we also obtain uniform coercivity, which is required in the Strang lemma.

Lemma 4.9 (Uniform coercivity). *Let the assumptions of Lemma 4.8 hold. Then, there are constants $c > 1$ and $\widetilde{\alpha} > 0$ such that for $n \geq cp$ there holds*

$$\widetilde{\alpha} \|v\|^2_{\widetilde{H}^s(\Omega)} \leq \widetilde{a}_n(v,v) \qquad \text{for all } \ v \in \mathcal{P}_0^p(\mathcal{T}). \tag{4.19}$$

Proof. The coercivity of $a(\cdot,\cdot)$, the triangle inequality and Lemma 4.8 give

$$\begin{aligned} \alpha \|v\|^2_{\widetilde{H}^s(\Omega)} &\leq a(v,v) \leq \widetilde{a}_n(v,v) + |a(v,v) - \widetilde{a}_n(v,v)| \\ &\leq \widetilde{a}_n(v,v) + C_s p^6 \rho^{2p-2n+1} \|v\|^2_{\widetilde{H}^s(\Omega)}. \end{aligned}$$

As the second term on the right-hand side tends to zero for $n \to \infty$, we may ensure for $n \geq cp$ with large enough constant $c > 1$ that

$$C_s p^6 \rho^{1-2(c-1)p} \leq \frac{\alpha}{2} \tag{4.20}$$

so that coercivity of $\widetilde{a}_n$ follows with coercivity constant $\widetilde{\alpha} := \alpha/2$. ■

Remark. The condition $n \geq cp$ can be weakened to $n \geq p + c_1 \log(p+1)$ with large enough constant c_1.

Chapter 5

Data-sparse representation

The stiffness matrix (3.11) is fully populated, resulting in $O(N^2)$ memory requirement (and also $O(N^2)$ complexity for the matrix-vector multiplication). A variety of matrix compression techniques are available to reduce the complexity of discretization of integral operators to logarithmic-linear complexity $O(N \log^\beta N)$ (for some $\beta \geq 0$). We mention the (fast) multipole method, [Rok85, GR97], panel clustering, panel clustering, [NK88, HN89, HS93, Sau92], wavelet compression techniques, [Rat98, Rat01, Sch98, vPSS97, Tau03, TW03], the mosaic-skeleton method, [Tyr00], the adaptive cross approximation (ACA) method, [Beb00], and the hybrid cross approximation (HCA), [BG05]. Many of these data-sparse methods can be understood as specific instances of $\mathcal{H}$-matrices, [Hac15, Bör10, GH03, Gra01, Hac99]. $\mathcal{H}$-matrices are a specific version of blockwise low-rank matrices. In this section, we will sketch how the fully populated system matrix can be approximated well by blockwise low-rank matrices such as $\mathcal{H}$-matrices or multipole expansions. For the many algorithmic aspects of $\mathcal{H}$-matrices, we refer to, e.g., the monograph [Hac15].

5.1 Hierarchical matrices

5.1.1 Separable kernels

Let $(\varphi_i)_{i=1}^N$ be the collection of basis functions. Consider a bilinear form

$$a(u,v) := \int_{\Omega\times\Omega} k(x,y)u(x)v(y)\,dy\,dx \tag{5.1}$$

with a *separable* kernel $k(\cdot,\cdot)$ of the form

$$k(x,y) = k_1(x)k_2(y). \tag{5.2}$$

Then, the stiffness matrix $\mathbf{A}$ with entries $\mathbf{A}_{ij} = a(\varphi_j, \varphi_i)$ has the form

$$\mathbf{A} = \mathbf{a}\mathbf{b}^\top, \qquad \mathbf{a}_i := \int_\Omega k_1(x)\varphi_i(x)\,dx, \qquad \mathbf{b}_i := \int_\Omega k_2(y)\varphi_i(y)\,dy.$$

That is, $\mathbf{A}$ is a rank-1 matrix. By storing only the vectors $\mathbf{a}$, $\mathbf{b}$, the matrix $\mathbf{A}$ can be represented with complexity $O(N)$ and also the matrix-vector multiplication has complexity $O(N)$. Similarly, if $k(\cdot,\cdot)$ has the form

$$k(x,y) = \sum_{\ell=1}^{r} k_{1,\ell}(x)k_{2,\ell}(y)$$

then $\mathbf{A}$ has the form

$$\mathbf{A} = \mathbf{a}\mathbf{b}^\top, \qquad \mathbf{a}_{i,\ell} := \int_\Omega k_{1,\ell}(x)\varphi_i(x)\,dx, \qquad \mathbf{b}_{i,\ell} := \int_\Omega k_{2,\ell}(y)\varphi_i(y)\,dy, \qquad i = 1,\dots,N, \quad \ell = 1,\dots,r,$$

i.e., it is a rank-r matrix. Storing it requires $O(Nr)$ complexity, which is avantageous if $r \ll N$.

5.1.2 Block approximation by separable kernels

One way to approximate $\mathbf{A}$ by a blockwise *low-rank* matrix is to approximate the kernel function $k(\cdot,\cdot)$ by separable kernels.
We call a *cluster* a collection of indices from $I := \{1,\dots,N\}$. For a cluster σ, we call an axis-parallel[1] "box" $B_\sigma \subset \mathbb{R}^d$ a bounding box, if

$$\cup_{i\in\sigma} \operatorname{supp} \varphi_i \subset B_\sigma,$$

i.e., of the support of all basis functions φ_i, $i \in \sigma$, is contained in B_σ.
Let σ, $\tau \subset \{1,\dots,N\}$ be two clusters with bounding boxes B_σ and B_τ. Assume that k is smooth on $B_\sigma \times B_\tau$. Then it can be approximated by a polynomial k_p of degree p, e.g., by tensor product interpolation.
To that end, given a set of interpolation points $(\xi_i)_{i=1}^p$ (i.e. Chebyshev or Gauss-Lobatto points), we define the 1D interpolation operators $I^x_{p-1} : C((-1,1)^2) \to P_{p-1}$ and $I^y_{p-1} : C((-1,1)^2) \to P_{p-1}$ by

$$(I^x_{p-1} f)(x,y) := \sum_{i=1}^p f(\xi_i, y)\ell_i(x), \qquad (I^y_{p-1} f)(x,y) := \sum_{i=1}^p f(x, \xi_i)\ell_i(y),$$

acting on the variable indicated in the superscript.

In a similar way as in Lemma 4.6, for analytic f, we get again exponential convergence for the interpolation error:

Lemma 5.1. *Let $f \in C^\infty((-1,1)^2)$ satisfy*

$$\|D_x^{m_1} D_y^{m_2} f\|_{L^\infty((-1,1)^2)} \le C_f \gamma_1^{m_1}\gamma_2^{m_2} m_1! m_2! \qquad \forall (m_1,m_2) \in \mathbb{N}_0^2.$$

Then there are constants C, $b > 0$ depending only on γ_1, γ_2 such that

$$\|f - I^x_p \circ I^y_p f\|_{L^\infty((-1,1)^2)} \le C C_f e^{-bp}.$$

In the present setting, this polynomial approximation is from the space

$$\mathcal{Q}_p := \operatorname{span}\{x^\alpha y^\beta \mid \alpha, \beta \in \mathbb{N}_0^d, |\alpha|_\infty, |\beta|_\infty \le p\},$$

where $|\alpha|_\infty := \max_{i=1,\dots,d} \alpha_i$. It therefore has the form

$$k_p(x,y) = \sum_{|\alpha|_\infty, |\beta|_\infty \le p} a_{\alpha,\beta} x^\alpha y^\beta, \tag{5.3}$$

[1]bounding boxes need not be axis-parallel, but we assume this to fix ideas

and we observe that k_p is a sum of separable functions (i.e., a sum of products of functions of x and y only). This sum has $r := (p+1)^{2d}$ terms so that replacing k with k_p produces an approximation $\widetilde{\mathbf{A}}|_{\sigma\times\tau}$ to the block $\mathbf{A}|_{\sigma\times\tau}$ with a rank-r matrix. The error introduced by approximating k by k_p can be controlled by

$$|\mathbf{A}_{ij} - \widetilde{\mathbf{A}}_{ij}| \leq \|k - k_p\|_{L^\infty(B_\sigma\times B_\tau)} \|\varphi_i\|_{L^2(\Omega)} \|\varphi_j\|_{L^2(\Omega)}, \qquad i \in \sigma, \quad j \in \tau. \tag{5.4}$$

In typical situations, the error $\|k - k_p\|_{L^\infty(B_\sigma\times B_\tau)}$ decays exponentially in p (see below). The discretization error of the method is typically $O(h^\gamma)$ for some $\gamma > 0$. Hence, selecting $p = |\log h|$ introduces errors into the stiffness matrix that are of the size of the discretization error. Noting that $p = |\log h|$ is rather small, significant savings can be achieved by approximating the block $\mathbf{A}|_{\sigma\times\tau}$ by a low-rank block.

Remark. The impact of approximating matrix entries on the computation of the solution u_h can be analyzed rigorously with the aid of Strang lemmas. Already estimates from general matrix analysis (cf. [GVL13, Lemma 2.6.1]) allow one to estimate the effect of approximating matrix entries in terms of the condition number of $\mathbf{A}$.

Remark. The representation (5.3) leads to a rank-r matrix approximation of the block $\mathbf{A}|_{\sigma\times\tau}$ with storage requirement $(|\sigma| + |\tau|)\cdot r$ and $r = (p+1)^{2d}$. The special structure of k_p permits the following, even more economical representation

$$\mathbf{A}|_{\sigma\times\tau} = \mathbf{a}\mathbf{S}\mathbf{b}^\top, \qquad \mathbf{a}_{i,\alpha} := \int_\Omega x^\alpha \varphi_i(x)\,dx, \qquad \mathbf{b}_{j,\beta} := \int_\Omega x^\beta \varphi_j(x)\,dx, \qquad \mathbf{S}_{\alpha,\beta} := a_{\alpha,\beta}$$

with storage requirement $|\sigma|(p+1)^d + |\tau|(p+1)^d + (p+1)^{2d}$.

Blockwise low-rank approximation for asymptotically smooth kernels

As a model, we consider the bilinear form[2]

$$b(u,v) := \int_{\Omega\times\Omega} |x-y|^{-\alpha} u(x) v(y)\,dy\,dx \tag{5.5}$$

for some $\alpha \in \mathbb{R}$.

Remark. The kernel is the prototype of an *asymptotically smooth* kernel, which is characterized by

$$|\partial_x^n \partial_y^m k(x,y)| \leq C\gamma^{n+m} n! m! |x-y|^{-\alpha-n-m} \qquad \forall n, m \in \mathbb{N}_0^d. \tag{5.6}$$

The ideas presented here can be directly applied to integral operators with kernels of such type.

Given $\eta > 0$, we introduce the *η-admissibility* condition as follows: A cluster pair (σ, τ) is *η-admissible* if their bounding boxes B_σ and B_τ satisfy the *admissibility condition*

$$\operatorname{dist}(B_\sigma, B_\tau) \geq \eta \max\{\operatorname{diam} B_\sigma, \operatorname{diam} B_\tau\} \tag{5.7}$$

for some user-chosen parameter $\eta > 0$. A cluster pair (σ, τ) is *inadmissible*, if they do not satisfy (5.7).

[2]structurally, the situation for the bilinear form for the fractional Laplacian is very similar

Next, consider a partition $\mathbb{P}$ of $I \times I$ (which will be constructed below). We decompose $\mathbb{P}$ as

$$\mathbb{P} = \mathbb{P}_{far} \dot{\cup} \mathbb{P}_{near},$$

where

$$\mathbb{P}_{far} = \{(\sigma, \tau) \in \mathbb{P} \,|\, (\sigma, \tau) \text{ is } \eta\text{-admissible}\}, \qquad \mathbb{P}_{near} := \mathbb{P} \setminus \mathbb{P}_{far}.$$

The $\mathcal{H}$-approximation of the stiffness matrix $\mathbf{A}$ is done as follows:

Definition 5.2 ($\mathcal{H}$-matrix approximation). *The $\mathcal{H}$-matrix approximation $\widetilde{\mathbf{A}}$ of the stiffness matrix $\mathbf{A}$ is obtained as follows:*

1. *for blocks $\sigma \times \tau \in \mathbb{P}_{near}$, one takes the exact block: $\widetilde{\mathbf{A}}|_{\sigma\times\tau} := \mathbf{A}|_{\sigma\times\tau}$*
2. *for blocks $\sigma \times \tau \in \mathbb{P}_{far}$, one replaces k by its tensor product polynomial approximation k_p on $B_\sigma \times B_\tau$ of degree p, which results in an rank $r = (p+1)^{2d}$ approximation of the matrix block.*

Let us now turn to the question of the error introduced by approximating k by k_p. Indeed, from Lemma 5.1 (actually, Lemma 5.1 discusses only tensor product interpolation in $\mathbb{R}^2$ but analogous results hold for tensor product interpolation in $\mathbb{R}^d$ for any d) we obtain for admissible blocks

$$\|k - k_p\|_{L^\infty(B_\sigma \times B_\tau)} \leq C(\operatorname{dist}(B_\sigma, B_\tau))^{-\alpha} e^{-bp}, \tag{5.8}$$

for some $b > 0$ that depends only on the admissibility parameter η. The key step for this approximation result is the assertion that k is *asymptotically smooth*[3] and that by scaling the boxes B_σ, B_τ to the reference configuration $(-1,1)^d$, one obtains a scaled function $\widehat{k}$ satisfying

$$|\partial^n_{(\xi,\eta)} \widehat{k}(\xi, \eta)| \leq C\gamma^n n! \operatorname{dist}(B_\sigma, B_\tau)^{-\alpha} \qquad n \in \mathbb{N}_0^{2d}$$

for some C, $\gamma > 0$. We observe that a similar mechanism has already been exploited in the proof of Lemma 4.3.
The estimate (5.8) shows exponential convergence for admissible blocks. For quasi-uniform meshes of mesh size h, the distance of two admissible blocks B_σ, B_τ is at least $O(h)$ so that one obtains

$$\|k - k_p\|_{L^\infty(B_\sigma \times B_\tau)} \leq Ch^{-\alpha} e^{-bp}.$$

Recalling that we will select $p = O(|\log h|)$, we see that the factor $h^{-\alpha}$ can be absorbed in the exponentially decaying factor e^{-bp}.

The discussion so far shows that the stiffness matrix for (5.5) can be approximated very well by blockwise low-rank matrices, provided the admissible blocks $\mathbb{P}_{far}$ satisfy (5.7). It remains to see that block partitions can be created in such a way that also the complexity is low.

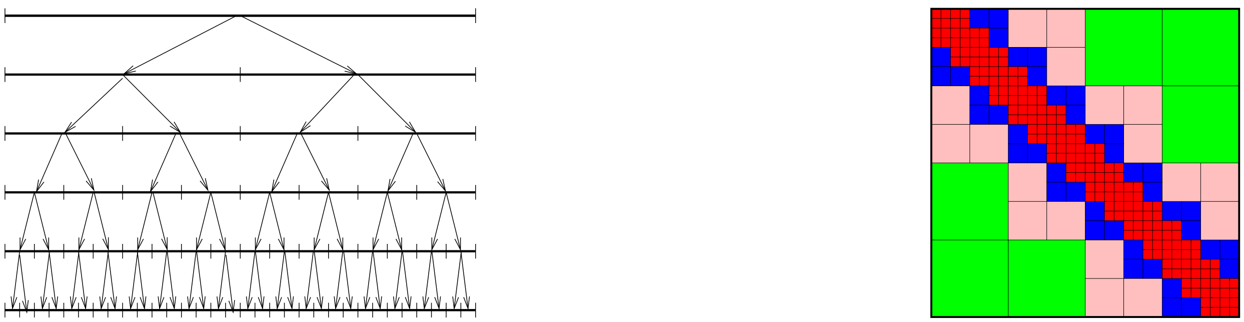


Figure 5.1: Left: organizing the degrees of freedom in a tree structure in 1D. Right: block structure for the 1D example

5.1.3 Creating the block partition $\mathbb{P}$

Clustering the degrees of freedom

Let $(\varphi_i)_{i=1}^N$ be the basis functions, which we assume to have small support (e.g., the classical hat functions). We start by *clustering* the indices $I := \{1, \dots, N\}$ by organizing them in a tree as follows:

1. Associate a "center" x_i with each φ_i, e.g., the barycenter of $\operatorname{supp} \varphi_i$.

2. Create an (infinite) 2^d-tree $\mathbb{T}_{\mathbb{R}^d}$ in $\mathbb{R}^d$ by recursively subdividing an axisparallel box that contains all x_i. See Fig. 5.1 for the binary tree in the 1D situation. The "root", i.e., the top node of the tree, corresponds to a single box and each node of the tree has 2^d sons. We say that the root has *level* 0 and define the level of the nodes inductively by setting, for a node of level ℓ, the level of its sons to be $\ell + 1$.

3. Create a 2^d-tree $\mathbb{T}_I$ for the degrees of freedom by combining those degrees of freedom in a node for which the corresponding x_i are in the same box created in Step 2. (Tie-breaking: for points x_i on the boundary of box, select arbitrarily.) Such a collection is called a *cluster*. The tree is pruned by removing nodes with less than $n_{\text{leaf}} \geq 1$ points, where n_{leaf} is a user-specified, fixed value. These last nodes of the tree are called *leaves*. The collection $I := \{1, \dots, N\}$ is the root of the tree with level 0. The level is defined inductively as follows: For a node σ with level ℓ, its sons have level $\ell + 1$. We denote by $\mathbb{T}_I^{(\ell)} := \{\sigma \in \mathbb{T}_I \mid \operatorname{level}(\sigma) = \ell\}$ the clusters of level ℓ.

For each cluster σ, we let B_σ be a bounding box for σ. For the complexity analysis we will later require the following assumption.

Assumption 5.3

(i) Bounding boxes are shape-regular, i.e., $(\operatorname{diam} B_\sigma)^d \leq C_1 |B_\sigma|$ *for all clusters* $\sigma \in \mathbb{T}_T$.

(ii) Bounding boxes on the same level are comparable in size:

$$\max_{\sigma \in \mathbb{T}_I^{(\ell)}} \operatorname{diam} B_\sigma \leq C_2 \min_{\sigma \in \mathbb{T}^{(\ell)}} \operatorname{diam} B_\sigma \qquad \forall \ell.$$

[3]for the 1D setting $k(x-y) = |x-y|^{-\alpha}$, one readily checks that $k^{(n)}(z) = (-\alpha)(-\alpha-1)\cdots(-\alpha-n)z^{-\alpha-n}$ and one therefore sees $|k^{(n)}(z)| \leq Cn!\gamma^n |z|^{-\alpha-n}$ for suitable γ depending on α.

(iii) finite overlap: for all $x \in \mathbb{R}^d$ and all ℓ there holds $|\{\sigma \in \mathbb{T}^{(\ell)} \,|\, x \in B_\sigma\}| \le C_{\rm ov}$.

Remark. Assumption 5.3 is reasonable: The centers x_i, $i \in \sigma$, are in boxes of the tree $\mathbb{T}_{\mathbb{R}^d}$. Note that these boxes of $\mathbb{T}_{\mathbb{R}^d}$ are shape regular and do not overlap. If one thinks of the φ_i as the classical hat functions, then, the supports of the φ_i will stick out only a little bit over the boundary of the boxes of $\mathbb{T}_{\mathbb{R}^d}$. Hence, the correct bounding boxes for the clusters will also be shape regular and satisfy the overlap condition.

Remark. If the nodes x_i are distributed rather evenly, then the cluster tree $\mathbb{T}_I$ has depth $O(\log N)$.

Matrix partitioning/Block cluster tree

Given the admissibility condition (5.7), one can construct a partition $\mathbb{P} = \mathbb{P}_{far} \cup \mathbb{P}_{near}$ of the index set $I \times I$ with the following with the following recursive algorithm started with `matrix_partition`$(\mathrm{root}, \mathrm{root}, \emptyset, \emptyset)$:

Algorithm 2 (Matrix partition)

function **matrix_partition**$(\sigma, \tau, \mathbb{P}_{far}, \mathbb{P}_{near})$
% input: clusters σ, τ
% output: partition of $\sigma \times \tau$ into admissible and remaining inadmissible blocks
if (σ, τ) satisfies (5.7) **then**
    add (σ, τ) to $\mathbb{P}_{far}$
else if $\mathrm{sons}(\sigma) \neq \emptyset$ **and** $\mathrm{sons}(\tau) \neq \emptyset$ **then**
    for all $(\sigma', \tau') \in \mathrm{sons}(\sigma) \times \mathrm{sons}(\tau)$ **do**
        call matrix_partition $(\sigma', \tau', \mathbb{P}_{far}, \mathbb{P}_{near})$
    end for
else% inadmissible block that cannot be split any further
    add (σ, τ) to $\mathbb{P}_{near}$
end if

The procedure in Algorithm 2 is as follows: starting with the pair $(\mathrm{root}, \mathrm{root})$ the algorithms checks whether a pair (σ, τ) is admissible in the sense of (5.7). If so, then this pair is added to the list $\mathbb{P}_{far}$ of admissible blocks. If not, then recursively all combinations of sons are checked for admissibility. The recursion stops if either admissibility is encountered or least one of σ, τ does not have any sons, in which case the pair (σ, τ) is put into the list $\mathbb{P}_{near}$ of inadmissible pairs.
The storage requirement is very conveniently controlled in terms of the *sparsity constant* of the partition $\mathbb{P}$ given by

$$C_{\rm sp} := \max\{C_{\rm sp}^{\rm row}, C_{\rm sp}^{\rm col}\}, \tag{5.9}$$

with

$$C_{\rm sp}^{\rm row} := \max_{\sigma \in \mathbb{T}_I} |\{\tau \in \mathbb{T}_I \,|\, (\sigma, \tau) \in \mathbb{P}\}|, \qquad C_{\rm sp}^{\rm col} := \max_{\tau \in \mathbb{T}_I} |\{\sigma \in \mathbb{T}_I \,|\, (\sigma, \tau) \in \mathbb{P}\}|. \tag{5.10}$$

Remark. The sparsity constant $C_{\rm sp}$ introduced above for block matrices can be viewed as a generalization of the matrix case:

$$C_{\rm row} := \max_{i=1,\dots,N} |\{j \,|\, A_{ij} \neq 0\}|, \qquad C_{\rm col} := \max_{j=1,\dots,N} |\{i \,|\, A_{ij} \neq 0\}|, \tag{5.11}$$

and $\max\{C_{\text{row}}, C_{\text{col}}\}$ measures the number of non-zero entries per row/column of the matrix A. The constant C_{sp} in (5.9) controls the number of blocks present in a block matrix corresponding to a row cluster σ or a column cluster τ.
The following lemma shows that the sparsity constant allows one to control the complexity of $\mathcal{H}$-matrices.

Lemma 5.4 (complexity). *The storage requirement N_{store} for the $\mathcal{H}$-matrix $\widetilde{\mathbf{A}}$ is*

$$N_{\text{store}} \leq C_{\text{sp}}\,(\text{depth}(\mathbb{T}_I)+1)\max\{n_{\text{leaf}}, r\}2N.$$

Proof. For $(\sigma,\tau) \in \mathbb{P}_{near}$ we have by Alg. 2 that σ is a leaf or τ is a leaf, i.e., $|\sigma| \leq n_{\text{leaf}}$ or $|\tau| \leq n_{\text{leaf}}$. Hence, $|\sigma||\tau| \leq n_{\text{leaf}}(|\sigma|+|\tau|)$. Together with the storage requirement $r(|\sigma|+|\tau|)$ for $(\sigma,\tau) \in \mathbb{P}_{far}$, we get

$$N_{\text{store}} = \sum_{(\sigma,\tau)\in\mathbb{P}_{far}} r(|\sigma|+|\tau|) + \sum_{(\sigma,\tau)\in\mathbb{P}_{near}} |\sigma||\tau| \leq \max\{n_{\text{leaf}}, r\} \sum_{(\sigma,\tau)\in\mathbb{P}} (|\sigma|+|\tau|). \tag{5.12}$$

The σ-part of this sum is estimated as follows:

$$\sum_{(\sigma,\tau)\in\mathbb{P}} |\sigma| = \sum_{\ell=0}^{\text{depth}(\mathbb{T}_I)} \sum_{\sigma\in\mathbb{T}^{(\ell)}} \sum_{\substack{\tau\in\mathbb{T}_I:\\(\sigma,\tau)\in\mathbb{P}}} |\sigma| = \sum_{\ell=0}^{\text{depth}(\mathbb{T}_I)} \sum_{\sigma\in\mathbb{T}^{(\ell)}} |\sigma| \sum_{\substack{\tau\in\mathbb{T}_I:\\(\sigma,\tau)\in\mathbb{P}}} 1 \leq \sum_{\ell=0}^{\text{depth}(\mathbb{T}_I)} \sum_{\sigma\in\mathbb{T}^{(\ell)}} |\sigma| C_{\text{sp}}.$$

We note that for each level ℓ we have

$$\sum_{\sigma\in\mathbb{T}^{(\ell)}} |\sigma| \leq |I| = N$$

so that we obtain

$$\sum_{(\sigma,\tau)\in\mathbb{P}} |\sigma| \leq C_{\text{sp}}\,(\text{depth}(\mathbb{T}_I)+1)\,N.$$

The τ term in (5.12) is handled analogously, and we arrive at the claim. ∎

For quasi-uniform meshes, one has $\text{depth}(\mathbb{T}_I) = O(\log N)$. Additionally, as discussed above, one typically selects $p = O(\log N)$ so that $r = O(\log^{2d} N)$ and the total storage requirement is $N_{\text{store}} = O(N \log^{1+2d} N)$.
As a last step, we show that C_{sp} is indeed bounded. For that, we need a structural assumption on the bounding boxes such as Assumption 5.3.[4]

[4] the proof of Lemma 5.5 follows [Bör17, Lemma 8], where the concept of block cluster tree $\mathbb{T}_{I\times I}$ is employed—we simplified the presentation to avoid that.

Lemma 5.5 (control of $C_{\rm sp}$). *Let Assumption 5.3 be valid. Then the following holds:*

(i) For σ, $\tau \in \mathbb{T}_I^{(\ell)}$ set

$$\mathcal{S}_\sigma^{\rm row} := \{\tau \in \mathbb{T}_I^{(\ell)} \,|\, (\sigma,\tau) \text{ is inadmissible}\},$$
$$\mathcal{S}_\tau^{\rm col} := \{\sigma \in \mathbb{T}_I^{(\ell)} \,|\, (\sigma,\tau) \text{ is inadmissible}\}.$$

Set

$$\widetilde{C}_{\rm sp,inad} := \max_{\sigma\in\mathbb{T}_I} \max\{|\mathcal{S}_\sigma^{\rm row}|, |\mathcal{S}_\sigma^{\rm col}|\}.$$

Then $C_{\rm sp} \le 2^d \widetilde{C}_{\rm sp,inad}$.

(ii) There is a constant $C > 0$ depending only on the constants of Assumption 5.3 and the fact that Alg. 2 is used such that $\widetilde{C}_{\rm sp,inad} \le C$. In particular, an upper bound on the sparsity constant $C_{\rm sp}$ depends only on the constants of Assumption 5.3.

Proof. *Proof of (i):* Given a non-root $(\sigma,\tau) \in \mathbb{P}$, the pair (σ^+,τ^+) of father clusters satisfies $\tau^+ \in \mathcal{S}_{\sigma^+}^{\rm row}$ by the construction in Alg. 2. Let $\operatorname{row}(\sigma) := \{\tau \in \mathbb{T}_I \,|\, (\sigma,\tau) \in \mathbb{P}\}$. Then

$$\operatorname{row}(\sigma) \subset \bigcup_{\tau^+\in\mathcal{S}_{\sigma^+}^{\rm row}} \operatorname{sons}(\tau^+),$$

so that

$$|\operatorname{row}(\sigma)| \le \sum_{\tau^+\in\mathcal{S}_{\sigma^+}^{\rm row}} |\operatorname{sons}(\tau^+)| \le 2^d \sum_{\tau^+\in\mathcal{S}_{\sigma^+}^{\rm row}} 1 \le 2^d |\mathcal{S}_{\sigma^+}^{\rm row}|.$$

This implies $C_{\rm sp}^{\rm row} \le 2^d \widetilde{C}_{sp,inad}$. Analogously, $C_{\rm sp}^{\rm col} \le 2^d \widetilde{C}_{sp,inad}$. This implies the statement.
Proof of (ii): Let $\sigma \in \mathbb{T}_I$ and $\tau \in \mathcal{S}_\sigma^{\rm row}$. By definition of $\mathcal{S}_\sigma^{\rm row}$ we have $\operatorname{level}(\sigma) = \operatorname{level}(\tau)$. By Assumption 5.3,(ii) we have $\operatorname{diam} B_\sigma \sim \operatorname{diam} B_\tau$. Also, because the pair (σ,τ) is inadmissible, we have

$$\operatorname{dist}(B_\sigma, B_\tau) \overset{\text{inadmiss.}}{\le} \eta^{-1} \max\{\operatorname{diam} B_\sigma, \operatorname{diam} B_\tau\} \overset{\operatorname{diam} B_\sigma \sim \operatorname{diam} B_\tau}{\lesssim} \operatorname{diam} B_\sigma.$$

By the triangle inequality, we conclude the existence of $c > 0$ such that

$$B_\tau \subset B_{c \operatorname{diam} B_\sigma}(m_\sigma), \tag{5.13}$$

where m_σ is an arbitrary point in B_σ (e.g., the barycenter). Hence, we get

$$\begin{aligned}
(\operatorname{diam} B_\sigma)^d |\mathcal{S}_\sigma^{\rm row}| &\overset{\operatorname{diam} B_\sigma \sim \operatorname{diam} B_\tau}{\lesssim} \sum_{\tau\in\mathcal{S}_\sigma^{\rm row}} (\operatorname{diam} B_\tau)^d \overset{\text{A. 5.3,(i)}}{\lesssim} \sum_{\tau\in\mathcal{S}_\sigma^{\rm row}} |B_\tau| \\
&\overset{(5.13)}{\lesssim} \int_{B_{c \operatorname{diam} B_\sigma}(m_\sigma)} \sum_{\tau\in\mathcal{S}_\sigma^{\rm row}} \chi_{B_\tau}(x)\, dx \\
&\overset{\text{A. 5.3,(iii)}}{\lesssim} C_{ov} \int_{B_{c \operatorname{diam} B_\sigma}(m_\sigma)} dx \lesssim C_{ov} (\operatorname{diam} B_\sigma)^d.
\end{aligned}$$

This shows $|\mathcal{S}_\sigma^{\mathrm{row}}| \lesssim C_{ov}$ for any $\sigma \in \mathbb{T}_I$. An analogous calculation shows $|\mathcal{S}_\sigma^{\mathrm{col}}| \lesssim C_{ov}$ for any $\sigma \in \mathbb{T}_I$. ■

Chapter 6

FEM for the Extension Problem - Spectral Fractional Laplacian

In the second chapter, we established that the integral fractional Laplacian and the spectral fractional Laplacian are indeed different operators. Therefore, deriving a numerical approximation to the spectral fractional Laplacian is also of interest.
Recalling the definition $(-\Delta)^s_\sigma u := \sum \lambda_k^s u_k \varphi_k$ with eigenvalues λ_k and eigenfunctions φ_k of the Dirichlet-Laplacian $(-\Delta)$ and the Fourier coefficients $u_k = \int u\varphi_k$, one can see that deriving a numerical method directly is rather hard.
Fortunately, Theorem 2.17, provides us with a different interpretation by

$$\lim_{y\to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -d_s(-\Delta)^s_\sigma u,$$

i.e., as Neumann-data of the solution $\mathcal{U}$ of the Caffarelli-Silvestre extension problem

$$\begin{aligned} \operatorname{div}(y^\alpha \nabla \mathcal{U}(x,y)) &= 0 && \text{in } \Omega \times (0,\infty) \\ \mathcal{U} &= 0 && \text{on } \partial\Omega \times (0,\infty) \\ \mathcal{U}(x,0) &= u(x) && \text{in } \Omega, \end{aligned}$$

where $\alpha = 1 - 2s \in (-1,1)$.
As this is an–albeit degenerate–elliptic problem, we can propose a finite element formulation to derive a numerical approximation.

For simplicity, from now on, we assume that Ω is convex. In order to shorten notation, we write $\mathcal{C} := \Omega \times (0,\infty)$ in the following.

As we want to derive a weak formulation for the extension problem, we need spaces with zero lateral boundary conditions, defined by

$$\mathring{H}^1(\mathcal{C}; y^\alpha) := \{w \in H^1(\mathcal{C}; y^\alpha) \ : \ w = 0 \text{ on } \partial\Omega \times (0,\infty)\}.$$

On this space, similarly to the non-weighted Sobolev space, a weighted Poincaré inequality holds

$$\|w\|_{L^2(\mathcal{C};y^\alpha)} \lesssim \|\nabla w\|_{L^2(\mathcal{C};y^\alpha)} \qquad \forall w \in \mathring{H}^1(\mathcal{C}; y^\alpha). \tag{6.1}$$

Moreover, we have a trace inequality for the boundary at $y = 0$. Denoting the trace onto $\Omega \times \{0\}$ by tr_Ω, we have that $\operatorname{tr}_\Omega \mathring{H}^1(\mathcal{C}; y^\alpha) = \widetilde{H}^s(\Omega)$ as well as

$$\|\operatorname{tr}_\Omega w\|_{\widetilde{H}^s(\Omega)} \lesssim \|w\|_{H^1(\mathcal{C};y^\alpha)}.$$

6.1 Weak Formulation

We multiply equation (2.29) by a test function $w \in \mathring{H}^1(\mathcal{C}; y^\alpha)$ and integrate by parts to obtain

$$\begin{aligned} 0 &= -\int_{\mathcal{C}} \operatorname{div}(y^\alpha \nabla \mathcal{U}) \cdot w dx dy = \int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla w dx dy - \int_{\partial \mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot n w \, dx dy \\ &= \int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla w \, dx dy + \int_{\Omega \times \{0\}} y^\alpha \partial_y \mathcal{U} \operatorname{tr}_\Omega w \, dx dy, \end{aligned}$$

where we used that the test function vanishes on $\partial\Omega \times (0, \infty)$. Using Theorem 2.17 as well as the fractional PDE, we obtain

$$\int_{\Omega \times \{0\}} y^\alpha \partial_y \mathcal{U} w \, dx dy = \langle -d_s(-\Delta)^s_\sigma u, \operatorname{tr}_\Omega w \rangle = -d_s \langle f, \operatorname{tr}_\Omega w \rangle.$$

Inserting this in the equation above gives the weak formulation of finding $\mathcal{U} \in \mathring{H}^1(\mathcal{C}; y^\alpha)$ such that

$$a_{\mathcal{C}}(\mathcal{U}, w) := \int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla w \, dx dy = d_s \langle f, \operatorname{tr}_\Omega w \rangle \qquad \forall w \in \mathring{H}^1(\mathcal{C}; y^\alpha). \tag{6.2}$$

By (6.1), the symmetric bilinear form $a_{\mathcal{C}}(\cdot, \cdot)$ actually induces a norm equivalent to $\|\cdot\|_{H^1(\mathcal{C};y^\alpha)}$. Hence, (6.2) is uniquely solvable by the Lax-Milgram Lemma.

6.2 Galerkin method

The Galerkin method for (6.2) is as follows: Given a closed subspace $V_N \subset \mathring{H}^1(\mathcal{C}; y^\alpha)$, the Galerkin approximation is: Find $\mathcal{U}_N \in V_N$ such that

$$a_{\mathcal{C}}(\mathcal{U}_N, w_N) = d_s \langle f, \operatorname{tr}_\Omega w_N \rangle \qquad \forall w_N \in V_N. \tag{6.3}$$

The Céa Lemma is applicable. We obtain existence, uniqueness, and quasi-optimality: there is $C > 0$ depending only on Ω and s such that

$$\|\mathcal{U} - \mathcal{U}_N\|_{H^1(\mathcal{C};y^\alpha)} \leq C \inf_{v \in V_N} \|\mathcal{U} - v\|_{H^1(\mathcal{C};y^\alpha)}. \tag{6.4}$$

Below, we will select V_N as a space of piecewise polynomials on certain meshes. Key to understanding the mesh layout is the regularity of the solution $\mathcal{U}$, which we study in the following section.

6.3 Regularity

Before studying a finite element method applied to the extension problem, we need to discuss the regularity of solutions of the extension problem. The main question what regularity one may expect in the extended variable y.

Sobolev spaces in terms of series representations

Let $(\varphi_k, \lambda_k) \in H_0^1(\Omega)\times\mathbb{R}$ be the eigenpairs of the Dirichlet Laplacian $(-\Delta)$ normalized to $\|\varphi_k\|_{L^2(\Omega)} = 1$ as introduced in Section 2.5.3. We note that the eigenpairs may be assumed to be sorted in ascending order, i.e.,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty. \tag{6.5}$$

We have for functions u expanded as $u(x) = \sum_k u_k\varphi_k(x)$ (i.e., $u_k = (u, \varphi_k)_{L^2(\Omega)}$)

$$\|u\|^2_{L^2(\Omega)} = \sum_k |u_k|^2, \tag{6.6}$$

$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_k \lambda_k |u_k|^2, \tag{6.7}$$

$$\|\Delta u\|^2_{L^2(\Omega)} = \sum_k \lambda_k^2 |u_k|^2. \tag{6.8}$$

(6.6), (6.7) are in fact special cases of the following characterization of the norm $\|\cdot\|_{\widetilde{H}^s(\Omega)}$:

$$\|u\|^2_{\widetilde{H}^s(\Omega)} \sim \sum_k \lambda_k^s |u_k|^2, \qquad s \in (0,1); \tag{6.9}$$

that is, the space $\widetilde{H}^s(\Omega)$, $s \in (0,1)$, could alternatively be characterized as the subspace of $L^2(\Omega)$ such that the right-hand side of (6.9) is finite[1].

Exercise 6.1 *By Section 2.5.1, the space $H^{-s}(\Omega)$ (equipped with $\|\cdot\|_{H^{-s}(\Omega)}$) is defined as the dual space of $\widetilde{H}^s(\Omega)$. Show for $s \in (0,1)$ and $f \in L^2(\Omega)$ with expansion $f = \sum_k f_k\varphi_k$ that*

$$\|f\|^2_{H^{-s}(\Omega)} \sim \sum_k \lambda_k^{-s} |f_k|^2.$$

Remark 6.2 *For $\sigma \geq 0$ one can define spaces $\mathbb{H}^\sigma(\Omega) := \{u = \sum_k u_k\varphi_k \in L^2(\Omega) \,:\, \|u\|^2_{\mathbb{H}^\sigma(\Omega)} := \sum_k u_k^2\lambda_k^\sigma < \infty\}$. One then defines the spaces $\mathbb{H}^{-\sigma}(\Omega) := (\mathbb{H}^\sigma(\Omega))'$ by duality. For smooth Ω and $\sigma \in \mathbb{N}_0$ one has (see, e.g., [Tho84, Lemma 3.1])*

$$\mathbb{H}^\sigma(\Omega) = \{u \in H^\sigma(\Omega) \,:\, \Delta^j u|_{\partial\Omega} = 0 \quad 0 \leq j < \sigma/2\}. \tag{6.10}$$

Hence, e.g., $H_0^1(\Omega) = \mathbb{H}^1(\Omega)$ and in fact $\widetilde{H}^s(\Omega) = \mathbb{H}^s(\Omega)$ for $0 \leq s \leq 1$. For large σ, however, the spaces $\mathbb{H}^\sigma(\Omega)$ and $\widetilde{H}^\sigma(\Omega)$ differ as illustrated in the following Example 6.3.

Example 6.3 *Consider $\Omega = (0,\pi)$ with (up to scaling) $\varphi_k(x) = \sin kx$ and $\lambda_k = k^2$.*

(a) The Gibbs phenomenon shows that smoothness of u does not necessarily imply rapid decay of the Fourier coefficients: The Fourier series of the function $u(x) = 1$ on $\Omega = (0,\pi)$ is $u(x) = \sum_k k^{-1}(1-(-1)^k)\sin(kx)$.

[1] The right-hand side of (6.9) is the square of the interpolation norm between $\sum_k |u_k|^2$ and $\sum_k \lambda_k^1 |u_k|^2$; the characterization (6.9) then follows from the fact that the spaces $\widetilde{H}^s$ are interpolation spaces between L^2 and H_0^1

(b) If $\sum_k u_k^2 \lambda_k^\sigma < \infty$, then clearly $u \in H^\sigma(0,\pi)$. Additionally, $u^{(2j)}(0) = 0$ for $0 \le j < \sigma/2$ since

$$|u^{(2j)}(0)| = |\sum_k u_k k^{2j}| \le \sqrt{\sum_k u_k^2 k^{2\sigma}} \sqrt{\sum_k k^{4(j-\sigma/2)}} \sim \underbrace{\sqrt{\sum_k u_k^2 \lambda_k^\sigma}}_{<\infty \, by\ assump.} \sqrt{\sum_k k^{4(j-\sigma/2)}}.$$

Since j, σ are integers, the assumption $j < \sigma/2$ implies $\sigma/2 - j \le -1/2$. Hence, the second sum converges.

(c) Let $u \in H^\sigma(\Omega)$ and $u^{(2j)}|_{\partial\Omega} = 0$ for $0 \le j < \sigma/2$. We show that u can be extended to a 2π-periodic function such that a) it is odd (i.e., the Fourier series has the form $u = \sum_k u_k \sin(kx)$) and b) it is in H^σ. For such a function, standard Fourier techniques (i.e., integration by parts in the computation of the Fourier coefficients) show $\sum_k |u_k|^2 \lambda_k^\sigma < \infty$. We illustrate the construction of such an extension near $x = 0$. The odd extension $\mathcal{E}u$ is given by $\mathcal{E}u(-x) := -u(x)$ for $x > 0$. For smooth functions u, the odd extension satisfies $(\mathcal{E}u)^{(2j+1)}(0-) = u^{(2j+1)}(0+)$ but $(\mathcal{E}u)^{(2j)}(0-) = -u^{(2j)}(0+)$. By assumption, $u^{(2j)}(0) = 0$ so that the odd extension $\mathcal{E}u$ is as smooth as u.

Series representations of the solution u

As a first step, we note that solutions of $(-\Delta)_\sigma^s u = f$ can be written as $u = \sum u_k \varphi_k$, where the Fourier coefficients of u and f satisfy the relation

$$u_k = \lambda_k^{-s} f_k \qquad \text{with } u_k = \int u\varphi_k, \text{ and } f_k = \int f\varphi_k. \tag{6.11}$$

Remark 6.4 *(6.11) shows a "shift theorem":*

$$\sum_k |u_k|^2 \lambda_k^{s+\varepsilon} = \sum_k |f_k|^2 \lambda_k^{-s+\varepsilon}$$

for any $\varepsilon \ge 0$ such that the right-hand side is finite. In the notation of Remark 6.2, the shift theorem takes the form $f \in \mathbb{H}^{-s+\varepsilon}(\Omega) \Longrightarrow u \in \mathbb{H}^{s+\varepsilon}(\Omega)$ for any $\varepsilon \ge 0$. That is: decay of the coefficients f_k ("regularity") implies decay of the coefficients u_k ("regularity").

Series representations of the extension $\mathcal{U}$

As observed in (2.31) in Section 2.5.4 the representation of the extension $\mathcal{U}$ is given explicitly as

$$\mathcal{U}(x,y) = \sum_{k=1}^\infty u_k \varphi_k(x)\psi_k(y), \qquad \psi_k(y) = \psi(\sqrt{\lambda_k} y), \qquad \psi(z) = c_s z^s K_s(z), \tag{6.12}$$

where K_s is a modified Bessel function. Bessel functions are well-studied objects. We have:

(i) K_s is positive for $s > -1$, [AS64, Chapter 9.6].

(ii) $K_s(z) = K_{-s}(z)$ for $s \in \mathbb{R}$, [AS64, Chap. 9.6].

(iii) $(z^s K_s(z))' = -z^s K_{1-s}(z)$ for $s \in \mathbb{R}$, [AS64, formula (9.6.28)].

(iv) $\lim_{z\to 0^+} K_s(z) z^s = (1/2)^{1-s}\Gamma(s)$ for $s > 0$, [AS64, (9.6.9)].

(v) For $z > 0$, we have the $z^{\min\{s,1/2\}} e^z K_s(z)$ is decreasing, [MS01, Thm. 5].

(vi) K_s solves the differential equation

$$z^2 y'' + zy' - (z^2 + s^2) y = 0.$$

Using properties (ii)–(iv) one can show

$$\psi_k'(y) \sim y^{-\alpha}, \qquad \psi_k''(y) \sim y^{-\alpha-1} \quad \text{for } y \to 0^+.$$

Additionally, property (v) shows that ψ_k decays exponentially for large arguments. Using properties (i)–(v) one can show the following:

Lemma 6.5 ([BMN+19, Lemma 5]). *For the function $\psi(z) = z^s K_s(z)$ there holds for $\delta > -1$, $\beta > -1 - 4s$, $\gamma < 2$ and all $\ell \in \mathbb{N}_0$*

$$\int_{z=0}^{\infty} z^{\delta} e^{\gamma z} |\psi(z)|^2 \, dz < \infty, \tag{6.13}$$

$$\int_{z=0}^{\infty} z^{\beta + 2\ell} e^{\gamma z} |\psi^{(\ell)}(z)|^2 \, dz < \infty. \tag{6.14}$$

Proof. See [BMN+19, Lemma 5] for details. The results of [BMN+19, Lemma 5] are sharper in that the dependence on ℓ in (6.14) is made explicit and shows that $\mathcal{U}$ is an analytic function of y. ∎

Employing (6.12) the properties of ψ given in Lemma 6.5 transfer to $\partial_y \mathcal{U}$ and $\partial_y^2 \mathcal{U}$. To that end, recall from (6.5) that $\lambda_k \geq \lambda_1 > 0$ and compute for $0 \leq \gamma < 1$

$$\begin{aligned}
\|e^{\gamma y \sqrt{\lambda_1}} \partial_y \mathcal{U}\|^2_{L^2(\mathcal{C}; y^\alpha)} &= \int_{\Omega \times (0,\infty)} y^\alpha e^{2\gamma y \sqrt{\lambda_1}} |\partial_y \mathcal{U}(x,y)|^2 \, dy \, dx \\
&= \sum_k |u_k|^2 \int_{y=0}^{\infty} e^{2\gamma y \sqrt{\lambda_1}} y^\alpha |\sqrt{\lambda_k}|^2 |\psi'(\sqrt{\lambda_k} y)|^2 \, dy \\
&\overset{z = \sqrt{\lambda_k} y}{=} \sum_k |u_k|^2 \lambda_k^{1-1/2} \int_{z=0}^{\infty} e^{2\gamma z \sqrt{\lambda_1/\lambda_k}} (z/\sqrt{\lambda_k})^\alpha |\psi'(z)|^2 \, dz \\
&\leq \sum_k |u_k|^2 \lambda_k^s \int_{z=0}^{\infty} e^{2\gamma z} z^\alpha |\psi'(z)|^2 \, dz \overset{\text{L. 6.5}}{\leq} C \sum_k |u_k|^2 \lambda_k^s \sim \|u\|^2_{\widetilde{H}^s(\Omega)} \lesssim \|f\|^2_{H^{-s}(\Omega)}.
\end{aligned}$$

By a similar calculation, we have for $\beta > 1 + 2\alpha$

$$\begin{aligned}
\int_{\Omega \times (0,\infty)} e^{2\gamma \sqrt{\lambda_1} y} y^\beta |\partial_y^2 \mathcal{U}(x,y)|^2 \, dx \, dy &= \sum_k |u_k|^2 |\sqrt{\lambda_k}|^4 \int_{y=0}^{\infty} e^{2\gamma \sqrt{\lambda_1} y} y^\beta |\psi''(\sqrt{\lambda_k} y)|^2 \, dy \\
&\overset{\beta > 1 + 2\alpha; \text{L. 6.5}}{\leq} C_\beta \sum_k |u_k|^2 \lambda_k^{2 - 1/2 - \beta/2}.
\end{aligned}$$

Selecting β of the form $\beta = \alpha + 2 - \varepsilon$ for $\varepsilon \in [0, 1-\alpha)$, we arrive at

$$\int_{\Omega\times(0,\infty)} e^{2\gamma\sqrt{\lambda_1}y} y^\beta |\partial_y^2 \mathcal{U}(x,y)|^2\, dx\, dy \le C \sum_k |u_k|^2 \lambda_k^{s+\varepsilon} \le C\|f\|^2_{H^{-s+\varepsilon}(\Omega)}. \tag{6.15}$$

(if $-s+\varepsilon \le 0$)

Remark. The use of the weight y^β in (6.15) is crucial for the case $s \le 1/4$: Since the asymptotics $\psi''(z) \sim z^{-\alpha-1}$ (as $z \to 0^+$) is sharp, one sees that

$$\int_{y=0}^{1} z^\beta |\psi''(z)|^2\, dz < \infty$$

if and only if $\beta > 1+2\alpha$. Since $1+2\alpha > \alpha$, we see that we should expect $\partial_y^2\mathcal{U} \in L^2(\mathcal{C}; y^\beta)\backslash L^2(\mathcal{C}; y^\alpha)$. The following theorem can be shown by calculations similar to the above ones:

Theorem 6.6. *Let $s \in (0,1)$ and $f \in \widetilde{H}^{1-s}(\Omega)$. Let $\mathcal{U} \in \mathring{H}^1(\mathcal{C}; y^\alpha)$ solve (6.2). Let $\widetilde{\varepsilon} \in [0,s)$, $\varepsilon \in [0, 1+s)$, $\varepsilon' \in [0, 1-s)$, and $\gamma \in [0,2)$. Then:*

$$\begin{aligned}
\|e^{\gamma\sqrt{\lambda_1}y}\nabla\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha-2\min\{\widetilde{\varepsilon},\varepsilon'\}})} &\lesssim \|f\|_{H^{-s+\min\{\widetilde{\varepsilon},\varepsilon'\}}(\Omega)},\\
\|e^{\gamma\sqrt{\lambda_1}y}\Delta_x\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha-2\varepsilon'})} &\lesssim \|f\|_{\widetilde{H}^{1-s+\varepsilon'}(\Omega)},\\
\|e^{\gamma\sqrt{\lambda_1}y}\partial_y^2\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2-2\widetilde{\varepsilon}})} &\lesssim \|f\|_{H^{-s+\widetilde{\varepsilon}}(\Omega)},\\
\|e^{\gamma\sqrt{\lambda_1}y}D_x\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2-2\varepsilon})} &\lesssim \begin{cases} \|f\|_{H^{-s+\varepsilon}(\Omega)} & \text{if } -s+\varepsilon \le 0\\ \|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)} & \text{if } -s+\varepsilon > 0,\end{cases}\\
\|e^{\gamma\sqrt{\lambda_1}y}D_x\partial_y^2\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+4-2\varepsilon)})} &\lesssim \begin{cases} \|f\|_{H^{-s+\varepsilon}(\Omega)} & \text{if } -s+\varepsilon \le 0\\ \|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)} & \text{if } -s+\varepsilon > 0,\end{cases}\\
\|e^{\gamma\sqrt{\lambda_1}y}\partial_y\Delta_x\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2-2\varepsilon})} &\lesssim \|f\|_{\widetilde{H}^{1-s+\varepsilon}(\Omega)}.
\end{aligned}$$

Proof. See [BMN+19, Thm. 1] (and [NOS15]). ■

Since Ω is assumed to be convex, we obtain a classical shift theorem, i.e., $\|v\|_{H^2(\Omega)} \lesssim \|\Delta v\|_{L^2(\Omega)}$ for $v \in H^2(\Omega)\cap H_0^1(\Omega)$. Therefore, the second statement of Theorem 6.6 provides

$$\|e^{\gamma\sqrt{\lambda_1}y}D_x^2\mathcal{U}\|^2_{L^2(\mathcal{C};y^\alpha)} \lesssim \|f\|^2_{\widetilde{H}^{1-s}(\Omega)} \tag{6.16}$$

for all second derivatives in x. A bound in the same norm holds for all mixed second order derivatives in x and y by the fourth statement of the previous theorem. However, looking at the result for the second derivative in y-direction shows much worse regularity that has to be compensated by a higher power y^β.

6.4 Galerkin discretization

(6.4) shows that it suffices to select the space V_N such that the solution $\mathcal{U}$ can be approximated well. The regularity assertions of Theorem 6.6 shows the following:

- The solution $\mathcal{U}$ decays exponentially as $y \to \infty$. This suggests that one may truncate the cylinder $\mathcal{C}$ to a finite cylinder $\mathcal{C}_\mathcal{Y} := \Omega \times (0, \mathcal{Y})$ for some $\mathcal{Y}$ and merely discretize $\mathcal{C}_\mathcal{Y}$.
- The solution has an algebraic singularity at $y = 0$. This suggests the use of meshes that are μ-graded in the direction of y.
- The transversal derivative $D_x^2 \mathcal{U}$ are controlled. This suggests that no refinement in the x-direction is required.[2]

A concrete way to create meshes and spaces is by (tensor) product constructions, i.e., given a regular, shape-regular mesh $\mathcal{T}_\Omega$ on Ω and a 1D mesh $\mathcal{T}_\mathcal{Y}$ on $(0, \mathcal{Y})$, the mesh on $\mathcal{C}_\mathcal{Y}$ is given by

$$\mathcal{T}_{\mathcal{C}_\mathcal{Y}} := \mathcal{T}_\Omega \times \mathcal{T}_\mathcal{Y} = \{K \times I \,:\, K \in \mathcal{T}_\Omega,\ \ I \in \mathcal{T}_\mathcal{Y}\}.$$

The finite element space on $\mathcal{T}_{\mathcal{C}_\mathcal{Y}}$ is taken as

$$\begin{aligned}
\mathcal{P}^1_\mathcal{Y}(\mathcal{T}_\mathcal{Y}) &:= \{u \in \mathcal{P}^1(\mathcal{T}_\mathcal{Y}) \,:\, u(\mathcal{Y}) = 0\}, \\
\mathcal{Q}^1(\mathcal{T}_{\mathcal{C}_\mathcal{Y}}) &:= \mathcal{P}^1_0(\mathcal{T}_\Omega) \otimes \mathcal{P}^1_\mathcal{Y}(\mathcal{T}_\mathcal{Y}) \\
&\quad = \{v \in C(\overline{\mathcal{C}_\mathcal{Y}}) \,:\, u|_{K \times I} \in \operatorname{span}\{x^\alpha y^j \,:\, |\alpha| \le 1, 0 \le j \le 1\}, \quad u(\cdot, \mathcal{Y}) = 0, \\
&\qquad\quad u(x,y) = 0 \quad (x,y) \in \partial\Omega \times (0, \mathcal{Y})\}.
\end{aligned}$$

The above discussion of the regularity suggests to employ the following meshes $\mathcal{T}_\Omega$ and $\mathcal{T}_\mathcal{Y}$:

Assumption 6.7
- *$\mathcal{T}_\Omega$ is a regular, shape-regular mesh on Ω; for simplicity, we assume it be a quasi-uniform mesh with mesh size h.*
- *Let $\mathcal{Y} > 1$. Let $\mathcal{T}_\mathcal{Y} = \{(x_i, x_{i+1}) \,|\, i = 0, \ldots, M-1\}$ be a mesh on $(0, \mathcal{Y})$ that is μ-graded on $(0,1)$ and quasi-uniform on $(1, \mathcal{Y})$. That is, the mesh $\mathcal{T}_\mathcal{Y}$ satisfies for all $I \in \mathcal{T}_\mathcal{Y}$*

$$\begin{aligned}
&h_I \sim h \operatorname{dist}(I, 0)^{1-1/\mu} && \text{if } 0 \notin \overline{I} \text{ and } I \subset (0,1), \\
&h_I \sim h^\mu && \text{if } 0 \in \overline{I}, \\
&h_I \sim h \text{ if } I \subset (1, \mathcal{Y}).
\end{aligned}$$

The approximation is conveniently achieved by tensor product approximation. To that end, let $\Pi^x : C^\infty(\overline{\Omega}) \to \mathcal{P}^1_0(\mathcal{T}_\Omega)$ and $\Pi^y : C^\infty([0, \mathcal{Y}]) \to \mathcal{P}^1_\mathcal{Y}(\mathcal{T}_\mathcal{Y})$ be two linear approximation operators. Their domains of definition will be enlarged shortly. The approximation of $\mathcal{U}$ is then taken to be $(\Pi^x \circ \Pi^y)\mathcal{U}$.

Exercise 6.8 *Let Π^x, Π^y be bounded on L^2. Show that $\Pi^x \circ \Pi^y \mathcal{U} \in \mathcal{Q}^1(\mathcal{T}_{\mathcal{C}_\mathcal{Y}})$. Show that $\Pi^x \circ \Pi^y \mathcal{U} = \Pi^y \circ \Pi^x \mathcal{U}$.*

The stability of the operators Π^x and Π^y plays a key role. We have:

[2] This is related to the fact that Ω is convex and that we consider piecewise linear approximation; for higher order approximations or non-convex geometries, mesh refinement towards corners and edges of Ω might be necessary.

Lemma 6.9. *Let Π^x be stable in L^2 and H^1, i.e.,*

$$\|\Pi^x u\|_{L^2(\Omega)} \le C_{L^2}\|u\|_{L^2(\Omega)}, \qquad |\Pi^x u|_{H^1(\Omega)} \le C_{H^1}|u|_{H^1(\Omega)}. \tag{6.17}$$

Then,

$$\|\nabla(\mathcal{U} - \Pi^x \circ \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} \le \|\nabla(\mathcal{U} - \Pi^x \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} + \sqrt{C_{L^2}^2 + C_{H^1}^2}\|\nabla(\mathcal{U} - \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)}.$$

Proof. We write

$$\mathcal{U} - \Pi^x \circ \Pi^y \mathcal{U} = \mathcal{U} - \Pi^x \mathcal{U} + \Pi^x(\mathcal{U} - \Pi^y \mathcal{U})$$

so that

$$\|\nabla(\mathcal{U} - \Pi^x \circ \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} \le \|\nabla(\mathcal{U} - \Pi^x \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} + \|\nabla \Pi^x(\mathcal{U} - \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)}.$$

We exploit the product structure of the norm $\|\cdot\|_{L^2(\mathcal{C};y^\alpha)}$ as well as the fact that ∂_y and Π^x commute to estimate

$$\|\partial_y \Pi^x(\mathcal{U} - \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} \le \|\Pi^x \partial_y(\mathcal{U} - \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} \le C_{L^2}\|\partial_y(\mathcal{U} - \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)}, \tag{6.18}$$

$$\|\nabla_x \Pi^x(\mathcal{U} - \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} \le C_{H^1}\|\nabla_x(\mathcal{U} - \Pi^y \mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)}. \tag{6.19}$$

Summing the squares of (6.18) and (6.19) yields the result. ■

Lemma 6.9 shows that we have to construct an operator Π^x that is stable in L^2 and H^1. We also require this operator to have the approximation properties

$$\|u - \Pi^x u\|_{L^2(\Omega)} \le Ch\|\nabla_x u\|_{L^2(\Omega)}, \qquad \|\nabla(u - \Pi^x u)\|_{L^2(\Omega)} \le Ch\|D_x^2 u\|_{L^2(\Omega)}. \tag{6.20}$$

An option is the Scott-Zhang operator discussed in Section 3.1.5. In the present case of a quasi-uniform mesh, even the L^2-projection satisfies this.

Exercise 6.10 *Let $\mathcal{T}_\Omega$ be a quasi-uniform mesh and Ω be convex. Then the L^2-projection Π^{L^2} onto $\mathcal{P}_0^1(\mathcal{T}_\Omega)$ satisfies*

$$\|\Pi^{L^2} u\|_{H^1(\Omega)} \le C\|u\|_{H^1(\Omega)}.$$

Hint: *use convexity of Ω to prove with a duality argument that $\|u - \Pi^{H^1} u\|_{L^2} \le Ch\|u\|_{H^1}$, where Π^{H^1} is the H^1-projection onto $\mathcal{P}_0^1(\mathcal{T}_\Omega)$. Use the polynomial inverse estimate $\|v\|_{H^1(\Omega)} \le Ch^{-1}\|v\|_{L^2(\Omega)}$ for all $v \in \mathcal{P}_0^1(\mathcal{T}_\Omega)$.*

The operator Π^y is a 1D operator. We define it elementwise as follows:

- $(\Pi^y u)|_I$ is the nodal interpolant of u if I does not touch 0.
- $(\Pi^y u)|_I$ interpolates in the midpoint $x_1/2$ and the right endpoint x_1 of the first element $I = (0, x_1)$.

- On the terminal element (x_{M-1}, x_M) ensure that $(\Pi^y u)(\mathcal{Y}) = 0$ by subtracting a linear function from the nodal interpolant:

$$
\begin{aligned}
(\Pi^y u)(y) &= u(x_{M-1})\frac{x_M - x}{x_M - x_{M-1}} + u(x_M)\frac{x - x_{M-1}}{x_M - x_{M-1}} - u(x_M)\frac{x - x_{M-1}}{x_M - x_{M-1}} \\
&= u(x_{M-1})\frac{x_M - x}{x_M - x_{M-1}}.
\end{aligned}
$$

For the operator Π^y, we have:

Lemma 6.11 ([BMN$^+$19, Lemma 9]). *Let $\alpha \in (-1,1)$ and let $\theta \in [0,1)$ and $\gamma > 0$. Let $\mathcal{Y} > 1$. Assume that the grading exponent μ satisfies*

$$\mu\theta \geq 1.$$

Assume that $u \in C^\infty(0,\infty)$ satisfies

$$\lim_{y\to\infty} |u(y)| = 0, \qquad \lim_{y\to\infty} |u'(y)| = 0,$$

and assume that the following right-hand sides are finite. Then for $j \in \{0,1\}$:

$$\|u - \Pi^y u\|_{L^2((0,\mathcal{Y});y^\alpha)} \leq C(h + \mathcal{Y}^{\alpha/2+(1-\theta)}e^{-\gamma\mathcal{Y}})\|e^{\gamma y}\partial_y u\|_{L^2((0,\infty);y^{\alpha+2(1-\theta)})}; \tag{6.21}$$

$$\|\partial_y^j(u - \Pi^y u)\|_{L^2((0,\mathcal{Y});y^\alpha)} \leq C(h^{2-j} + h^{1/2-j}h^{-j}\mathcal{Y}^{\alpha/2+2(1-\theta)}e^{-\gamma\mathcal{Y}})\|e^{\gamma y}\partial_y^2 u\|_{L^2((0,\infty);y^{\alpha+2(1-\theta)})}. \tag{6.22}$$

The number of elements is $O(h^{-1}(1+\mathcal{Y}))$.

Proof. In the interest of brevity, we will sketch the proof of (6.21) only. The proof of (6.22) proceeds along similar lines; see [BMN$^+$19, Lemma 5].
In principle, the proof proceeds by studying the error on each element. To that end, one distinguishes 4 types of elements: the first element $(0, x_1)$, the elements in $(0,1)$, the elements in $(1,\mathcal{Y})$, and the terminal element. For each element, the approximation result is obtained by a scaling argument. Details can be found in [BMN$^+$19, Lemma 9].
The first element $(0, x_1)$*:* As often when dealing with weighted norms, a Hardy inequality is the technical tool. We have by [DL93, Chap. 2, Thm. 3.1] the *Hardy inequality*

$$\int_0^1 y^\alpha \left|\int_y^1 v(t)\,dt\right|^2 dy \leq (\alpha+1)^{-2}\int_0^1 y^{\alpha+2}|v(t)|^2\,dt. \tag{6.23}$$

Let $\widehat{u}(\xi) := u(\xi x_1)$ be the pull-back of $u|_{(0,x_1)}$ to $(0,1)$ and $\widehat{\Pi}^y\widehat{u}$ be the interpolant of $\widehat{u}$ in the points

$1/2$ and 1. Then $\widehat{u}(\xi) - \widehat{\Pi}^y \widehat{u}(\xi) = \int_\xi^1 (\widehat{u} - \widehat{\Pi^y \widehat{u}})'(t)\,dt$. Hence,

$$\begin{aligned}
\int_{\xi=0}^1 \xi^\alpha |(\widehat{u} - \widehat{\Pi}^y \widehat{u})(\xi)|^2\,d\xi &= \int_{\xi=0}^1 \xi^\alpha \left| \int_\xi^1 (\widehat{u} - \widehat{\Pi}^y \widehat{u})'(t)\,dt \right|^2 d\xi \\
&\le 2 \int_{\xi=0}^1 \xi^\alpha \left| \int_\xi^1 \widehat{u}'(t)\,dt \right|^2 d\xi + 2 \int_{\xi=0}^1 \xi^\alpha \left| \int_\xi^1 (\widehat{\Pi}^y \widehat{u})'(t)\,dt \right|^2 d\xi \\
&\overset{(6.23)}{\le} \frac{2}{(\alpha+1)^2} \left[\int_{\xi=0}^1 \xi^{\alpha+2} \left(|\widehat{u}'(\xi)|^2 + |(\widehat{\Pi}^y \widehat{u})'(\xi)|^2 \right) d\xi \right].
\end{aligned}$$

We note the representation

$$(\widehat{\Pi}^y \widehat{u})'(\xi) = 2 \int_{t=1/2}^1 u'(t)\,dt$$

so that we arrive at

$$\int_{\xi=0}^1 \xi^\alpha |(\widehat{u} - \Pi^y \widehat{u})(\xi)|^2\,d\xi \le C \int_{\xi=0}^1 \xi^{\alpha+2} |\widehat{u}'(\xi)|^2\,d\xi \overset{(1-\theta)\le 1}{\le} C \int_{\xi=0}^1 \xi^{\alpha+2(1-\theta)} |\widehat{u}'(\xi)|^2\,d\xi.$$

An affine scaling then provides

$$\|u - \Pi^y u\|_{L^2((0,x_1);y^\alpha)} \le C h_1^{1-(1-\theta)} \|u'\|_{L^2((0,x_1);y^{\alpha+2(1-\theta)})} = C h_1^\theta \|u'\|_{L^2((0,x_1);y^{\alpha+2(1-\theta)})}.$$

In view of $h_1 \sim h^\mu$ and the assumption $\theta\mu \ge 1$, we obtain the desired estimate.

The elements $I \subset (0,1)$ with $0 \notin \overline{I}$: The procedure is as in the analysis of the μ-graded mesh in Theorem 3.20. In contrast to Theorem 3.20, we have to account for the weight y^α. However, for each I with $0 \notin \overline{I}$ we have

$$\max_{y \in I} y^\alpha \le C \min_{y \in I} y^\alpha \tag{6.24}$$

so that it can be viewed as a constant elementwise. Proceeding as in the proof of Theorem 3.20 and using $1 - 1/\mu \ge 1 - \theta$, we get

$$\|u - \Pi^y u\|_{L^2((x_1,1);y^\alpha)} \le h^1 \|u'\|_{L^2((x_1,1);y^{\alpha+2(1-\theta)})}.$$

The elements $I \subset (1,\mathcal{Y})$ with $\mathcal{Y} \notin \overline{I}$: Denote by (x_{M-1}, x_M) the last element with $x_M = \mathcal{Y}$. We employ standard estimates for the piecewise linear interpolant, exploiting again (6.24) to arrive at

$$\|u - \Pi^y u\|_{L^2((1,x_{M-1});y^\alpha)} \le C h^1 \|u'\|_{L^2((1,x_{M-1});y^\alpha)} \le C h^1 \|u'\|_{L^2((1,x_{M-1});y^{\alpha+2(1-\theta)})}.$$

We note that the weight $e^{\gamma y}$ has not been used so far.

The terminal element $I = (x_{M-1}, x_M)$: We have to modify the linear interpolant so as to ensure $(\Pi^y u)(\mathcal{Y}) = 0$. This is crudely achieved by subtracting from the linear interpolant the linear function

$$u(\mathcal{Y}) \frac{x - x_{M-1}}{h_M}, \qquad h_M := x_N - x_{M-1} \sim h.$$

It remains to bound $u(\mathcal{Y})$. To that end, we use $\lim_{y\to\infty} |u(y)| = 0$ to write

$$\begin{aligned}
|u(\mathcal{Y})| = \left| \int_{y=\mathcal{Y}}^\infty u'(t)\,dt \right| &= \left| \int_{y=\mathcal{Y}}^\infty e^{-\gamma y} y^{-(\alpha+2(1-\theta))/2} e^{\gamma y} y^{(\alpha+2(1-\theta))/2} u'(y)\,dy \right| \\
&\le \sqrt{\int_{y=\mathcal{Y}}^\infty e^{-2\gamma y} y^{-(\alpha+2(1-\theta))}\,dy} \sqrt{\int_{y=\mathcal{Y}}^\infty e^{2\gamma y} y^{\alpha+2(1-\theta)} (u'(y))^2\,dy} \\
&\lesssim e^{-\gamma\mathcal{Y}} \mathcal{Y}^{-(\alpha+2(1-\theta))/2} \|u'\|_{L^2(\mathcal{Y},\infty;y^{\alpha+2(1-\theta)})}.
\end{aligned}$$

In total, we conclude that

$$\|u - \Pi^y u\|_{L^2((x_{M-1},x_M);y^\alpha)} \lesssim (h + e^{-\gamma\mathcal{Y}}\mathcal{Y}^{-(\alpha/2+(1-\theta)})\|e^{\gamma y}u'\|_{L^2(x_{M-1},\infty);y^{2+2(1-\theta)})},$$

which finishes the proof. ∎

Theorem 6.12. *Let Ω be convex. Let $f \in \widetilde{H}^{1-s}(\Omega)$ and $\mu s > 1$.*
There are constants c, $C > 0$ such that the following holds: If $\mathcal{Y} \geq c|\log h|$, then

$$\|\operatorname{tr}_\Omega \mathcal{U} - \operatorname{tr}_\Omega \mathcal{U}_N\|_{\widetilde{H}^s(\Omega)} \leq C\|\mathcal{U} - \mathcal{U}_N\|_{\mathring{H}^1(\mathcal{C};y^\alpha)} \leq C \inf_{V\in V_N} \|\mathcal{U} - V\|_{\mathring{H}^1(\mathcal{C};y^\alpha)} \leq Ch,$$

and the problem size satisfies $\dim V_N \leq Ch^{-(d+1)}|\log h|$.

Proof. We fix $\theta \in (0, s)$ such that still $\theta\mu \geq 1$.
By Lemma 6.9 we have

$$\|\mathcal{U} - \mathcal{U}_N\|_{\mathring{H}^1(\mathcal{C};y^\alpha)} \leq C\left[\|\nabla(\mathcal{U} - \Pi^x\mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} + \|\nabla(\mathcal{U} - \Pi^y\mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)}\right] =: I + II.$$

From Theorem 6.6 we get the following estimates $\theta < s$

$$\|e^{\gamma\sqrt{\lambda_1}y}D_x^2\mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)} \overset{(6.16)}{\lesssim} \|f\|_{\widetilde{H}^{1-s}(\Omega)}, \tag{6.25}$$

$$\|e^{\gamma\sqrt{\lambda_1}y}D_x\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2(1-\theta)})} \overset{\text{Thm. 6.6, }\varepsilon = s}{\lesssim} \|f\|_{L^2(\Omega)}, \tag{6.26}$$

$$\|e^{\gamma\sqrt{\lambda_1}y}D_x\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)} \overset{\text{Thm. 6.6, }\varepsilon = 1}{\lesssim} \|f\|_{\widetilde{H}^{1-s}(\Omega)}, \tag{6.27}$$

$$\|e^{\gamma\sqrt{\lambda_1}y}\partial_y^2\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2(1-\theta)})} \overset{\text{Thm. 6.6,}\widetilde{\varepsilon} = \theta < s}{\lesssim} \|f\|_{L^2(\Omega)}, \tag{6.28}$$

$$\|e^{\gamma\sqrt{\lambda_1}y}\nabla\mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)} \overset{\text{Thm. 6.6,}\widetilde{\varepsilon} = 0}{\lesssim} \|f\|_{H^{-s}(\Omega)}. \tag{6.29}$$

Writing $\nabla = (D_x, \partial_y)$, we split the term I further:

$$I \leq \|D_x(\mathcal{U} - \Pi^x\mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} + \|\partial_y(\mathcal{U} - \Pi^x\mathcal{U})\|_{L^2(\mathcal{C};y^\alpha)} =: I_a + I_b.$$

From the product structure of the norm and the fact that ∂_y and Π^x commute

$$I_a \lesssim h\|D_x^2\mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)} \overset{(6.20),(6.25)}{\lesssim} h\|f\|_{L^2(\Omega)}$$

$$I_b \lesssim \|\partial_y\mathcal{U} - \Pi^x\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)} \overset{(6.20)}{\lesssim} h\|D_x\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)} \overset{(6.27)}{\lesssim} h\|f\|_{\widetilde{H}^{1-s}(\Omega)}.$$

To estimate II, we first claim that truncating $\mathcal{C}$ at $\mathcal{Y}$ introduces an error $O(h)$ if $\mathcal{Y} \geq c|\log h|$ if c is sufficiently large. To see this, we compute

$$\begin{aligned}\|\nabla(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\mathcal{C};y^\alpha)} &= \|\nabla(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} + \|\nabla(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\Omega\times(\mathcal{Y},\infty);y^\alpha)} \\ &= \|\nabla(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} + \|\nabla\mathcal{U}\|^2_{L^2(\Omega\times(\mathcal{Y},\infty);y^\alpha)} \\ &\overset{(6.29)}{\lesssim} \|\nabla(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} + C\mathcal{Y}^{-\alpha}e^{-2\gamma\sqrt{\lambda_1}\mathcal{Y}}\|f\|^2_{H^{-s}(\Omega)}.\end{aligned}$$

Since $\lambda_1 > 1$ is fixed, we may select $c > 0$ such that $\mathcal{Y} \geq c|\log h|$ implies $e^{-\gamma\sqrt{\lambda_1}\mathcal{Y}}\|f\|_{H^{-s}(\Omega)} \leq Ch\|f\|_{H^{-s}(\Omega)}$.
We write using $\nabla = (D_x, \partial_y)$

$$\|\nabla(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} = \|D_x(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} + \|\partial_y(\mathcal{U} - \Pi^y\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} =: II_a + II_b.$$

Next, we may use that D_x and Π^y commute to estimate

$$\begin{aligned} II_a &\lesssim \|D_x\mathcal{U} - \Pi^y D_x\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} \overset{(6.21)}{\lesssim} (h + \mathcal{Y}^{-\alpha/2+(1-\theta)}e^{-\gamma\sqrt{\lambda_1}\mathcal{Y}})\|e^{\gamma\sqrt{\lambda_1}y}D_x\partial_y\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^{\alpha+2(1-\theta)})} \\ &\overset{\mathcal{Y}>1;\mathcal{Y}\sim|\log h|}{\lesssim} h\|e^{\gamma\sqrt{\lambda_1}y}D_x\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2(1-\theta)})} \overset{(6.26)}{\lesssim} h\|f\|_{L^2(\Omega)}, \\ II_b &\overset{(6.22)}{\lesssim} (h + h^{-1}\mathcal{Y}^{-\alpha/2+(1-\theta)}e^{-\gamma\sqrt{\lambda_1}\mathcal{Y}})\|e^{\gamma\sqrt{\lambda_1}y}\partial_y^2\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^{\alpha+2(1-\theta)})} \overset{(6.28),\mathcal{Y}\sim|\log h|,(6.29)}{\lesssim} h\|f\|_{L^2(\Omega)}. \end{aligned}$$

In summary, we have arrive at the desired $O(h)$ convergence. The problem size is given by $|\mathcal{T}_\Omega| \cdot |\mathcal{T}_\mathcal{Y}| \sim h^{-d}h^{-1}(1 + \mathcal{Y}) \sim h^{-(d+1)}(1 + |\log h|)$. ■

6.5 Sparse grids

The convergence in Theorem 6.12 is the hoped-for $O(h)$ behavior. The problem size N, however, is not $O(h^{-d})$ but (up to logarithmic terms) $O(h^{-(d+1)})$. One can improve this complexity estimate with *sparse grids* ideas under slightly more stringent regularity assumptions. A key ingredient is that not a single approximation space is available but a hierarchy. We assume to have:

- A sequence of meshes $(\mathcal{T}_\Omega^\ell)_\ell$ of mesh size $h_\ell \sim 2^{-\ell}$ such that the spaces $\mathcal{P}^1_0(\mathcal{T}^\ell_\Omega)$, $\ell = 0, 1, \ldots,$ are nested. The corresponding approximation operators Π^x are denoted by Π^x_ℓ.
- A sequence of meshes $(\mathcal{T}_\mathcal{Y}^\ell)_\ell$ that are μ-graded meshes as described in Assumption 6.7 with h replaced by $h_\ell \sim 2^{-\ell}$. The spaces $\mathcal{P}^1_\mathcal{Y}(\mathcal{T}^\ell_\mathcal{Y})$, $\ell = 0, 1, \ldots,$ are assumed to be nested. The corresponding approximation operators Π^y are denoted by Π^y_ℓ.

Remark 6.13 *Sequences of quasi-uniform meshes with the desired nestedness properties are obtained by uniform ("red") refinement. Graded meshes in the interval $(0, 1)$ can be created by points $(i/N)^\mu$, $i = 0, \ldots, N$. Taking $N = 2^\ell$ shows the desired nestedness. Nested sequences of graded meshes can also be created in the context of NVB meshes, [GHS16].*

We set

$$\Pi^x_{-1} := 0, \qquad \Pi^y_{-1} := 0.$$

Given a terminal level $L \in \mathbb{N}_0$, the approximation result of Section 6.4 is done by the operator $\Pi^x_L \circ \Pi^y_L$. The operators Π^x_L and Π^y_L admit telescoping sum representations:

$$\Pi^x_L = \sum_{\ell=0}^{L}(\Pi^x_\ell - \Pi^x_{\ell-1}) =: \sum_{\ell=0}^{L} Q^x_\ell, \qquad \Pi^y_L = \sum_{\ell=0}^{L}(\Pi^y_\ell - \Pi^y_{\ell-1}) =: \sum_{\ell=0}^{L} Q^y_\ell,$$

where we introduced the operators $Q_\ell^x := \Pi_\ell^x - \Pi_{\ell-1}^y$ and $Q_\ell^y = \Pi_\ell^y - \Pi_{\ell-1}^y$. The basic idea is to approximate

$$\Pi_L^x \circ \Pi_L^y = (\sum_{\ell_x=0}^{L} Q_{\ell_x}^x) \circ (\sum_{\ell_y=0}^{L} Q_{\ell_y}^y) = \sum_{\ell_x,\ell_y=0}^{L} Q_{\ell_x}^x \circ Q_{\ell_y}^y \approx \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y\le L} Q_{\ell_x}^x \circ Q_{\ell_y}^y.$$

Numerically, one can realize a Galerkin method based on the *sparse grids space*

$$\mathcal{Q}_L^{\rm sp} := \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y\le L} \mathcal{P}_0^1(\mathcal{T}_\Omega^{\ell_x}) \otimes \mathcal{P}_{\mathcal{Y}}^1(\mathcal{T}_{\mathcal{Y}}^{\ell_y}).$$

To see that the approximation from the subspace $\mathcal{Q}_L^{\rm sp} \subset \mathcal{P}_0^1(\mathcal{T}_\Omega^L) \otimes \mathcal{P}_{\mathcal{Y}}^1(\mathcal{T}_{\mathcal{Y}})$ leads to reduced complexity, we estimate the dimension of $\mathcal{Q}_L^{\rm sp}$:

$$\begin{aligned}\dim \mathcal{Q}_L^{\rm sp} &\le \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y\le L} \dim \mathcal{P}_0^1(\mathcal{T}_\Omega^{\ell_x}) \cdot \dim \mathcal{P}_{\mathcal{Y}}^1(\mathcal{T}_{\mathcal{Y}}^{\ell_y}) \\ &\le C \sum_{\ell_x=0}^{L} \sum_{\ell_y=0}^{L-\ell_x} h_{\ell_x}^{-d} h_{\ell_y}^{-1} \le C \sum_{\ell_x=0}^{L} h_{\ell_x}^{-d} \sum_{\ell_y=0}^{L-\ell_x} 2^{\ell_y} \le C \sum_{\ell_x=0}^{L} h_{\ell_x}^{-d} 2^{L-\ell_x} \le C \sum_{\ell_x=0}^{L} 2^{\ell_x d} 2^{L-\ell_x} \\ &\le C 2^L 2^{L(d-1)} \le C 2^{Ld} \sim h_L^{-d} \sim \dim \mathcal{P}_0^1(\mathcal{T}_\Omega^L).\end{aligned}$$

That is, the complexity of the Galerkin method could be that of the discretization in space only and the increased problem size due to the extra variable y is not present any more! We next show in Theorem 6.15 that the reduction in problem size does hardly affect the convergence rate. Before doing so, let us illustrate in a simple setting the mechanism:

Example 6.14 *Consider sequences of quasi-uniform triangulations $\mathcal{T}_\Omega^\ell$ and $\mathcal{T}_{\Omega'}^\ell$ of domains Ω, Ω' with mesh sizes $h_\ell \sim 2^{-\ell}$. Let $I_\ell^x : C(\overline{\Omega}) \to \mathcal{P}^1(\mathcal{T}_\Omega^\ell)$ and Let $I_\ell^y : C(\overline{\Omega}) \to \mathcal{P}^1(\mathcal{T}_\Omega^\ell)$ be the piecewise linear interpolation operators and $Q_\ell^x := I_\ell^x - I_{\ell-1}^x$ and $Q_\ell^y := I_\ell^y - I_{\ell-1}^y$. Then for $Q_L^{\rm sp} := \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y\le L} Q_{\ell_x}^x \circ Q_{\ell_y}^y$ we have*

$$\|u - Q_L^{\rm sp} u\|_{L^2(\Omega\times\Omega')} \le C h_L^2 |\log h_L| \|D_x^2 D_y^2 u\|_{L^2(\Omega\times\Omega')}.$$

To see this we note that upon setting $I_{-1}^x = 0$ and $I_{-1}^y = 0$ we have the telescoping sum representations $\sum_\ell (I_\ell^x - I_{\ell-1}^x) = \mathrm{I}$ and $\sum_\ell (I_\ell^y - I_{\ell-1}^y) = \mathrm{I}$ so that also $\sum_{\ell_x,\ell_y=0}^\infty Q_{\ell_x}^x \circ Q_{\ell_y}^y = \mathrm{I}$. Hence,

$$\|u - Q_L^{\rm sp} u\|_{L^2(\Omega\times\Omega')} = \Big\| \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} Q_{\ell_x}^x \circ Q_{\ell_y}^y u \Big\|_{L^2(\Omega\times\Omega')} \le \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} \|Q_{\ell_x}^x \circ Q_{\ell_y}^y u\|_{L^2(\Omega\times\Omega')}.$$

The triangle inequality gives

$$\|Q_{\ell_x}^x v\|_{L^2(\Omega)} \le \|I_{\ell_x}^x v - v\|_{L^2(\Omega)} + \|I_{\ell_x-1}^x v - v\|_{L^2(\Omega)} \lesssim h_{\ell_x}^2 \|D_x^2 v\|_{L^2(\Omega)}$$

and an analogous estimate for $Q_{\ell_y}^y v$. Hence, we get

$$\begin{aligned}\|u - Q_L^{\rm sp} u\|_{L^2(\Omega\times\Omega')} &\lesssim \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} h_{\ell_x}^2 \|D_x^2 Q_{\ell_y}^y u\|_{L^2(\Omega\times\Omega')} \lesssim \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} h_{\ell_x}^2 h_{\ell_y}^2 \|D_x^2 D_y^2 u\|_{L^2(\Omega\times\Omega')} \\ &\lesssim \|D_x^2 D_y^2 u\|_{L^2(\Omega\times\Omega')} \left[\sum_{\ell_x=0}^{L} h_{\ell_x}^2 \sum_{\ell_y=L+1-\ell_x}^{\infty} h_{\ell_y}^2 + \sum_{\ell_x=L+1}^{\infty} h_{\ell_x}^2 \sum_{\ell_y=0}^{\infty} h_{\ell_y}^2 \right].\end{aligned}$$

Since $h_{\ell_x} \sim 2^{-\ell_x}$ and $h_{\ell_y} \sim 2^{-\ell_y}$, we arrive at

$$\begin{aligned}
\|u - Q_L^{\mathrm{sp}} u\|_{L^2(\Omega\times\Omega')} &\lesssim \|D_x^2 D_y^2 u\|_{L^2(\Omega\times\Omega')} \left[\sum_{\ell_x=0}^{L} 2^{-2\ell_x} 2^{2(L+1-\ell_x)} + 2^{-2(L+1)}\right] \\
&\lesssim \|D_x^2 D_y^2 u\|_{L^2(\Omega\times\Omega')} (L+1) 2^{-2(L+1)} \lesssim h_L^2 |\log h_L| \|D_x^2 D_y^2 u\|_{L^2(\Omega\times\Omega')}.
\end{aligned}$$

It is worth noting that, while the approximation is (up to $|\log h_L|$) as good as from the full space $\mathcal{P}^1(\mathcal{T}_\Omega^L) \otimes \mathcal{P}^1(\mathcal{T}_{\Omega'}^L)$, the regularity requirement is more stringent: From an approximation theoretical point of view, one would expect the requirement $u \in H^2(\Omega\times\Omega')$ (i.e., control of second derivatives) to obtain $O(h_L^2)$ instead of the condition $D_x^2 D_y^2 u \in L^2$ (i.e., control of mixed second derivatives).

Theorem 6.15. *Let $f \in \widetilde{H}^{-s+\nu}(\Omega)$ for some $\nu \in (1, 1+s)$. Assume the grading parameter μ for the meshes $\mathcal{T}_{\mathcal{Y}}^\ell$ satisfies*

$$\mu(\nu - 1) \geq 1.$$

Then there are constants c, $C > 0$ such that for $\mathcal{Y} \geq c|\log h|$ there holds

$$\inf_{V \in \mathcal{Q}_L^{\mathrm{sp}}} \|\mathcal{U} - V\|_{\mathring{H}^1(\mathcal{C};y^\alpha)} \leq C h_L |\log h_L|$$

and the problem size satisfies $\dim \mathcal{Q}_L \leq C h_L^{-d}$.

Proof. *1. step (preliminaries)* We estimate $\mathcal{U} - \mathcal{Q}_L^{\mathrm{sp}}\mathcal{U}$. To that end, we assume that the sequences of meshes are infinite. Since $\sum_{\ell_x=0}^\infty Q_{\ell_x}^x = \mathrm{I}$ and $\sum_{\ell_y=0}^\infty Q_{\ell_y}^y = \mathrm{I}$ (on appropriate function spaces), we have $\sum_{\ell_x,\ell_y=0}^\infty Q_{\ell_x}^x \circ Q_{\ell_y}^y = \mathrm{I}$ (on appropriate function spaces). Hence, the error is

$$\mathcal{U} - \mathcal{Q}_L^{\mathrm{sp}}\mathcal{U} = \sum_{\ell_x,\ell_y :\, \ell_x+\ell_y > L} (Q_{\ell_x}^x \circ Q_{\ell_y}^y)\mathcal{U}.$$

We have to estimate

$$\|\nabla(\mathcal{U} - \mathcal{Q}_L^{\mathrm{sp}}\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} = \|D_x(\mathcal{U} - \mathcal{Q}_L^{\mathrm{sp}}\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} + \|\partial_y(\mathcal{U} - \mathcal{Q}_L^{\mathrm{sp}}\mathcal{U})\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} =: I^2 + II^2.$$

With the triangle iequality, we have for any norm/seminorm $\|\cdot\|$

$$\|Q_{\ell_x}^x \mathcal{U}\| = \|\Pi_{\ell_x}^x - \Pi_{\ell_x-1}^x \mathcal{U}\| \leq \|\Pi_{\ell_x}^x - \mathcal{U}\| + \|\mathcal{U} - \Pi_{\ell_x-1}^x \mathcal{U}\|.$$

Since $h_{\ell_x} \sim h_{\ell_x-1}$, these two terms are of the same order and will be treated simultaneously below as it was done in Example 6.14.
2. step (estimating II) From Theorem 6.6 we have

$$\|e^{\gamma\sqrt{\lambda_1}y} D_x \partial_y^2 \mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+4-2\varepsilon})} \leq C\|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)}, \qquad 1 < \varepsilon < 1+s < 2, \tag{6.30}$$

$$\|e^{\gamma\sqrt{\lambda_1}y} D_x \partial_y \mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2-2\varepsilon})} \leq C\|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)}, \qquad 1 < \varepsilon < 1+s < 2. \tag{6.31}$$

For $c > 0$ sufficiently large and $\mathcal{Y} \geq c|\log h|$ we can recast (6.22) as

$$\begin{aligned}\|\partial_y(u - \Pi^y u)\|_{L^2(L^2(0,\mathcal{Y};y^\alpha)} &\lesssim (h + h^{-1/2}\mathcal{Y}^{-\alpha/2+(1-\theta)}e^{-\gamma\sqrt{\lambda_1}\mathcal{Y}})\|e^{\gamma\sqrt{\lambda_1}y}\partial_y^2 u\|_{L^2(0,\infty;y^{\alpha+2(1-\theta)})} \\ &\lesssim h\|e^{\gamma\sqrt{\lambda_1})y}\partial_y^2 u\|_{L^2(0,\infty;y^{\alpha+2(1-\theta)})}. \end{aligned} \tag{6.32}$$

For II, we compute with (6.30) and (6.32) (taking $h = h_L$ there)

$$\begin{aligned} II &\leq \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} \|\partial_y Q^x_{\ell_x} \circ Q^y_{\ell_y}\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} = \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} \|Q^x_{\ell_x} \circ \partial_y Q^y_{\ell_y}\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} \\ &\lesssim \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} h_{\ell_x}\|D_x\partial_y Q^y_{\ell_y}\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} = \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} h_{\ell_x}\|\partial_y Q^y_{\ell_y} D_x\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} \\ &\overset{(6.32),\theta=\varepsilon-1}{\lesssim} \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} h_{\ell_x}h_{\ell_y}\|e^{\gamma\sqrt{\lambda_1}y}\partial_y^2 D_x\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^{\alpha+4-2\varepsilon})} \\ &\overset{(6.30)}{\lesssim} \|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)}\left[\sum_{\ell_x=0}^{L}\sum_{\ell_y=L+1-\ell_x}^{\infty} 2^{-\ell_x-\ell_y} + \sum_{\ell_x=L+1}^{\infty}\sum_{\ell_y=0}^{\infty} 2^{-\ell_x-\ell_y}\right] \\ &\lesssim L2^{-L}\|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)} \sim h_L|\log h_L|\|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)}. \end{aligned}$$

2. step (estimating I) For I, we first provide some regularity: From the last statement of Theorem 6.6 and the classical shift theorem for the Laplacian we get for $0 \leq \nu < 1 + s$ that $\|e^{\gamma\sqrt{\lambda_1}y}D_x^2\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+2-2\nu})} \lesssim \|f\|_{\widetilde{H}^{1-s+\nu}(\Omega)}$. Selecting $\nu = \varepsilon - 1 \in (0, s)$ we arrive at

$$\|e^{\gamma\sqrt{\lambda_1}y}D_x^2\partial_y\mathcal{U}\|_{L^2(\mathcal{C};y^{\alpha+4-2\varepsilon})} \lesssim \|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)}. \tag{6.33}$$

We estimate

$$\begin{aligned} I &\leq \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} \|D_x Q^x_{\ell_x} \circ Q^y_{\ell_y}\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} = \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} \|D_x Q^x_{\ell_x} \circ Q^y_{\ell_y}\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} \\ &\lesssim \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} h_{\ell_x}\|D_x^2 Q^y_{\ell_y}\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} \\ &\overset{(6.21),\theta=\varepsilon-1}{\lesssim} \sum_{\ell_x,\ell_y\,:\,\ell_x+\ell_y>L} h_{\ell_x}h_{\ell_y}\|e^{\gamma\sqrt{\lambda_1}y}\partial_y D_x^2\mathcal{U}\|_{L^2(\mathcal{C}_\mathcal{Y};y^{\alpha+4-2\varepsilon})} \\ &\overset{(6.33)}{\lesssim} L2^{-L}\|f\|_{\widetilde{H}^{-s+\varepsilon}(\Omega)}. \end{aligned}$$

Putting the estimates together concludes the proof. ■

Remark 6.16 *A Galerkin method based on the space $\mathcal{Q}_L^{\mathrm{sp}}$ would require a basis of $\mathcal{Q}_L^{\mathrm{sp}}$. An alternative is the use of the* combination formula, *[HPS13], which combines several Galerkin solutions: Let $\mathcal{U}_{\ell_x,\ell_y} \in \mathcal{P}^1_0(\mathcal{T}^{\ell_x}_\Omega) \otimes \mathcal{P}^1_\mathcal{Y}(\mathcal{T}^{\ell_y}_\mathcal{Y})$ be the Galerkin approximation to $\mathcal{U}$, i.e.,*

$$a_\mathcal{C}(\mathcal{U}_{\ell_x,\ell_y}, V) = a_\mathcal{C}(\mathcal{U}, V) = \langle f, V\rangle \qquad \forall V \in \mathcal{P}^1_0(\mathcal{T}^{\ell_x}_\Omega) \otimes \mathcal{P}^1_\mathcal{Y}(\mathcal{T}^{\ell_y}_\mathcal{Y}).$$

Then, the Galerkin approximation $\mathcal{U}_L^{\mathrm{sp}} \in \mathcal{Q}_L^{\mathrm{sp}}$ *is given by*

$$\mathcal{U}_L^{\mathrm{sp}} = \sum_{\ell=0}^{L} (\mathcal{U}_{\ell,L-\ell} - \mathcal{U}_{\ell-1,L-\ell}),$$

with the understanding that $\mathcal{U}_{-1,\ell} = 0$.

Chapter 7

Dunford–Taylor Approach

7.1 The Dunford-Taylor Definition

The starting point for the Dunford-Taylor calculus is Cauchy's integral formula, which states that

$$f(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z - \zeta} dz, \tag{7.1}$$

where the contour $\mathcal{C}$ is a rectifiable Jordan curve oriented such that f is holomorphic on the left of $\mathcal{C}$ and ζ is also on the left of $\mathcal{C}$.
The Dunford-Taylor calculus generalizes (7.1) to the case of operator by defining $f(T)$ for linear operators T on a Hilbert space by

$$f(T) := \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z - T)^{-1} dz; \tag{7.2}$$

here, the spectrum of T lies on the left of the contour $\mathcal{C}$ and does not touch the contour. For more details about the Dunford-Taylor calculus, we refer to [Yos80].

7.1.1 The Balakrishnan formula for the Spectral Fractional Laplacian

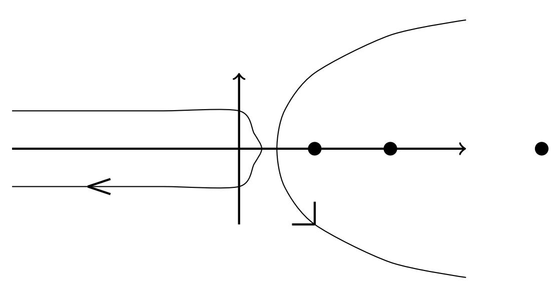


Figure 7.1: Contour in the definition of $((-\Delta)^s_\sigma)^{-1}$: "standard" contour on the right. Deformed to a cover twice the negative real axis on the left. Spectrum of $-\Delta$ is given by solid dots.

Using (7.2) for the negative Laplacian $(-\Delta)$, we may define the solution operator for $(-\Delta)^s_\sigma u = f$ as

$$u = (-\Delta)^{-s}_\sigma f = \frac{1}{2\pi i}\int_{\mathcal{C}} z^{-s}(z+\Delta)^{-1} f dz,$$

where the contour is chosen such that the negative real axis (as well as a neighborhood of the origin) is on the right side (see Fig. 7.1). (A good, concise exposition of defining fractional powers of operators is given in [RR04, Sec. 12.4].) Deforming the contour onto the negative real axis (see Fig. 7.1) gives the so-called Balakrishnan formula

$$(-\Delta)^{-s}_\sigma f = \frac{\sin(s\pi)}{\pi}\int_0^\infty \mu^{-s}(\mu-\Delta)^{-1} f d\mu. \tag{7.3}$$

Remark 7.1 *(7.3) can be obtained by "elementary" means without the Dunford-Taylor calculus. Proceeding formally (think of finite sums and later check convergence in the appropriate sense)*[1]

$$\begin{aligned}\frac{\sin(\pi s)}{\pi}\int_{\mu=0}^\infty \mu^{-s}(\mu-\Delta)^{-1} f\, d\mu &= \frac{\sin(\pi s)}{\pi}\sum_k f_k\varphi_k \int_{\mu=0}^\infty \mu^{-s}(\mu+\lambda_k)^{-1}\, d\mu \\ &\overset{\mu=\lambda_k t}{=} \frac{\sin(\pi s)}{\pi}\sum_k f_k\varphi_k\lambda_k^{-s}\int_{\mu=0}^\infty t^{-s}(t+1)^{-1}\, dt \\ &= \sum_k f_k\lambda_k^{-s}\varphi_k = \left((-\Delta)^s_\sigma\right)^{-1} f.\end{aligned}$$

7.1.2 The sinc-quadrature technique

The Balakrishnan formula (7.3) suggests a numerical method:
In the following, we present a numerical method based on the Balakrishnan formula, which consists of two parts:

- Quadrature for the integral in the variable μ;
- FEM in the variables x to approximate $(\mu-\Delta)^{-1}$ in the quadrature points μ.

Since sinc-quadrature is usually formulated for integration over $\mathbb{R}$ instead of $(0,\infty)$, we perform a change of variables: Using the transformation $\mu = e^y$ in (7.3), we get

$$u = (-\Delta)^{-s}_\sigma f = \frac{\sin(s\pi)}{\pi}\int_{-\infty}^\infty e^{(1-s)y}(e^y\,\mathrm{I}-\Delta)^{-1} f\, dy = \frac{\sin(s\pi)}{\pi}\int_{-\infty}^\infty e^{-sy}(\mathrm{I}-e^{-y}\Delta)^{-1} f\, dy. \tag{7.4}$$

We will discretize the integral $\int_{-\infty}^\infty$ by *sinc-quadrature*, which here is simply the trapezoidal rule.

[1] The integral $\int_{t=0}^\infty t^{-s}(t+1)^{-1}\, dt = \frac{\pi}{\sin(\pi s)}$ can be evaluated by contour integration taking the following closed loop: just above the positive real axis from ε to R, then on a circle $Re^{-i\varphi}$, $\varphi\in(0,2\pi)$, then back just below the real axis and on a small circle of radius ε around the origin. One selects the positive real real axis as the branch cut for x^{-s}. The integrand has a pole at $x=-1$ and the residue is $2\pi i x^{-s}$ at $x=-1$.

Sinc quadrature for scalar-valued functions

Good references for sinc techniques are [Ste93, LB92]. The *sinc*-function is defined by

$$\operatorname{sinc}(z) := \begin{cases} \frac{\sin \pi z}{\pi z} & z \neq 0 \\ 1 & z = 0 \end{cases}. \tag{7.5}$$

Lemma 7.2. *If* $\operatorname{supp} \mathcal{F}(F) \subset [-\pi/h, \pi/h]$ *for some* $h > 0$ *and* $F \in L^1(\mathbb{R})$ *then for all* $z \in \mathbb{C}$

$$F(z) = \sum_{j \in \mathbb{Z}} F(jh) \operatorname{sinc}\left(\frac{z - jh}{h}\right). \tag{7.6}$$

Proof. *1. step:* We claim:

$$e^{-izt} = \sum_j e^{ijht} \operatorname{sinc}\left(\frac{z + jh}{h}\right), \qquad z \in \mathbb{C}, \quad t \in (-\pi/h, \pi/h). \tag{7.7}$$

To see (7.7) we extend the function $e^{-iz\cdot}|_{(-\pi/h,\pi/h)}$ as a $2\pi/h$-periodic function. The Fourier series of this functions converges pointwise and is given by[2]

$$\begin{aligned} e^{-izt} &= \sum_j e^{ijht} \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-ijhx} e^{-izx}\, dx = \sum_j e^{-ijht} \frac{h}{2\pi} \frac{-1}{i(jh+z)} \left[e^{-\pi/h(jh+z)} - e^{\pi/h(jh+z)} \right] \\ &= \sum_j e^{ijht} \frac{h}{2\pi} \frac{2i \sin(\pi(jh+z)/h)}{i(jh+z)} = \sum_j e^{ijht} \operatorname{sinc}((z+jh)/h). \end{aligned}$$

2. step: Since the (periodized) function $t \mapsto e^{-izt}$ is smooth with jumps at $j\pi/h$, $j \in \mathbb{Z}$, the Fourier series converges uniformly on compact subsets of $(-\pi/h, \pi/h)$ and the partial sums are uniformly bounded.[3] This will allow us to interchange summation and integration later.
3. step: By the Paley-Wiener theorem the function F is in fact an entire function on $\mathbb{C}$. In particular, it is smooth so that pointwise evalulations are admissible.
4. step: We use the Fourier inversion formula

$$\begin{aligned} \sqrt{2\pi} F(z) &\overset{\operatorname{supp} \mathcal{F}(F) \subseteq [-\pi/h,\pi/h]}{=} \int_{\zeta \in (-\pi/h,\pi/h)} \mathcal{F}(F)(\zeta) e^{iz\zeta}\, d\zeta \\ &\overset{(7.7)}{=} \int_{\zeta \in (-\pi/h,\pi/h)} \mathcal{F}(F)(\zeta) \sum_j e^{ijh\zeta} \operatorname{sinc}\left(\frac{-z+jh}{h}\right) \\ &\overset{\text{Step 2}}{=} \sum_j \operatorname{sinc}\left(\frac{-z+jh}{h}\right) \int_{\zeta \in (-\pi/h,\pi/h)} \mathcal{F}(F)(\zeta) e^{ijh\zeta} \\ &\overset{\text{Fourier inv. form.}}{=} \sum_j \operatorname{sinc}\left(\frac{-z+jh}{h}\right) \sqrt{2\pi} F(jh). \end{aligned}$$

[2] we assume $z + jh \neq 0$ for all $j \in \mathbb{Z}$

[3] this is effectively a consequence of the Gibbs phenomenon, which states that the overshoot and the undershoot are ca. 9%.

In view of the fact that sinc is an even function, the claim of the lemma is proved. ■

Corollary 7.3. *Under the assumptions of Lemma 7.2 there holds*

$$\int_{\mathbb{R}} F(z)\,dz = h\sum_{j\in\mathbb{Z}} F(jh).$$

Proof. One simply evaluates $\int_{\mathbb{R}} \operatorname{sinc}((z-jh)/h) = h$. To see this, one ultimately has to evaluate

$$\int_{x=0}^{\infty} \frac{\sin x}{x}\,dx = \int_{x=0}^{\infty}\int_{y=0}^{\infty} \sin x \exp(-xy)\,dydx = \int_{y=0}^{\infty}\int_{x=0}^{\infty} \sin x \exp(-xy)\,dxdy = \int_{y=0}^{\infty} \frac{1}{1+y^2}\,dy = \frac{\pi}{2}.$$

■

Remark. If the function F additionally decays sufficiently fast at $\pm\infty$, then the sum in Corollary 7.3 can be truncated. This shows that a truncated trapezoidal rule can be quite accurate.

Since bandlimited functions are entire by the Paley-Wiener theorem, one is interested in enlarging the class of functions for the represention (7.6) holds—at least approximately.
The following theorem show that the series (7.6) can be a very good approximation:

Theorem 7.4 ([LB92, Thm. 2.13], [Ste93, Chap. 3]). *Let F be holomorophic in the strip $D_d := \{z \in \mathbb{C} : \operatorname{Im} z < d\}$. Assume*

$$N_d^2(F) := \sup_{0<y<d}\left\{\left(\int_{x\in\mathbb{R}} |F(x+iy)|^2\,dx\right)^{1/2} + \left(\int_{x\in\mathbb{R}} |F(x-iy)|^2\,dx\right)^{1/2}\right\} < \infty,$$

$$\int_{y=-d}^{d} |F(x+iy)|\,dy = \mathcal{O}(|x|^a) \qquad \text{as } |x|\to\infty \quad \text{for some } a\in[0,1).$$

Then,

$$F(x) - \sum_{j\in\mathbb{Z}} F(jh)\operatorname{sinc}\left(\frac{x-jh}{h}\right) = \varepsilon(z,h) \tag{7.8}$$

with

$$\|\varepsilon(\cdot,h)\|_{L^\infty(\mathbb{R})} \le \frac{N_d^2(F)}{2(\pi d)^{1/2}\sinh(\pi d/h)} = \mathcal{O}(e^{-\pi d/h}). \tag{7.9}$$

Proof. We will not prove the theorem, although the proof is not very difficult. Essentially, it follows from the residue theorem applied to the function

$$G(z) := \sin(\pi x/h)\frac{F(z)}{(z-x)\sin(\pi z/h)}$$

by taking contours as the boundary of the rectangles

$$R_n := \{z\in\mathbb{C}\,|\, z = x+iy, |y| < y_n, -(n+1/2)d < x < (n+1/2)d\}, \qquad y_n := d - 1/n.$$

One notices that G has poles at x and jh, $j \in \mathbb{Z}$. ■

Remark. It is worth noting that h is still a parameter at ones disposal. Thus, by making h smaller, the accuracy of the sinc approximation improves very quickly.

We actually need a result for the quadrature error:

Theorem 7.5 ([LB92, Thm. 2.20]). *Under the assumptions of Theorem 7.4 there holds for the quadrature error*

$$\left| \int_{\mathbb{R}} F(x)\, dx - h \sum_{j \in \mathbb{Z}} F(jh) \right| \leq \frac{N_d^2(F)}{2 \sinh(\pi d/h)} e^{-\pi d/h} = \mathcal{O}(e^{-2\pi d/h}).$$

In practice, the infinite sum has to be truncated. A typical case is that F decays exponentially. In this case, the choice

$$Q^M(F) := h \sum_{|j| \leq M} F(jh), \qquad h := 1/\sqrt{M} \tag{7.10}$$

is particularly useful:

Corollary 7.6. *Let F satisfy the assumptions of Theorem 7.5 and additionally $|F(x)| \leq C_F e^{-a|x|}$ for some C_F, $a > 0$. Then there are constants C, $b > 0$ such that for all $M \in \mathbb{N}$*

$$\left| \int_{\mathbb{R}} F(x)\, dx - Q^M(F) \right| \leq C e^{-b\sqrt{M}}.$$

Proof. One estimates

$$\begin{aligned}\left| \int_{\mathbb{R}} F(x)\, dx - Q^M(F) \right| &\leq \left| \int_{\mathbb{R}} F(x)\, dx - h \sum_{j \in \mathbb{Z}} F(jh) \right| + \sum_{|j| > M} h|F(jh)| \leq C e^{-2\pi d/h} + 2C_F h \sum_{j > M} e^{-ajh} \\ &\leq C e^{-2\pi d/h} + \frac{2C_F h}{1 - e^{-aMh}} e^{-aMh}.\end{aligned}$$

The choice $h = 1/\sqrt{M}$, which we made in the definition of the quadrature Q^M leads to the claim.[4] ■

Remark. The sinc-quadrature has many parameters to improve the convergence behavior of the method. A more general technique would be to select

$$Q^{M^-,M^+}(F) := h \left(\sum_{j=-M^-}^{0} F(jh) + \sum_{j=1}^{M+} F(jh) \right)$$

with $M^- = a^-/\sqrt{h}$ and $M^+ = a^+/\sqrt{h}$ for two parameters a^+, a^- that better balance the error contributions.

[4] A better choice would be to ensure $2\pi d/h = aMh$, i.e., $h = \sqrt{2\pi d/(aM)}$.

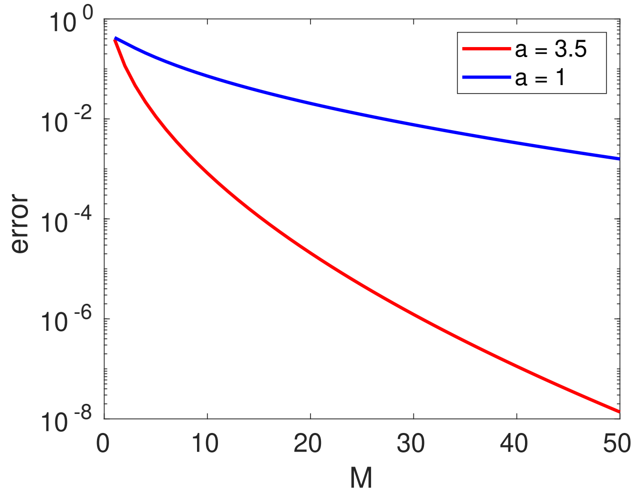


Figure 7.2: Sinc quadrature error; see Example 7.7.

Example 7.7 *Typically, integrals are transformed prior to an application of the sinc-quadrature. For example, the integral* $\int_0^\infty (1+x^2)^{-1}\,dx = \frac{\pi}{2}$ *can be evaluated as*

$$\int_{x=0}^{\infty} \frac{1}{1+x^2}\,dx \overset{x=e^{at}}{=} \int_{-\infty}^{\infty} \frac{ae^{at}}{1+e^{2at}}\,dt.$$

The sinc-quadrature error $\pi/2 - Q^M(F_a)$ *with* $F_a = ae^{at}/(1+e^{1at})$ *for two different values of* a *is given in Fig. 7.2 indicating that the choice of the transformation affects the actual performance of the sinc-quadrature* Q^M.

Sinc quadrature: semidiscretization

We discretize (7.4) by the sinc-quadrature Q^M of (7.10). That is, we approximate

$$\begin{aligned}\frac{\pi}{\sin(\pi s)} u &= \int_{y=-\infty}^{\infty} e^{(1-s)y}(e^y\,\mathrm{I} - \Delta)^{-1} f\,dy = \int_{y=-\infty}^{\infty} e^{-sy}(\mathrm{I} - e^{-y}\Delta)^{-1} f\,dy \\ &\approx Q^M(e^{-sy}(\mathrm{I} - e^{-y}\Delta)^{-1} f) = h \sum_{|j|\le M} e^{-sy_j}(\mathrm{I} - e^{-y_j}\Delta)^{-1} f\end{aligned}$$

with

$$y_j = jh, \qquad h = 1/\sqrt{M}.$$

We abbreviate

$$c_B := \frac{\sin(\pi s)}{\pi}.$$

For the semidiscretization error $((-\Delta)^s_\sigma)^{-1} - c_B Q^M(e^{-sy}(\mathrm{I} - e^{-y}\Delta)^{-1})$, we have the following result:

Lemma 7.8. *There exists a $C > 0$, depending only on s and Ω such that*

$$\begin{aligned}
&\|((-\Delta)^s_\sigma)^{-1} - c_B Q^M(e^{-sy}(\mathrm{I} - e^{-y}\Delta)^{-1}) f\|_{\widetilde{H}^s(\Omega)} \\
&\leq C\left(e^{-\frac{\pi^2}{2}\sqrt{M}}\|f\|_{H^{-s}(\Omega)} + e^{-\frac{s}{2}\sqrt{M}}\|f\|_{L^2(\Omega)} + e^{-(1-s)\sqrt{M}}\|f\|_{L^2(\Omega)}\right).
\end{aligned}$$

Proof. Since the operator $(-\Delta)$ has an ONB $(\varphi_k)_k$, we perform the analysis in that basis.

1. step (reduction to scalar quadrature): Let us first consider the approximation

$$Q_h(f) := h \sum_{j \in \mathbb{Z}} e^{-sy_j}(\mathrm{I} - e^{-y_j}\Delta)^{-1} f.$$

Expanding $f = \sum_k f_k \varphi_k$, we see that we have

$$\begin{aligned}
((-\Delta)^s_\sigma)^{-1} f &= c_B \sum_k f_k \varphi_k \int_{y=-\infty}^{\infty} e^{-sy}(1 + \lambda_k e^{-y})^{-1}\, dy = c_B \sum_k f_k \varphi_k I(\lambda_k), \\
c_B Q_h(f) &= c_B \sum_k f_k \varphi_k h \sum_j e^{-sy_j}(1 + \lambda_k e^{-y_j}) = c_B \sum_k f_k \varphi_k \widetilde{I}_h(\lambda_k),
\end{aligned}$$

where we used the abbreviations

$$\begin{aligned}
I(\lambda) &:= \int_{-\infty}^{\infty} e^{-sy}(1 + \lambda e^{-y})\, dy, \\
\widetilde{I}_h(\lambda) &:= h \sum_{j \in \mathbb{Z}} e^{-sy_j}(1 + \lambda e^{-y_j})\, dy.
\end{aligned}$$

Then the error is given by

$$\|((-\Delta)^s_\sigma)^{-1} f - c_B Q_h(f)\|^2_{\widetilde{H}^s(\Omega)} = c_B^2 \sum_k \lambda_k^s |f_k|^2 |I(\lambda_k) - \widetilde{I}_h(\lambda_k)|^2.$$

Step 2 (analysis of the scalar quadrature): It can be checked that

$$\sup_{\lambda > 0} \lambda^s \sup_{d \in (-\pi/2, \pi/2)} \int_{y=-\infty}^{\infty} \left| \frac{e^{-s(y+id)}}{1 + \lambda e^{-(y+id)}} \right| dy \leq C.$$

From Theorem 7.5 we therefore get

$$|I(\lambda_k) - \widetilde{I}_h(\lambda_k)| \leq C \lambda_k^{-s} e^{-\pi^2/h}$$

and conclude

$$\|((-\Delta)^s_\sigma)^{-1} f - c_B Q_h(f)\|^2_{\widetilde{H}^s(\Omega)} \leq c_B^2 \sum_k \lambda_k^s |f_k|^2 \lambda_k^{-2s} e^{-\pi^2/h} \leq C \|f\|^2_{H^{-s}(\Omega)} e^{-\pi^2/h}.$$

Step 3: Since the quadrature formula is a finite sum, we have to estimate

$$\sum_k \lambda_k^s |f_k|^2 \left| h \sum_{|j|>M} e^{-sy_j}(1+\lambda_k e^{-y_j})^{-1} \right|^2 .$$

To estimate this, we need more regularity of f than just $f \in H^{-s}(\Omega)$. By comparing a (Riemann) sum with the corresponding integral, we obtain

$$h \sum_{j>M} \frac{e^{-sy_j}}{1+\lambda e^{-y_j}} \lesssim \int_{\sqrt{M}}^{\infty} \frac{e^{-sy}}{1+\lambda e^{-y}}\, dy \overset{x=\lambda e^{-y}}{\lesssim} e^{-s\sqrt{M}} \min\{1, (\lambda e^{-\sqrt{M}})^{-1}\}, \tag{7.11}$$

$$h \sum_{j<-M} \frac{e^{-sy_j}}{1+\lambda e^{-y_j}} \lesssim \int_{-\infty}^{-\sqrt{M}} \frac{e^{-sy}}{1+\lambda e^{-y}}\, dy \overset{x=\lambda e^{-y}}{\lesssim} \lambda^{-s} \min\{1, (\lambda e^{\sqrt{M}})^{s-1}\}. \tag{7.12}$$

We estimate further

$$\begin{aligned}
\sum_k \lambda_k^s |f_k|^2 \left| h \sum_{j>M} \frac{e^{-sy_j}}{1+\lambda e^{-y_j}} \right|^2 &\lesssim \sum_k \lambda_k^s |f_k|^2 e^{-2s\sqrt{M}} \min\{1, (\lambda_k e^{-\sqrt{M}})^{-2}\} \\
&\lesssim \sum_{k:\ \lambda_k e^{-\sqrt{M}} \le 1} \lambda_k^s |f_k|^2 e^{-2s\sqrt{M}} + \sum_{k:\ \lambda_k e^{-\sqrt{M}} > 1} \lambda_k^s |f_k|^2 e^{-2s\sqrt{M}} (\lambda_k e^{-\sqrt{M}})^{-2} \\
&= e^{-s\sqrt{M}} \sum_{k:\ \lambda_k e^{-\sqrt{M}} \le 1} (\lambda_k e^{-\sqrt{M}})^s |f_k|^2 + e^{-s\sqrt{M}} \sum_{k:\ \lambda_k e^{-\sqrt{M}} > 1} (\lambda_k e^{-\sqrt{M}})^{s-2} |f_k|^2 \\
&\le 2e^{-s\sqrt{M}} \|f\|^2_{L^2(\Omega)}
\end{aligned}$$

and, using that $\lambda_k \ge \lambda_1 > 0$ we see that the minimum in (7.12) is actually always attained for $(\lambda_k e^{\sqrt{M}})^{s-1}$ so that

$$\begin{aligned}
\sum_k \lambda_k^s |f_k|^2 \left| h \sum_{j<-M} \frac{e^{-sy_j}}{1+\lambda e^{-y_j}} \right|^2 &\lesssim \sum_k \lambda_k^s |f_k|^2 \lambda_k^{-2s} (\lambda_k e^{\sqrt{M}})^{2(s-1)} = \sum_k \lambda_k^{s-2} |f_k|^2 e^{-2(1-s)\sqrt{M}} \\
&\lesssim e^{-2(1-s)\sqrt{M}} \|f\|^2_{L^2(\Omega)}.
\end{aligned}$$

■

Sinc quadrature: fully discrete method

The semidiscrete approximation is given by

$$c_B Q^M(e^{-sy}(\mathrm{I} - e^{-y}\Delta))f = c_B h \sum_{|j|\le M} e^{-sy_j} \underbrace{(\mathrm{I} - e^{-y_j}\Delta)^{-1} f}_{=:u_j \in H_0^1(\Omega)}.$$

A fully discrete method can be obtained by approximating the functions u_j by a numerical method, e.g., the FEM. The function $u_j \in H_0^1(\Omega)$ is the solution of

$$\int_\Omega e^{-y_j} \nabla u_j \cdot \nabla v + u_j v = \int_\Omega f v \qquad \forall v \in H_0^1(\Omega). \tag{7.13}$$

Given a finite-dimensional $V_j \subset H_0^1(\Omega)$, the Galerkin approximation $u_{j,N} \in V_j$ is given by requiring

$$\int_\Omega e^{-y_j} \nabla u_{j,N} \cdot \nabla v + u_{j,N} v = \int_\Omega f v \qquad \forall v \in V_j \tag{7.14}$$

and one sets

$$u_N := c_B h \sum_{|j| \le M} e^{-s y_j} u_{j,N}. \tag{7.15}$$

Concerning the error, we have with $b := \min\{s/2, 1-s, \pi^2/2\}$

$$\|u - u_N\|_{\widetilde{H}^s(\Omega)} \lesssim e^{-b\sqrt{M}} \|f\|_{L^2(\Omega)} + h \sum_{|j| \le M} \|u_j - u_{j,N}\|_{\widetilde{H}^s(\Omega)},$$

where the first term is the semidiscretization error of Lemma 7.8 whereas the second term arises from the triangle inequality. The analysis of the method is thus reduced to the Galerkin error analysis. An example of the type of results that can be obtained is:

Proposition 7.9 ([BP15]). *Let Ω be convex, $s \in (0,1)$ and $f \in \widetilde{H}^{2-2s}(\Omega)$. Then, if the functions $u_{j,N}$ are approximated using piecewise linears on a mesh of mesh size H, there holds*

$$\|u - u_N\|_{\widetilde{H}^s(\Omega)} \le C |\log H| \left(H^{2-s} \|f\|_{\widetilde{H}^{2-2s}(\Omega)} + e^{-b\sqrt{M}} \|f\|_{\widetilde{H}^s(\Omega)} \right).$$

Remark. The assumptions on f are not very realistic in practice, since (for $s < 1/2$) the function f needs to vanish on $\partial\Omega$; reduced rates of convergence (in terms of H) then result.
For smooth data f not necessarily vanishing on $\partial\Omega$, see the *hp*-FEM discussed in [BMS23].

The error analysis for $u_j - u_{j,N}$ hinges on understanding the regularity of the solution u_j, which solves, in strong form,

$$-e^{-y_j} \Delta u_j + u_j = f \quad \text{in } \Omega, \qquad u_j|_{\partial\Omega} = 0. \tag{7.16}$$

Two cases may arise:

1. $y_j \le 0$: Then (7.16) can alternatively be written as

   $$-\Delta u_j + e^{y_j} u_j = e^{y_j} f.$$

   Since $0 < e^{y_j} \le 1$ (and possibly even very small if $y_j << -1$), we see that u_j is the solution of a *regular perturbation* of $-\Delta$. Thus, a standard space V_j of piecewise polynomials on a quasi-uniform mesh will be appropriate.

2. $y_j > 0$: Then for $e^{-y_j} << 1$ problem (7.16) is a *singularly perturbed* equation. The solution features a boundary layer that needs to be resolved for accurate solutions. See the following Example 7.10. Special meshes can be designed to produce accurate numerical approximations.

Example 7.10 *Consider for $\varepsilon > 0$ the problem*

$$-\varepsilon^2 u'' + u = f \quad \text{on } \Omega = (0,1) \qquad u|_{\partial\Omega} = 0. \tag{7.17}$$

The solution plot in Fig. 7.3 shows clearly that the solution features boundary layers *at the endpoints. That is, the solution varies sharply in an $O(\varepsilon)$ neighborhood of the endpoint. A FEM thus needs small elements near the boundary to resolve these layers. See [Mel02, Chap 2] and the references there for more details.*

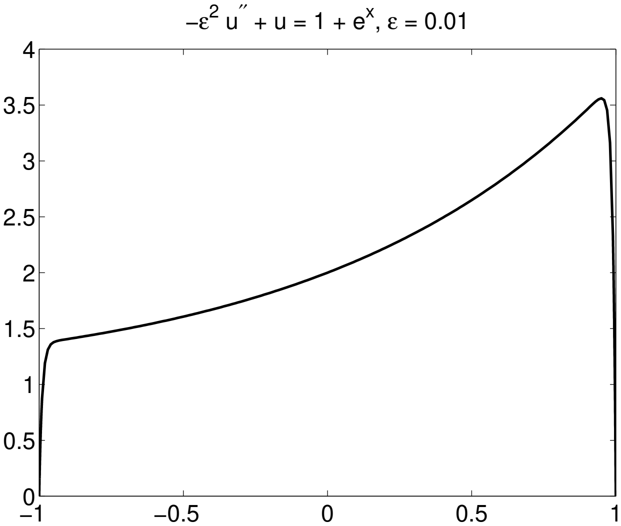


Figure 7.3: Solution plot for Example 7.10

7.1.3 The Integral Fractional Laplacian

A direct definition like (7.3) is not possible for the integral fractional Laplacian, as the spectrum does not need to be positive and the Dunford-Taylor calculus can not be justified.
However, using the Fourier definition of the fractional Laplacian, we can derive a different representation. We have with Plancherel's formula

$$a(u,w) = C(d,s)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{d+2s}}dx\,dy = \int_{\mathbb{R}^d}(-\Delta)^s u(x)w(x)dx \tag{7.18}$$

$$= \int_{\mathbb{R}^d}\mathcal{F}((-\Delta)^s u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta = \int_{\mathbb{R}^d}|\zeta|^{2s}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta. \tag{7.19}$$

Using Parseval's theorem, we can write

$$\int_{\mathbb{R}^d}\frac{|\zeta|^2}{1+\mu^2|\zeta|^2}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta = \int_{\mathbb{R}^d}(-\Delta)(I-\mu^2\Delta)^{-1}u(x)\,w(x)dx,$$

which leads, using the transformation $t = \mu|\zeta|$ to

$$\begin{aligned} C_s\int_0^\infty \mu^{1-2s}\int_{\mathbb{R}^d}(-\Delta)(I-\mu^2\Delta)^{-1}u(x)\,w(x)dx\,d\mu &= C_s\int_{\mathbb{R}^d}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}\int_0^\infty\frac{|\zeta|^2\mu^{1-2s}}{1+\mu^2|\zeta|^2}d\mu\,d\zeta \\ &= C_s\int_{\mathbb{R}^d}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}\int_0^\infty\frac{|\zeta|^{2s}t^{1-2s}}{1+t^2}dt\,d\zeta \\ &= C_s\frac{\pi}{2\sin(\pi s)}\int_{\mathbb{R}^d}|\zeta|^{2s}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta. \end{aligned}$$

Choosing $C_s = \frac{2\sin(\pi s)}{\pi}$ and inserting this into (7.18), we obtain a different representation for the bilinear form $a(\cdot,\cdot)$

$$a(u,w) = \frac{2\sin(\pi s)}{\pi}\int_0^\infty \mu^{1-2s}\int_{\mathbb{R}^d}(-\Delta)(I-\mu^2\Delta)^{-1}u(x)\,w(x)dx\,d\mu.$$

Again, sinc-quadrature can be employed to approximate the integral on the right-hand side. However, this leads to a non-conforming method with approximative bilinear forms, which can be analyzed by Strang-type estimates. For details, we refer to [BLP19].

Appendix A

Appendix

A.1 Gauss-Lobatto interpolation

Quadrature or best-approximation errors are usually estimated using certain interpolation operators.
Here, we consider Lagrange interpolation in some given nodes x_i. Classical examples are Chebyshev interpolation (with nodes x_i being the zeros of the Chebyshev polynomials), Gauss interpolation (with nodes x_i being the zeros of the Legendre polynomials) and Gauss-Lobatto interpolation (with nodes x_i being the extrema of the Legendre polynomials including the endpoints).
Chebyshev and Gauss-Lobatto points are optimal in the sense that the corresponding Lebesgue constants only grow logarithmically.
We now show an exponential convergence result by means of complex extension.
For $\rho > 1$, we define the Bernstein ellipse $\mathcal{E}_\rho$ by

$$\mathcal{E}_\rho := \{z \in \mathbb{C} : |z-1| + |z+1| < \rho + \rho^{-1}\}. \tag{1.1}$$

Proposition A.1. *Let u be holomorphic on $\mathcal{E}_{\widetilde{\rho}}$, $\widetilde{\rho} > 1$. Then, for every $1 < \rho < \widetilde{\rho}$, we have*

$$\inf_{v \in P_n} \|u - v\|_{L^\infty(-1,1)} \leq \frac{2}{\rho - 1} \rho^{-n} \|u\|_{L^\infty(\mathcal{E}_\rho)}. \tag{1.2}$$

Moreover, there holds

$$\inf_{v \in P_n} \|u' - v'\|_{L^\infty(-1,1)} \leq \rho^{-n} \|u\|_{L^\infty(\mathcal{E}_\rho)}. \tag{1.3}$$

Proof. We consider the 2π-periodic function $\widehat{u} := u \circ \cos : [-\pi, \pi] \to \mathbb{R}$, and express it by its Fourier series (since $\widehat{u}$ is even, no terms involving sin appear)

$$\widehat{u}(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos(\ell t) \quad a_\ell := \frac{1}{\pi} \int_{-\pi}^{\pi} \widehat{u}(t) \cos(\ell t) dt.$$

Setting $\cos(t) = x$, one obtains an expansion of u in terms of Chebyshev polynomials

$$u(x) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell C_\ell(x).$$

Since $|C_\ell(x)| \leq 1$, there holds

$$|u(x) - \frac{a_0}{2} - \sum_{\ell=1}^{n} a_\ell C_\ell(x)| \leq \sum_{\ell=n+1}^{\infty} |a_\ell|.$$

As $\frac{a_0}{2} + \sum_{\ell=1}^{n} a_\ell C_\ell(x) \in P_n$, we only need a suitable estimate for $|a_\ell|$. With the substitution $z = e^{it}$, we obtain

$$\begin{aligned} a_\ell &= \frac{1}{\pi}\int_{-\pi}^{\pi} u(\cos(t))\cos(\ell t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\mathrm{Re}\,(e^{it}))\mathrm{Re}\,(e^{i\ell t})dt \\ &= \frac{1}{\pi}\int_{|z|=1} u(\mathrm{Re}\,(z))\mathrm{Re}\,(z^\ell)\frac{1}{iz}dz. \end{aligned}$$

For $z \in \partial B_1(0)$, there holds

$$\mathrm{Re}\,(z) = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}\left(z + \frac{|z|^2}{z}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Consequently, with the Joukowsky-Transformation $J : z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$, this gives

$$\begin{aligned} a_\ell &= \frac{1}{2\pi i}\int_{|z|=1} u\left(Jz\right)\frac{z^\ell + z^{-\ell}}{z}dz \\ &= \frac{1}{2\pi i}\int_{|z|=1} u\left(Jz\right)z^{\ell-1}dz + \frac{1}{2\pi i}\int_{|z|=1} u\left(Jz\right)z^{-(\ell+1)}dz. \end{aligned}$$

We use the Cauchy integral theorem on the annulus $B_{1/\rho,\rho}(0)$. As the Joukouwsky transformation mappes each circle with radius ρ' onto the ellipse $\mathcal{E}_{\rho'}$, the function $w(z) := u\left(\frac{1}{2}(z + \frac{1}{z})\right) = u \circ J(z)$ is analytic in $B_{1/\rho,\rho}(0)$. Consequently the integrands in the integrals above are analytic in $B_{1/\rho,\rho}(0)$ and we may change the path of integration to $|z| = 1/\rho$ in the first integral and to $|z| = \rho$ in the second integral. This gives

$$\begin{aligned} a_\ell &= \frac{1}{2\pi i}\int_{|z|=\rho^{-1}} w(z)z^{\ell-1}dz + \frac{1}{2\pi i}\int_{|z|=\rho} w(z)z^{-(\ell+1)}dz \\ &\leq \frac{1}{2\pi}2\pi\|u\|_{L^\infty(\mathcal{E}_\rho)}\frac{1}{\rho}\rho^{-(\ell-1)} + \frac{1}{2\pi}2\pi\|u\|_{L^\infty(\mathcal{E}_\rho)}\rho\rho^{-(\ell+1)} \\ &= 2\rho^{-\ell}\|u\|_{L^\infty(\mathcal{E}_\rho)}. \end{aligned}$$

This imples

$$|u(x) - \frac{a_0}{2} - \sum_{\ell=1}^{n} a_\ell C_\ell(x)| \leq \sum_{\ell=n+1}^{\infty} |a_\ell| \leq 2\|u\|_{L^\infty(\mathcal{E}_\rho)}\sum_{\ell=n+1}^{\infty} \rho^{-\ell} = 2\frac{\rho^{-n}}{\rho - 1}\|u\|_{L^\infty(\mathcal{E}_\rho)},$$

which gives the first statement.
The estimate for the derivative follows by approximating u' by a polynomial of degree $k-1$ and integration. ■

The previous proposition gives an exponential approximation result for holomorphic functions. In practice, for real valued functions $u \in C^\infty(-1,1)$, it is therefore of interest to provide a criterium whether a function has a holomorphic extension, which is the subject of the following lemma from [AM15, BLM05].

Lemma A.2. *Let $u \in C^\infty(-1,1)$. Then, the following statements are equivalent*

1. *There exist constants C, γ (which may depend on u) such that*
$$\|u^{(n)}\|_{L^2(-1,1)} \leq C\gamma^n n! \qquad \forall n \in \mathbb{N}_0.$$
2. *There exists $\rho > 1$ such that u has a holomorphic extension to $\mathcal{E}_\rho$ with*
$$\|u\|_{L^\infty(\mathcal{E}_\rho)} < \infty.$$

Proof. The implication 2. $\implies$ 1. follows from the Cauchy integral formula for derivatives.
For the converse implication 1. $\implies$ 2., we start with the 1D Sobolev embedding $C([-1,1]) \hookrightarrow H^1(-1,1)$, which implies by assumption

$$\|u^{(n)}\|_{L^\infty(-1,1)} \leq C\|u^{(n)}\|_{H^1(-1,1)} \leq \widehat{C}\widehat{\gamma}^n n! \qquad \forall n \in \mathbb{N}_0.$$

As a consequence, the Taylor series of u around any point $\zeta \in [-1,1]$ converges in an open ball $B_{1/\widehat{\gamma}}(\zeta) \subset \mathbb{C}$ of radius $1/\widehat{\gamma}$ (e.g. by the quotient criterium). Thus, u has a analytic extension to $\bigcup_{\zeta\in(-1,1)} B_{1/\widehat{\gamma}}(\zeta) =: G_{\widehat{\gamma}}$ and Taylor expansion on the slightly reduced subset $G_{\widehat{\gamma}(1+\epsilon)}$ for $\epsilon > 0$ gives for any $x \in G_{\widehat{\gamma}(1+\epsilon)}$ and sufficiently close $\zeta \in (-1,1)$

$$|u(x)| = |\sum_{\ell=0}^{\infty} \frac{u^{(\ell)}(\zeta)}{\ell!}(x-\zeta)^\ell| \lesssim C_u \sum_{\ell=0}^{\infty}(1+\epsilon)^{-\ell} = C_u \frac{1+\epsilon}{\epsilon}.$$

By elementary calculations, one observes that the Bernstein ellipse $\mathcal{E}_{\rho_\epsilon}$ with $\rho_\epsilon := 1 + 1/(\widehat{\gamma}(1+\epsilon))$ satisfies $\mathcal{E}_{\rho_\epsilon} \subset G_{\widehat{\gamma}(1+\epsilon)}$. In particular, u is analytic on $\mathcal{E}_{\rho_\epsilon}$ and the bound on the L^∞-norm holds. ∎

Note that by affine transformation to the reference interval $(-1,1)$, the previous corollary also holds on an arbitrary interval $J \subset \mathbb{R}$.

Bibliography

[AB17] G. Acosta and J.P. Borthagaray. A fractional Laplace equation: regularity of solutions and finite element approximations. *SIAM J. Numer. Anal.*, 55(2):472–495, 2017.

[AG17] M. Ainsworth and C. Glusa. Aspects of an adaptive finite element method for the fractional Laplacian: a priori and a posteriori error estimates, efficient implementation and multigrid solver. *Comput. Methods Appl. Mech. Engrg.*, 327:4–35, 2017.

[AM15] T. Apel and J.M. Melenk. Interpolation and quasi-interpolation in h- and hp-version finite element spaces (extended version). Technical Report 39, Institute of Analysis and Scientific Computing, TU Wien, 2015.

[AM17] M. Ainsworth and Z. Mao. Analysis and approximation of a fractional Cahn-Hilliard equation. *SIAM J. Numer. Anal.*, 55(4):1689–1718, 2017.

[AM18] T. Apel and J.M. Melenk. Interpolation and quasi-interpolation in h- and hp-version finite element spaces. In E. Stein, R. de Borst, and T.J.R. Hughes, editors, *Encyclopedia of Computational Mechanics*, pages 1–33. John Wiley & Sons, Chichester, UK, second edition, 2018. extended preprint at `http://www.asc.tuwien.ac.at/preprint/2015/asc39x2015.pdf`.

[AS64] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[BBN+18] A. Bonito, J.P. Borthagaray, R.H. Nochetto, E. Otárola, and A.J. Salgado. Numerical methods for fractional diffusion. *Comput. Vis. Sci.*, 19(5-6):19–46, 2018.

[Beb00] M. Bebendorf. Approximation of boundary element matrices. *Numer. Math.*, 86(4):565–589, 2000.

[BFM+23] B. Bahr, M. Faustmann, C. Marcati, J.M. Melenk, and C. Schwab. Exponential convergence of hp-FEM for the integral fractional Laplacian in 1D. In *Spectral and high order methods for partial differential equations ICOSAHOM 2020+1*, volume 137 of *Lect. Notes Comput. Sci. Eng.*, pages 291–306. Springer, Cham, [2023] ©2023.

[BFM24] B. Bahr, M. Faustmann, and J.M. Melenk. An implementation of hp-fem for the fractional laplacian. *arXiv:2407.11482*, 2024.

[BG05] S. Börm and L. Grasedyck. Hybrid cross approximation of integral operators. *Numer. Math.*, 101(2):221–249, 2005.

[BLM05] Steffen Börm, Maike Löhndorf, and Jens M. Melenk. Approximation of integral operators by variable-order interpolation. *Numer. Math.*, 99(4):605–643, 2005.

[BLN22] J.P. Borthagaray, W. Li, and R.H. Nochetto. Fractional elliptic problems on lipschitz domains: Regularity and approximation. *arXiv e-prints*, page arXiv:2212.14070, 2022.

[BLP19] A. Bonito, W. Lei, and J.E. Pasciak. Numerical approximation of the integral fractional Laplacian. *Numer. Math.*, 142(2):235–278, 2019.

[BM97] C. Bernardi and Y. Maday. Spectral methods. In *Handbook of numerical analysis, Vol. V*, volume V of *Handb. Numer. Anal.*, pages 209–485. North-Holland, Amsterdam, 1997.

[BMN$^+$19] L. Banjai, J.M. Melenk, R.H. Nochetto, E. Otárola, A.J. Salgado, and C. Schwab. Tensor FEM for spectral fractional diffusion. *Found. Comput. Math.*, 19(4):901–962, 2019.

[BMS23] L. Banjai, J.M. Melenk, and Ch. Schwab. Exponential convergence of *hp* FEM for spectral fractional diffusion in polygons. *Numer. Math.*, 153(1):1–47, 2023.

[BN23a] J.P. Borthagaray and R.H. Nochetto. Besov regularity for the Dirichlet integral fractional Laplacian in Lipschitz domains. *J. Funct. Anal.*, 284(6):Paper No. 109829, 33, 2023.

[BN23b] J.P. Borthagaray and R.H. Nochetto. Constructive approximation on graded meshes for the integral fractional Laplacian. *Constr. Approx.*, 57:463–487, 2023.

[Bör10] S. Börm. *Efficient numerical methods for non-local operators*, volume 14 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2010. $\mathcal{H}^2$-matrix compression, algorithms and analysis.

[Bör17] S. Börm. Directional $\mathcal{H}^2$-matrix compression for high-frequency problems. *Numer. Linear Algebra Appl.*, 24(6):e2112, 19, 2017.

[BP15] A. Bonito and J.E. Pasciak. Numerical approximation of fractional powers of elliptic operators. *Math. Comp.*, 84(295):2083–2110, 2015.

[BS08] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[BS21] T. Betcke and M.W. Scroggs. Bempp-cl: A fast python based just-in-time compiling boundary element library. *Journal of Open Source Software*, 6(59):2879, 2021.

[BV16] C. Bucur and E. Valdinoci. *Nonlocal diffusion and applications*, volume 20 of *Lecture Notes of the Unione Matematica Italiana*. Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[CFPP14] C. Carstensen, M. Feischl, M. Page, and D. Praetorius. Axioms of adaptivity. *Comput. Math. Appl.*, 67(6):1195–1253, 2014.

[Cia13] P. Ciarlet, Jr. Analysis of the Scott-Zhang interpolation in the fractional order Sobolev spaces. *J. Numer. Math.*, 21(3):173–180, 2013.

[Clé75] P. Clément. Approximation by finite element functions using local regularization. *RAIRO Anal. Numérique*, 9:77–84, 1975.

[CS07] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[CS12] Alexey Chernov and Christoph Schwab. Exponential convergence of Gauss-Jacobi quadratures for singular integrals over simplices in arbitrary dimension. *SIAM J. Numer. Anal.*, 50(3):1433–1455, 2012.

[CS16] L.A. Caffarelli and P.R. Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(3):767–807, 2016.

[CvPS15] A. Chernov, T. von Petersdorff, and C. Schwab. Quadrature algorithms for high dimensional singular integrands on simplices. *Numer. Algorithms*, 70(4):847–874, 2015.

[DL93] R.A. DeVore and G.G. Lorentz. *Constructive Approximation.* Springer Verlag, 1993.

[DLMF] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.1.1 of 2021-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[Eva98] L.C. Evans. *Partial Differential Equations.* American Mathematical Society, 1998.

[Fae00] B. Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. I. The two-dimensional case. *IMA J. Numer. Anal.*, 20(2):203–234, 2000.

[Fae02] B. Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. II. The three-dimensional case. *Numer. Math.*, 92(3):467–499, 2002.

[FMMS22] M. Faustmann, C. Marcati, J.M. Melenk, and C. Schwab. Weighted analytic regularity for the integral fractional Laplacian in polygons. *SIAM J. Math. Anal.*, 54(6):6323–6357, 2022.

[FMMS23] M. Faustmann, C. Marcati, J.M. Melenk, and C. Schwab. Exponential convergence of *hp*-FEM for the integral fractional Laplacian in polygons. *SIAM J. Numer. Anal.*, 61(6):2601–2622, 2023.

[FMP19] M. Faustmann, J.M. Melenk, and D. Praetorius. Quasi-optimal convergence rate for an adaptive method for the integral fractional laplacian. Technical report, 2019. arXiv: 1912.09160.

[FMP21] M. Faustmann, J.M. Melenk, and D. Praetorius. Quasi-optimal convergence rate for an adaptive method for the integral fractional Laplacian. *Math. Comp.*, 90(330):1557–1587, 2021.

[GH03] L. Grasedyck and W. Hackbusch. Construction and arithmetics of $\mathcal{H}$-matrices. *Computing*, 70:295–334, 2003.

[GHS16] F.D. Gaspoz, C.-J. Heine, and K.G. Siebert. Optimal grading of the newest vertex bisection and H^1-stability of the L_2-projection. *IMA J. Numer. Anal.*, 36(3):1217–1241, 2016.

[GR97] L. Greengard and V. Rokhlin. A new version of the fast multipole method for the Laplace in three dimensions. In *Acta Numerica 1997*, pages 229–269. Cambridge University Press, 1997.

[Gra01] L. Grasedyck. *Theorie und Anwendungen Hierarchischer Matrizen.* PhD thesis, Universität Kiel, 2001.

[Gri11] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[Gru15] G. Grubb. Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators. *Adv. Math.*, 268:478–528, 2015.

[GVL13] G.H. Golub and C.F. Van Loan. *Matrix computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, fourth edition, 2013.

[Hac99] W. Hackbusch. A sparse matrix arithmetic based on $\mathcal{H}$-matrices. Part I: Introduction to $\mathcal{H}$-matrices. *Computing*, 62:89–108, 1999.

[Hac15] W. Hackbusch. *Hierarchical matrices: algorithms and analysis*, volume 49 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2015.

[HN89] W. Hackbusch and Z.P. Nowak. On the fast matrix multiplication in the boundary element method by panel clustering. *Numer. Math.*, 54:463–491, 1989.

[HPS13] H. Harbrecht, M. Peters, and M. Siebenmorgen. Combination technique based k-th moment analysis of elliptic problems with random diffusion. *J. Comput. Phys.*, 252:128–141, 2013.

[HS93] W. Hackbush and S.A. Sauter. On the efficient use of the Galerkin method to solve Fredholm integral equations. In *Proceedings of ISNA '92—International Symposium on Numerical Analysis, Part I (Prague, 1992)*, volume 38, pages 301–322, 1993.

[KM19] M. Karkulik and J.M. Melenk. $\mathcal{H}$-matrix approximability of inverses of discretizations of the fractional Laplacian. *Adv. Comput. Math.*, 45(5-6):2893–2919, 2019.

[KPP13] M. Karkulik, D. Pavlicek, and D. Praetorius. On 2D newest vertex bisection: optimality of mesh-closure and H^1-stability of L_2-projection. *Constr. Approx.*, 38(2):213–234, 2013.

[Kwa17] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.*, 20(1):7–51, 2017.

[LB92] J. Lund and K.L. Bowers. *Sinc methods for quadrature and differential equations.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[LPG+20] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, Mark M. Meerschaert, M. Ainsworth, and G.E. Karniadakis. What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys.*, 404:109009, 62, 2020.

[McL00] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[Mel02] J.M. Melenk. *hp finite element methods for singular perturbations*, volume 1796 of *Lecture Notes in Mathematics.* Springer Verlag, 2002.

[MS01] K.S. Miller and S.G. Samko. Completely monotonic functions. *Integral Transform. Spec. Funct.*, 12(4):389–402, 2001.

[NK88] Z. P. Novak and U. Khakbush. Complexity of the method of panels. In *Computational processes and systems, No. 6 (Russian)*, pages 233–244. "Nauka", Moscow, 1988.

[NOS15] R.H. Nochetto, E. Otárola, and A.J. Salgado. A PDE approach to fractional diffusion in general domains: a priori error analysis. *Found. Comput. Math.*, 15(3):733–791, 2015.

[Rat98] A. Rathsfeld. A wavelet algorithm for the boundary element solution of a geodetic boundary value problem. *Comput. Methods Appl. Mech. Engrg.*, 157(3-4):267–287, 1998. Seventh Conference on Numerical Methods and Computational Mechanics in Science and Engineering (NMCM 96) (Miskolc).

[Rat01] A. Rathsfeld. On a hierarchical three-point basis in the space of piecewise linear functions over smooth surfaces. In *Problems and methods in mathematical physics (Chemnitz, 1999)*, volume 121 of *Oper. Theory Adv. Appl.*, pages 442–470. Birkhäuser, Basel, 2001.

[Rok85] V. Rokhlin. Rapid solution of integral equations of classical potential theory. *J. Comp. Phys.*, 60:187–207, 1985.

[ROS14] X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014.

[RR04] M. Renardy and R.C. Rogers. *An introduction to partial differential equations*, volume 13 of *Texts in Applied Mathematics.* Springer-Verlag, New York, second edition, 2004.

[Sau92] S.A. Sauter. *Über die effiziente Verwendung des Galerkinverfahrens zur Lösung Fredholmscher Integralgleichungen.* PhD thesis, Universität Kiel, 1992.

[Sch98] R. Schneider. *Multiskalen- und Wavelet-Matrixkompression.* Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 1998. Analysisbasierte Methoden zur effizienten Lösung großer vollbesetzter Gleichungssysteme. [Analysis-based methods for the efficient solution of large nonsparse systems of equations].

[SS11] S.A. Sauter and C. Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011.

[ST10] P.R. Stinga and J.L. Torrea. Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differential Equations*, 35(11):2092–2122, 2010.

[Ste93] F. Stenger. *Numerical methods based on sinc and analytic functions*, volume 20 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1993.

[Ste08a] O. Steinbach. *Numerical approximation methods for elliptic boundary value problems*. Springer, New York, 2008. Finite and boundary elements, Translated from the 2003 German original.

[Ste08b] R. Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77(261):227–241, 2008.

[Sti10] P.R. Stinga. *fractional powers of second order partial differential operators: extension problem and regularity theory*. PhD thesis, 2010.

[SV14] R. Servadei and E. Valdinoci. On the spectrum of two different fractional operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 144(4):831–855, 2014.

[SZ90] L.R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[Tau03] J. Tausch. Sparse BEM for potential theory and Stokes flow using variable order wavelets. *Comput. Mech.*, 32(4-6):312–318, 2003.

[Tho84] V. Thomée. *Galerkin Finite Element Methods for Parabolic Problems*, volume 1024 of *Lecture Notes in Math.* Springer Verlag, 1984.

[TW03] J. Tausch and J. White. Multiscale bases for the sparse representation of boundary integral operators on complex geometry. *SIAM J. Sci. Comput.*, 24(5):1610–1629, 2003.

[Tyr00] E.E. Tyrtyshnikov. Incomplete cross approximation in the mosaic-skeleton method. *Computing*, 64(4):367–380, 2000. International GAMM-Workshop on Multigrid Methods (Bonn, 1998).

[vPSS97] T. von Petersdorff, C. Schwab, and R. Schneider. Multiwavelets for second-kind integral equations. *SIAM J. Numer. Anal.*, 34(6):2212–2227, 1997.

[Yos80] K. Yosida. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, sixth edition, 1980.