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On full linear convergence and optimal complexity of adaptive FEM with inexact solver

slides

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joint work with M. Brunner, M. Feischl, A. Miraçi, D. Praetorius, J. Streitberger

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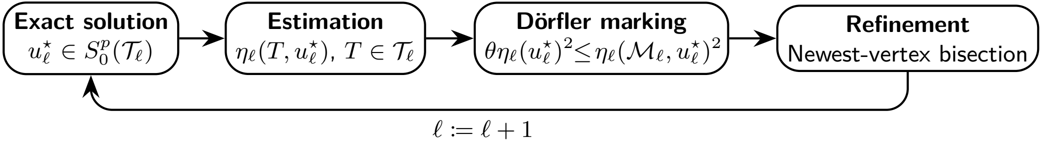
NumPDEs Workgroup on Numerics of PDEs

Standard AFEM algorithm

Input: initial triangulation $\mathcal{T}_0$, adaptivity parameter $0 < \theta \leq 1$



Exact solution
$u_\ell^\star \in S_0^p(\mathcal{T}_\ell)$

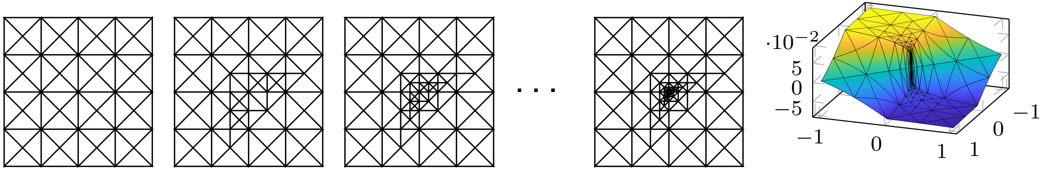
Estimation
$\eta_\ell(T, u_\ell^\star),\ T \in \mathcal{T}_\ell$

Dörfler marking
$\theta \eta_\ell(u_\ell^\star)^2 \leq \eta_\ell(\mathcal{M}_\ell, u_\ell^\star)^2$

Refinement
Newest-vertex bisection

$\ell := \ell + 1$

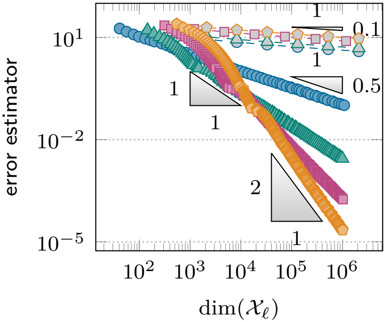
Output: triangulations $\mathcal{T}_\ell$ with corresponding discrete solution $u_\ell^\star$ and estimators $\eta_\ell(u_\ell^\star)$



Kellogg: *Applicable Anal.*, 4 (1974)

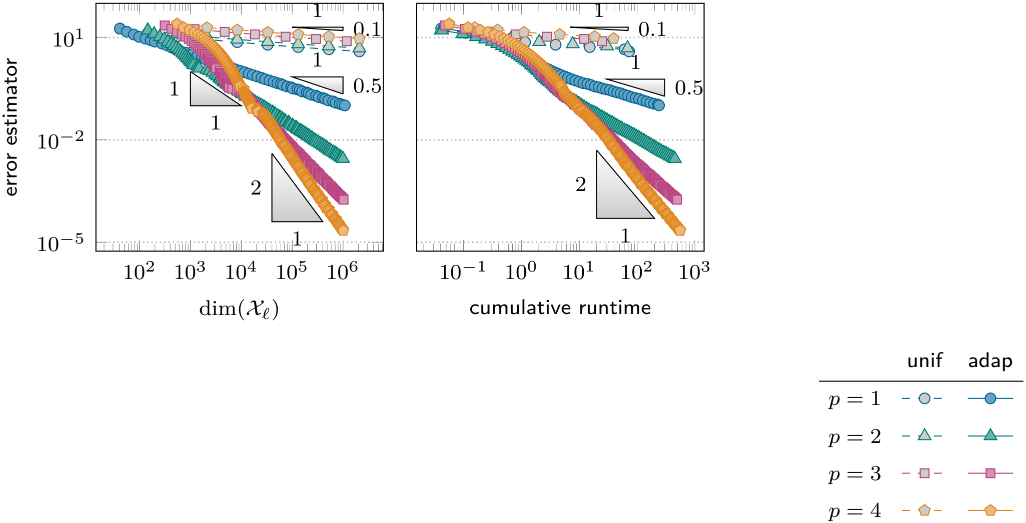
Bringmann, Miraçi and Praetorius: arXiv: 2404.07126 (2024)

How to measure convergence?

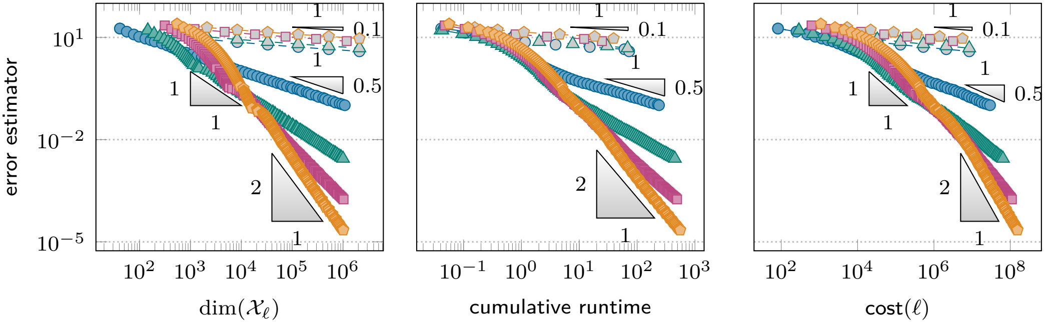


	unif	adap
$p = 1$	-○-	—●—
$p = 2$	-△-	—▲—
$p = 3$	-□-	—■—
$p = 4$	-⬠-	—⬟—

How to measure convergence?



How to measure convergence?



Linear complexity algorithms ensure

$$\text{cumulative runtime} \approx \sum_{\ell'=0}^{\ell} K[\ell'] \dim(\mathcal{X}_{\ell'}) \approx \sum_{\ell'=0}^{\ell} K[\ell'] \, \#\mathcal{T}_{\ell'} =: \mathrm{cost}(\ell)$$

	unif	adap
$p = 1$	- ○ -	—●—
$p = 2$	- △ -	—▲—
$p = 3$	- □ -	—■—
$p = 4$	- ⬠ -	—⬟—

Pioneering works on complexity of AFEM

Stopping criteria for inexact AFEM

[Becker–Johnson–Rannacher, *Computing*, 1995] algebra error control for MG
[Chaillou–Suri, *JCAM*, 2007] balancing of error components
[Arioli–Georgoulis–Loghin, *SISC*, 2013] stopping of algebraic solver in AFEM

AFEM with optimal complexity

[Binev–Dahmen–DeVore, *Numer. Math.*, 2004] with coarsening
[Stevenson, *FoCM*, 2007] for optimal convergence of final iterates
[Dahmen–Rohwedder–Schneider–Zeiser, *Numer. Math.*, 2008] for eigenvalue problem
[Becker–Mao, *M2AN*, 2009] for separate marking algorithm
[Carstensen–Gedicke, *SINUM*, 2012] for eigenvalue problem

Our main focus: Parameter-robust convergence

Abstract discrete scheme

Variational problem	Find $u^\star \in \mathcal{X} = H^1_0(\Omega)$ with $b(u^\star, v) = F(v)$ for all $v \in \mathcal{X}$
Conforming triangulation	$\mathcal{T}_H$ of bounded polytopal Lipschitz domain $\Omega \subset \mathbb{R}^d$
Discrete subspaces	$\mathcal{X}_H = \mathsf{S}^p_0(\mathcal{T}_H) \subset \mathcal{X}$ with fixed polynomial degree $p \in \mathbb{N}$
Discrete problem	Find $u^\star_H \in \mathcal{X}_H$ with $b(u^\star_H, v_H) = F(v_H)$ f.a. $v_H \in \mathcal{X}_H$
Iterative solver	$u^k_H := \Psi_H(u^\star_H; u^{k-1}_H)$ with $\Psi_H(u^\star_H)\colon \mathcal{X}_H \to \mathcal{X}_H$

A posteriori error estimation

Discretization error $\eta_H \colon \mathcal{T}_H \times \mathcal{X}_H \to \mathbb{R}_{\geq 0}$ for exact discrete solution $u_H^\star \in \mathcal{X}_H$

$$|||u^\star - u_H^\star||| \lesssim \eta_H(u_H^\star) \lesssim |||u^\star - u_H^\star||| + \operatorname{osc}_H(u_H^\star)$$

Solver error Contraction $|||u_H^\star - u_H^k||| \leq q|||u_H^\star - u_H^{k-1}|||$, $0 < q < 1$ ensures

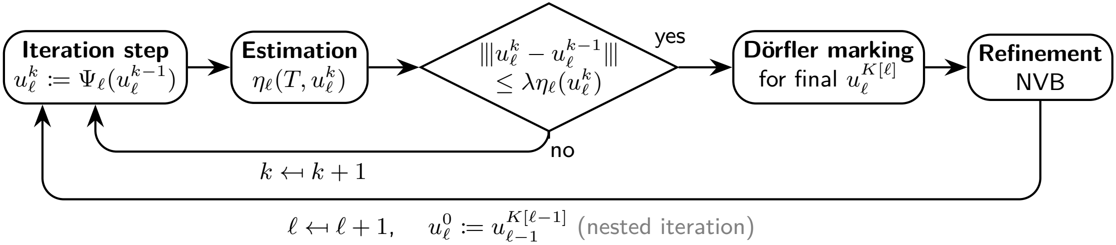
$$\frac{1-q}{q}|||u_H^\star - u_H^k||| \leq |||u_H^k - u_H^{k-1}||| \leq (1+q)\,|||u_H^\star - u_H^{k-1}|||$$

Stability of the estimator yields

$$|\eta_H(u_H^k) - \eta_H(u_H^\star)| \leq C_{\mathsf{stab}}\,|||u_H^\star - u_H^k|||$$

Adaptive FEM with inexact solvers

Input: $\mathcal{T}_0$, initial iterate u_0^0, adaptivity parameter $0 < \theta \leq 1$, stopping parameter $0 < \lambda$



Iteration step
$u_\ell^k := \Psi_\ell(u_\ell^{k-1})$

Estimation
$\eta_\ell(T, u_\ell^k)$

$|||u_\ell^k - u_\ell^{k-1}||| \leq \lambda \eta_\ell(u_\ell^k)$

yes

Dörfler marking
for final $u_\ell^{K[\ell]}$

Refinement
NVB

no

$k \mapsfrom k+1$

$\ell \mapsfrom \ell + 1, \quad u_\ell^0 := u_{\ell-1}^{K[\ell-1]}$ (nested iteration)

Output: $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$, $(u_\ell^k)_{(\ell,k) \in \mathcal{Q}}$ ordered by the step counter $|\ell, k| := \sum_{\ell'=0}^{\ell-1} K[\ell'] + k$

Pfeiler and Praetorius: *Math. Comp.*, 89 (2020)

Stevenson: *Math. Comp.*, 77 (2008)

Parameter-robust full R-linear convergence

Define quasi-error $\mathsf{H}_\ell^k := |||u_\ell^\star - u_\ell^k||| + \eta_\ell(u_\ell^k)$

Theorem Assume that the estimator η_ℓ satisfies stability, reduction, and reliability as well as (generalized) quasi-orthogonality of the exact discrete solutions. There exist $0 < q_{\mathsf{lin}} < 1$ and $C_{\mathsf{lin}} > 0$ such that for all $(\ell, k), (\ell', k') \in \mathcal{Q}$ with $|\ell', k'| < |\ell, k|$ (i.e., $u_{\ell'}^{k'}$ was computed earlier than u_ℓ^k)

$$\mathsf{H}_\ell^k \le C_{\mathsf{lin}}\, q_{\mathsf{lin}}^{|\ell,k|-|\ell',k'|}\, \mathsf{H}_{\ell'}^{k'}$$

This holds without any assumptions on the parameters $0 < \theta \le 1$ and $\lambda > 0$!

Gantner, Haberl, Praetorius and Schimanko: *Math. Comp.*, 90 (2021)

Feischl: *Math. Comp.*, 91 (2022)

Bringmann, Feischl, Miraçi, Praetorius and Streitberger: arXiv: 2311.15738 (2023)

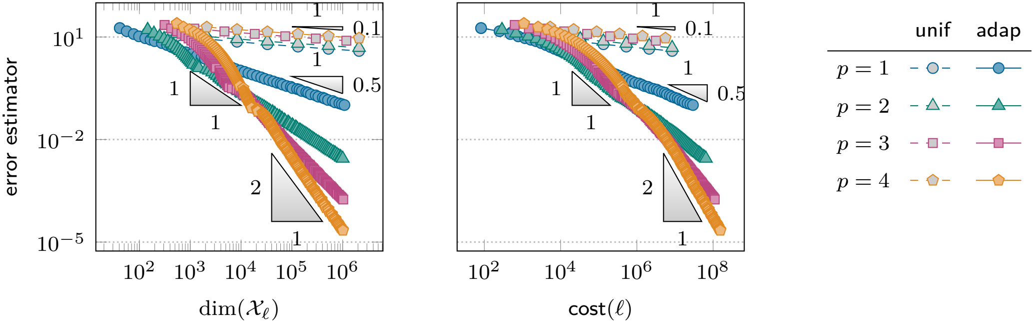
Corollary – Equivalence of rates and complexity

Assume $M(s) \coloneqq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\,\mathsf{H}_\ell^k < \infty$ for some $s > 0$

$$\begin{aligned}
\mathrm{cost}(\ell,k) \coloneqq \sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} &\le M(s)^{1/s} \sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} (\mathsf{H}_{\ell'}^{k'})^{-1/s}\\
&\le M(s)^{1/s} C_{\mathsf{lin}}^{1/s} \sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} (q_{\mathsf{lin}}^s)^{|\ell',k'|-|\ell,k|}\,(\mathsf{H}_\ell^k)^{-1/s}\\
&\le \frac{M(s)^{1/s} C_{\mathsf{lin}}^{1/s}}{1-q_{\mathsf{lin}}^{1/s}} (\mathsf{H}_\ell^k)^{-1/s}
\end{aligned}$$

Hence $M(s) \le \sup_{(\ell,k)\in\mathcal{Q}} \mathrm{cost}(\ell,k)^s\,\mathsf{H}_\ell^k \lesssim M(s)$

Equivalence of rates and complexity



Optimal complexity

Perturbation argument For $0 < \lambda < \lambda^\star$ sufficiently small

$$(1 - \lambda/\lambda^\star)\,\eta_\ell(u_\ell^K) \le \eta_\ell(u_\ell^\star) \le (1 + \lambda/\lambda^\star)\,\eta_\ell(u_\ell^K)$$

Theorem For $0 < \theta < \theta^\star$ and $0 < \lambda < \lambda^\star$ sufficiently small

$$\|u^\star\|_{\mathbb{A}_s} := \sup_{N \in \mathbb{N}_0} (N+1)^s \min_{\substack{\mathcal{T}_{\mathrm{opt}} \in \texttt{refine}(\mathcal{T}_0) \\ |\mathcal{T}_{\mathrm{opt}}| - |\mathcal{T}_0| \le N}} \eta_{\mathrm{opt}}(u_{\mathrm{opt}}^\star)$$

$$\lesssim \sup_{(\ell,k)\in\mathcal{Q}} \mathrm{cost}(\ell,k)^s \mathsf{H}_\ell^k \lesssim \max\left\{\|u^\star\|_{\mathbb{A}_s}, \mathsf{H}_0^0\right\}$$

Bringmann, Feischl, Miraçi, Praetorius and Streitberger: arXiv: 2311.15738 (2023)

Bringmann, Miraçi and Praetorius: arXiv: 2404.07126 (2024)

Iterative linearization

Strongly monotone and **Lipschitz continuous** quasi-linear PDE

$$(\mu(|\nabla u|^2)\nabla u, \nabla v)_{L^2(\Omega)} = F(v) \quad \text{for all } v \in \mathcal{X}$$

Kačanov iteration $u_\ell^{k+1,\star} := \Phi_\ell(u_\ell^{k,\underline{j}})$ defined by

$$(\mu(|\nabla u_\ell^{k,\underline{j}}|)\nabla \Phi_\ell(u_\ell^{k,\underline{j}}), \nabla v_\ell)_{L^2(\Omega)} = F(v_\ell) \quad \text{for all } v_\ell \in \mathcal{X}_\ell$$

- Haberl, Praetorius, Schimanko and Vohralík: *Numer. Math.*, 147 (2021)
- Gantner, Haberl, Praetorius and Schimanko: *Math. Comp.*, 90 (2021)
- Becker, Brunner, Innerberger, Melenk and Praetorius: *M2AN*, 57 (2023)
- Brunner, Praetorius and Streitberger: (2024)
- Miraçi, Praetorius and Streitberger: arXiv: 2401.17778 (2024)

Iterative symmetrization

General second-order linear elliptic problem

$$b(u^\star, v) := (\boldsymbol{A}\nabla u^\star, \nabla v)_{L^2(\Omega)} + (\boldsymbol{b}\cdot\nabla u^\star + c u^\star, v)_{L^2(\Omega)} = F(v) \quad \text{for all } v \in \mathcal{X}$$

Zarantonello iteration Ellipticity and boundedness of $b(\cdot,\cdot)$ with respect to $|||\cdot||| = [(\boldsymbol{A}\nabla\cdot, \nabla\cdot)_{L^2(\Omega)}]^{1/2}$ ensures contraction of $u_\ell^{k+1,\star} := \Phi_\ell(u_\ell^{k,\underline{j}})$ defined by

$$(\nabla\Phi_\ell(u_\ell^{k,\underline{j}}), \nabla v_\ell)_{L^2(\Omega)} = (\nabla u_\ell^{k,\underline{j}}, \nabla v_\ell)_{L^2(\Omega)} + \delta\,[F(v_\ell) - b(u_\ell^{k,\underline{j}}, v_\ell)] \quad \text{for all } v_\ell \in \mathcal{X}_\ell$$

for sufficiently small damping parameter $0 < \delta \le \delta^\star$

Brunner et al. *IMAJNA*, in print (2023)

Goal-oriented adaptivity

Dual problem Find $z^\star \in \mathcal{X}$ with $b(v, z^\star) = G(v)$ for all $v \in \mathcal{X}$

Goal-error estimate The discrete goal functional

$$G_H(u_H, z_H) \coloneqq G(u_H) + [F(z_H) - a(u_H, z_H)]$$

leads to the improved estimate

$$|G(u^\star) - G_H(u_H^\star, z_H^\star)| \le |||u^\star - u_H^\star||| \; |||z^\star - z_H^\star||| \lesssim \eta_H(u_H^\star)\zeta_H(z_H^\star)$$

📄 Becker, Gantner, Innerberger and Praetorius: *Numer. Math.*, 153 (2023)

📄 Bringmann, Brunner, Praetorius and Streitberger: *JNUM* (2024)

Thank you for your attention!

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FWF Österreichischer Wissenschaftsfonds

Publications & Preprints

P. Bringmann, M. Feischl, A. Miraçi, D. Praetorius and J. Streitberger (2023). *On full linear convergence and optimal complexity of adaptive FEM with inexact solver*. arXiv: `2311.15738`

P. Bringmann, A. Miraçi and D. Praetorius (2024). ‘Iterative solvers in adaptive FEM’. *Error Control, Adaptive Discretizations, and Applications*. Ed. by F. Chouly, S. P. Bordas, R. Becker and P. Omnes. *AAM*. Accepted. arXiv: `2404.07126`

P. Bringmann, M. Brunner, D. Praetorius and J. Streitberger (2024). *Optimal complexity of goal-oriented adaptive FEM for nonsymmetric linear elliptic PDEs*. *JNUM* 2024. Accepted for publication. Preprint under `arxiv.org:2312.00489`

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