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AKNUM: Non-local opertors

Markus Faustmann

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About me

Markus Faustmann, Institute ASC

- studied at TU Wien (PhD 2015)
- since 2021 permanent at ASC TU Wien (senior scientist)
- research focus: numerical analysis of PDEs, non-local operators
- teaching: lectures for mathematicians (numerics, analysis), CSE
  - special lectures: numerics on quantum computers, non-local operators, hierarchical matrices

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- office: DA 04 L 02 (Freihaus, 4th floor green)
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Formalities

Format: Blackboard Lecture

Lecture

- 2 academic hours = 1h30 per week
- date: Wednesday 13:15-14:45
  - room: small seminar room DA04G10 (Freihaus, 4th floor green, inside the institute)

- lecture notes on my website (will be updated)
  - https://www.tuwien.at/mg/asc/faustmann/lehre/nichtlokale-operatoren
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Assumed Knowledge

Course is in the curriculum

- Technical Mathematics (066 394), elective, AKNUM catalogue

It is assumed that the hearer has basic knowledge on

- Finite element methods (e.g. done in Numerics of partial differential equations: stationary problems – 101.506)
- PDE analysis (e.g. done in Partial Differential Equations – 101.803)
- Numerical integration (e.g. done in Numerical analysis – 101.320)

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Contents of the lecture

Local Operators

Local operators - definition

X, Y ... function spaces of functions $u : \mathbb{R}^d \to \mathbb{R}$

$\mathcal{A} : X \to Y$ is local, if for $u \in X$, $x \in \mathbb{R}^d$

$$(\mathcal{A}u)(x) \qquad \text{only depends on values} \qquad u|_{B_\varepsilon(x)} \ \ \forall \varepsilon > 0$$

Example: differential operators, e.g. Laplacian on $X = C^2(\mathbb{R}^d)$

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(x)$$

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Non-local Operators

Non-local operators

$\mathcal{A} : X \to Y$ is non-local, if definition of local (prev. slide) does not hold.

In other words: $(\mathcal{A}u)(x)$ does not only depend on values of u close to x.

Toy example: integral operator on $X = L^2(0,1)$, $x \in (0,1)$ arbitrary

$$\mathcal{A}u(x) := \int_0^1 (x-y)u(y)\,dy$$

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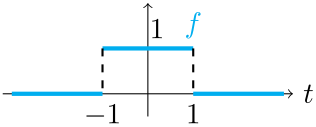
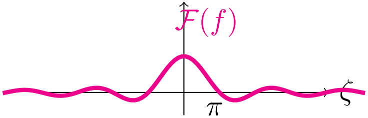
Non-local Operators - Examples

- The Fourier transform

$$\mathcal{F}(f)(\zeta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\zeta t} dt$$





- Physics examples: gravity, quantum entanglement, Coulomb potentials, . . .

- This lecture: singular integral operators of convolution type

$$Au(x) := \int k(x-y)u(y)\, dy, \qquad k \ldots \text{ kernel fct., singular for } x = y$$

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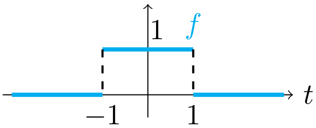

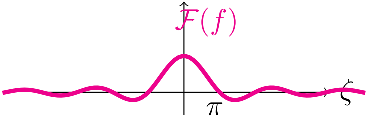

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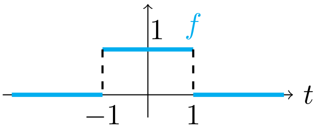

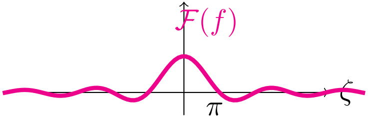

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Integral Operators of Convolution Type - Example 1

Reformulation of PDE on unbounded domain (very difficult for FEM only!)

$$-\Delta u = f \qquad \text{in } \mathbb{R}^d \backslash \Omega$$
$$u|_{\partial\Omega} = g \qquad \text{on } \partial\Omega$$

Green's formula $\Longrightarrow$ representation formula for solution

$$u(x) = \widetilde{N} f(x) + \widetilde{V} \partial_n u(x) - \widetilde{K} g(x) \quad \text{for } x \in \Omega,$$

with non-local operators Newton, single-layer, double-layer potential ($G \dots$ fundamental sol.)

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$$g(x) = N_0 f(x) + V\phi(x) - (K - 1/2)g(x) \quad \text{for } x \in \partial\Omega$$

Can be solved with Galerkin method (”boundary element method” – BEM)

Advantage: equation posed on $\partial\Omega$ (bounded!!)

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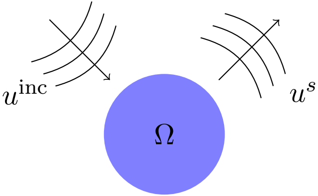
Wave Scattering

Problem: Send wave $u^{\mathrm{inc}}(x) = e^{ikx}$ onto object Ω – compute the scattered wave u^s



PDE for $u = u^{\mathrm{inc}} + u^s$: Helmholtz equation + Sommerfeld radation condition

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^d \backslash \Omega$$

$$\partial_r u^s - iku^s = o(r^{-1}) \qquad r = |x|$$

Corresponding integral equation

$$u(x) = u^{\mathrm{inc}} - \int_{\partial\Omega} G(x-y)\partial_n u(y)dy \qquad G(x-y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

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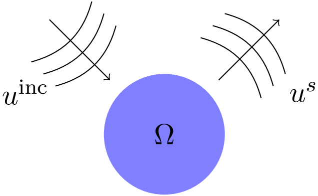

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Wave Scattering – Simulation

Simulation: bempp (Univ. College London)

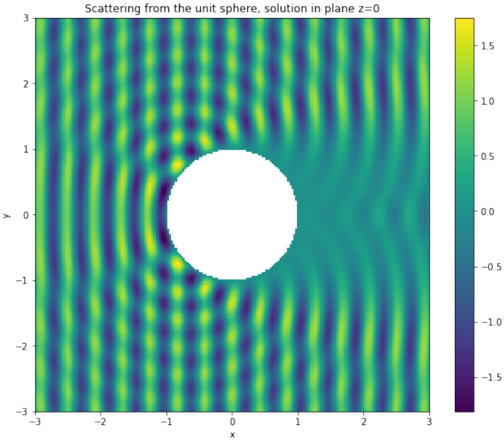

Figure: Scattering at unit sphere ($d = 3$), solution plot in the plane $z = 0$

Fractional Differential Operators

Main model problem of this lecture

$$(-\Delta)^s \qquad \text{for} \quad s \in (0,1)$$

Formal definition – plenty ideas in literature:

- Fourier transform: $\mathcal{F}^{-1}(|\zeta|^{2s}\,\mathcal{F}u)$
- Integral fractional Laplacian:

$$(-\Delta)^s u(x) := C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy$$

- Spectral fractional Laplacian: (λ_k, φ_k) … eigenpair for Laplacian

$$(-\Delta)^s_\sigma u := \sum_{k=1}^{\infty} \lambda_k^s u_k \varphi_k, \qquad u_k := \int_\Omega u \varphi_k dx$$

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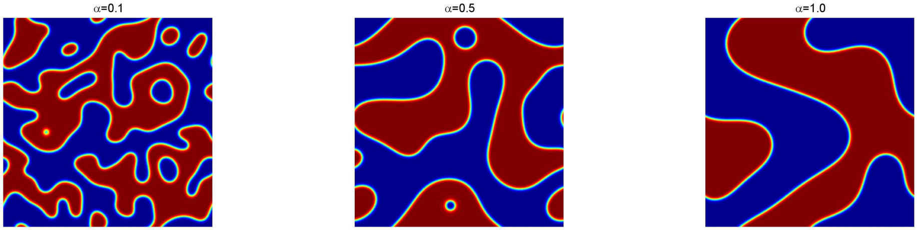
Fractional Models – Including Non-local Effects

Gives additional parameter for more accurate models (e.g. physics, finance,...)

Example: phase separation by (non-local) Cahn-Hilliard equation

$$\partial_t u + (-\Delta)^{\alpha}_{\sigma}(-\Delta u + f(u)) = 0$$



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- Are different definitions of $(-\Delta)^s$ equivalent?
- How to do a reasonable FEM for PDEs of the form $(-\Delta)^s u = f$?
- How to derive bounds for the FEM error and improve convergence rates?
- Are solutions to fractional PDEs smooth?
- How to deal with fully populated matrices arising from non-locality?

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