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Functional a-posteriori error estimates for BEM

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Slides

TU WIEN ASC INSTITUTE OF ANALYSIS AND SCIENTIFIC COMPUTING

NumPDEs Workgroup on Numerics of PDEs

FWF Österreichischer Wissenschaftsfonds

Introduction

Model problem

PDE model problem

- $\Delta u = 0$ in $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$
- $u = g$ on $\Gamma = \partial\Omega$

indirect BEM ansatz

- $u = \widetilde{V}\phi$ in Ω with unknown $\phi \in H^{-1/2}(\Gamma)$

direct BEM ansatz

- $u = \widetilde{V}\phi - \widetilde{K}g$ in Ω with unknown $\phi \in H^{-1/2}(\Gamma)$

Boundary element method

indirect BEM

- solve $V\phi = g$ on Γ
- compute $\phi_h \approx \phi$ by Galerkin / collocation / ...
- obtain approximation $u \approx u_h := \widetilde{V}\phi_h$

direct BEM

- solve $V\phi = (K + 1/2)g$ on Γ
- compute $\phi_h \approx \phi$ by Galerkin / collocation / ...
- obtain approximation $u \approx u_h := \widetilde{V}\phi_h - \widetilde{K}g$

- sought: $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- known: $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

- note: This observation is independent of how ϕ_h is computed!

Quantity of interest?

- integral density ϕ?
  - makes sense for direct BEM, where $\phi_h \approx \phi = \partial_n u$

  $\Longrightarrow$ $\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)}$ should be controlled / small
  - Stephan, Carstensen '95ff., Heuer, Maischak, Funken, Praetorius, ... / Faermann '00, '02 / 'Steinbach '00 / Dahmen, Harbrecht, Schneider '06 / Bakry, Pernet, Collino '17 / ...

- point value $u(x)$?
  - particular strength of BEM!

  $\Longrightarrow$ error control by product of primal and dual errors (resp. estimators)
  - Feischl, Praetorius et al. '16 / Bakry '17 / Harbrecht, Moor '19

- potential u?
  - e.g., in exterior domains

  $\Longrightarrow$ $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ should be controlled / small
  - focus of the present talk!

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numer. Math.*, 147 (2021)

Functional error estimates

Functional error identity

Theorem

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

$$\Longrightarrow \quad \boxed{\max_{\substack{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) \\ \nabla \cdot \boldsymbol{\tau} = 0}} \underline{\mathfrak{M}}(\boldsymbol{\tau}; u_h|_\Gamma, g) \; = \; \|\nabla(u - u_h)\|_\Omega^2 \; = \min_{\substack{w \in H^1(\Omega) \\ w|_\Gamma = g - u_h|_\Gamma}} \overline{\mathfrak{M}}(\nabla w)}$$

- $\underline{\mathfrak{M}}(\boldsymbol{\tau}; u_h|_\Gamma, g) := 2\,\langle g - u_h|_\Gamma \,,\, \boldsymbol{\tau}|_\Gamma \cdot \boldsymbol{n}\rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2$
- $\overline{\mathfrak{M}}(\nabla w) := \|\nabla w\|_\Omega^2$

- obtain lower bound by choosing $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)$ with $\nabla \cdot \boldsymbol{\tau} = 0$
- obtain upper bound by choosing $w \in H^1(\Omega)$ with $w|_\Gamma = g - u_h|_\Gamma$
- note: bounds come with known constant 1

Kurz, Pauly, Praetorius, Repin, Sebastian: *Numer. Math.*, 147 (2021)

Proof of upper bound

Upper bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega = \min_{\substack{w \in H^1(\Omega)\\ w|_\Gamma = g - u_h|_\Gamma}} \|\nabla w\|_\Omega$$

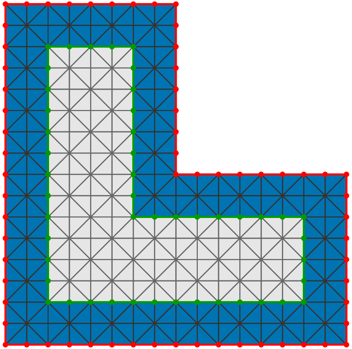
- $v \in H^1(\Omega)$ with $v|_\Gamma = g = u|_\Gamma$
- $\|\nabla(u - u_h)\|_\Omega^2 = \underbrace{\langle \nabla(u - v)\,,\, \nabla(u - u_h)\rangle_\Omega}_{=0} + \langle \nabla(v - u_h)\,,\, \nabla(u - u_h)\rangle_\Omega$

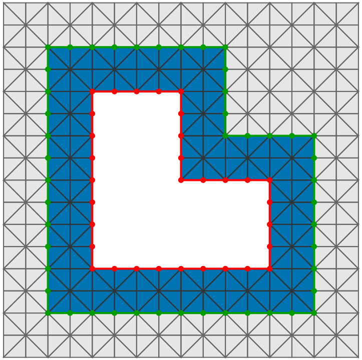
$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega \le \|\nabla(v - u_h)\|_\Omega$$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega = \min_{\substack{v \in H^1(\Omega)\\ v|_\Gamma = g}} \|\nabla(v - u_h)\|_\Omega = \min_{\substack{w \in H^1(\Omega)\\ w|_\Gamma = g - u_h|_\Gamma}} \|\nabla w\|_\Omega$$

Computable upper bound 1/3

- goal: cheap computation of good $w \in H^1(\Omega)$ with $w|_\Gamma = g - u_h|_\Gamma$
- approach: employ FEM on boundary layer (to approx. $u - u_h \in H^1$)





- problem: FEM functions cannot satisfy continuous trace $g - u_h|_\Gamma$
- solution: allow for data oscillation terms

Computable upper bound 2/3

Perturbed upper bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$ and $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$
- $\mathcal{S}_h \subset H^1(\omega)$ FEM space on boundary layer $\omega \subseteq \Omega$
- $J_h : H^{1/2}(\Gamma) \to \{v_h|_\Gamma \, : \, v_h \in \mathcal{S}_h\}$

$$\Longrightarrow \quad \|\nabla(u-u_h)\|_\Omega \le \min_{\substack{w \in H^1(\Omega) \\ w|_\Gamma = J_h(g - u_h|_\Gamma)}} \|\nabla w\|_\Omega + \|(1-J_h)(g-u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}$$

- known: $\|\nabla(u-u_h)\|_\Omega = \min_{\substack{\overline{w} \in H^1(\Omega) \\ \overline{w}|_\Gamma = g - u_h|_\Gamma}} \|\nabla \overline{w}\|_\Omega$
- choose $v \in H^1(\Omega)$ with $\Delta v = 0$ and $v|_\Gamma = (1-J_h)(g-u_h|_\Gamma)$

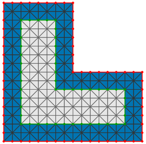
$$\Longrightarrow \quad \|\nabla(u-u_h)\|_\Omega \le \min_{\substack{\overline{w} \in H^1(\Omega) \\ \overline{w}|_\Gamma = g - u_h|_\Gamma}} \|\nabla(\overline{w}-v)\|_\Omega + \|\nabla v\|_\Omega$$

- choose $w := \overline{w} - v$

Computable upper bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$ and $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$
- $\mathcal{S}_h \subset H^1(\omega)$ FEM space on boundary layer $\omega \subseteq \Omega$
- $J_h : H^{1/2}(\Gamma) \to \{ v_h|_\Gamma \,:\, v_h \in \mathcal{S}_h \}$
- compute FEM solution $w_h \in \mathcal{S}_h^\star := \{ v_h \in \mathcal{S}_h \,:\, v_h|_{\partial\omega\setminus\Gamma} = 0 \}$ s.t.
  - $w_h|_\Gamma = J_h(g - u_h|_\Gamma)$
  - $\langle \nabla w_h \,,\, \nabla v_h \rangle_\omega = 0 \quad \forall w_h \in \mathcal{S}_h^\star$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega \le \|\nabla w_h\|_\omega + \|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}$$



- $w_h|_{\partial\omega\setminus\Gamma} = 0$ allows to extend to $w_h \in H^1(\Omega)$
- later: $\mathcal{S}_h = \mathcal{S}^1(\mathcal{T}_h|_\omega)$ p/w affine, globally continuous

Proof of lower bound

Lower bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

$$\Longrightarrow \quad \max_{\substack{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) \\ \nabla\cdot\boldsymbol{\tau}=0}} \Big(2\,\langle g - u_h|_\Gamma \,,\, \boldsymbol{\tau}|_\Gamma \cdot \boldsymbol{n} \rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2 \Big) \;=\; \|\nabla(u-u_h)\|_\Omega^2$$

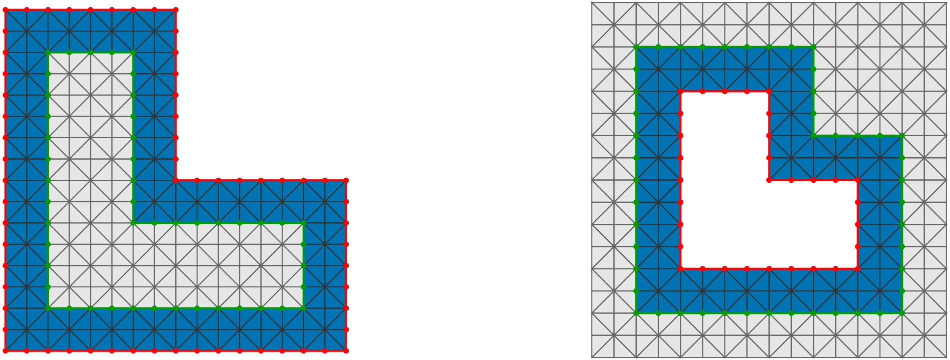
- $\|x\|_{\mathcal{H}}^2 = \max_{y\in\mathcal{H}} \big(2\langle x\,,\,y\rangle_{\mathcal{H}} - \|y\|_{\mathcal{H}}^2\big)$ in any Hilbert space $\mathcal{H}$
- $\nabla(u-u_h) \in \mathcal{H} := \big\{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega) \,:\, \nabla\cdot\boldsymbol{\sigma} = 0\big\}$

$$\Longrightarrow \quad \|\nabla(u-u_h)\|_\Omega^2 = \max_{\boldsymbol{\tau}\in\mathcal{H}} \Big(2\,\langle \nabla(u-u_h)\,,\,\boldsymbol{\tau}\rangle_\Omega - \|\boldsymbol{\tau}\|_\Omega^2\Big)$$

$+$ integration by parts

Computable lower bound 1/2

- goal: cheap computation of good $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{\tau} = 0$
- approach: mixed FEM on boundary layer (for $\nabla(u - u_h) \in \boldsymbol{H}(\mathrm{div})$)



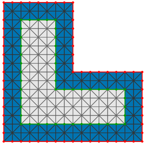
Computable lower bound 2/2

Computable lower bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$ and $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$
- $\mathcal{RT}^q(\mathcal{T}_h) \subset \boldsymbol{H}(\mathrm{div}, \omega)$ Raviart–Thomas space on boundary layer $\omega \subseteq \Omega$
- $\mathcal{RT}^q_\star(\mathcal{T}_h) := \big\{\boldsymbol{\sigma}_h \in \mathcal{RT}^q \, : \, \boldsymbol{\sigma}_h|_{\partial\omega\setminus\Gamma} \cdot \boldsymbol{n} = 0\big\}$
- compute FEM solution $(\boldsymbol{\tau}_h, p_h) \in \mathcal{RT}^q_\star(\mathcal{T}_h) \times \mathcal{P}^q(\mathcal{T}_h)$ s.t.
  - $\langle \boldsymbol{\tau}_h \, , \, \boldsymbol{\sigma}_h \rangle_\omega + \langle \nabla \cdot \boldsymbol{\sigma}_h \, , \, p_h \rangle_\omega = \langle g - u_h|_\Gamma \, , \, \boldsymbol{\sigma}_h|_\Gamma \cdot \boldsymbol{n} \rangle_\Gamma \qquad \forall \boldsymbol{\sigma}_h \in \mathcal{RT}^q_\star(\mathcal{T}_h)$
  - $\langle \nabla \cdot \boldsymbol{\tau}_h \, , \, q_h \rangle_\omega = 0 \qquad \forall q_h \in \mathcal{P}^q(\mathcal{T}_h)$

$$\Longrightarrow \quad \Big(2\,\langle g - u_h|_\Gamma \, , \, \boldsymbol{\tau}_h|_\Gamma \cdot \boldsymbol{n} \rangle_\Gamma - \|\boldsymbol{\tau}\|_\omega{}^2\Big) \quad \le \quad \|\nabla(u - u_h)\|_\Omega^2$$

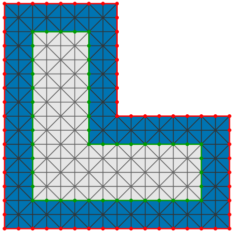
- $\boldsymbol{\tau}_h|_{\partial\omega\setminus\Gamma} \cdot \boldsymbol{n} = 0$ allows to extend to $\boldsymbol{\tau}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$
- note: unique solution, but unclear if $0 \le \mathrm{LHS}$

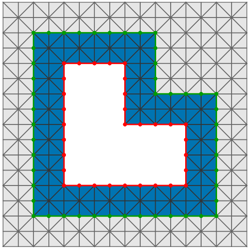


Adaptive algorithm

Comments

- error bounds are independent of ω and Ω
- no reason to fix boundary layer $\omega \subseteq \Omega$
- ω should be shrunken to preserve dimension reduction of BEM
- for numerics:
  - start with triangulation $\mathcal{T}_h$ of some boundary layer $\widehat{\omega} \subset \Omega$
  - always extract $\mathcal{T}_h^\Gamma := \mathcal{T}_h|_\Gamma$
  - always extract $\mathcal{T}_h^\omega := \mathcal{T}_h|_\omega$ as, e.g., 2nd order patch of Γ





Adaptive algorithm

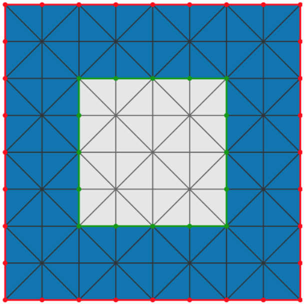
Iterate the following loop, until $\|w_h\|_\omega \le \text{tolerance}$

1. extract BEM triangulation $\mathcal{T}_h^\Gamma := \mathcal{T}_h|_\Gamma$
2. extract $\mathcal{T}_h^\omega := \{T \in \mathcal{T}_h \,:\, \exists T' \in \mathcal{T}_h, \quad T' \cap \Gamma \neq \emptyset \neq T \cap T'\}$
3. define boundary layer $\omega := \bigcup \mathcal{T}_h^\omega$
4. compute (Galerkin) BEM approximation $\phi \approx \phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
5. compute $J_h(g - u_h|_\Gamma)$ and data oscillations $\mathrm{osc}_h(T)$ for all $T \in \mathcal{T}_h^\omega$
6. compute FEM solution $w_h \in \mathcal{S}^1(\mathcal{T}_h^\omega)$ for the majorant $\|w_h\|_\omega$
7. assemble error estimator $\eta_h(T)^2 = \|w_h\|_T^2 + \mathrm{osc}_h(T)^2$ for $T \in \mathcal{T}_h^\omega$
8. choose $\mathcal{M}_h \subseteq \mathcal{T}_h^\omega$ s.t. $\theta \sum_{T \in \mathcal{T}_h^\omega} \eta_h(T)^2 \le \sum_{T \in \mathcal{M}_h} \eta_h(T)^2$

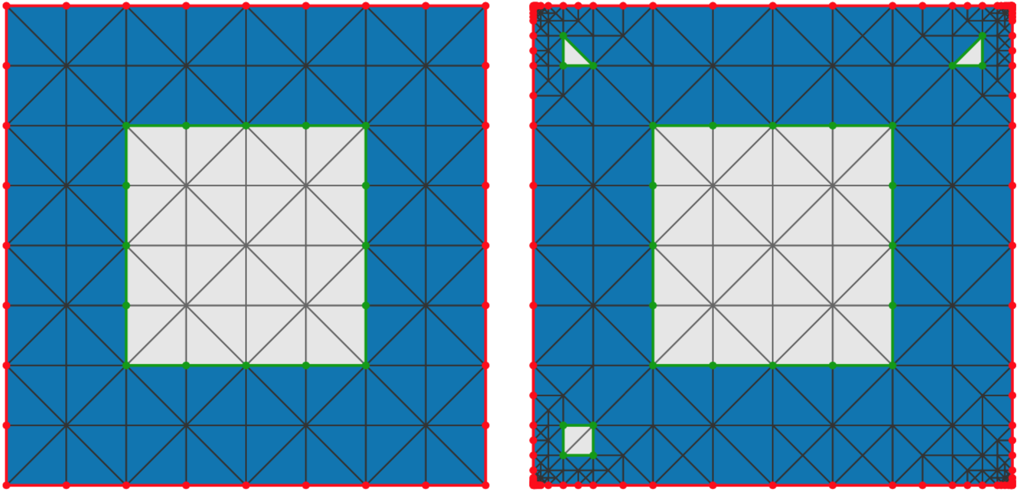
Indirect BEM:
smooth potential, square domain

Smooth potential

- $\Omega = [0, 1/2]^2$
- $u(x) = \cosh(x_1)\cos(x_2)$
- indirect BEM formulation $V\phi = u|_\Gamma$
- Galerkin BEM with $\phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
- note: u is smooth, but ϕ is non-smooth

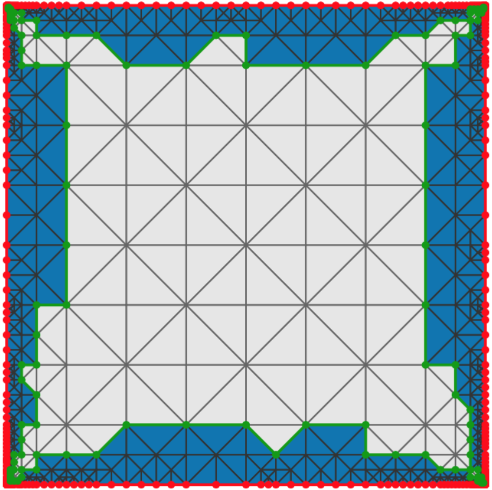


Adaptively generated meshes 1/2

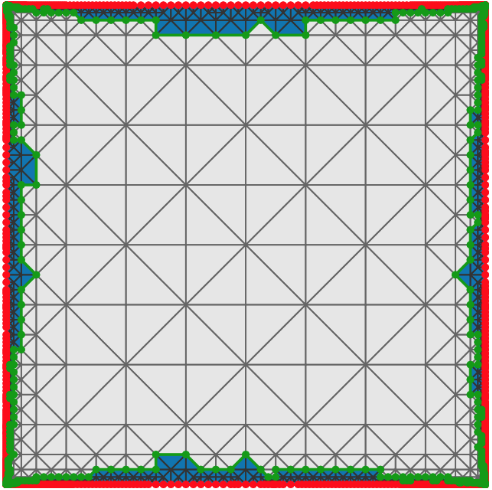


a) $\#\mathcal{T}_h^S = 72$, $\#\mathcal{F}_h^\Gamma = 32$, $\ell = 0$ b) $\#\mathcal{T}_h^S = 314$, $\#\mathcal{F}_h^\Gamma = 119$, $\ell = 17$

Adaptively generated meshes 2/2

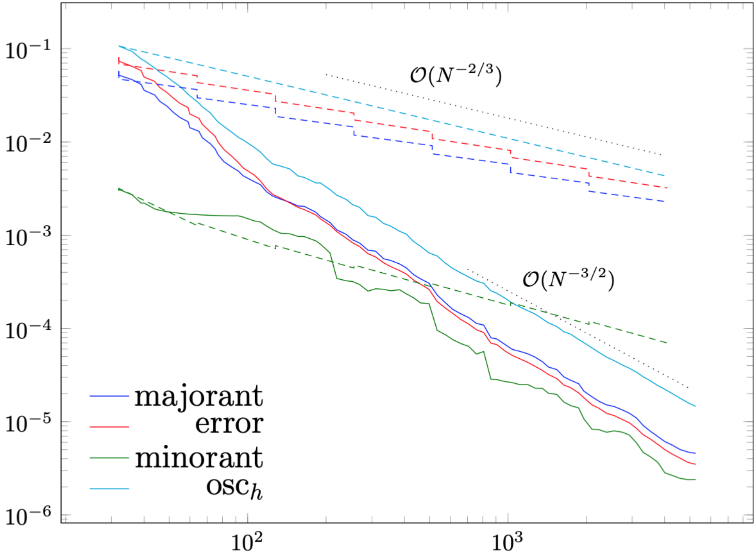


c) $\#\mathcal{T}_h^S = 923$, $\#\mathcal{F}_h^\Gamma = 352$, $\ell = 27$

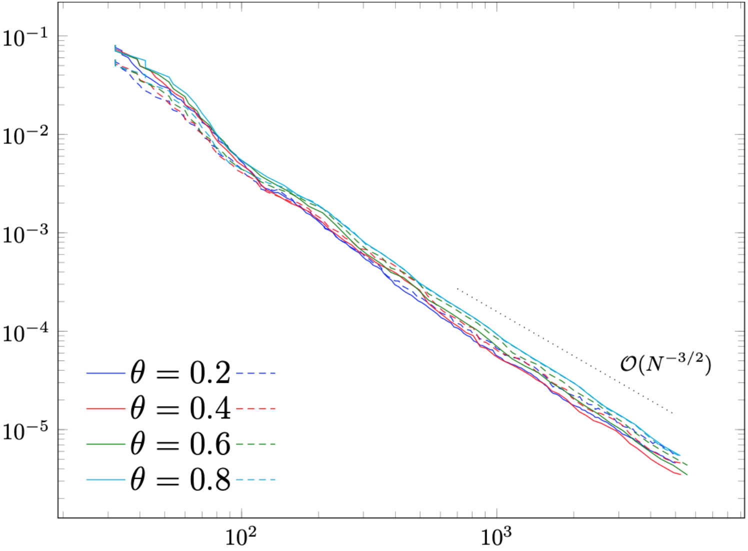


d) $\#\mathcal{T}_h^S = 3176$, $\#\mathcal{F}_h^\Gamma = 1148$, $\ell = 38$

Numerical results 1/3



Numerical results 2/3



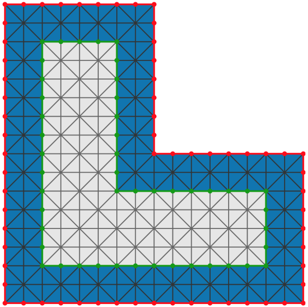
Numerical results 3/3

ℓ	$\#\mathcal{F}_h^\Gamma$	$\frac{\#\mathcal{T}_h^S}{\#\mathcal{F}_h^\Gamma}$	$\mathrm{dof}(\mathcal{T}_h^S)$	$\Vert\nabla(u-u_h)\Vert_{L^2(\Omega)}$	$\Vert\nabla w_h\Vert_{L^2(S)}$	$\frac{\Vert\nabla w_h\Vert_{L^2(S)}}{\Vert\nabla(u-u_h)\Vert_{L^2(\Omega)}}$
0	32	2.25	15	$8.01e-2$	$5.75e-2$	0.71
4	40	2.33	28	$4.97e-2$	$3.58e-2$	0.72
10	59	2.44	60	$2.33e-2$	$1.65e-2$	0.71
16	77	2.66	103	$9.95e-3$	$7.29e-3$	0.73
22	112	2.60	148	$3.81e-3$	$3.48e-3$	0.91
28	165	2.82	234	$1.88e-3$	$2.03e-3$	1.08
34	253	2.83	343	$8.27e-4$	$8.92e-4$	1.08
40	383	2.81	512	$4.18e-4$	$4.91e-4$	1.18
46	575	2.70	707	$1.66e-4$	$1.89e-4$	1.14
52	860	2.63	978	$6.96e-5$	$7.94e-5$	1.14
58	1072	2.61	1389	$3.92e-5$	$4.92e-5$	1.25
64	1869	2.61	2008	$2.04e-5$	$2.55e-5$	1.25
70	2748	2.58	2803	$1.06e-5$	$1.34e-5$	1.27
76	4007	2.55	3976	$5.00e-6$	$6.12e-6$	1.22
80	5259	2.53	5077	$3.50e-6$	$4.58e-6$	1.31

Indirect BEM: non-smooth potential

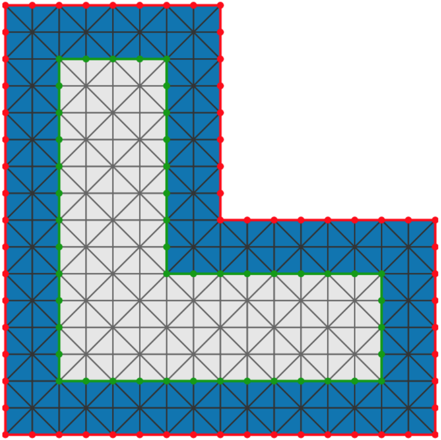
Non-smooth potential

- $\Omega \subsetneqq [0,1/2]^2$ L-shaped domain
- $u(x) = r^{2/3}\cos(2\varphi/3)$ w.r.t. re-entrant corner
- indirect BEM formulation $V\phi = u|_\Gamma$
- Galerkin BEM with $\phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
- note: u is non-smooth and ϕ is non-smooth



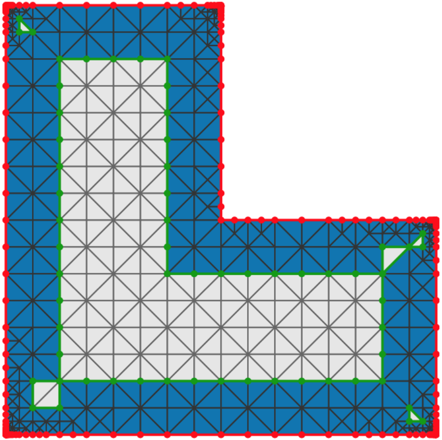
Adaptively generated meshes 1/2

$\ell = 0$



$\#\mathcal{T}_h^S = 168, \ \#\mathcal{F}_h^\Gamma = 64$

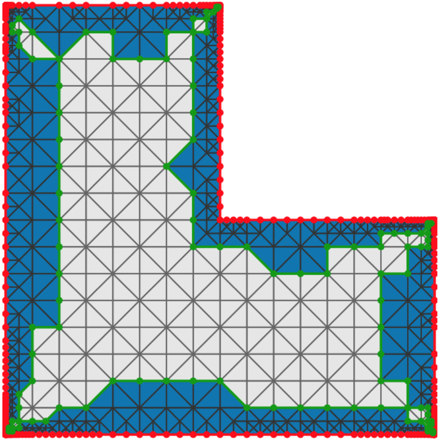
$\ell = 20$



$\#\mathcal{T}_h^S = 514, \ \#\mathcal{F}_h^\Gamma = 201$

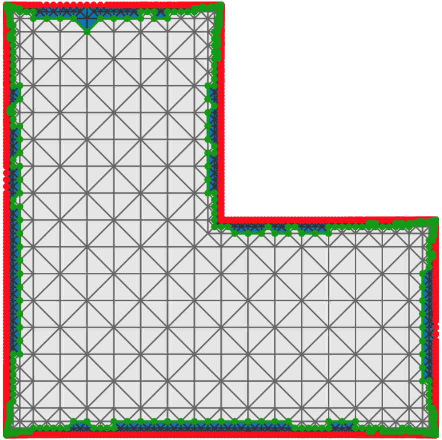
Adaptively generated meshes 2/2

$\ell = 26$



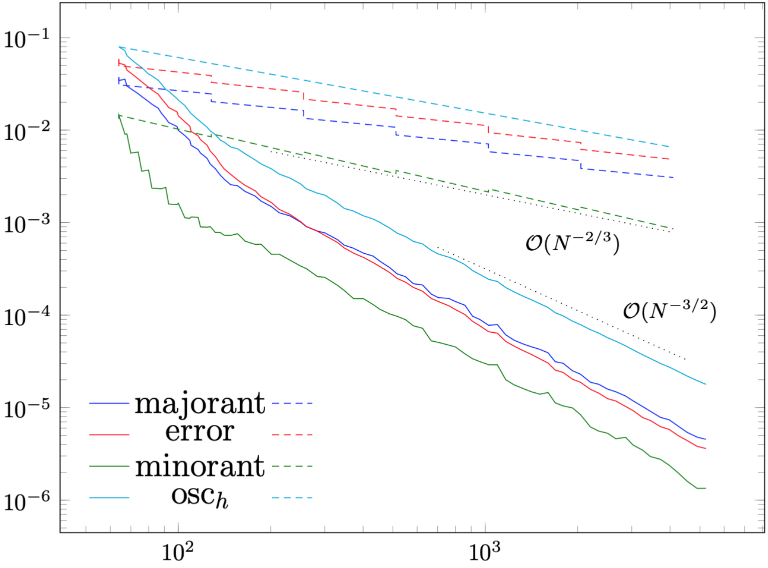
$\#\mathcal{T}_h^S = 919, \ \#\mathcal{F}_h^\Gamma = 368$

$\ell = 38$



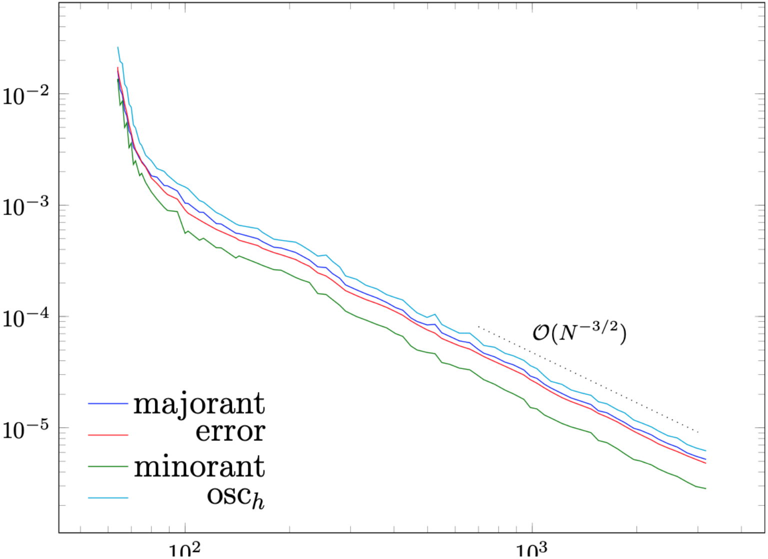
$\#\mathcal{T}_h^S = 3177, \ \#\mathcal{F}_h^\Gamma = 1216$

Numerical results



Direct BEM:
non-smooth potential

Numerical results 1/2



Numerical results 2/2

ℓ	$\#\mathcal{F}_h^\Gamma$	$\frac{\#\mathcal{T}_h^S}{\#\mathcal{F}_h^\Gamma}$	$\mathrm{dof}(\mathcal{T}_h^S)$	$\|\nabla(u-u_h)\|_{L^2(\Omega)}$	$\|\nabla w_h\|_{L^2(S)}$	$\frac{\|\nabla w_h\|_{L^2(S)}}{\|\nabla(u-u_h)\|_{L^2(\Omega)}}$
0	64	2.63	33	$1.74e-2$	$1.61e-2$	0.93
6	69	2.62	37	$5.30e-3$	$4.64e-3$	0.88
12	77	2.69	42	$2.20e-3$	$2.22e-3$	1.01
18	100	2.57	46	$9.06e-4$	$1.04e-3$	1.15
24	140	2.50	79	$5.07e-4$	$5.60e-4$	1.10
30	208	2.58	159	$3.25e-4$	$3.76e-4$	1.16
36	279	2.61	248	$1.88e-4$	$2.21e-4$	1.18
42	404	2.59	381	$1.10e-4$	$1.20e-4$	1.09
48	549	2.62	531	$6.35e-5$	$7.14e-5$	1.12
54	780	2.67	790	$3.93e-5$	$4.33e-5$	1.10
60	1085	2.67	1131	$2.28e-5$	$2.46e-5$	1.08
66	1550	2.73	1695	$1.35e-5$	$1.42e-5$	1.05
72	2203	2.71	2372	$7.78e-6$	$7.92e-6$	1.02
78	3166	2.74	3442	$4.80e-6$	$5.20e-6$	1.08

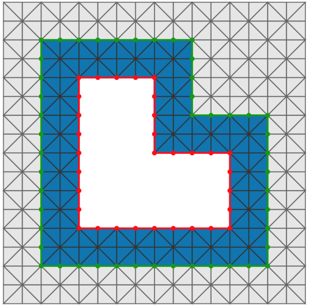
Direct exterior BEM: non-smooth potential

Non-smooth potential

- $\mathbb{R}^2 \backslash \Omega \subsetneq [0, 1/2]^2$ L-shaped domain
- $g \equiv 1 = (1/2 - K)1$ on Γ

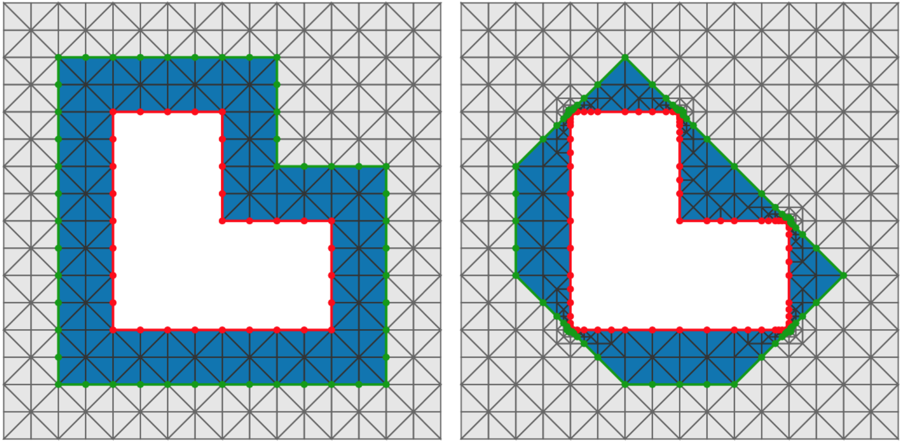
$\Longrightarrow$ indirect BEM $=$ direct BEM

- BEM formulation $V\phi = 1$
- Galerkin BEM with $\phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
- note: u is non-smooth and ϕ is non-smooth



Adaptively generated meshes 1/2

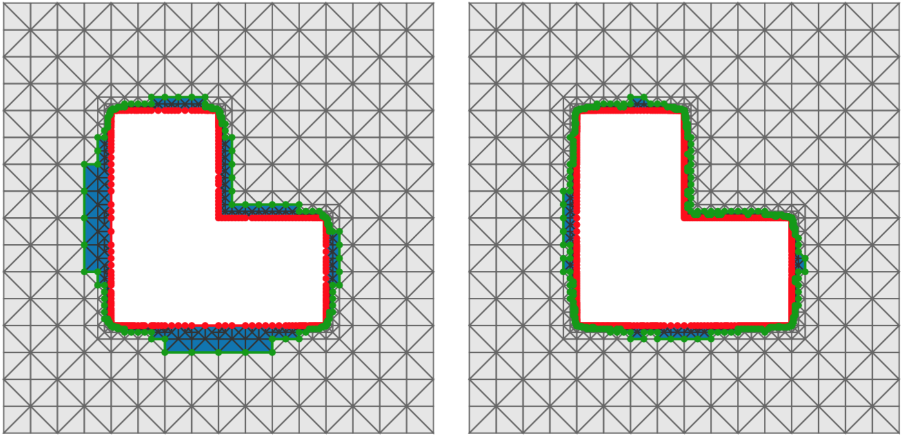
$\ell = 0$ $\qquad\qquad$ $\ell = 13$



$\#\mathcal{T}_h^S = 120,\ \#\mathcal{F}_h^\Gamma = 32$ $\qquad$ $\#\mathcal{T}_h^S = 347,\ \#\mathcal{F}_h^\Gamma = 113$

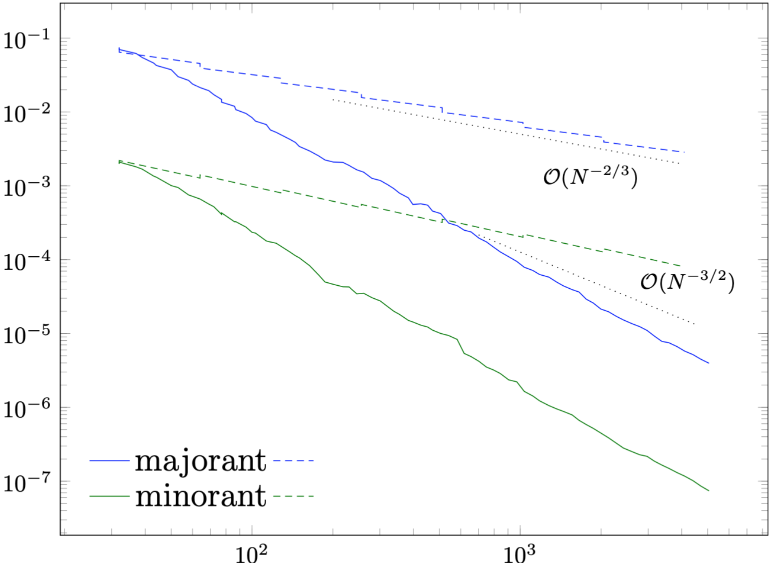
Adaptively generated meshes 2/2

$\ell = 27$ $\qquad$ $\ell = 35$



$\#\mathcal{T}_h^S = 1316,\ \#\mathcal{F}_h^\Gamma = 461$ $\qquad$ $\#\mathcal{T}_h^S = 2891,\ \#\mathcal{F}_h^\Gamma = 1024$

Numerical results



Conclusion

What did I present?

- new philosophy for a-posteriori error estimation for BEM
  - it is hardly needed that $\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \approx \mathrm{small}$
  - it is rather needed that $\|\nabla(u - u_h)\|_\Omega \approx \mathrm{small}$
- simple FEM-based strategy for a-posteriori error estimation
  - $2\,\langle g - u_h|_\Gamma\,,\, \boldsymbol{\tau}_h|_\Gamma \cdot \boldsymbol{n}\rangle_\Gamma - \|\boldsymbol{\tau}\|_\omega^2 \leq \|\nabla(u - u_h)\|_\Omega^2 \leq \big(\|\nabla w_h\|_\omega + \mathrm{osc}_h\big)^2$
  - known constants 1 in lower and upper estimate
- upper bound can drive an adaptive algorithm
- empirically $\|\nabla(u - u_h)\|_\Omega \approx \|\nabla w_h\|_\omega$ for lowest-order 2D Galerkin BEM
  - i.e., osc_h only employed for refinement, but negligible for error estimation
- analysis works for arbitrary $\phi_h \approx \phi$ and 2D / 3D
  - Galerkin / collocation
  - exact / inexact solution

What remains open?

Empirical experiments:

- How can we include inexact solvers into the adaptive algorithm?
- How does the new strategy perform for 3D BEM?
  - Galerkin BEM vs. collocation
  - standard discretization vs. isogeometric analysis
  - higher order BEM
  - anisotropic triangulations

Numerical analysis:

- Can we guarantee (optimal) convergence of the adaptive algorithm?
- Can we extend the argument to other relevant PDEs?
- Can we guarantee that $\text{lower bound} \simeq \text{upper bound}$?

Thank you for your attention!

Kurz, Pauly, Praetorius, Repin, Sebastian

Functional a posteriori error estimates for boundary element methods

Numer. Math., 147 (2021)

How to localize the data oscillations?

- computable upper bound $\|\nabla(u - u_h)\|_\Omega \leq \|\nabla w_h\|_\omega + \mathrm{perturbation}$
- $\mathrm{perturbation} := \|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)} \lesssim \mathrm{osc}_h$

Known upper bound for $\|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}$

- standard-FEM to compute $w_h \in \mathcal{S}^p(\mathcal{T}_h|_\omega)$
- J_h one of the following operators
  - H^1-stable $L^2(\Gamma)$-orthogonal projection
  - L^2-variant of Scott–Zhang projection
  - nodal interpolation for $p = 1$ and $d = 2$
- $g \in H^1(\Omega)$
- $\phi_h \in L^2(\Gamma)$

$\Longrightarrow$ $g - u_h|_\Gamma \in H^1(\Gamma)$

- $\|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)} \leq C\,\|h^{1/2}\nabla\big[(1 - J_h)(g - u_h|_\Gamma)\big]\|_{L^2(\Gamma)}$

Aurada, Feischl, Kemetmüller, Page, Praetorius: *ESAIM:M2AN*, 47 (2013)

Functional error identity

Theorem

- $u \in \boldsymbol{H}(\mathrm{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ with $\nabla \times (\nabla \times \boldsymbol{u}) = 0$, $\nabla \cdot \boldsymbol{u} = 0$, and $\boldsymbol{n} \times \boldsymbol{u}|_\Gamma = \boldsymbol{g}$
- $u_h \in \boldsymbol{H}(\mathrm{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ with $\nabla \times (\nabla \times \boldsymbol{u}_h) = 0$, $\nabla \cdot \boldsymbol{u}_h = 0$

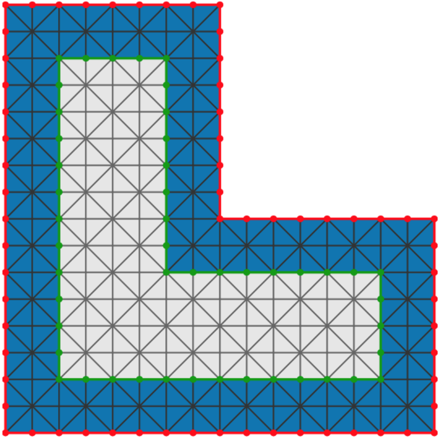
$$\Longrightarrow \boxed{\max_{\substack{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) \\ \nabla \times \boldsymbol{\tau} = 0}} \underline{\mathfrak{M}}(\boldsymbol{\tau}; \boldsymbol{u}_h, \boldsymbol{g}) = \|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|_\Omega^2 = \min_{\substack{\boldsymbol{w} \in \boldsymbol{H}(\mathrm{curl},\Omega) \\ \boldsymbol{n} \times \boldsymbol{w}|_\Gamma = \boldsymbol{g} - \boldsymbol{n} \times \boldsymbol{u}_h|_\Gamma}} \overline{\mathfrak{M}}(\boldsymbol{w})}$$

- $\underline{\mathfrak{M}}(\boldsymbol{\tau}; \boldsymbol{u}_h, \boldsymbol{g}) := 2\,\langle \boldsymbol{g} - \boldsymbol{n} \times \boldsymbol{u}_h|_\Gamma\,,\, \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\tau}|_\Gamma)\rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2$
- $\overline{\mathfrak{M}}(\boldsymbol{w}) := \|\nabla \times \boldsymbol{w}\|_\Omega^2$

Indirect BEM:
smooth potential, L-shaped domain

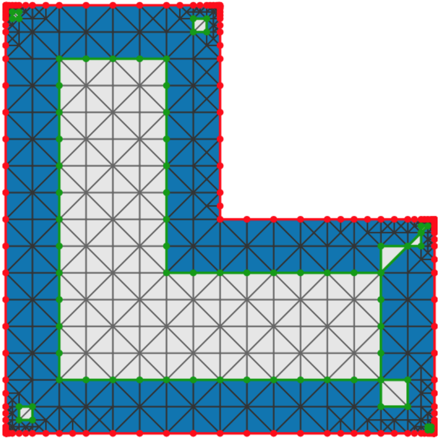
Adaptively generated meshes 1/2

$\ell = 0$



$\#\mathcal{T}_h^S = 168,\ \#\mathcal{F}_h^\Gamma = 64$

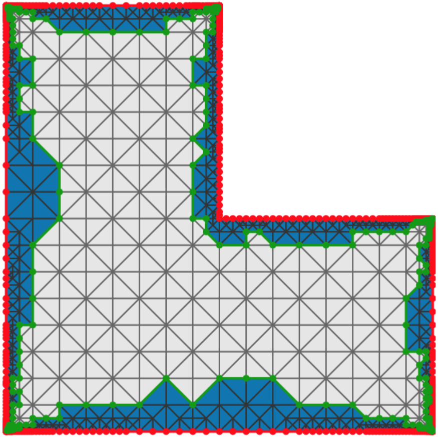
$\ell = 17$



$\#\mathcal{T}_h^S = 532,\ \#\mathcal{F}_h^\Gamma = 193$

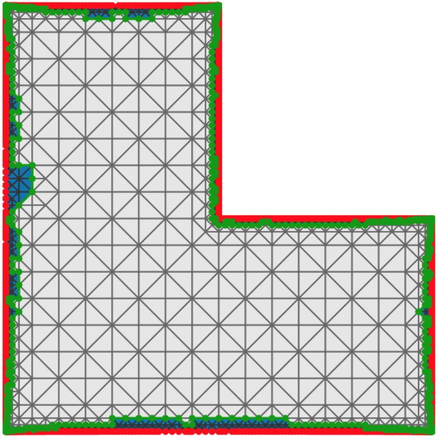
Adaptively generated meshes 2/2

$\ell = 27$



$\#\mathcal{T}_h^S = 1505, \ \#\mathcal{F}_h^\Gamma = 562$

$\ell = 38$



$\#\mathcal{T}_h^S = 4943, \ \#\mathcal{F}_h^\Gamma = 1835$

Numerical results