A MULTI-PARAMETER SINGULAR PERTURBATION ANALYSIS OF THE ROBERTSON MODEL

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I. Why are we interested?

- The classical **Robertson model** is given by

$$\begin{aligned} \dot{x} &= -k_1 x + k_3 y z \\ \dot{y} &= k_1 x - k_2 y^2 - k_3 y z \\ \dot{z} &= k_2 y^2, \end{aligned} \tag{1}$$

with $k_1 = 4 \cdot 10^{-2}$, $k_2 = 3 \cdot 10^7$, $k_3 = 10^4$ and initial value $(x, y, z)^T = (1, 0, 0)^T$.

- Simple qualitative dynamics of (1): Convergence to equilibrium $(0, 0, 1)^T$
$\implies$ Interested in multi-scale structure of solutions, see Figure 1.

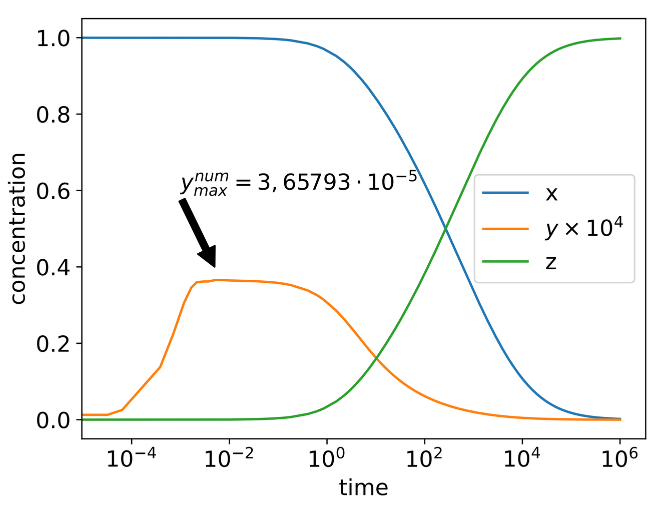


Figure 1: Numerical simulation of (1).

- Dynamics on widely different time scales for $0 < k_1, k_3 \ll k_2$.
- Logarithmic time scale in Figure 1.
- 3 phases of reaction: Fast - intermediate - slow.
- Similar phenomena observed in many chemical and biological systems.
- **Now:** Asymptotic analysis with GSPT.

II. A two-parameter singular perturbation problem

- Use conservation of mass $x(t) = c - y(t) - z(t)$ for all $t \geq 0$ and $c > 0$.
- Assume $k_1, k_3 \ll k_2$, change to fast time $\tau = k_2 t$, define $\frac{k_1}{k_2} =: \varepsilon_1$, $\frac{k_3}{k_2} =: \varepsilon_2$

$$\begin{aligned} y' &= \varepsilon_1 (c - y - z) - y^2 - \varepsilon_2 y z \\ z' &= y^2, \end{aligned} \tag{2}$$

where $0 < \varepsilon_1, \varepsilon_2 \ll 1$, with initial value $O = (0, 0)^T$ and equilibrium $Q = (0, c)^T$.

- The layer problem ($\varepsilon_1 = \varepsilon_2 = 0$, Figure 2)

$$\begin{aligned} y' &= -y^2 \\ z' &= y^2 \end{aligned} \tag{3}$$

has a very degenerate (double zero eigenvalue) critical manifold $\mathcal{S} = \{(y, z)^T \in \mathbb{R}^2 : y = 0\}$.
- Detailed asymptotic structure depends sensitively on $(\varepsilon_1, \varepsilon_2)^T \approx (0, 0)^T$.

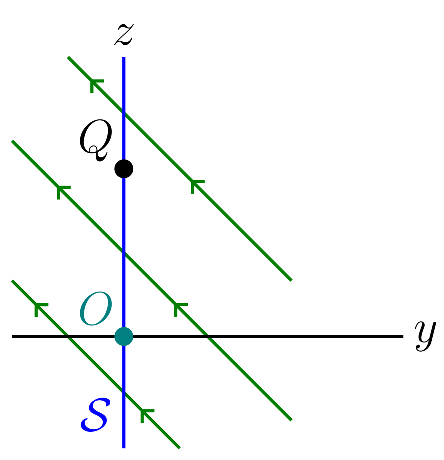
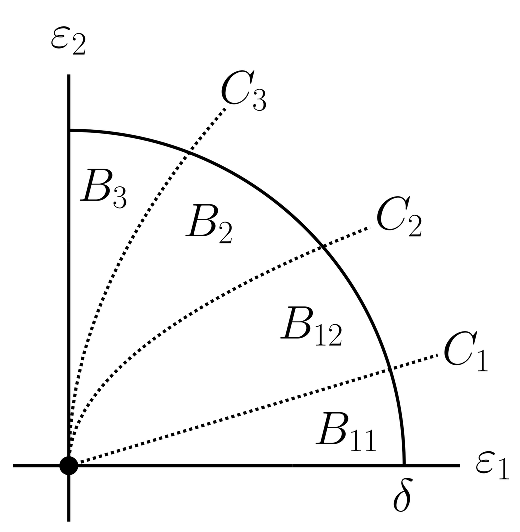


Figure 2: Orbits of (3).

III. Main result

1. In $\varepsilon_1{}^2 + \varepsilon_2{}^2 < \delta$, with $\delta > 0$ there exist four regions B_{11}, B_{12}, B_2, B_3 corresponding to different slow-fast structures of (2).
2. For each B_{11}, B_{12}, B_2, B_3 there exists a different type of singular orbit γ_0 connecting $O = (0, 0)^T$ to $Q = (0, c)^T$.
3. Solutions converge to γ_0 in Hausdorff distance as $(\varepsilon_1, \varepsilon_2)^T \to (0, 0)^T$ in B_{11}, B_{12}, B_2, B_3.



IV. Perform two parameter blow-ups to apply GSPT

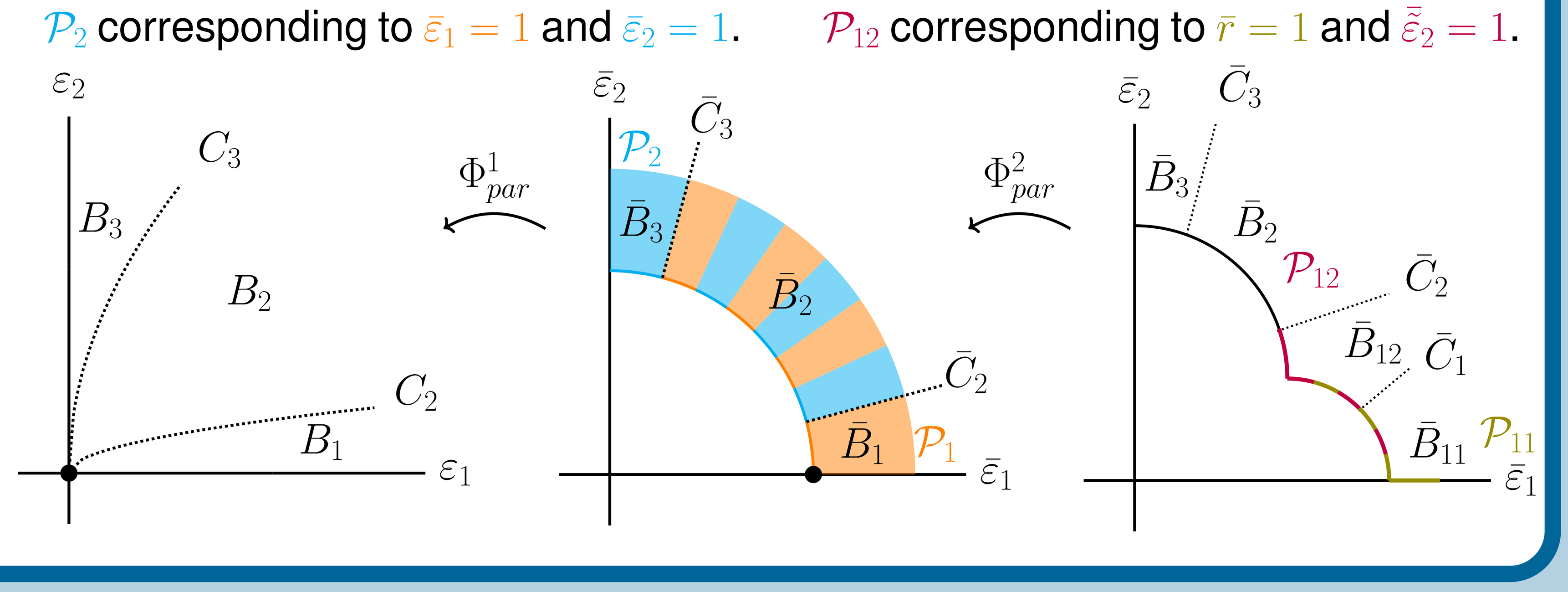
- Three regions B_1, B_2 and B_3 corresponding to $\varepsilon_2^2 \ll \varepsilon_1$, $\varepsilon_1 \approx \varepsilon_2^2$, $\varepsilon_1 \ll \varepsilon_2^2$.
- Separated by $C_2 : \varepsilon_1 = \beta_2 \varepsilon_2^2$ and $C_3 : \varepsilon_1 = \beta_3 \varepsilon_2^2$ with $0 < \beta_3 < \beta_2$.
- Blow-up transformation Φ^1_{par} given by

$$\begin{aligned} \varepsilon_1 &= r^2 \bar{\varepsilon}_1 \\ \varepsilon_2 &= r \bar{\varepsilon}_2, \end{aligned}$$

with $r \in [0, \infty)$ and $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) \in \mathbb{S}^1$.
- Analysis in directional charts $\mathcal{P}_1$ and $\mathcal{P}_2$ corresponding to $\bar{\varepsilon}_1 = 1$ and $\bar{\varepsilon}_2 = 1$.
- Blow-up transformation Φ^2_{par} given by

$$\begin{aligned} r &= s \bar{r} \\ \bar{\varepsilon}_2 &= s \bar{\bar{\varepsilon}}_2. \end{aligned}$$

with $s \in [0, \infty)$ and $(\bar{r}, \bar{\bar{\varepsilon}}_2) \in \mathbb{S}^1$.
- Analysis in directional charts $\mathcal{P}_{11}$ and $\mathcal{P}_{12}$ corresponding to $\bar{r} = 1$ and $\bar{\bar{\varepsilon}}_2 = 1$.

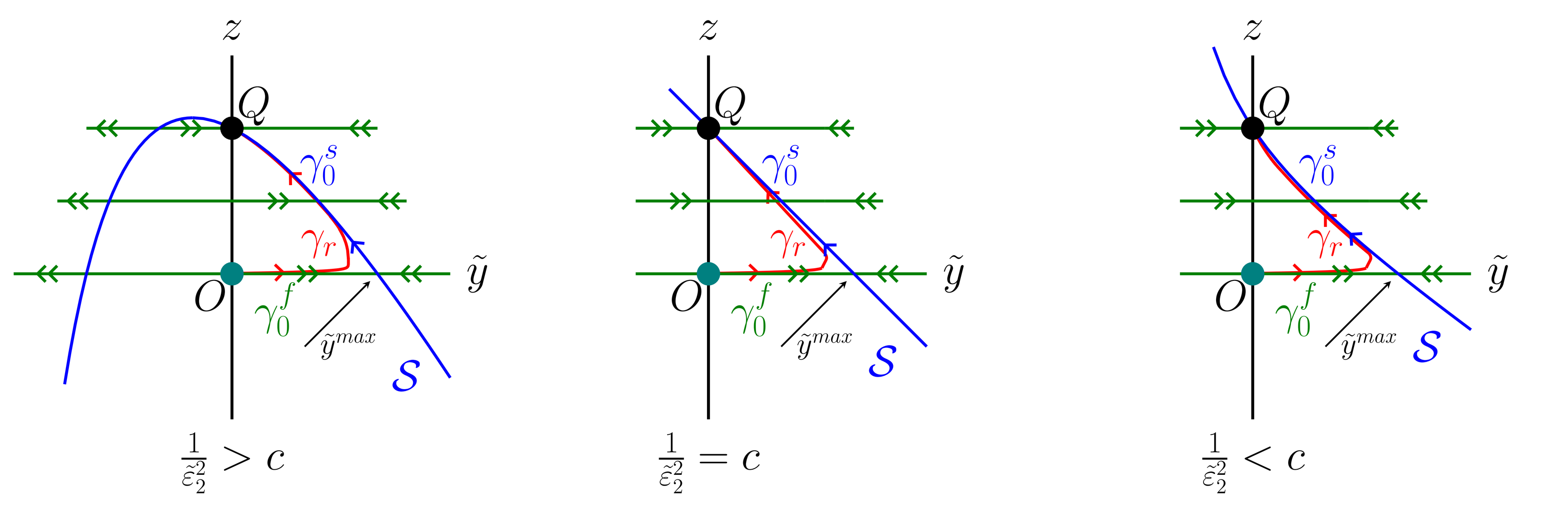


V. Analysis in region $\bar{B}_2$, i.e., $\tilde{\varepsilon}_2 \in [1/\sqrt{\beta_2}, 1/\sqrt{\beta_3}]$

- Rescaling $y = r\tilde{y}$ and chart $\mathcal{P}_1$ ($\varepsilon_1 = r^2$, $\varepsilon_2 = r\tilde{\varepsilon}_2$) gives:

$$\begin{aligned} \tilde{y}' &= c - r\tilde{y} - z - \tilde{y}^2 - \tilde{\varepsilon}_2 \tilde{y} z \\ z' &= r\tilde{y}^2. \end{aligned} \tag{4}$$

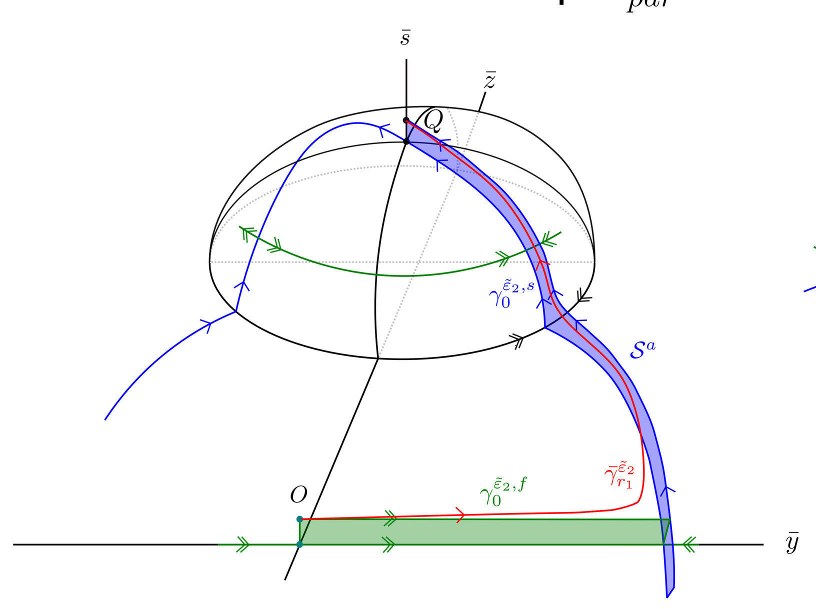
- Slow-fast in r with critical manifold $\mathcal{S} = \{(\tilde{y}, z)^T \in \mathbb{R}^2 : c - z - \tilde{y}^2 - \tilde{\varepsilon}_2 \tilde{y} z = 0\}$.



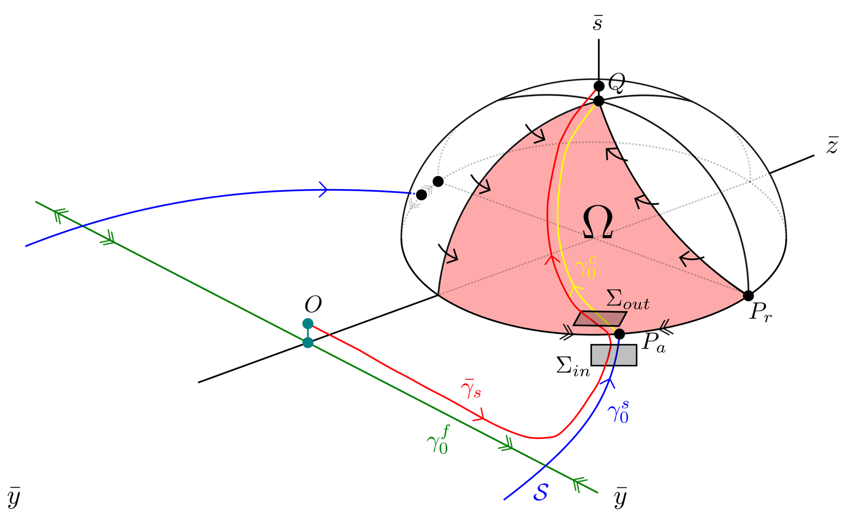
- Fenichel: $\exists r_0 > 0 \, \forall r \in (0, r_0] \, \exists$ orbit γ_r, $\mathcal{O}(r)$-close to $\gamma_0 := \gamma_0^f \cup \gamma_0^s$.
- Validate numerics: $y^{max} = \sqrt{\varepsilon_1 c} + \mathcal{O}(\varepsilon_1) \approx 3,651 \cdot 10^{-5} + \mathcal{O}(10^{-9})$.

VI. Fold point of $\mathcal{S}$ for $\varepsilon_2 = 0$ in region $\bar{B}_1$

- Normal hyperbolicity of $\mathcal{S}$ is lost at $Q = (\tilde{y}, z, \tilde{\varepsilon}_2)^T = (0, c, 0)^T$ in (4)
$\implies$ Parameter blow-up Φ^2_{par} and blow-up $\tilde{y} = \sigma\bar{\tilde{y}}$, $z = c + \sigma^2 \bar{z}$, and $s = \sigma\bar{s}$.



Region B_{12}



Region B_{11}

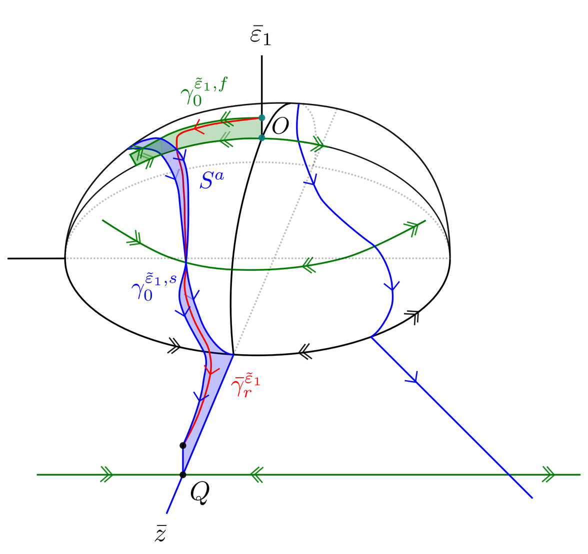
- 2-D normally attracting critical manifold $\mathcal{S}^a$.
- Family of singular orbits $\gamma_0^{\tilde{\varepsilon}_2} = \gamma_0^{\tilde{\varepsilon}_2,s} \cup \gamma_0^{\tilde{\varepsilon}_2,f}$ perturbes for $\tilde{\varepsilon}_2 \in (0, 1/\sqrt{\beta_2}]$ and $0 < r_1 \ll 1$ to $\bar{\gamma}_{r_1}^{\tilde{\varepsilon}_2}$.
- Singular orbit $\gamma_0 = \gamma_0^f \cup \gamma_0^s \cup \gamma_0^c$ connecting O and Q.
- Existence of perturbed orbit $\bar{\gamma}_s$ for $0 < s \ll 1$: Analysis of classical fold point and trapping region Ω.

VII. Transcritical bifurcation of $\mathcal{S}$ for $\tilde{\varepsilon}_1$ in region $\bar{B}_3$

- Studied in chart $\mathcal{P}_2$ ($\varepsilon_1 = r^2 \tilde{\varepsilon}_1$, $\varepsilon_2 = r$):

$$\begin{aligned} \tilde{y}' &= \tilde{\varepsilon}_1 (c - r\tilde{y} - z) - \tilde{y}^2 - \tilde{y} z \\ z' &= r\tilde{y}^2. \end{aligned}$$

- Critical manifold $\mathcal{S} : \tilde{\varepsilon}_1 (c - z) - \tilde{y}^2 - \tilde{y} z = 0$ not hyperbolic at $O = (\tilde{y}, z, \tilde{\varepsilon}_1)^T = (0, 0, 0)^T$.
$\implies$ Blow-up $\tilde{y} = \sigma\bar{\tilde{y}}$, $z = \sigma\bar{z}$, and $\tilde{\varepsilon}_1 = \sigma^2 \bar{\tilde{\varepsilon}}_1$.
- 2-D normally attracting critical manifold $\mathcal{S}^a$.
- Family of singular orbits $\gamma_0^{\tilde{\varepsilon}_1} = \gamma_0^{\tilde{\varepsilon}_1,s} \cup \gamma_0^{\tilde{\varepsilon}_1,f}$ perturbes for $\tilde{\varepsilon}_1 \in (0, \beta_3]$ and $0 < r \ll 1$ to $\bar{\gamma}_r^{\tilde{\varepsilon}_1}$.



Region B_3

VIII. Conclusion and Outlook

- Using GSPT and the blow-up method we obtained a **full asymptotic analysis** of the Robertson model for $k_1, k_3 \ll k_2$.
- To deal with the singular structures associated with the **multiple small parameters**, we introduced suitable **blow-up transformations in parameter space**.
- Case study highlights the **potential of combining GSPT with blow-up in parameter space** for analyzing multi-parameter singular perturbation problems.
- We believe that this approach is applicable to more complicated problems and has the potential to lead to a **framework for the analysis of multi-parameter singular perturbations**.
- For **details** see the preprint:

[1] L. Baumgartner and P. Szmolyan. *A multi-parameter singular perturbation analysis of the Robertson model.* 2024. arXiv: 2407.04008 [math.DS].