What the heck is GSPT?

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02.04.2024

GSPT

- **G**eometric
  - Dynamical systems approach of singular perturbations.
  - Transform "difficult/singular equations" in "simple geometries" to "easier/regular equations" in "more complicated geometries". (D. Grieser)
- **S**ingular **P**erturbation
  - **Wikipedia:** A singular perturbation problem, is a problem containing a small parameter $\varepsilon \ll 1$ such that the solution cannot be approximated uniformly by an asymptotic expansion.
  - Singularly perturbed problems are generally characterized by dynamics operating on multiple scales.
- **T**heory
  - Better used as a **T**oolbox.
  - Adapt and extend it to new problems.

Goal:

Want to prove statements which hold for $\varepsilon \ll 1$, i.e.,

$\exists\, \varepsilon_0 > 0\, \forall \varepsilon \in (0, \varepsilon_0) : \ \ldots$

Perturbations

Full problem: $F(x, \varepsilon) = 0$ with solution x_ε, $0 < \varepsilon \ll 1$.
Limit problem: $F(x, 0) = 0$ with solution x_0.

Regular perturbation	Singular perturbation
▪ $x_\varepsilon \to x_0$ smoothly. ▪ Convergent expansion $x_\varepsilon = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$. ▪ Implicit function theorem.	▪ x_ε may develop singularities. ▪ No smooth approximation by a single limit problem. ▪ Several scalings needed.

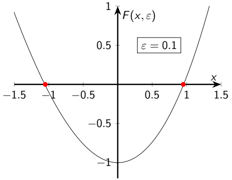
- "Decompose" problem into some easier limit problems.
- "Connect" results to see the whole picture.

Singular Perturbation 1 (Zoom out)

Example (1)

Find the roots of $F(x,\varepsilon) = \varepsilon x^3 + x^2 - 1$ *where* $\varepsilon \ll 1$.

- $\varepsilon = 0$: $x_0 = \pm 1$.
- Plug in expansion $x_\varepsilon = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$ $\implies x_\varepsilon^\pm = \pm 1 - \varepsilon \frac{1}{2} \pm \varepsilon^2 \frac{5}{8} + O(\varepsilon^3)$.



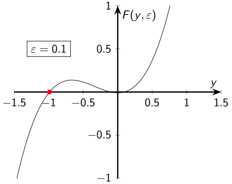
What about the third solution?

- Rescaling $y = \varepsilon x \implies F(y,\varepsilon) = y^3 + y^2 - \varepsilon^2 = 0$.
- $\varepsilon = 0$: $y_0 = -1, 0$.

Expansion $y(\varepsilon) = -1 + \varepsilon y_1 + \varepsilon^2 y_2 + O(\varepsilon^3)$

$$\implies y(\varepsilon) = -1 + \varepsilon^2 + O(\varepsilon^3)$$

$$\implies x(\varepsilon) = \frac{y(\varepsilon)}{\varepsilon} = -\frac{1}{\varepsilon} + \varepsilon + O(\varepsilon^2).$$



Slow-Fast Systems in Standard Form

Fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, $0 < \varepsilon \ll 1$.

Slow time scale t

$$\begin{cases} \varepsilon \dot{x} &= f(x, y, \varepsilon) \\ \dot{y} &= g(x, y, \varepsilon) \end{cases}$$

$$\overset{\tau = t/\varepsilon}{\longleftrightarrow}$$

Fast time scale τ

$$\begin{cases} x' &= f(x, y, \varepsilon) \\ y' &= \varepsilon g(x, y, \varepsilon) \end{cases}$$

- For $\varepsilon > 0$ the two systems are equivalent.
- Two limiting problems for $\varepsilon = 0$:

Reduced problem

$$\begin{cases} 0 &= f(x, y, 0) \\ \dot{y} &= g(x, y, 0) \end{cases}$$

Layer problem

$$\begin{cases} x' &= f(x, y, 0) \\ y' &= 0 \end{cases}$$

Slow Manifold S

Reduced problem

$$\begin{cases} 0 &= f(x,y,0) \\ \dot{y} &= g(x,y,0) \end{cases}$$

Layer problem

$$\begin{cases} x' &= f(x,y,0) \\ y' &= 0 \end{cases}$$

- Dynamics on slow manifold $S := \{(x,y) : f(x,y,0) = 0\}$.
- Fast variables x "slaved" to slow variables y.

- S is a "manifold" of equilibria.
- Slow variables y act as parameters.

How can we connect these two limits?

- Fenichel Theory (1979):
  If S satisfies some regularity conditions, then for $\varepsilon \ll 1$ it perturbs to S_ε with similar properties.

Van der Pol Oscillator

Reduced problem

$$\begin{cases} 0 & = y - \frac{x^3}{3} + x \\ \dot{y} & = \alpha - x \end{cases}$$

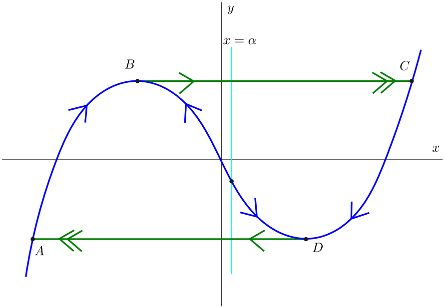
VdP oscillator

$$\begin{cases} \varepsilon \dot{x} & = y - \frac{x^3}{3} + x \\ \dot{y} & = \alpha - x \end{cases}$$

Layer problem

$$\begin{cases} x' & = y - \frac{x^3}{3} + x \\ y' & = 0 \end{cases}$$

- Slow manifold $S : y = \frac{x^3}{3} - x$.
- S is attracting if $|x| > 1$.
- S is repelling if $|x| < 1$.



Van der Pol Oscillator

Reduced problem

$$\begin{cases} 0 &= y - \frac{x^3}{3} + x \\ \dot{y} &= \alpha - x \end{cases}$$

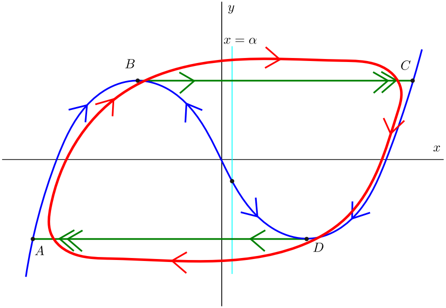
VdP oscillator

$$\begin{cases} \varepsilon \dot{x} &= y - \frac{x^3}{3} + x \\ \dot{y} &= \alpha - x \end{cases}$$

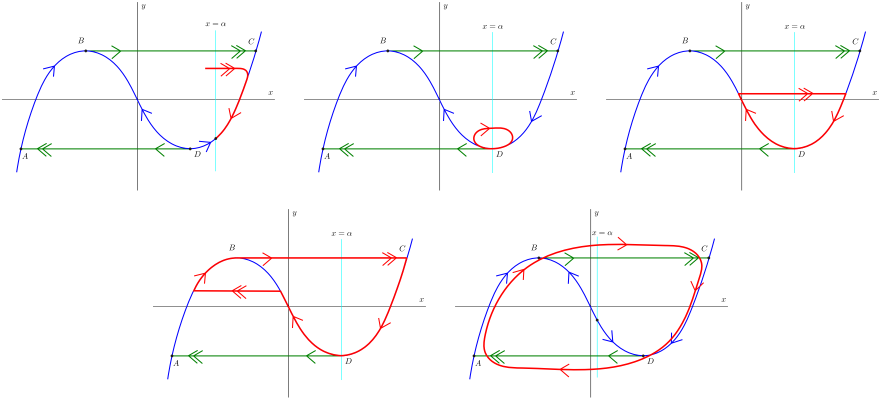
Layer problem

$$\begin{cases} x' &= y - \frac{x^3}{3} + x \\ y' &= 0 \end{cases}$$

- Fenichel theory and the blow-up method guarantee the existence of a limit cycle γ_ε for $\varepsilon \ll 1$ and $-1 < \alpha < 1$.
- Fold points at $x = \pm 1$ $\implies$ blow-up to desingularize.
- $\alpha \approx \pm 1 \implies$ canard explosion/canard cycles.



Canards



Canards