An Introduction to (evolutionary) Game Theory

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Table of Contents

1 Bi-matrix Games

2 Replicator Dynamics

3 Cooperation

A Game

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Ingredients of a game

- **Players:** Who are the individuals whose behaviour we want to analyze?
- **Actions:** How can the players act?
- **Order of moves:** In which order do players act?
- **Information:** What do players know when taking an action?
- **Payoffs:** How does a player's fate depend on other players?

Second prize sealed bid auction

- **Players:** The n bidders.
- **Actions:** Their bid $b_i \in (0, \infty)$.
- **Order of moves:** Simultaneously.
- **Information:** Their own true values v_i of the good.
- **Payoffs:**
  If $b_i > \max_{j \neq i} b_j$:
  $\pi_i(b_1, ..., b_n) = v_i - \max_{j \neq i} b_j$.
  Else it is zero.

Bi-matrix games

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- Two players (player 1 and player 2) can choose among $n \in \mathbb{N}$, respectively $m \in \mathbb{N}$, strategies.
- The players move simultaneously and know all aspects of the game (except co-player's decision).

	Action 1	...	Action m
Action 1	a_{11}, b_{11}	...	a_{1m}, b_{1m}
...	...		...
Action n	a_{n1}, b_{n1}	...	a_{nm}, b_{nm}

- The matrix $A = (a_{ij})$ is called the payoff matrix of player 1.
- The matrix $B = (b_{ij})$ is called the payoff matrix of player 2.

Mixed strategies

- Let x_i be the probability of player 1 choosing action i. Let $x = (x_i)$ be the corresponding column vector. If $x = e_i$ then we say it is a pure strategy.
- Analogously we define the vector $y = (y_i)$ for player 2.
- In particular, the vectors x and y must be an element of a simplex $S_k := \{z \in \mathbb{R}^k : z_i \geq 0, \sum_{i=1}^{k} z_i = 1\}$.
- If both players use mixed strategies, we can write their respective (expected) payoffs as

$$\pi_1(x, y) = x^T Ay = \sum_{i,j} x_i a_{ij} y_j$$

$$\pi_2(x, y) = x^T By = \sum_{i,j} x_i b_{ji} y_j.$$

Prisoner's Dilemma

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- Suppose there are two players who need to decide whether or not to do a favor to the other player. A player who does a favor pays a cost of 1 and provides a benefit of 3 to the co-player.

	Cooperate	Defect
Cooperate	2,2	−1,3
Defect	3,−1	0,0

- Suppose the first player uses the (mixed) strategy to cooperate with 50% probability, whereas the second player uses the (pure) strategy of always to defect. ($x = (0.5, 0.5)^T$, $y = (0, 1)^T$)

$$\pi_1(x, y) = 0.5 \cdot 2 \cdot 0 + 0.5 \cdot (-1) \cdot 1 + 0.5 \cdot 3 \cdot 0 + 0.5 \cdot 0 \cdot 1 = -0.5$$
$$\pi_2(x, y) = 0.5 \cdot 2 \cdot 0 + 0.5 \cdot 3 \cdot 1 + 0.5 \cdot (-1) \cdot 0 + 0.5 \cdot 0 \cdot 1 = 1.5$$

- What would you do?

'Solutions' of a game

- A strategy e_i of player 1 is called (strictly) dominated by a strategy x, if

$$\pi_1(e_i, e_j) < \pi_1(x, e_j) \quad \forall j = 1, ..., m.$$

- A strategy e_j of player 2 is called (strictly) dominated by a strategy y, if

$$\pi_2(e_i, e_j) < \pi_1(e_i, y) \quad \forall i = 1, ..., n.$$

- Suppose player 1 feels bad when getting a favour without returning a favour.

	Cooperate	Defect
Cooperate	2,2	−1,3
Defect	−1,−1	0,0

'Solutions' of a game

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- A matrix game is called dominance solveable if (iterated) elimination of dominated strategies leads to a unique strategy profile (e_i, e_j).
- Such a profile does not always have to exist.
- A strategy profile (x^*, y^*) is called Nash equilibrium, if played by both players neither of them has an incentive to deviate:

$$\pi_1(x^*, y^*) \geq \pi_1(x, y^*) \quad \forall x \in S_n$$

$$\pi_2(x^*, y^*) \geq \pi_2(x^*, y) \quad \forall y \in S_m$$

- John Nash proved 1951, that for all Bi-matrix games such a strategy profile exists.

Replicator Dynamics

- Consider an infinitely large population with n strategies/traits.
- Let $x \in \mathbb{R}^n$ be the strategy/trait distribution in the population and $A \in M^{n \times n}(\mathbb{R})$ the payoff/fitness matrix, where a_{ij} denotes the fitness consequence of an individual with trait i meeting an individual with trait j.
- If interaction occurs randomly the average fitness of an individual with trait i is $f_i(x) = (Ax)_i$
- The average fitness of the population is given by $\bar{f}(x) = x^T Ax$.
- The replicator equation is then given by the following system of ODEs:

$$\dot{x}_i = x_i\Big(f_i(x) - \bar{f}(x)\Big) = x_i\Big((Ax)_i - x^T Ax\Big) \quad i = 1, \dots, n. \tag{1}$$

Replicator Dynamics

$$\dot{x}_i = x_i\Big((Ax)_i - x^T Ax\Big) \quad i = 1, \dots, n.$$

- Observations:
  1. The simplex S_n and its boundary faces are left invariant and therefore solutions exist for all times.
  2. Dominated traits go extinct, i.e.,

$$(Ax)_i < (Ax)_j \quad \forall x \in S_n \text{ and } x_j(0) > 0 \quad \Rightarrow \quad \lim_{t\to\infty} x_i(t) = 0$$

This is quickly checked by the quotient rule:

$$\dot{\left(\frac{x_i}{x_j}\right)} = \frac{x_i}{x_j}\Big((Ax)_i - (Ax)_j\Big) < -\delta\frac{x_i}{x_j}$$

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	2	−1
Defect	3	0

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$$

- We have two traits (cooperate/defect). Denote their densities by $x \in [0, 1]$ and $(1 - x)$, respectively.
- Their average fitnesses are given by $f_1 = 2x - (1 - x)$ and $f_2 = 3x$, which results in $\bar{f} = 2x^2 - x(1 - x) + 3x(1 - x) = 2x^2 + 2x(1 - x) = 2x$ and

$$\dot{x} = x(x - 1)$$

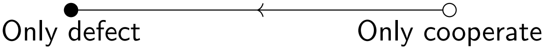

Figure: (REP) of the prisoner's dilemma

Cooperation

- Classical game theory as well as evolutionary game theory tell us that we should not observe cooperation.
- Most likely the games that people play are not covered by the simple game of prisoner's dilemma.
- **Direct reciprocity:** Individuals may interact with each other more than once.
- **Indirect reciprocity:** My actions might influence how third parties treat me.
- **Kin selection:** Individuals may be related to each other.
- ...

Iterated prisoner's Dilemma

	C	D
C	$b-c$	$-c$
D	b	0

$\xrightarrow{\delta}$

	C	D
C	$b-c$	$-c$
D	b	0

$\xrightarrow{\delta}$ $\dots$

IPD

- **Players:** Two individuals.
- **Actions:** New round with probability δ. In each round C or D.
- **Order of moves:** Simultaneously.
- **Information:** All history.
- **Payoffs:** In each round t: $\pi_i(t)$. For IPD weighted mean: $\pi_i = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\pi_i(t)$

Possible strategies:

- ALLC
- ALLD
- Tit-for-Tat (TFT)
- ...

TFT vs. ALLD

	C	D
C	$b-c$	$-c$
D	b	0

$\xrightarrow{\text{iterate}}$

	TFT	ALLD
TFT	$\frac{b-c}{1-\delta}$	$-c$
ALLD	b	0

- Can TFT hold itself against ALLD?
- Yes, if $\frac{b-c}{1-\delta} > b$. In this case, TFT is not dominated anymore.
- This is the case iff $\delta > \frac{c}{b}$. This is a necessary condition for the evolution of cooperation.
- So direct reciprocity can only lead to cooperation if the chance of meeting again is higher than the cost to benefit ratio of the favor.