UE Mengenlehre SoSe2024

lena.wallner@tuwien.ac.at

Session 2

updated on March 16 at 13:55

1) (a) Let a and b be sets. Show that $a \times b$ is a set.
   Bonus: Can you do it in ZF-(Replacement) and in ZF-(Power)?
   (b) Let F be a function. Show that F is a set if and only if $dom(F)$ is a set.

2) Let $<_i$ be a wellorder with $dom(<_i) = D_i$ for $i = 0, 1$. We define addition on wellorders by

$$dom(<_0 \oplus <_1) := \big(D_0 \times \{0\}\big) \cup \big(D_1 \times \{1\}\big)$$

and for $x \in D_i$ and $y \in D_j$,

$$\big((x, i), (y, j)\big) \in \ (<_0 \oplus <_1) \Leftrightarrow (i = j \wedge x <_i y) \vee (i = 0 \wedge j = 1).$$

   (a) Show that $\oplus$ is associative but not commutative.
   (b) Let α and β be ordinals and $\alpha \leq \beta$. Show that there is a unique ordinal γ such that $\alpha + \gamma = \beta$.

3) Let $\alpha > 0$ be an ordinal. Show that there are unique nonzero natural numbers $n, k_1, \dots, k_n$ and there are unique ordinals $\beta_n < \cdots < \beta_1 \leq \alpha$ such that

$$\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n.$$

This representation is called Cantor's Normal Form.

4) (a) Let A be a class. Show that there is a class HA such that

$$x \in HA \Leftrightarrow x \in A \wedge x \subseteq HA.$$

   HA is the class of hereditarily-in-A set.
   (b) Let Tr be the class of transitive sets. Show that the following are equivalent
      i. $x \in Ord$,
      ii. $x \in Tr \wedge x \subseteq Tr$,
      iii. $x \in HTr$.