Mass-lumped high-order cell methods for the time-dependent Maxwell's equations

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Abstract

Our goal is the efficient numerical approximation of solutions to the time-domain linear Maxwell system. Methods like the classical finite-difference time-domain method rely on the approximation of the electric and the magnetic field on two interlaced (Cartesian) grids respectively.

Our method expands this idea to general triangular/tetrahedral meshes by defining a dual grid using the barycentric subdivision of the primal mesh. The resulting primal and dual ements are both composed of the same set of quadrilateral (in 2d) and hexahedral (in 3d) *cells*. We use Lagrangian polynomial basis functions with respect to tensor product integration points on the unit square/cube which are mapped to the physical cells and suitably combined to obtain differing, element-wise conforming bases on the dual and primal mesh respectively. This approach provides generically computable, stable high order bases.

Using lumped mass matrices with respect to the defining points of the basis functions leads to block diagonal mass matrices with block-size independent of the polynomial degree. Thus the combination of the spatial discretization with explicit time-stepping schemes yields an efficient method with very little memory requirement.

Keywords: time-domain Maxwell equations, cell method, discontinuous Galerkin, dual grids, mass lumping

1 Introduction

Consider the time-dependent Maxwell equations in their first-order form on a suitable domain $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$ for an electric field $E \in \mathrm{L}^2(\mathrm{H}(\mathrm{curl}; \Omega); [0, T])$ and a magnetic field $H \in \mathrm{L}^2(\mathrm{H}(\mathrm{curl}; \Omega); [0, T])$, such that:

$$\varepsilon \frac{\partial E}{\partial t} = \operatorname{curl} H - J, \quad \mu \frac{\partial H}{\partial t} = -\operatorname{curl} E, \quad (1)$$

in $\Omega \times (0, T)$ with initial conditions $E(\cdot, 0) \in \mathrm{H}(\mathrm{curl}; \Omega)$, $H(\cdot, 0) \in \mathrm{H}(\mathrm{curl}; \Omega)$, and homogeneous boundary conditions $E \times n_\Omega = 0$, on $\partial\Omega \times (0, T)$. Here $\varepsilon, \mu \in L^\infty(\Omega)$ are the dielectric permittivity and magnetic permeability, assumed to be scalar valued and time-invariant for simplicity of exposition. The vector field $J(x, t)$ is the electric current density (a suitable a priori known source of the system) and $T > 0$ is the final simulation time.

The most used choice for the space discretization is finite differences (usually the second order accurate version, on staggered grids). Nevertheless, due to the growing popularity of discontinuous Galerkin (DG) Finite Element Methods (FEM) there has been a revitalized enthusiasm in using variational methods to discretize the Maxwell system in a way that leads to block diagonal mass matrices even on unstructured grids.

The presented work falls into this framework and, while building originally on low order Finite Integration Techniques (FIT), is instead high-order accurate like the DG approach and is based on the manuscripts [1, 2].

2 Definition of dual meshes and spaces

For the ease of presentation we focus on the case $d = 2$ and (1) in TM form, i.e., H is scalar and E is a vector field and the two curl-operators are the vectorial and scalar ones respectively.

2.1 Barycentric mesh refinement

We assume that a triangulation $\mathcal{T}$ of Ω consisting of disjoint open triangles is given such that $\bigcup_{T \in \mathcal{T}} \overline{T} = \overline{\Omega}$. We construct the dual mesh by connecting the barycenter of each primal cell T to the midpoints of its edges which results in the so-called dual edges. The dual cells $\tilde{T} \in \tilde{\mathcal{T}}$ are then given by the smallest subsets of Ω bounded by dual edges and subsets of $\partial\Omega$ (cf. Figure 3).

Each primal/dual cell is composed of disjoint quadrilateral cells $K \in \mathcal{K}$.

2.2 Local spaces on primal/dual cells

We define (local) spaces and basis functions on each primal/dual cell by means of the positively oriented bilinear mapping $F_K : [0,1]^2 \to \overline{K}$ which maps vertices to vertices and the origin to the (unique) vertex of the primal mesh for each cell K and denote by J_K its Jacobian matrix.

We choose to define the discrete space for the scalar quantity H to be locally conforming on the primal mesh:

$$X_P^{\mathrm{grad}} := \{h \in L^2(\Omega) : h|_T \in H^1(T)\,\forall T \in \mathcal{T}\} \cap \{h \in L^2(\Omega) : h|_K \circ F_K \in \Pi_P\,\forall K \in \mathcal{K}\},$$

where Π_P denotes the set of polynomials over $\mathbb{R}^2$ of maximum degree $P \in \mathbb{N}$ in each variable.

Conversely the space for the vectorial quantity E is chosen to consist of (covariantly) mapped polynomials on each cell K and to be locally conforming on the dual mesh:

$$\tilde{X}_P^{\mathrm{curl}} := \{e \in L^2(\Omega) : e|_{\tilde{T}} \in H^{\mathrm{curl}}(\tilde{T})\,\forall \tilde{T} \in \tilde{\mathcal{T}}\} \cap \{e \in L^2(\Omega)^2 : J_K^\top e|_K \circ F_K \in \Pi_P^2\,\forall K \in \mathcal{K}\}.$$

Note that the choice of defining the spaces on the dual/primal mesh can be interchanged in a straightforward way.

2.3 Basis functions

For a given polynomial order $P \in \mathbb{N}$ let $\xi_0, \dots, \xi_P$ be the integration points of the according Gauss-Radau rule with $\xi_P = 1$ and $\tilde{\xi}_0, \dots, \tilde{\xi}_P$ the Gauss-Radau rule with $\tilde{\xi}_0 = 1$. On each given cell K we may construct local basis functions of the scalar and vectorial space respectively by using the appropriate push-forwards of the Lagrangian polynomials $\ell_i, \tilde{\ell}_i$ of the quadrature points defined above, i.e.,

$$\begin{aligned} h_{i,j}^K &= (\ell_i \circ \xi)\,(\ell_j \circ \eta)\,, \\ e_{i,j,k}^K &= \left(\tilde{\ell}_i \circ \xi\right)\left(\tilde{\ell}_j \circ \eta\right)\left(J_K^{-\top}\right) \circ F_K^{-1} \vec{e}_k, \end{aligned} \tag{2}$$

where $(\xi, \eta)^\top := F_K^{-1}$ for $i, j = 0, \dots, P$, $k = 1, 2$. To obtain the basis functions on each primal/dual element the (cell-)local basis functions are glued together in a suitable way to ensure conformity (cf. Figure 3).

3 Discrete formulation

We use the semi-discrete weak formulation

$$\int_\Omega \varepsilon \frac{\partial E}{\partial t} e = \sum_{\tilde{T} \in \tilde{\mathcal{T}}_h} \left(\int_{\tilde{T}} H \operatorname{curl} e + \int_{\partial \tilde{T}} e \cdot H \times n_{\tilde{T}} \right),$$

$$\int_\Omega \mu \frac{\partial H}{\partial t} h = \sum_{K \in \mathcal{T}_h} \left(- \int_T E \cdot \operatorname{curl} h + \int_{\partial T} E \cdot h \times n_T \right),$$

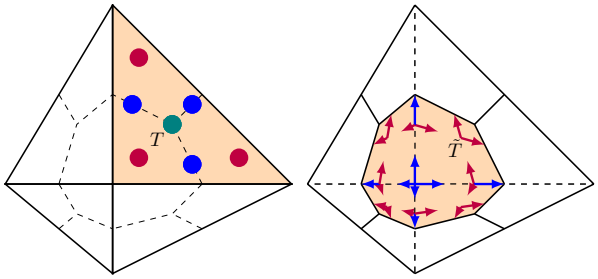

Figure 1: DoFs of X_1^{grad} (left) and $\tilde{X}_1^{\mathrm{curl}}$ (right).

for all test functions $e \in \tilde{X}_P^{\mathrm{curl}}, h \in X_P^{\mathrm{grad}}$. Note that applying integration by parts on each cell K immediately yields that the resulting system is skew-symmetric.

4 Mass lumping

We employ a leap-frog time-stepping scheme to the semi-discrete system above. Thus each time-step requires one application of each of the inverse mass matrices as well as the discrete curl-operators. We use the Gauss-Radau quadrature rules which were used in the construction of the basis functions to approximate the mass integrals, i.e., the polynomials in (2) are evaluated only at points where they take the values zero or one. Due to the presence of the Jacobian (matrices) in the integrals this leads to block diagonal mass matrices where the block-sizes only depend on the structure of the mesh and the spatial dimension but are independent of the polynomial degree.

5 Concluding remarks

We present numerical results which show that the method yields spectrally accurate approximations of the underlying eigenvalue problems. Moreover we demonstrate the applicability of our method to large-scale problems with very little memory requirement. The construction presented above can be generalized to higher dimensions in a straightforward way.

References

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