Computing resonances of a wind instrument using a Krylov solver based on filtered time domain solutions

Lothar Nannen[1,*], Markus Wess[1]
[1]Institute of Analysis and Scientific Computing, TU Wien, Austria
*Email: lothar.nannen@tuwien.ac.at

Abstract

We present a method for computing eigenvalues and eigenvectors of a generalized Hermitian, matrix eigenvalue problem. The work is focused on large scale eigenvalue problems, where the application of a direct inverse is out of reach. Instead, an explicit time domain integrator for the corresponding wave problem is combined with a proper filtering and a Krylov iteration in order to solve for eigenvalues within a given region of interest. To demonstrate the efficiency we report numerical results of acoustic resonances in a hunting horn

Keywords: eigenvalue problem, Krylov method, filtering of time domain solutions

1 Introduction

We consider the computation of eigenpairs (ω^2, v) to the generalized matrix eigenvalue problem

$$Sv = \omega^2 Mv \tag{1}$$

with sparse Hermitian, positive (semi-)definite matrices S and M generated by a finite element discretization of a Laplacian eigenvalue problem. We are interested in large scale problems where non-extremal or clustered eigenvalues are sought.

Following the ideas in [1, 2] for Helmholtz solvers we construct an orthonormal basis of a Krylov space $\mathcal{K}_m(C, r_0)$ of small dimension m with a random starting vector r_0 and a matrix C, which is never computed explicitly. Instead, the application of C is realized by filtered time domain solutions. This corresponds to a polynomial filtering of $M^{-1}S$ with a filter depending on a weight function and the explicit time-stepping scheme for the wave equation. Approximating (1) in this Krylov space leads to eigenvalue approximations to eigenvalues ω^2 with largest absolute values of the filtered eigenvalues. Using suitable weight functions the method can be focused to approximate eigenvalues within a chosen region of interest. Note, that in contrast to shift-and-invert methods no direct solver is needed.

2 Krylov space

In order to construct an orthonormal basis of $\mathcal{K}_m(C, r_0)$ we define the matrix application $r \mapsto Cr$ implicitly. Let $y(\cdot; r) : [0, \infty) \to \mathbb{R}^N$ be for any $r \in \mathbb{R}^N$ the solution to the semi-discrete wave problem $M\ddot{y}(t; r) = -Sy(t; r)$ for $t > 0$ with initial values $y(0; r) = r$ and $\dot{y}(0; r) = 0$. Moreover, for a given piece wise continuous weight function $\alpha : [0, \infty) \to \mathbb{R}$ with compact support we define the weighted time integral operator $\Pi_\alpha : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\Pi_\alpha r := \int_0^\infty \alpha(t) y(t; r)\, dt. \tag{2}$$

Defining the continuous filter function $\beta_\alpha(s) := \int_0^\infty \alpha(t) \cos(ts)\, dt$ it is straightforward to show that $(\beta_\alpha(\omega), v)$ is an eigenpair of Π_α if (ω^2, v) is an eigenpair of (1). Using the Fourier transform it can be shown, that for a finite time interval $[0, T]$ the filter β_α approximates the characteristic function of an interval $[a, b]$ if

$$\alpha(t) := \frac{4}{\pi t} \sin\left(\frac{t}{2}(b - a)\right) \cos\left(\frac{t}{2}(b + a)\right)$$

for $t \in (0, T]$ and $\alpha(0) := \frac{2(b-a)}{\pi}$ is chosen. We discretize (2) using the rectangle rule and the solution $y(t; r)$ to the semi-discrete wave problem using the following explicit time-stepping

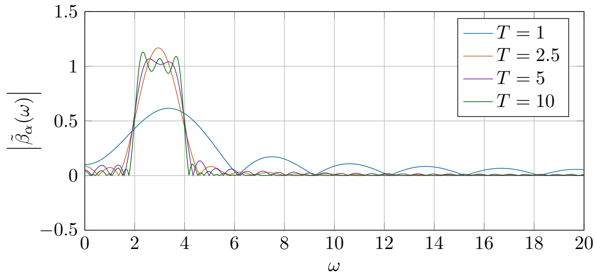
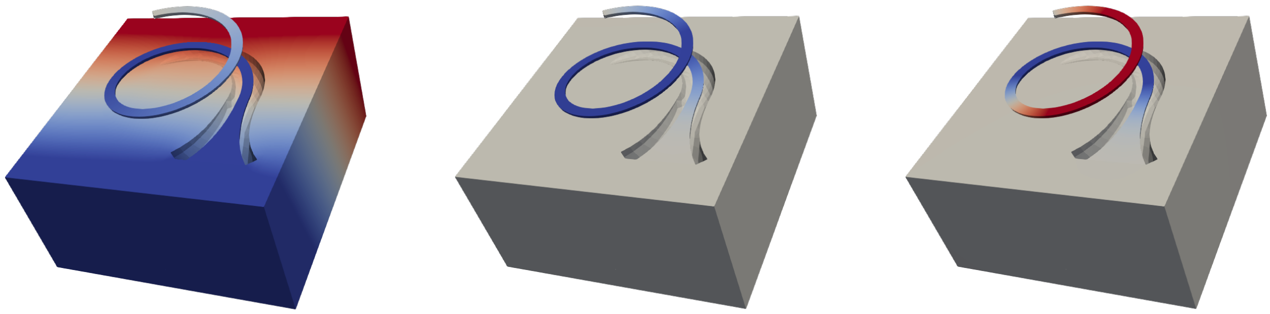


Figure 1: Discrete filter functions for fixed timestep $\tau = 0.025$, the target interval $[a, b] = [2, 4]$, and varying end times T.



(a) $\omega_2^{\text{room}} \approx 1.308$ (b) $\omega_0^{\text{horn}} \approx 2.315$ (c) $\omega_1^{\text{horn}} \approx 4.624$

Figure 2: Eigenfunctions to a background resonance (left) and the base resonance and first harmonic (right). The colors blue and red correspond to higher/lower values while grey corresponds to zero.

scheme: For a step-size τ and $\ell = 1, \dots, L-1$ we define

$$y_{\ell+1}(r) = -\tau^2 M^{-1} S y_\ell(r) + 2y_\ell(r) - y_{\ell-1}(r), \tag{3}$$

with starting values $y_{-1}(r) := y_0(r) := r$. Note, that the application of M^{-1} can be done efficiently if e.g. mass lumping is used. Finally, the desired matrix application is given by

$$Cr := \sum_{\ell=0}^{L} \tau\alpha(\ell\tau) y_\ell(r). \tag{4}$$

Diagonalization of (3) results into

$$q_{\ell+1}(\omega) = \left(2 - \tau^2\omega^2\right) q_\ell(\omega) - q_{\ell-1}(\omega) \tag{5}$$

with starting values $q_{-1}(\omega) := q_0(\omega) := 1$ for those $\omega \geq 0$ such that ω^2 is an eigenvalue of (1). q_ℓ is a polynomial in ω^2 and it can be shown that $C = \tilde{\beta}_\alpha(\sqrt{M^{-1}S})$ with the discrete filter

$$\tilde{\beta}_\alpha(\omega) := \sum_{\ell=0}^{L} \tau\alpha(\tau\ell) q_\ell(\omega). \tag{6}$$

Fig.1 gives examples of discrete filter functions for different numbers of time steps L. The computational costs of the method mainly depend on the number of time steps L per Krylov iteration times the dimension m of the Krylov space. Sharp approximations of the characteristic function of an interval of interest reduce m but require a large number of time steps L. Hence, a balance has to be found depending on the number of eigenvalues in the vicinity of the interval of interest. However, (6) can be computed with negligible computational costs in a preprocessing step. Hence, an optimization of the method parameters for a given problem is easy to carry out experimentally.

3 Resonance frequencies of a hunting horn

In [3] numerical examples including studies to choose the method parameters are performed. We report here the results of a hunting horn in a closed room. This example is challenging, since the dimension $N \approx 1.245 \cdot 10^6$ of the problem is too large for a direct solver on a standard desktop computer to be feasible and since the sought resonances of the horn lie in a region where the background resonances of the room are already quite dense. Moreover, methods to compute the smallest eigenvalues like the LOBPCG method cannot be used, since there are many resonances smaller than the sought ones.

Fig.2 shows eigenfunctions to one background resonance, the base resonance and the first harmonic of the horn. Note, that the physical resonance frequencies

$$343\frac{\omega_0^{\text{horn}}}{2\pi} \approx 126.38Hz, \quad 343\frac{\omega_1^{\text{horn}}}{2\pi} \approx 252.43Hz$$

correspond to notes a little higher than a great and small B respectively. This is a reasonable result given the fact that our horn has a total length of $1.479m$, which is in a similar range as a trumpet tuned in B flat.

References

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