Radial perfectly matched layers and infinite elements for the anisotropic wave equation

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Abstract

It is well known [1] that anisotropic materials can lead to instabilities for perfectly matched layer (PML) methods, and in particular a successful modification of PMLs to treat general anisotropic elastodynamic problems is still an open problem. To study this question we consider radial PMLs for the scalar anisotropic wave equation. We discuss the origin of the instabilities of convenient PMLs, which can be traced back to an (additional) essential spectrum of the Laplace transformed problem. Following [2] we show that a suitable complex frequency shifted PML scaling removes the former troublesome spectrum. However, this approach does not permit to increase the damping constant and we are left without a meaningful mechanism to decrease the truncation error. As a remedy we apply truncation free approximations such as Hardy space infinite elements and certain "exact" PML methods. We report computational studies confirming the stability of the new numerical methods.

Keywords: perfectly matched layers, Hardy space infinite elements, anisotropic wave equation

1 The anisotropic wave equation

Let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ be symmetric strictly positive definite. We look for $u\colon \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{p}\colon \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbb{R}_0^+ := \{t \in \mathbb{R} : t \geq 0\}$, s.t.

$$\partial_t u = \operatorname{div} \mathbf{p} + f \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}^2, \tag{1a}$$

$$\mathbf{A}^{-1}\partial_t \mathbf{p} = \nabla u \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}^2, \tag{1b}$$

$$u(0,\mathbf{x}) = 0, \quad \mathbf{p}(0,\mathbf{x}) = 0 \text{ in } \mathbb{R}^2, \tag{1c}$$

where the source f is sufficiently regular, $f(0,\cdot) = 0$, and $f(t,\cdot)$ is compactly supported inside a bounded domain Ω_f for all $t > 0$.

2 Complex frequency shifted PMLs

To apply radial PMLs to (1), we start by rewriting it in the Laplace domain. For $v \in L^1(\mathbb{R})$ s.t. $v(t) = 0$ for $t < 0$ (i.e., v is a causal function) the Laplace transform is defined by

$$\hat{v}(s) := (\mathcal{L}v)(s) := \int_0^{+\infty} \mathrm{e}^{-st} v(t)dt,$$

$s \in \mathbb{C}^+ := \{s \in \mathbb{C}\colon \operatorname{Re} s > 0\}$. Next we rewrite the problem in the polar coordinates $(r,\phi) \in \mathbb{R}_0^+ \times [0, 2\pi)$. Denoting by $\nabla_{r,\phi} v = (\partial_r, r^{-1}\partial_\phi)^\top v$, $\operatorname{div}_{r,\phi} \boldsymbol{v} = r^{-1}(\partial_r(rv_r) + \partial_\phi v_\phi)$, we obtain the following Laplace-domain counterpart of (1):

$$s^2\hat{u} - \operatorname{div}_{r,\phi}(\mathbf{A}^\phi \nabla_{r,\phi}\hat{u}) = s\hat{f} \quad \text{in } \mathbb{R}^2,$$

with $\mathbf{A}^\phi := \mathbf{R}_\phi^\top \mathbf{A} \mathbf{R}_\phi$ and the rotation matrix $\mathbf{R}_\phi := \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. The radial PMLs are based on a change of variables

$$r_\sigma(s,r) := r + \frac{1}{s+\gamma}\int_0^r \sigma(r')dr',$$

with the layer starting at the radius $R_{\mathrm{pml}} > 0$ s.t. $\Omega_f \subset B_{R_{\mathrm{pml}}}$, damping (absorption) parameter $\sigma(r) = 0$ for $r < R_{\mathrm{pml}}$, $\sigma(r) = \sigma_c > 0$ for $r > R_{\mathrm{pml}}$ and frequency shift $\gamma \geq 0$. Here the choice $\gamma = 0$ correponds to standard PMLs. Introducing auxiliary unknowns and reverting the Laplace transform ($s \mapsto \partial_t$) then leads to the system

$$\begin{aligned}
\partial_t u^\sigma &= -(\sigma+\tilde{\sigma})u^\sigma + (\gamma(\sigma+\tilde{\sigma}) - \sigma\tilde{\sigma})v \\
&\quad + \gamma\sigma\tilde{\sigma}w + \operatorname{div}\mathbf{p}^\sigma + f, \\
\partial_t v &= u^\sigma - \gamma v, \\
\partial_t w &= v - \gamma w, \\
\mathbf{A}^{-1}\partial_t \mathbf{p}^\sigma &= (\sigma - \tilde{\sigma})\left(\Pi_\perp \mathbf{A}^{-1}\Pi_\perp \mathbf{p}^\sigma - \Pi_\parallel \mathbf{A}^{-1}\Pi_\parallel \mathbf{p}^\sigma \right. \\
&\quad \left. - \Pi_\perp \mathbf{A}^{-1}\Pi_\perp \mathbf{q} + \Pi_\parallel \mathbf{A}^{-1}\Pi_\parallel \mathbf{q}\right) + \nabla u^\sigma, \\
\partial_t \mathbf{q} &= (\tilde{\sigma}+\gamma)\left(\Pi_\parallel \mathbf{p}^\sigma - \Pi_\parallel \mathbf{q}\right) \\
&\quad + (\sigma+\gamma)\left(\Pi_\perp \mathbf{p}^\sigma - \Pi_\perp \mathbf{q}\right),
\end{aligned} \tag{2}$$

with homogeneous initial conditions, where $\tilde{\sigma}(r) := r^{-1}\int_{R_{\mathrm{pml}}}^{r}\sigma(r')\,dr'$, and $\Pi_{\|}(\mathbf{x})$, $\Pi_{\perp}(\mathbf{x})$ are the orthogonal projections on to the spaces $\mathrm{span}\{\mathbf{x}\}$, $(\mathrm{span}\{\mathbf{x}\})^{\perp}$ for $\mathbf{x}\in\mathbb{R}^2\setminus\{0\}$ respectively. For $\gamma = 0$ we are able to prove that the Laplace transformed ($\partial_t \mapsto s$) system (2) admits an essential spectrum in $\mathbb{C}^+$, which leads to unstable solutions. On the other hand, there exists a constant $\nu_0 > 0$ such that for $\sigma_c/\gamma < \nu_0$ and smooth enough f the solutions to (2) are stable, i.e., $\|u^{\sigma}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}$ grows only polynomial in $T > 0$.

3 Truncation free discretizations

The convenient approach to approximate (2) is to truncate $\mathbb{R}^2$ to a ball $B_{R_{\mathrm{pml}}+L}$, $L > 0$ together with imposing a homogeneous boundary condition, and to subsequently discretize with finite elements or finite differences. At last a time stepping is performed. To decrease the truncation error to a neglectable level one increases the damping parameter σ_c – however, the condition $\sigma_c/\gamma < \nu_0$ prevents us from doing so. As a remedy we apply discretizations which do not involve a domain truncation. First we consider are so-called exact PML methods, which apply a real domain transformation $\mathbb{R}^2 \to B_{R_{\mathrm{pml}}+L}$ before the finite element discretization. Secondly we apply Hardy space infinite elements (HSIEs) [4].

4 Computational experiments

Fig. 1 confirms the claimed spectral properties of an exact PML discretization associated to (2). In Fig. 2 and Fig. 3 we compare the errors of exact PMLs vs. HSIEs for an (isotropic) 1D Bessel equation. We observe much better results for HSIEs than for exact PMLs.

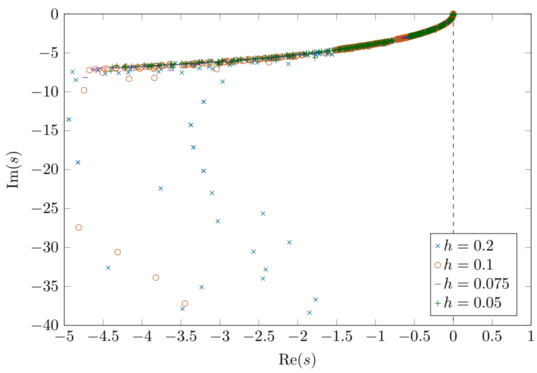

Figure 1: Spectrum of a complex frequency shifted truncation free PML.

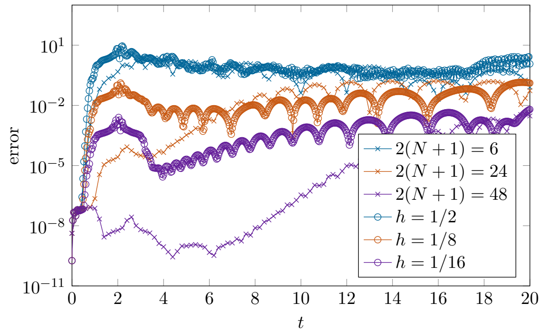

Figure 2: Comparison of the errors of HSIEs and truncation free PMLs, with respect to time.

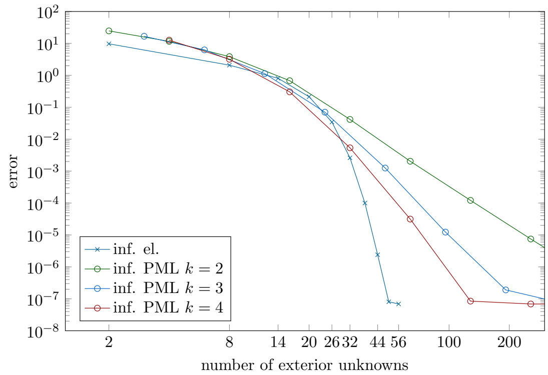

Figure 3: Comparison of the errors of HSIEs and truncation free PMLs, measured at a fixed time.

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