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Session 12

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Definition. A set $A \subseteq \omega^\omega$ is **Wadge reducible**[1] to $B \subseteq \omega^\omega$, $A \leq_{\mathrm{W}} B$, iff $A = f^{-1}[B]$ for some continuous $f\colon \omega^\omega \to \omega^\omega$.

Definition. A set C **separates** a set A from B if $A \subseteq C$ and $C \cap B = \emptyset$.

1) (a) Show that all the pointclasses in the projective hierarchy are closed under countable unions and intersections. That is, if $\langle A_k \mid k < \omega \rangle \in \Gamma^\omega$ then $\bigcup_{k<\omega} A_k$, $\bigcap_{k<\omega} A_k \in \Gamma$ for $\Gamma = \mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ or $\mathbf{\Delta}^1_n$ and $1 \leq n < \omega$.

   HINT: In the case $\Gamma = \mathbf{\Sigma}^1_1$, make use of the representation of analytic sets in terms of projections of trees on $\omega \times \omega$.

   (b) Conclude that $\mathbf{\Delta}^1_1$ contains all Borel sets.

2) (a) Suppose $\langle A_i \mid i \in I \rangle$, $\langle B_j \mid j \in J \rangle$ are sets and $C_{i,j}$ separates A_i from B_j for $i \in I, j \in J$. Show that $\bigcup_{i\in I} \bigcap_{j \in J} C_{i,j}$ separates $\bigcup_{i \in I} A_i$ from $\bigcup_{j\in J} B_j$.

   (b) Show that disjoint analytic sets are separated by Borel sets, i.e. if $A, B \in \mathbf{\Sigma}^1_1$ are disjoint then there is a Borel set C with $A \subseteq C$ and $C \cap B = \emptyset$.

   HINT: Note that if T is a tree on $\omega \times \omega$ then for any $n < \omega$, $p[T] = \bigcup_{s \in T_n} p[T_s]$ where $T_s = \{t \in T \mid t \leq_T s \vee s \leq_T t\}$ for $s \in T$. Assuming (b) fails, use (a) repeatedly to construct a real in $A \cap B$ to reach a contradiction.

   (c) Conclude that every $\mathbf{\Delta}^1_1$-set is Borel.

[1] Of course this is just continuous reducibility. Set Theorists use this term instead in honor of William W. Wadge who was the first to do Exercise 3.

3) Assume $\mathrm{ZF}+\mathrm{AD}$.

(a) Show that for any $A, B \subseteq \omega^\omega$ we have $A \leq_W B$ or $B \leq_W \omega^\omega \setminus A$.

HINT: Consider the game

I	n_0		n_1		$\dots$
II		m_0		m_1	$\dots$

in which I wins iff $n_0{}^\frown n_1{}^\frown \cdots \in B \Leftrightarrow m_0{}^\frown m_1{}^\frown \cdots \in A$.

(b) Conclude that $\leq_{\mathrm{pc}}$ is a preorder[2] on all pointclasses where $\Gamma \leq_{\mathrm{pc}} \Sigma$ iff $\Gamma \cup \neg\Gamma \subseteq \Sigma \cup \neg\Sigma$ for pointclasses Γ, Σ.

Let $\preceq_{\mathrm{pc}}$ be the linear order that is induced by $\leq_{\mathrm{pc}}$ after identifying Σ with $\neg\Sigma$ for any pointclass Σ. It is a theorem of Martin-Monk that the strict order $\prec_{\mathrm{pc}}$ is wellfounded (under AD). You may assume this result in the final exercise.

4) Work in $\mathrm{ZF}+\mathrm{AD}$.

(a) Construct a surjective map $x \mapsto f_x$ from ω^ω onto all continuous functions $f\colon \omega^\omega \to \omega^\omega$.

(b) Suppose $A \subseteq \omega^\omega$. Define

$$J(A) = \begin{cases} \{x \oplus y \mid x \in A \wedge f_y(y) \notin A\}, & \text{if } A \notin \{\emptyset, \omega^\omega\} \text{ and} \\ C, & \text{if } A \in \{\emptyset, \omega^\omega\}, \end{cases}$$

where C is any subset of ω^ω such that $C \neq \emptyset, \omega^\omega$.

Show that $A <_{\mathrm{W}} J(A)$, that is $A \leq_{\mathrm{W}} J(A)$ and $J(A) \not\leq_{\mathrm{W}} A$.

(c) Let $\Theta \coloneqq \{\alpha \in \mathrm{Ord} \mid \exists f\colon \omega^\omega \to \alpha \text{ surjective}\}$. Show that $\mathrm{otp}(\prec_{\mathrm{pc}}) = \Theta + 1$.

HINT: First show that for $A, B \subseteq \omega^\omega$, $\Gamma_A = \{f_x^{-1}[A] \mid x \in \omega^\omega\}$ is a pointclass and $A <_{\mathrm{W}} B$ implies $\Gamma_A \leq_{\mathrm{pc}} \Gamma_B$ and $\Gamma_B \not\leq_{\mathrm{pc}} \Gamma_A$. To show that the ordertype is $\geq \Theta$, suppose $f\colon \omega^\omega \to \alpha$ is surjective and define a sequence $\langle A_\beta \mid \beta < \alpha\rangle$ so that $A_\beta = J(\{x \oplus y \mid f(x) < \beta \wedge y \in A_{f(x)}\})$.

[2] That is, it satisfies the axioms of a linear order except to anti-symmetry.