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Optimal interplay of adaptive refinement and iterative solvers for elliptic PDEs

Dirk Praetorius

joint work with P. Bringmann, G. Gantner, A. Miraçi, J. Streitberger

Slides

TU WIEN ASC INSTITUTE OF ANALYSIS AND SCIENTIFIC COMPUTING

NumPDEs Workgroup on Numerics of PDEs

FWF Österreichischer Wissenschaftsfonds

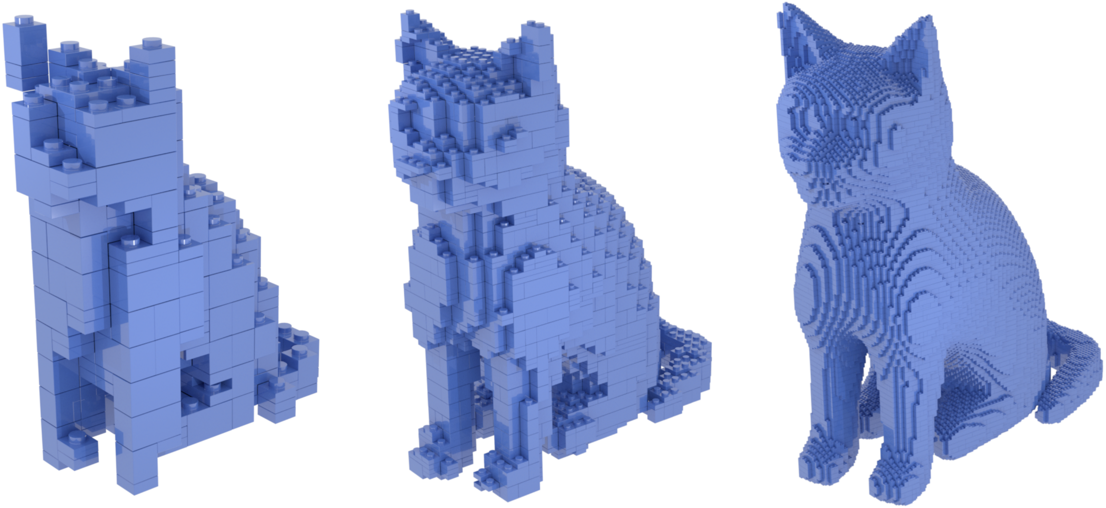
What is all about?

Overall aim

The ultimate goal of any numerical scheme is to compute a discrete solution with error below a prescribed tolerance at, up to a multiplicative (and potentially small) constant, the minimal computational cost.

- requires: resolve singularities
- requires: balance error components
  1. discretization
  2. linearization
  3. inexact solution
  4. quadrature
- talk will deal with discretization – linearization – inexact solution

Like building with Lego bricks



Courtesy of Michael Feischl

Well-known AFEM algorithm

Input: initial mesh $\mathcal{T}_0$, adaptivity parameter $0 < \theta \leq 1$

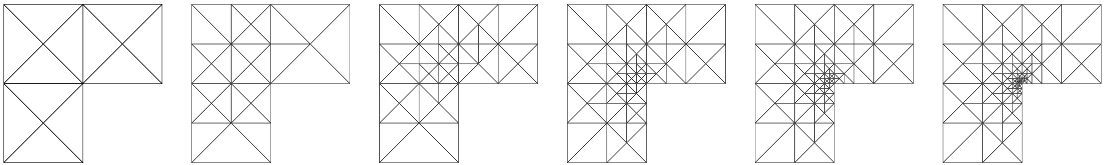
For each $\ell = 0, 1, 2, \ldots$, do

- **SOLVE:** compute discrete solution $u_\ell^\star$
- **ESTIMATE:** compute error indicators $\eta_\ell(T, u_\ell^\star)$ for all $T \in \mathcal{T}_\ell$
- **MARK:** determine quasi-minimal $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^\star)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^\star)^2$
- **REFINE:** employ newest vertex bisection $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

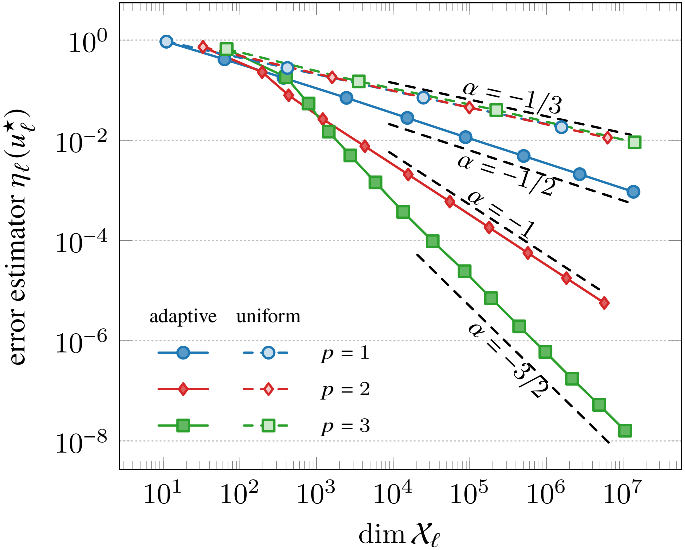
Output: Discrete solutions $u_\ell^\star$, corresponding estimators $\eta_\ell(u_\ell^\star)$

Example: AFEM for Laplace model problem

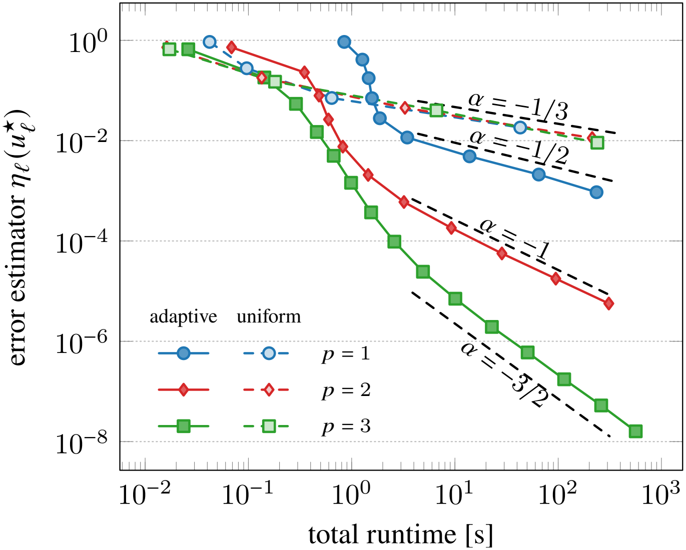
- solve $-\Delta u = 0$ in Ω subject to appropriate boundary conditions
- prescribed exact solution $u(x) = r^{2/3}\sin(2\varphi/3)$
- SOLVE – ESTIMATE – MARK – REFINE driven by residual error estimator $\eta_\ell(u_\ell^\star)$



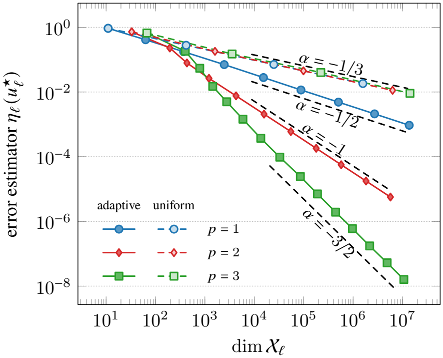
Convergence wrt. dofs

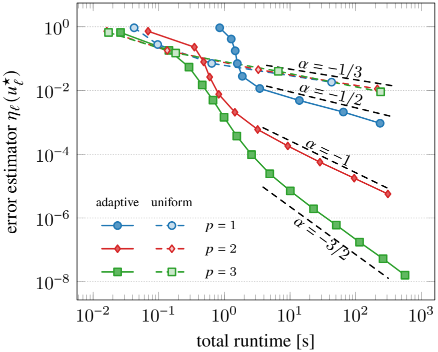


Convergence wrt. time (adaptivity pays off!)



Describing convergence rates





- clear: $\dim \mathcal{X}_\ell \simeq \#\mathcal{T}_\ell$ for fixed p
- $\mathfrak{R}(\alpha) := \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell < \infty$
- assume: $\texttt{work}(\mathcal{T}_{\ell'}) \simeq \#\mathcal{T}_{\ell'}$ for SEMR
- $\widehat{\mathfrak{R}}(\alpha) := \sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big)^\alpha \eta_\ell < \infty$

Rates = complexity?

- $\mathfrak{R}(\alpha) := \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell < \infty$ describes rate wrt. $\dim \mathcal{X}_\ell$
- $\widehat{\mathfrak{R}}(\alpha) := \sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big)^\alpha \eta_\ell < \infty$ describes rate wrt. (idealized) total cost/runtime

What is needed to guarantee $\mathfrak{R}(\alpha) \overset{\checkmark}{\leq} \widehat{\mathfrak{R}}(\alpha) \overset{?}{\lesssim} \mathfrak{R}(\alpha)$ for all $\alpha > 0$?

- note: independent of optimal rates!

R-linear convergence is sufficient

R-linear convergence $\quad \eta_\ell \leq C q^{\ell-\ell'} \eta_{\ell'} \quad \forall\, 0 \leq \ell' \leq \ell$

$$\Longrightarrow \quad \mathfrak{R}(\alpha) := \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell \leq \sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big)^\alpha \eta_\ell \leq \frac{C}{(1-q^{1/\alpha})^\alpha} \mathfrak{R}(\alpha)$$

Proof:

- $(\#\mathcal{T}_{\ell'})\, \eta_{\ell'}^{1/\alpha} \leq \mathfrak{R}(\alpha)^{1/\alpha} \quad \forall\, \ell'$ — definition of $\mathfrak{R}(\alpha)^{1/\alpha}$

$$\Longrightarrow \quad \sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \leq \mathfrak{R}(\alpha)^{1/\alpha} \sum_{\ell'=0}^{\ell} \eta_{\ell'}^{-1/\alpha}$$

- $\displaystyle\sum_{\ell'=0}^{\ell} \eta_{\ell'}^{-1/\alpha} \leq C^{1/\alpha} \Big(\sum_{\ell'=0}^{\ell} q^{(\ell-\ell')/\alpha} \Big) \eta_\ell^{-1/\alpha} \leq \frac{C^{1/\alpha}}{1-q^{1/\alpha}}\, \eta_\ell^{-1/\alpha}$ — R-linear convergence

$$\Longrightarrow \quad \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big) \eta_\ell^{1/\alpha} \leq \mathfrak{R}(\alpha)^{1/\alpha} \frac{C^{1/\alpha}}{1-q^{1/\alpha}} \quad \forall\, \ell$$

Characterization of R-linear convergence

Proposition (CFPP'14)

For any $\alpha > 0$, there holds equivalence:

1. $\eta_\ell \le C q^{\ell-\ell'} \eta_{\ell'} \quad \forall\, 0 \le \ell' \le \ell$ — R-linear convergence

2. $\sum_{\ell'=\ell}^{\infty} \eta_{\ell'}^{\alpha} \le C_\alpha\, \eta_\ell^{\alpha} \quad \forall\, \ell \ge 0$ — tail summability

3. $\sum_{\ell'=0}^{\ell} \eta_{\ell'}^{-1/\alpha} \le C'_\alpha\, \eta_\ell^{-1/\alpha} \quad \forall\, \ell \ge 0$ — inverse tail summability

- proof by elementary calculus

Carstensen, Feischl, Page, Praetorius: *Comput. Math. Appl.*, 67 (2014)

R-linear convergence is (essentially) necessary

optimal rate and cost $0 < \inf_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha \eta_\ell \le \widehat{\mathfrak{R}}(\alpha) := \sup_{\ell\in\mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}\Big)^\alpha \eta_\ell < \infty$

$\Longrightarrow$ R-linear convergence $\eta_\ell \le C q^{\ell-\ell'} \eta_{\ell'} \quad \forall\, 0 \le \ell' \le \ell$

Proof:

- $(\#\mathcal{T}_{\ell'})\, \eta_{\ell'}^{1/\alpha} \gtrsim 1 \quad \forall\, \ell'$ lower estimate in assumption

$\Longrightarrow \quad \sum_{\ell'=0}^{\ell} \eta_{\ell'}^{-1/\alpha} \lesssim \sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}$

- $\Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'}\Big) \eta_\ell^{1/\alpha} \le \widehat{\mathfrak{R}}(\alpha)^{1/\alpha}$ upper estimate in assumption

$\Longrightarrow \quad \sum_{\ell'=0}^{\ell} \eta_{\ell'}^{-1/\alpha} \le \widehat{\mathfrak{R}}(\alpha)^{1/\alpha}\, \eta_\ell^{-1/\alpha} \quad \forall\, \ell$

- proved: inverse tail summability ($\Longleftrightarrow$ R-linear convergence)

Take home messages, part I

1. AFEM analysis should address rates wrt. complexity/time instead of dofs

2. R-linear convergence is key to relate #dofs and complexity/time (and essentially necessary)

3. needs linear cost of all modules SOLVE – ESTIMATE – MARK – REFINE
   - SOLVE is critical (beyond 1D)
   - ESTIMATE is clear (with idealized quadrature)
   - MARK is known (e.g., Stevenson 2007 with binning, Pfeiler–Praetorius 2020 for minimal $\mathcal{M}_\ell$)
   - REFINE is known (e.g., Binev–Dahmen–DeVore 2004, Stevenson 2008)

4. our current developments focus on parameter-robust convergence
   - older works rely on a perturbation argument (and thus are not parameter-robust)
   - e.g., Stevenson 2007, Carstensen–Gedicke 2012

AFEM with contractive solver

Symmetric linear elliptic PDE

- symmetric linear elliptic PDE

$$
\begin{aligned}
-\mathrm{div}\,(\boldsymbol{A}\nabla u^\star) &= f \quad \text{in } \Omega \subset \mathbb{R}^d \\
u^\star &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

- $u^\star \in H^1_0(\Omega)$ s.t. $\quad a(u^\star, v) = F(v) \quad \forall\, v \in H^1_0(\Omega)$

- $u^\star_\ell \in \mathcal{X}_\ell$ s.t. $\qquad a(u^\star_\ell, v_\ell) = F(v_\ell) \quad \forall v_\ell \in \mathcal{X}_\ell := \mathcal{P}^p(\mathcal{T}_\ell) \cap H^1_0(\Omega)$

- $\eta_\ell(v_\ell)$ standard residual error estimator, evaluated at $v_\ell \in \mathcal{X}_\ell$

- energy norm $\|v\|^2 := a(v, v)$

Contractive iterative solver

- let $0 < \kappa < 1$
- consider contractive solver for discrete linear systems

$$\|u_\ell^\star - u_\ell^k\| \le \kappa\,\|u_\ell^\star - u_\ell^{k-1}\| \quad \text{for all} \quad k \in \mathbb{N}$$

- nothing but triangle inequality

$$\Longrightarrow \quad \frac{1-\kappa}{\kappa}\,\|u_\ell^\star - u_\ell^k\| \le (1-\kappa)\,\|u_\ell^\star - u_\ell^{k-1}\| \le \|u_\ell^k - u_\ell^{k-1}\| \le (1+\kappa)\,\|u_\ell^\star - u_\ell^{k-1}\|$$

- assumptions
  1. h-robustness: contraction constant $0 < \kappa < 1$ is $\mathcal{T}_\ell$-independent
  2. linear cost: cost $\mathcal{O}(\#\mathcal{T}_\ell)$ per solver step (also $\mathcal{O}(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'})$ is OK)

Contractive iterative solver / examples

- example solvers:
  - PCG with local multi-level additive Schwarz/BPX preconditioner is h-robust [CNX'12]
  - geometric multigrid (h-robust [WZ'17], hp-robust [IMPS'24])

- important properties
  - h-robustness: contraction constant $0 < \kappa < 1$ is $\mathcal{T}_\ell$-independent
  - linear cost: cost $\mathcal{O}(\#\mathcal{T}_\ell)$ per solver step (also $\mathcal{O}(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'})$ is OK)

Chen, Nochetto, Xu: *Numer. Math.*, 120 (2012)

Wu, Zheng: *Appl. Numer. Math.*, 113 (2017)

Innerberger, Miraçi, Praetorius, Streitberger: *ESAIM Math. Model. Numer. Anal.*, 58 (2024)

What is optimal interplay?

- adaptive mesh-refinement and iterative solver have optimal interplay

1. guaranteed convergence — for any choice of parameters
2. rates $=$ complexity — for any choice of parameters
3. optimal complexity — for appropriate choices of parameters

Stopping criterion for algebraic solver

- $$\|u^\star - u_\ell^k\| \le \|u^\star - u_\ell^\star\| + \|u_\ell^\star - u_\ell^k\| \overset{\text{reliability}}{\lesssim} \eta_\ell(u_\ell^\star) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{solver}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$$

- idea: equilibrate $\eta_\ell(u_\ell^k)$ and $\|u_\ell^k - u_\ell^{k-1}\|$

$\Longrightarrow$ stop algebraic solver for $K = k$ as soon as $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$

  - clear: small $0 < \lambda \ll 1 \quad \Longrightarrow \quad u_\ell^K \approx u_\ell^\star$ (can be handled by perturbation analysis)

- nested iteration: $u_\ell^0 := u_{\ell-1}^K$

$\Longrightarrow$ a-posteriori error control for all u_ℓ^k but u_0^0

AFEM with contractive solver

Input: initial mesh $\mathcal{T}_0$, initial guess u_0^0, adaptivity parameters $0 < \theta \le 1$, $\lambda > 0$

For each $\ell = 0, 1, 2, \ldots$ do

- **SOLVE & ESTIMATE:** For $k = 1, 2, 3, \ldots, K$, **repeat**
  - compute u_ℓ^k
  - compute $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

  until $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$

- **MARK:** choose quasi-minimal $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t. $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$

- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^0 := u_\ell^K$,

Output: Discrete solutions u_ℓ^k, corresponding estimator $\eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$

- note: number of solver steps $K = K(\ell)$ might vary with ℓ
- note: each step of the (nested) loops has cost $\mathcal{O}(\#\mathcal{T}_\ell)$

Index set $\mathcal{Q}$

- $\mathcal{Q} := \big\{(\ell, k) \in \mathbb{N}_0^2 \,:\, u_\ell^k \text{ computed by algorithm}\big\}$
- $|\ell, k| \;:= \#\big\{(\ell', k') \in \mathcal{Q} \,:\, u_{\ell'}^{k'} \text{ computed earlier than } u_\ell^k\big\}$
- note: $|\cdot,\cdot| : \mathcal{Q} \to \mathbb{N}_0$ bijection

$\Longrightarrow$ cost to compute u_ℓ^k is proportional to $\displaystyle\sum_{\substack{(\ell',k')\in\mathcal{Q}\\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} = k \cdot \#\mathcal{T}_\ell + \sum_{\ell'=0}^{\ell-1} K(\ell') \cdot \#\mathcal{T}_{\ell'}$

- recall quasi-error $\quad \Delta_\ell^k := \|u^\star - u_\ell^k\| + \eta_\ell^k(u_\ell^k)$
  - known: $\quad \Delta_\ell^\star \le C q^{\ell-\ell'} \Delta_{\ell'}^\star \qquad \forall\, 0 \le \ell' \le \ell \qquad$ AFEM with exact solver [CKNS'08]
  - goal: $\quad \Delta_\ell^k \le C q^{|\ell,k|-|\ell',k'|} \Delta_{\ell'}^{k'} \qquad \forall\, 0 \le |\ell', k'| \le |\ell, k| \qquad$ AFEM with contractive solver

Cascón, Kreuzer, Nochetto, Siebert: *SIAM J. Numer. Anal.*, 46 (2008)

Main result 1: Robust full R-linear convergence

Theorem (GHPS'21, BFMPS'23$^+$)

- arbitrary u_0^0, arbitrary $0 < \theta \le 1$, arbitrary $\lambda > 0$

$\Longrightarrow$ exists $0 < q < 1$ such that $\Delta_\ell^k \lesssim q^{|\ell,k|-|\ell',k'|}\Delta_{\ell'}^{k'} \quad \forall\, |\ell',k'| \le |\ell,k|$

- note: R-linear convergence $\Longleftrightarrow$ tail summability $\displaystyle\sum_{\substack{(\ell,k)\in\mathcal{Q}\\|\ell,k|>|\ell',k'|}} \Delta_\ell^k \lesssim \Delta_{\ell'}^{k'}$

- proof in [GHPS'21] exploits Pythagorean identity
  - and proves contraction extending [CKNS'08]

- new proof in [BFMPS'23] relies only on general quasi-orthogonality
  - and thus extends to general $\inf$-$\sup$-stable problems

Cascón, Kreuzer, Nochetto, Siebert: *SIAM J. Numer. Anal.*, 46 (2008)

Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

Bringmann, Feischl, Miraçi, Praetorius, Streitberger: arXiv: 2311.15738 (2023)

Main result 2: rates = complexity

Corollary (GHPS'21, BFMPS'23$^+$)

- assumptions for full R-linear convergence
- $\alpha > 0$

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^\alpha \, \Delta_\ell^k \simeq \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \Delta_\ell^k$$

- direct consequence of introductory considerations on R-linear convergence

Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

Bringmann, Feischl, Miraçi, Praetorius, Streitberger: arXiv: 2311.15738 (2023)

Main result 3: Optimal complexity

Theorem (GHPS'21)

- arbitrary $\alpha > 0$
- $\|u^\star\|_{\mathbb{A}_\alpha} := \sup_{N \geq \#\mathcal{T}_0} N^\alpha \big(\min_{\#\mathcal{T}_{\rm opt} \leq N} \eta_{\rm opt}(u^\star_{\rm opt}) \big) < \infty$
- sufficiently small $0 < \theta < 1$ and sufficiently small $\lambda > 0$

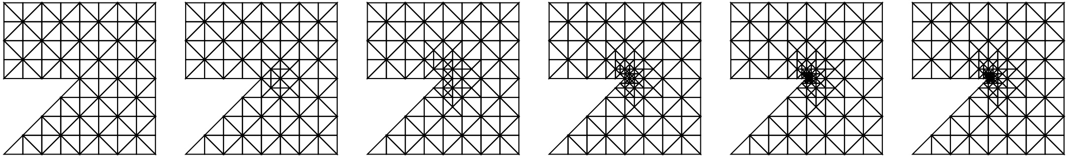
$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_\alpha} \lesssim \sup_{(\ell,k) \in \mathcal{Q}} \Big(\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ |\ell',k'| \leq |\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^\alpha \Delta_\ell^k \lesssim \max\big\{ \|u^\star\|_{\mathbb{A}_\alpha}, \Delta_0^0 \big\}$$

- proof by perturbation argument
  1. $0 < \theta \ll 1 \quad \Longrightarrow \quad$ optimal rates for AFEM with exact solver
  2. $0 < \lambda \ll \theta \quad \Longrightarrow \quad u_\ell^K \approx u_\ell^\star$ (and Dörfler marking essentially coincides)

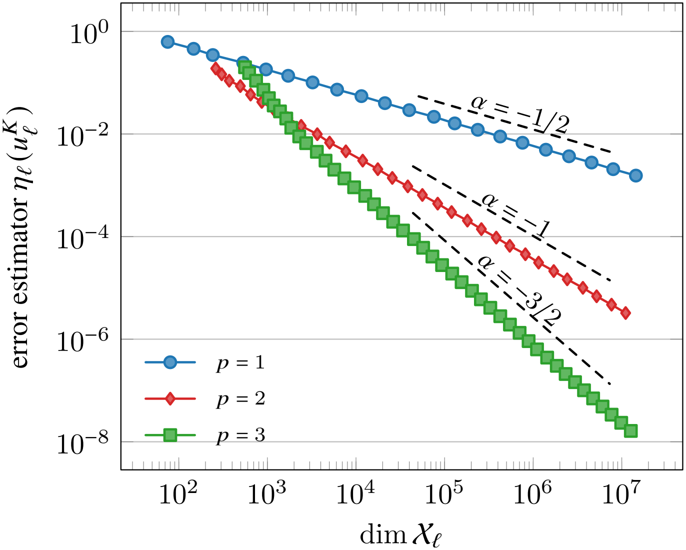
Gantner, Haberl, Praetorius, Schimanko: *Math. Comp.*, 90 (2021)

Example: AFEM with inexact solver

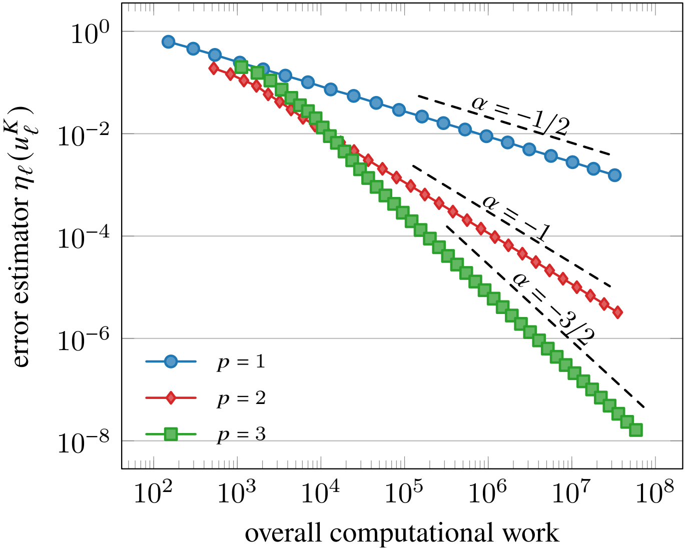
- solve $-\Delta u = 1$ in Ω subject to Dirichlet boundary conditions
- SOLVE & ESTIMATE – MARK – REFINE driven by residual error estimator $\eta_\ell(u_\ell^\star)$



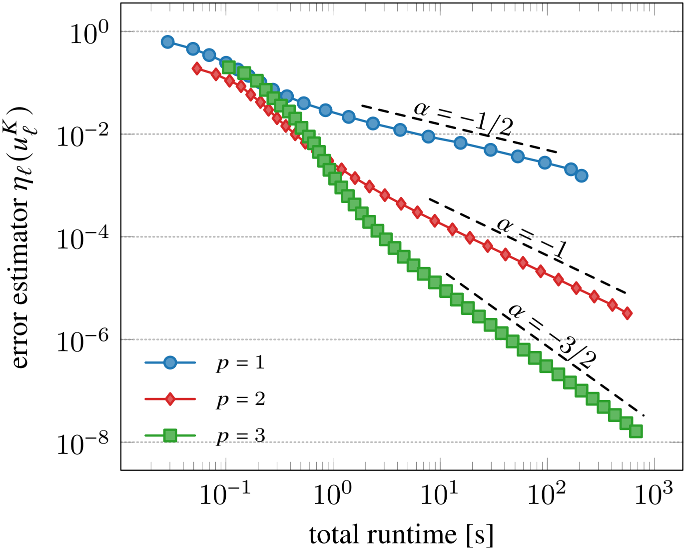
Convergence wrt. dofs ($\theta = 0.5$, $\lambda = 0.7$)

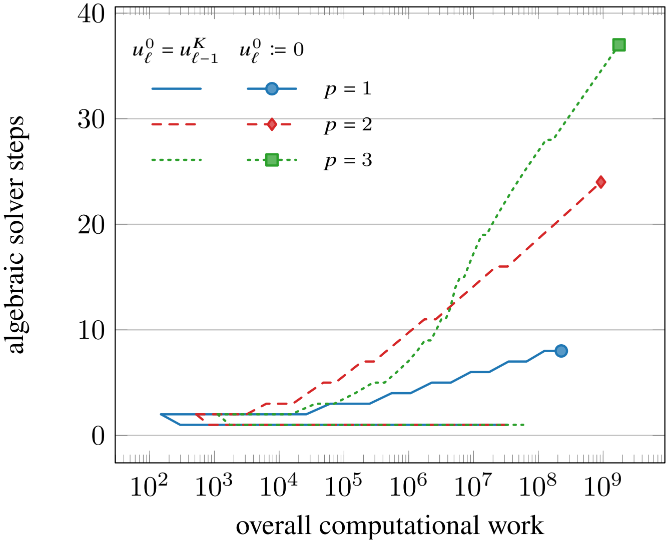


Convergence wrt. overall cost ($\theta = 0.5$, $\lambda = 0.7$)



Convergence wrt. time ($\theta = 0.5$, $\lambda = 0.7$)




40
30
20
10
0
algebraic solver steps
$u_\ell^0 = u_{\ell-1}^K$
$u_\ell^0 := 0$
$p = 1$
$p = 2$
$p = 3$
10^2 10^3 10^4 10^5 10^6 10^7 10^8 10^9
overall computational work

Take home messages, part II

1. can include contractive solver into AFEM algorithm / analysis

2. solver termination criterion by equibalance of error contributions

3. rigorous proof of optimal interplay of adaptive mesh-refinement and inexact solver
   - parameter-robust full R-linear convergence of quasi-error
   - parameter-robust rates = complexity
   - optimal rates/complexity by perturbation analysis for AFEM with exact solver

4. necessity of nested iteration

A nonlinear model problem

Model problem

Strong formulation with scalar nonlinearity

$$-\mathrm{div}\,\big(\mu(|\nabla u^\star|^2)\nabla u^\star\big) = f \quad \text{in } \Omega$$
$$u^\star = 0 \quad \text{on } \partial\Omega$$

- model example: $M\,(t-s) \le \mu(t^2)t - \mu(s^2)s \le L\,(t-s) \qquad \forall\, 0 \le s \le t$

Weak formulation

- find $u^\star \in H^1_0(\Omega)$ s.t. $\langle \mu(|\nabla u^\star|^2)\nabla u^\star\,,\,\nabla v\rangle_{L^2(\Omega)} = \langle f\,,\,v\rangle_{L^2(\Omega)} \quad \forall\, v \in H^1_0(\Omega)$

- note: excludes p-Laplacian

Zeidler: (1990)

Discretize – Linearize – Solve

Weak formulation

- find $u^\star \in H^1_0(\Omega)$ s.t. $\langle \mu(|\nabla u^\star|^2)\nabla u^\star \,,\, \nabla v\rangle_{L^2(\Omega)} = \langle f \,,\, v\rangle_{L^2(\Omega)} \quad \forall\, v \in H^1_0(\Omega)$

Discrete formulation

- find $u^\star_\ell \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u^\star_\ell|^2)\nabla u^\star_\ell \,,\, \nabla v_\ell\rangle_{L^2(\Omega)} = \langle f \,,\, v_\ell\rangle_{L^2(\Omega)} \quad \forall\, v_\ell \in \mathcal{X}_\ell$

Kačanov linearization (linearized discrete formulation)

- given $u^{k,\star}_\ell \in \mathcal{X}_\ell$, find $u^{k+1,\star}_\ell \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u^{k,\star}_\ell|^2)\nabla u^{k+1,\star}_\ell \,,\nabla v_\ell\rangle_{L^2(\Omega)} = \langle f \,,\, v_\ell\rangle_{L^2(\Omega)} \,\forall\, v_\ell \in \mathcal{X}_\ell$

Linearized discrete formulation (with inexact $u^{k,J}_\ell$)

- given $u^{k,J}_\ell \in \mathcal{X}_\ell$, find $u^{k+1,\star}_\ell \in \mathcal{X}_\ell$ s.t. $\langle \mu(|\nabla u^{k,J}_\ell|^2)\nabla u^{k+1,\star}_\ell \,,\nabla v_\ell\rangle_{L^2(\Omega)} = \langle f \,,\, v_\ell\rangle_{L^2(\Omega)} \,\forall\, v_\ell \in \mathcal{X}_\ell$

Available AFEM results in this setting

- linear convergence and optimal rates with exact solver [GMZ'12]
- full R-linear conv. and optimal rates with linearization + exact algebraic solver [HPW'21]
- full R-linear conv. and optimal complexity with linearization + contractive solver
  - analysis based on perturbation argument [HPSV'21]
  - parameter-robust R-linear convergence [MPS'24$^+$]

Garau, Morin, Zuppa: *Numer. Math: Theory, Meth. Appl.*, 5 (2012)

Heid, Praetorius, Wihler: *Comput. Methods Appl. Math.*, 21 (2021)

Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

Miraçi, Praetorius, Streitberger: arXiv: 2401.17778 (2024)

Termination of algebraic solver

Equilibration criterion [HPSV'21]

- stop algebraic solver if $\|u_\ell^{k,J} - u_\ell^{k,J-1}\| \le \mu\,[\,\eta_\ell(u_\ell^{k,J}) + \|u_\ell^{k,J} - u_\ell^{k-1,J}\|\,]$
- stop linearization if $\|u_\ell^{K,J} - u_\ell^{K-1,J}\| \le \lambda\,\eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ full R-linear convergence for arbitrary λ but sufficiently small μ

Energy-based criterion [MPS'24⁺]

- stop algebraic solver if $\|u_\ell^{k,J} - u_\ell^{k-1,J}\|^2 \lesssim \mathbb{D}^2(u_\ell^{k,J}, u_\ell^{k-1,J})$
- stop linearization if $\mathbb{D}^2(u_\ell^{K,J}, u_\ell^{K-1,J})^{1/2} \le \lambda\,\eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ parameter-robust full R-linear convergence for arbitrary λ — optimal interplay!

Haberl, Praetorius, Schimanko, Vohralík: *Numer. Math.*, 147 (2021)

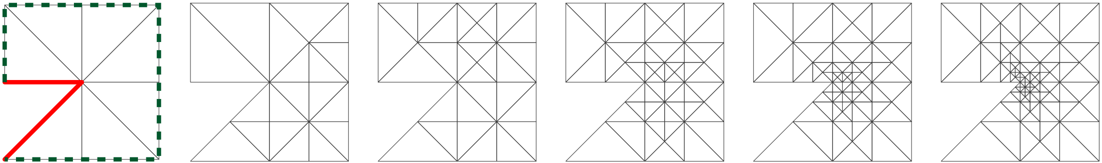
Heid, Praetorius, Wihler: *Comput. Methods Appl. Math.*, 21 (2021)

Miraçi, Praetorius, Streitberger: arXiv: 2401.17778 (2024)

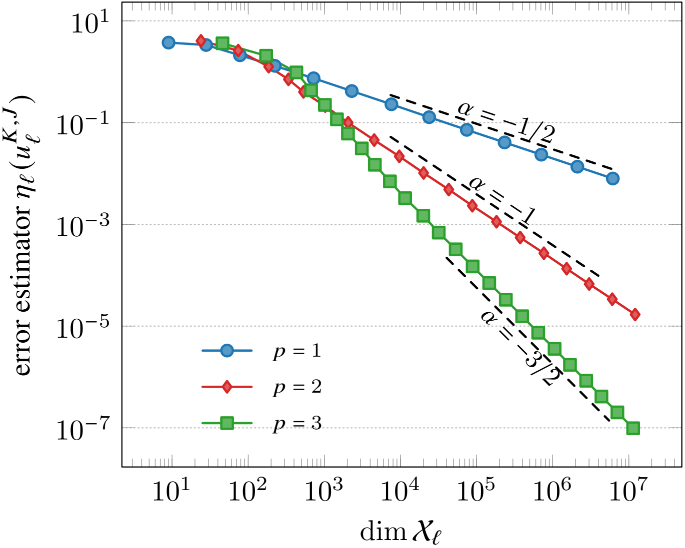
mixed BVP

$$
\begin{aligned}
-\mathrm{div}\big(\mu(|\nabla u^\star|)\,\nabla u^\star\big) &= f \quad \text{in } \Omega \\
\mu(|\nabla u^\star|)\,\nabla u^\star \cdot \boldsymbol{n} &= g \quad \text{on } \Gamma_N \\
u^\star &= 0 \quad \text{on } \Gamma_D
\end{aligned}
$$

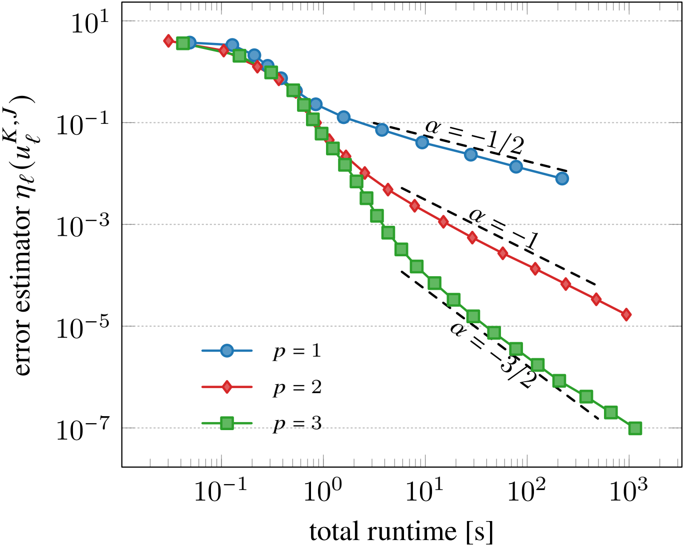
- $\mu(t) := 2 + \dfrac{1}{\sqrt{1+t^2}} \implies M = 2, \quad L = 3$ wrt. $\|\!|\cdot|\!\| = \|\nabla(\cdot)\|_{L^2(\Omega)}$



Optimal convergence wrt. dofs $(\theta = 0.3,\ \lambda = 0.7)$



Optimal convergence wrt. time $\quad (\theta = 0.3,\ \lambda = 0.7)$



Take home messages, part III

1. full R-linear convergence for (strongly monotone) energy minimization problems
   - i.e., "contraction" independently of mesh-refinement, linearization, or algebraic solver step
   - AFEM with optimal interplay of mesh-refinement and iterative solver

2. certain extensions (other linearization strategies, semilinear PDEs, goal-oriented AFEM)

3. current restrictions: $p = 1$ for quasilinear, only simple nonlinearities

4. excludes p-Laplacian
   - linear convergence and optimal rates with exact solution [Belenki–Diening–Kreuzer '12]
   - Kacanov-type lin. [Diening–Fornasier–Tomasi–Wank '20], [Balci–Diening–Storn '23], [Heid '23]

Thank you for your attention!

Gantner, Haberl, Praetorius, Schimanko
Rate optimality of adaptive finite element methods with respect to overall computational costs
Math. Comp., 90 (2021)

Bringmann, Feischl, Miraçi, Praetorius, Streitberger
On full linear convergence and optimal complexity of adaptive FEM with inexact solver
Preprint, arXiv: 2311.15738 (2023)

Miraçi, Praetorius, Streitberger
Parameter-robust full linear convergence and optimal complexity of adaptive iteratively linearized FEM for nonlinear PDEs
Preprint, arXiv: 2401.17778 (2024)

Thanks, Joscha! Hope to see you all at CMAM 2026!

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