UE Mengenlehre SoSe2024

lena.wallner@tuwien.ac.at

Session 11

updated on June 7 at 15:58

Definition. Suppose κ is an uncountable cardinal.

(i). A tree T is a **κ-tree** if $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.

(ii). κ has the **tree property** if every κ-tree has a cofinal branch.

(iii). κ is **weakly compact** if κ is inaccessible and has the tree property.

(iv). A **κ-model** is a transitive $M \models \mathrm{ZFC}^-$ of size κ with $\kappa + 1 \subseteq M$ and ${}^{<\kappa}M \subseteq M$ (i.e. ${}^{\alpha}M \subseteq M$ for $\alpha < \kappa$).

Definition. The **Borel hierarchy** consisting of $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha$ for $\alpha < \omega_1$ on ω^ω is defined as follows:

- $\mathbf{\Sigma}^0_0 \coloneqq \{A \subseteq \omega^\omega \mid A \text{ is clopen}\}$,
- $\mathbf{\Pi}^0_\alpha \coloneqq \neg\mathbf{\Sigma}^0_\alpha \coloneqq \{A \subseteq \omega^\omega \mid \omega^\omega \setminus A \in \mathbf{\Sigma}^0_\alpha\}$,
- $\mathbf{\Sigma}^0_\alpha \coloneqq \{\bigcup_{n<\omega} A_n \mid (A_n)_{n<\omega} \in (\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta)^\omega\}$,
- $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$.

Definition. A pointclass $\Gamma \subseteq \mathcal{P}(\omega^\omega)$ has a **universal set** if there is[1] $U \in \Gamma$, $U \subseteq (\omega^\omega)^2$ so that for every $A \in \Gamma$, $A \subseteq \omega^\omega$, there is some $x \in \omega^\omega$ with $A = U_x \coloneqq \{y \mid (x, y) \in U\}$.

[1]Note that ω^ω is homeomorphic to $(\omega^\omega)^2$, so (abusing notation) for $A \subseteq (\omega^\omega)^2$, we write $A \in \Gamma$ iff $\pi[A] \in \Gamma$ for any/all homeomorhpisms $\pi\colon (\omega^\omega)^2 \to \omega^\omega$.

1) Let κ be an inaccessible cardinal. Show that for every $A \subseteq \kappa$ there is a κ-model M with $A \in M$.

2) Let κ be an uncountable cardinal. Show that the following are equivalent.

i. κ is weakly compact.

ii. Whenever $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{A}| = \kappa$ then there is a $<\kappa$-closed filter $\mathcal{F}$ on κ which measures every $A \in \mathcal{A}$.

iii. For every $A \subseteq \kappa$ there is a κ-model M with $A \in M$, a transitive N and a nontrivial elementary $j \colon M \to N$ with $\operatorname{crit}(j) = \kappa$.

3) Suppose κ is weakly compact.

(a) Show that if $A \subseteq \kappa$ so that $A \cap \alpha \in L$ for all $\alpha < \kappa$ then $A \in L$.

(b) Show that $L \models$ "κ is weakly compact".

4) (a) Show that for $1 \leq \alpha < \omega_1$, the pointclasses $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ have universal sets.

(b) Suppose Γ is a pointclass which admits a universal set. Prove that

$$\Gamma \neq \neg\Gamma := \{A \subseteq \omega^\omega \mid \omega^\omega \setminus A \in \Gamma\}.$$

(c) Show that for $1 \leq \alpha < \beta < \omega_1$ the following relations hold:

$$\begin{array}{ccccc} & & \mathbf{\Sigma}^0_\alpha & & \\ & \subsetneq & & \subsetneq & \\ \mathbf{\Delta}^0_\alpha & & & & \mathbf{\Delta}^0_\beta \\ & \subsetneq & & \subsetneq & \\ & & \mathbf{\Pi}^0_\alpha & & \end{array}$$