Optimal cost of AFEM for linear elliptic PDEs

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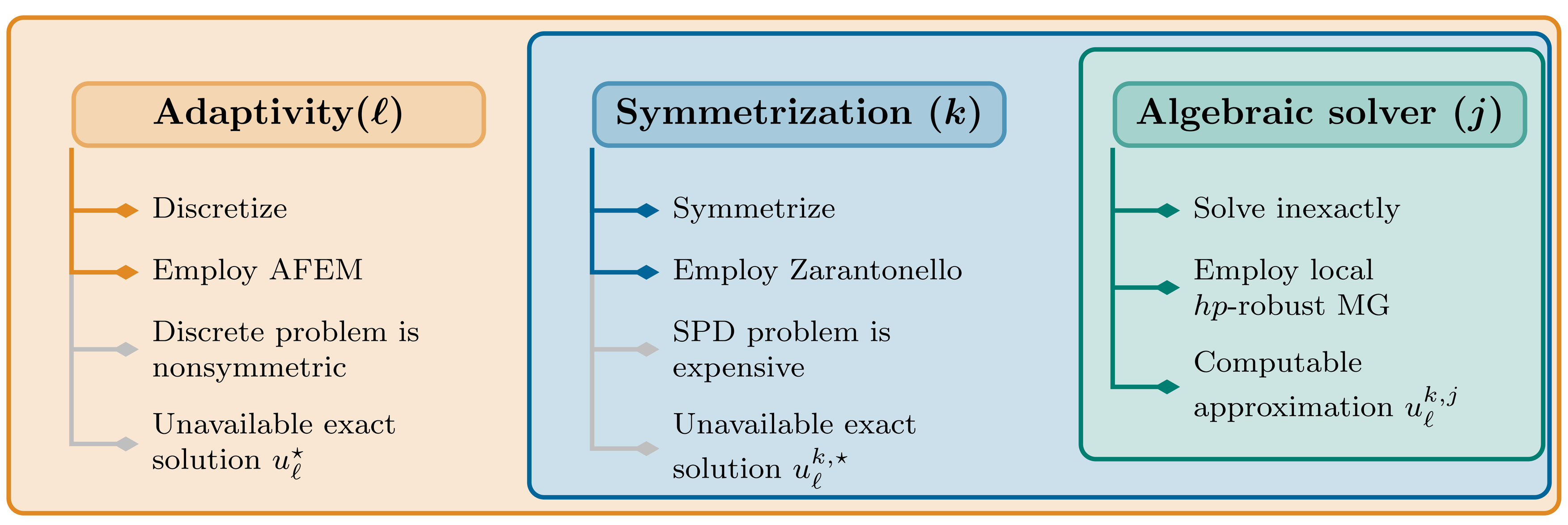
Motivation

Adaptive numerical schemes aim at approximating an unknown exact PDE solution $u^\star$ with *optimal convergence rates* and *minimal computational cost*, i.e., optimal complexity. Given an initial mesh $\mathscr{T}_0$, we say that $u^\star$ is approximable at rate $s > 0$, i.e., $\|u^\star\|_{\mathbb{A}_s(\mathscr{T}_0)} < \infty$, if there exist optimal refinements of $\mathscr{T}_0$ along which the (quasi-) error decreases with rate s. However, optimal complexity for *nonsymmetric* linear elliptic PDEs remained an open question due to the lack of a *contractive* algebraic solver suitable for the variational structure of the PDE. Closing this gap is the main contribution of the presented work.



Abstract variational formulation:

$(\mathbb{V}, a)$ Hilbert space, $F \in \mathbb{V}'$ right-hand side, $\mathscr{K} : \mathbb{V} \to \mathbb{V}'$ compact perturbation

Find $u^\star \in \mathbb{V}$ such that $\quad b(u^\star, v) := a(u^\star, v) + \langle \mathscr{K} u^\star\,,\, v\rangle = F(v) \quad$ for all $v \in \mathbb{V}$,

where $b(\cdot,\cdot)$ is elliptic and continuous (Lax–Milgram)

Adaptive algorithm

Approach: adaptivity, symmetrization, algebraic solver

Input: initial triangulation $\mathscr{T}_0$, $\quad u_0^{0,0} := 0, \quad 0 < \theta \le 1, \quad \delta > 0, \quad \lambda_{\mathsf{sym}}, \lambda_{\mathsf{alg}} > 0$

for $\ell = 0, 1, 2, \dots$ repeat

SOLVE & ESTIMATE

for $k = 1, 2, \dots, K$ repeat

set up SPD system with (expensive) unavailable solution $\; u_\ell^{k,\star} \in \mathbb{V}_\ell \;$ to

$$a(u_\ell^{k,\star}, v_\ell) = a(u_\ell^{k,j-1}, v_\ell) + \delta\big[F(v_\ell) - b(u_\ell^{k,j-1}, v_\ell)\big] \quad \text{for all } v_\ell \in \mathbb{V}_\ell.$$

for $j = 1, 2, \dots, J$ repeat

compute $\; u_\ell^{k,j} := u_\ell^{k,j-1} + \mathtt{hpMG}(u_\ell^{k,j-1}) \;$ by solver below $(\star)$

compute $\; \eta_\ell(T, u_\ell^{k,j}) \;$ for all $T \in \mathscr{T}_\ell$

until $\; |||u_\ell^{k,j} - u_\ell^{k,j-1}||| \le \lambda_{\mathsf{alg}}\big[\lambda_{\mathsf{sym}}\,\eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k-1,J}|||\big]$

$\longrightarrow$ equilibrate algebraic error

until $\; |||u_\ell^{k,J} - u_\ell^{k-1,J}||| \le \lambda_{\mathsf{sym}}\,\eta_\ell(u_\ell^{k,J})$

$\longrightarrow$ equilibrate symmetrization error

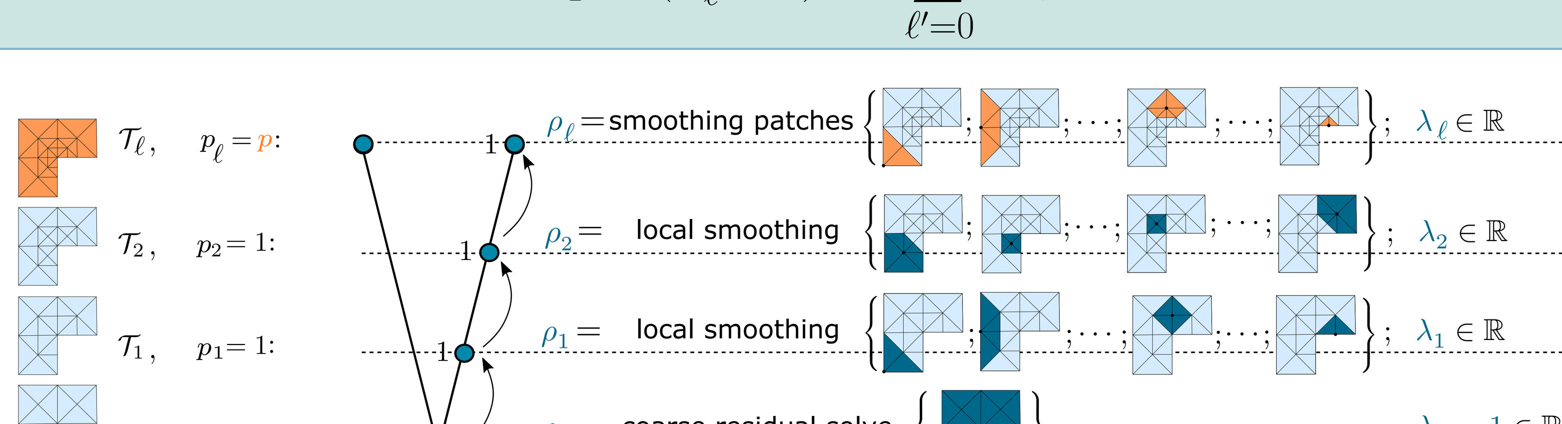
MARK choose $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ such that $\quad \theta \sum_{T\in\mathscr{T}_\ell} \eta_\ell(T, u_\ell^{K,J})^2 \le \sum_{T\in\mathscr{M}_\ell} \eta_\ell(T, u_\ell^{K,J})^2$

REFINE $\mathscr{T}_{\ell+1} := \mathtt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell), \quad u_{\ell+1}^{0,0} := u_\ell^{K,J} \;\longrightarrow$ nested iteration

Output: discrete solutions $u_\ell^{k,j}$ and estimators $\eta_\ell(u_\ell^{k,j})$ for indices (ℓ,k,j) in the set $\mathscr{Q}$ with ordering $|\ell,k,j| := \#\{(\ell',k',j') \in \mathscr{Q} \mid u_{\ell'}^{k',j'}$ computed earlier than $u_\ell^{k,j}\}$

Algebraic solver: hp-robust multigrid

$(\star)$ Compute solver update $\quad \mathtt{hpMG}(u_\ell^{k,j-1}) := \sum_{\ell'=0}^{\ell} \lambda_{\ell'}\rho_{\ell'} \quad$ by V-cycle multigrid:



Ensures: L-robustness by *local* smoothing on $\mathbb{P}_1$-levels,
p-robustness by *block* smoothing on the $\mathbb{P}_p$-level

Main results

Full linear convergence: For δ, λ_{alg} sufficiently small and $\ell_0 \le \ell'$ sufficiently large, the quasi-error $\Delta_\ell^{k,j} := |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell(u_\ell^{k,j})$ satisfies

$$\Delta_\ell^{k,j} \lesssim q_{\mathsf{lin}}^{|\ell,k,j| - |\ell',k',j'|}\,\Delta_{\ell'}^{k',j'} \quad \text{with fixed} \quad 0 < q_{\mathsf{lin}} < 1$$

Rate = complexity: Convergence rate s with respect degrees of freedom is equivalent to convergence rate s with respect to computational cost

$$\sup_{(\ell,k,j)\in\mathscr{Q}} (\#\mathscr{T}_\ell)^s \Delta_\ell^{k,j} \simeq \sup_{(\ell,k,j)\in\mathscr{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathscr{Q} \\ |\ell',k',j'| \le |\ell,k,j|}} \#\mathscr{T}_{\ell'} \Big)^s \Delta_\ell^{k,j}$$

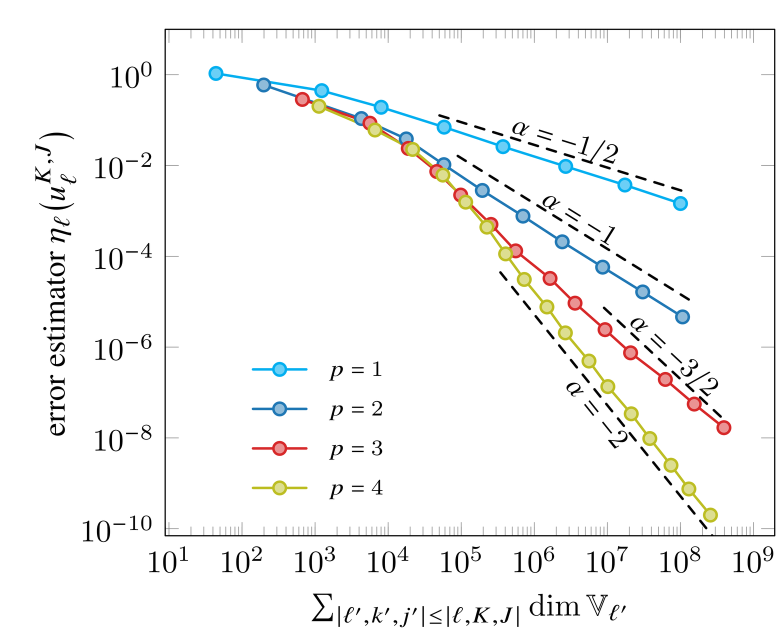
Optimal complexity: Optimal convergence rate s with respect to the computational cost for adaptivity parameters θ, δ, λ_{sym}, λ_{alg} being sufficiently small

$$\sup_{(\ell,k,j)\in\mathscr{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathscr{Q} \\ |\ell',k',j'| \le |\ell,k,j|}} \#\mathscr{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} \simeq \|u^\star\|_{\mathbb{A}_s(\mathscr{T}_0)}$$

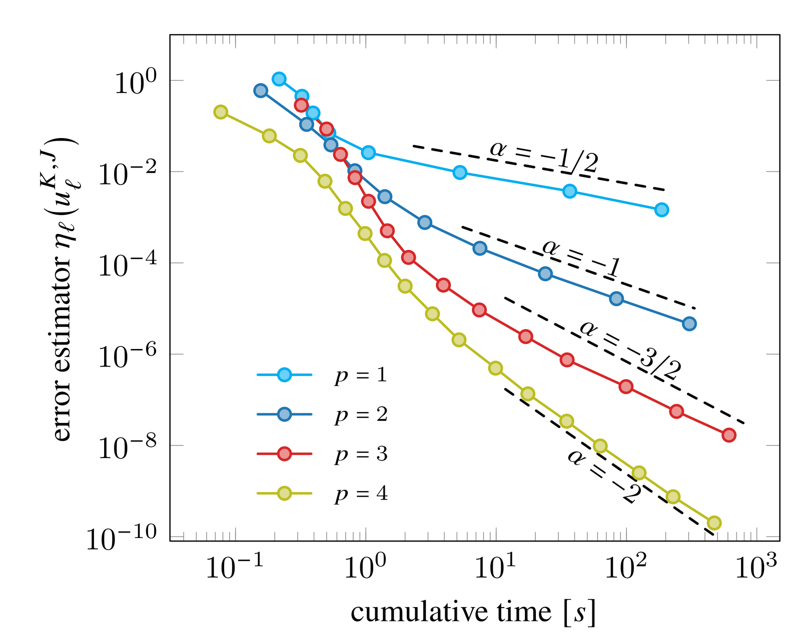
Numerical experiments

Conforming Courant FEM for the solution of

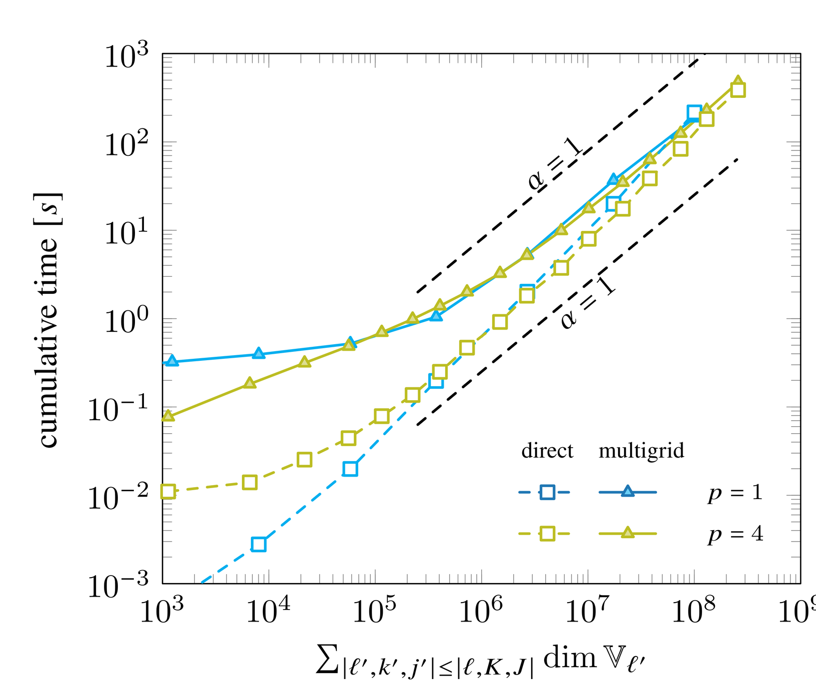
$$-\Delta u^\star + \mathrm{id}\cdot\nabla u^\star + u^\star = 1 \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega$$



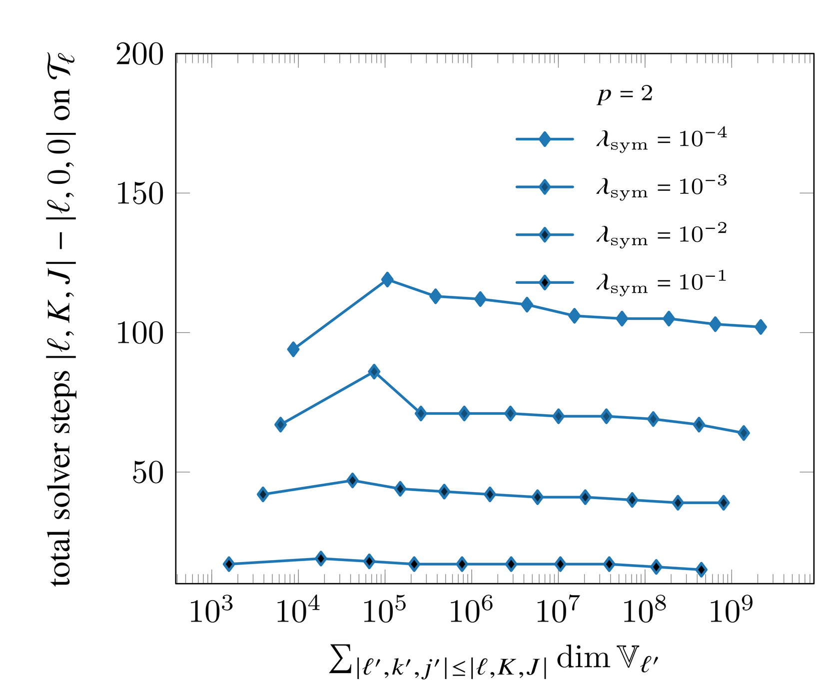
(a) Optimal complexity (in theory)



(b) Optimal complexity (in practice)



(c) Linear complexity of hpMG



(d) Robustness of inexact solve

References

Maximilian Brunner, Pascal Heid, Michael Innerberger, Ani Miraçi, Dirk Praetorius, Julian Streitberger: *Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs.* IMA J. Numer. Anal. (2023)

Michael Innerberger, Ani Miraçi, Dirk Praetorius, Julian Streitberger: *hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs.* arXiv:2210.10415 (2022)

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