

A multilevel algebraic error estimator and the corresponding iterative solver with p -robust behavior*

15TH U.S. NATIONAL CONGRESS ON COMPUTATIONAL MECHANICS

Austin, Texas, USA, July 28 - August 1, 2019

ANI MIRAÇI , JAN PAPEŽ, MARTIN VOHRALÍK

Inria Paris & École des Ponts, France

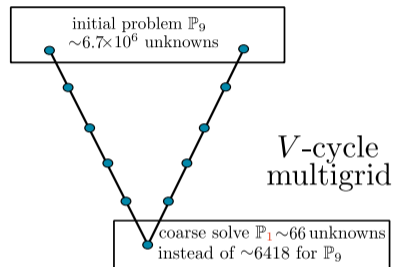


European Research Council
Established by the European Commission

*Miraçi, Papež, and Vohralík. "A multilevel algebraic error estimator and the corresponding iterative solver with p -robust behavior". HAL preprint 02070981, 2019.

CONTEXT

- ▶ We address the issue of large linear systems of type $Ax = b$ arising from finite element method of order p discretizations.
- ▶ The approach is of *geometric multigrid-type* : V-cycle MG(ν_1, ν_2), where ν_1, ν_2 , are the pre- and post-smoothing steps (ex. Jacobi, Gauss-Seidel, block Jacobi etc.).



References

- Hackbusch. "Multi-grid methods and applications". 1985.
- Pavarino. "Additive Schwarz methods for the p -version finite element method". 1994.
- Schöberl et al. "Additive Schwarz preconditioning for p -version triangular and tetrahedral finite elements". 2008.
- Kanschat. "Robust smoothers for high-order discontinuous Galerkin discretizations of advection-diffusion problems". 2008.
- Antonietti et al. "A uniform additive Schwarz preconditioner for high-order discontinuous Galerkin approximations of elliptic problems". 2017.
- Antonietti and Pennesi. "V-cycle multigrid algorithms for discontinuous Galerkin methods on non-nested polytopic meshes". 2019.
- Sundar, Stadler, and Biros. "Comparison of multigrid algorithms for high-order continuous finite element discretizations". 2015.

OVERVIEW

- ▶ Setting: finite element method of *order* p for the Poisson problem.
- ▶ Multilevel construction of an *algebraic residual lifting* to define:
 1. an *a posteriori algebraic error estimator*
 2. an *iterative linear solver*
- ▶ Main results:
 1. The a posteriori estimator is a **two-sided p -robust bound** on the algebraic error
 2. The iterative solver **contracts the error p -robustly** on each iteration
- ▶ Numerical results

FINITE ELEMENT DISCRETIZATION, ALGEBRAIC SYSTEM

Setting: $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, an open bounded polytope, $f \in L^2(\Omega)$ a source term.

Poisson problem: find $u \in H_0^1(\Omega)$ such that $(\nabla u, \nabla v) = (f, v)$, $\forall v \in H_0^1(\Omega)$.

Fix $p \geq 1$ and define

$$V_J^p = \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega),$$

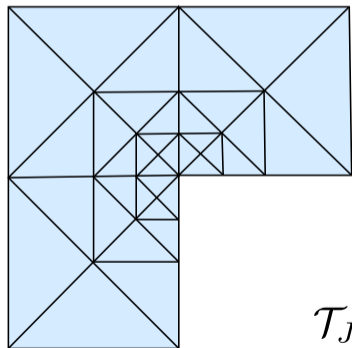
where $\mathbb{P}_p(\mathcal{T}_J) = \{v_J \in L^2(\Omega), v_J \in \mathbb{P}_p(K) \forall K \in \mathcal{T}_J\}$.

Discrete problem: Find $u_J \in V_J^p$ such that

$$(\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \quad (\text{FE})$$

Introducing a basis of V_J^p , then the problem can be rewritten as $\mathbb{A}_J \mathbf{U}_J = \mathbf{F}_J$.

We work with the *basis-independent* functional formulation (FE).

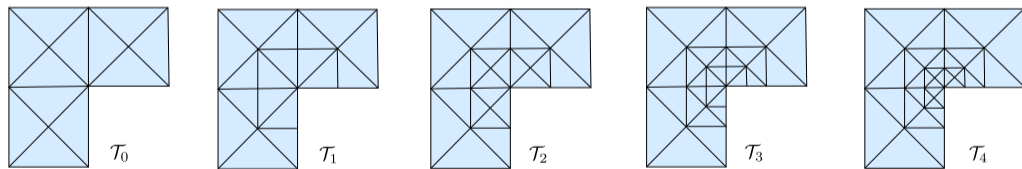

 \mathcal{T}_J

A HIERARCHY OF MESHES AND SPACES

Assumptions on $\{\mathcal{T}_j\}_{0 \leq j \leq J}$

- ▶ *Shape regularity*: The ratio element diameter over the diameter of the largest ball inscribed in the element is bounded for all elements by $\kappa_{\mathcal{T}} > 0$.
- ▶ *Strength of refinement*: For any $j \in \{1, \dots, J\}$, and for all $K \in \mathcal{T}_{j-1}$, $K^* \in \mathcal{T}_j$, such that $K^* \subset K$, h_{K^*} and h_K are comparable.

Example: A mesh hierarchy with $J = 4$, associated spaces with $p' \in \{1, \dots, p\}$



$$V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega) \quad V_1^{p'} = \mathbb{P}_{p'}(\mathcal{T}_1) \cap H_0^1(\Omega) \quad V_2^{p'} = \mathbb{P}_{p'}(\mathcal{T}_2) \cap H_0^1(\Omega) \quad V_3^{p'} = \mathbb{P}_{p'}(\mathcal{T}_3) \cap H_0^1(\Omega) \quad V_4^p = \mathbb{P}_p(\mathcal{T}_4) \cap H_0^1(\Omega)$$

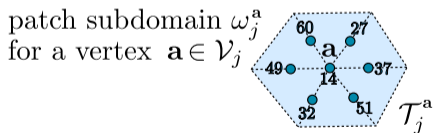
Note: We can have very general meshes (*highly refined meshes* are also allowed).
However, our theoretical results *depend* on the number of refinements J .

PATCHES

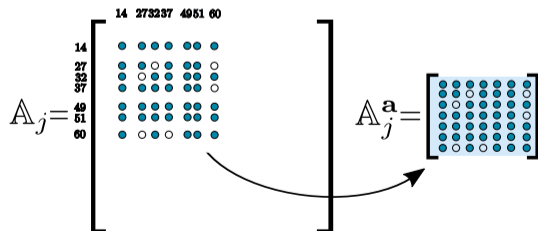
Let \mathcal{V}_j be the set of vertices of the mesh \mathcal{T}_j , $j \in \{1, \dots, J\}$. Given a vertex $\mathbf{a} \in \mathcal{V}_j$, we denote

- ▶ $\mathcal{T}_j^{\mathbf{a}}$ the patch of elements sharing vertex \mathbf{a}
- ▶ $\omega_j^{\mathbf{a}}$ the corresponding patch subdomain
- ▶ $V_j^{\mathbf{a}}$ the associated local space

Example: Geometric (left) and algebraic (right) representation of localizing the problem for $p' = p = 2$, $j \in \{1, \dots, J - 1\}$ and a patch composed of 6 elements:

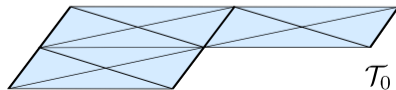


$$V_j^{\mathbf{a}} = \mathbb{P}_{p'}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$$



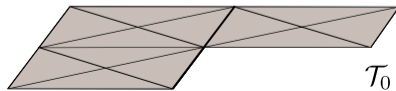
MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

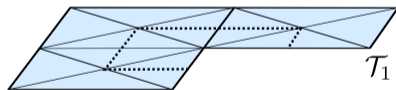
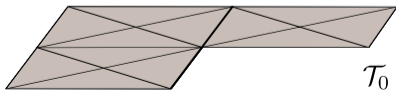
$j = 0 :$



MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

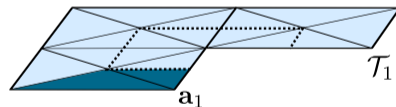
$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$



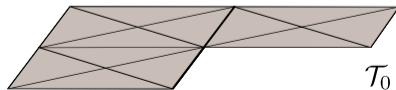
MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$ $j = 1 :$  $j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$ 

MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$$j = 1 : \underbrace{\rho_{1, \mathbf{a}_1}^i}_{\in V_1^{\mathbf{a}_1}}$$

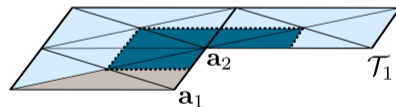


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

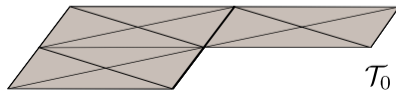


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$$j = 1 : \underbrace{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i}_{\in V_1^{\mathbf{a}_2}}$$

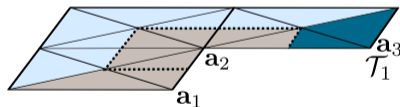


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

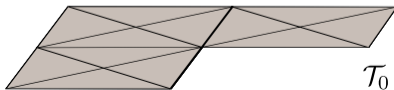


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$$j = 1 : \rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \underbrace{\rho_{1,\mathbf{a}_3}^i}_{\in V_1^{\mathbf{a}_3}}$$

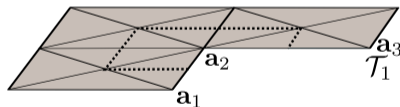


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

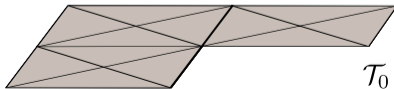


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$$j = 1 : \rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots$$

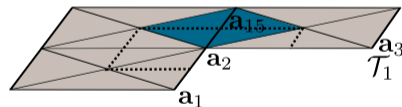


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

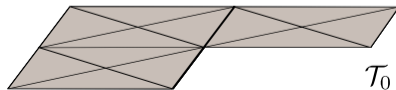


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$$j = 1 : \rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \underbrace{\rho_{1,\mathbf{a}_{15}}^i}_{\in V_1^{\mathbf{a}_{15}}}$$

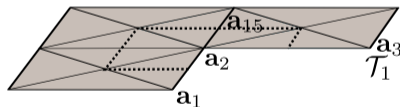


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

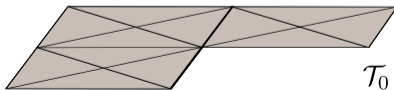


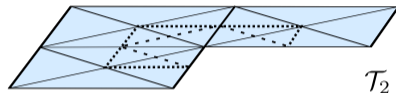
MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

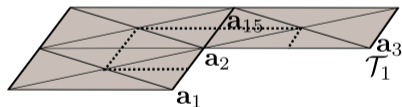
$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$

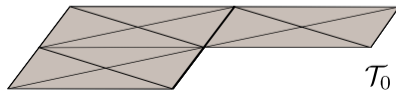


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$



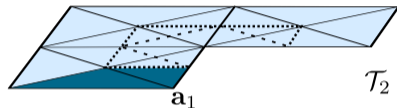
MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$ $j = 2 :$ 

$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$


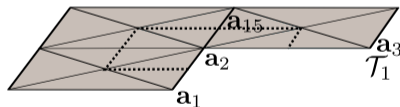
$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

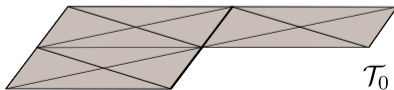
$$j = 2 : \underbrace{\rho_{2, \mathbf{a}_1}^i}_{\in V_2^{\mathbf{a}_1}}$$



$$j = 1 : \frac{\rho_{1, \mathbf{a}_1}^i + \rho_{1, \mathbf{a}_2}^i + \rho_{1, \mathbf{a}_3}^i + \dots + \rho_{1, \mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$

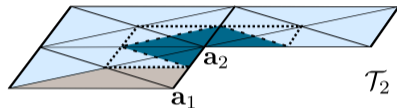


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

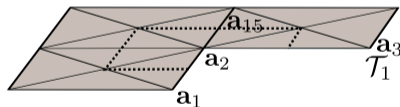


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

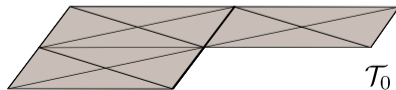
$$j = 2 : \underbrace{\rho_{2,\mathbf{a}_1}^i + \rho_{2,\mathbf{a}_2}^i}_{\in V_2^{\mathbf{a}_2}}$$



$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$

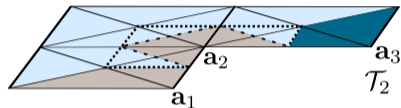


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

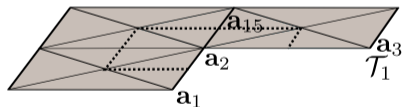


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

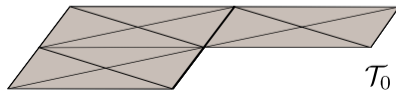
$$j = 2 : \rho_{2,\mathbf{a}_1}^i + \rho_{2,\mathbf{a}_2}^i + \underbrace{\rho_{2,\mathbf{a}_3}^i}_{\in V_2^{\mathbf{a}_3}}$$



$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$

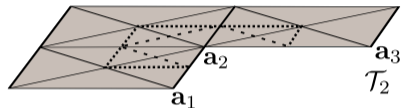


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

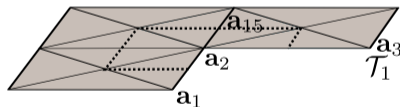


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

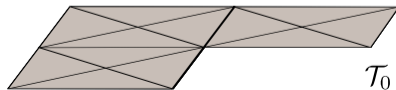
$$j = 2 : \rho_{2,\mathbf{a}_1}^i + \rho_{2,\mathbf{a}_2}^i + \rho_{2,\mathbf{a}_3}^i + \dots$$



$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$

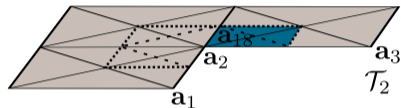


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

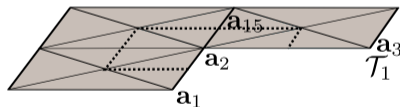


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

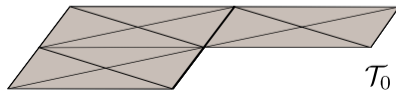
$$j = 2 : \rho_{2,\mathbf{a}_1}^i + \rho_{2,\mathbf{a}_2}^i + \rho_{2,\mathbf{a}_3}^i + \dots + \underbrace{\rho_{2,\mathbf{a}_{18}}^i}_{\in V_2^{\mathbf{a}_{18}}}$$



$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$

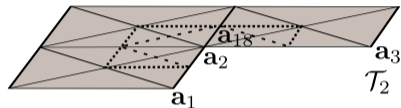


$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$

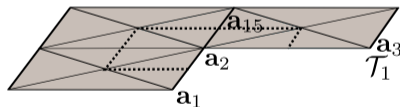


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

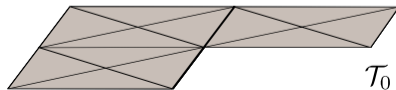
$$j = 2 : \frac{\rho_{2,\mathbf{a}_1}^i + \rho_{2,\mathbf{a}_2}^i + \rho_{2,\mathbf{a}_3}^i + \dots + \rho_{2,\mathbf{a}_{18}}^i}{J(d+1)} \in V_2^p$$



$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$



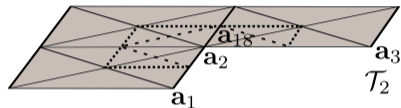
$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$



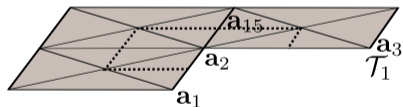
MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$$\rho_{2,\text{alg}}^i = \rho_0^i + \sum_{j=1}^2 \frac{\sum_{a \in \mathcal{V}_j} \rho_{j,a}^i}{J(d+1)}$$

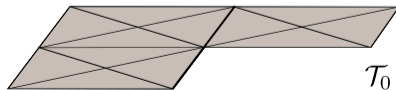
$$j = 2 : \frac{\rho_{2,\mathbf{a}_1}^i + \rho_{2,\mathbf{a}_2}^i + \rho_{2,\mathbf{a}_3}^i + \dots + \rho_{2,\mathbf{a}_{18}}^i}{J(d+1)} \in V_2^p$$



$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$



$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$



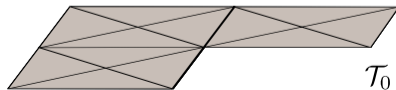
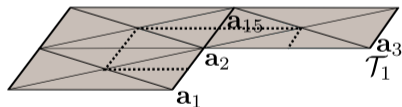
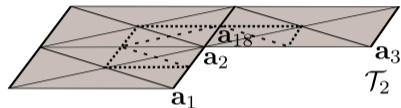
MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$$\rho_{2,\text{alg}}^i = \rho_0^i + \sum_{j=1}^2 \frac{\sum_{a \in \mathcal{V}_j} \rho_{j,a}^i}{J(d+1)} \in V_2^p$$

$$j = 2 : \frac{\rho_{2,\mathbf{a}_1}^i + \rho_{2,\mathbf{a}_2}^i + \rho_{2,\mathbf{a}_3}^i + \dots + \rho_{2,\mathbf{a}_{18}}^i}{J(d+1)} \in V_2^p$$

$$j = 1 : \frac{\rho_{1,\mathbf{a}_1}^i + \rho_{1,\mathbf{a}_2}^i + \rho_{1,\mathbf{a}_3}^i + \dots + \rho_{1,\mathbf{a}_{15}}^i}{J(d+1)} \in V_1^{p'}$$

$$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$$



MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL¹

Let $u_j^i \in V_j^p$ be arbitrary. We define its associated *algebraic residual lifting*.

Coarse solve: Define $\rho_0^i \in V_0$ by: $(\nabla \rho_0^i, \nabla v_0) = (f, v_0) - (\nabla u_j^i, \nabla v_0)$, $\forall v_0 \in V_0$.

Construction: Consider $\rho_{j,\text{alg}}^i \in V_j^p$

$$\rho_{j,\text{alg}}^i = \rho_0^i + \sum_{j=1}^J \frac{\sum_{\mathbf{a} \in \mathcal{V}_j} \rho_{j,\mathbf{a}}^i}{J(d+1)},$$

where for all $j = \{1, \dots, J\}$, $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$:

$$(\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\nabla u_j^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}, \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

Remark: $\rho_{j,\text{alg}}^i$ approximates the algebraic error $u_j - u_j^i$ by

- ▶ a V-cycle MG(0,1) with piecewise affine coarse solve
- ▶ the smoother is *damped additive Schwarz / block Jacobi* associated to the patches

¹Papež et al. “Sharp algebraic and total a posteriori error bounds for h and p finite elements via a multilevel approach”. HAL preprint 01662944, 2017.

Definition 1 (Multilevel a posteriori estimator)

Let $u_J^i \in V_J^p$ be **arbitrary**, and let $\rho_{J,\text{alg}}^i$ be the associated algebraic residual lifting.

Set $\eta_{\text{alg}}^i := \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|}$, or else $\eta_{\text{alg}}^i := 0$ if $\rho_{J,\text{alg}}^i = 0$.

Definition 2 (Multilevel solver)

1. Initialize $u_J^0 \in V_0$ as the solution of $(\nabla u_J^0, \nabla v_0) = (f, v_0)$, $\forall v_0 \in V_0$.
2. Let $i \geq 0$. Set $u_J^{i+1} := u_J^i + \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|^2} \rho_{J,\text{alg}}^i$, or else $u_J^{i+1} := u_J^i$ if $\rho_{J,\text{alg}}^i = 0$.

Remark: Note that the *step size* plays a decisive role:

- ▶ it is determined by a *line search* optimization in the direction of the lifting
- ▶ without it, the solver would become to MG(0,1) with block Jacobi smoothing

MAIN RESULTS

Theorem 1 (p -robust reliable and efficient bound on the algebraic error)

Let $u_J^i \in V_J^p$ be **arbitrary**, let η_{alg}^i be the associated a posteriori estimator. There holds

- reliability: $\|\nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i$
- efficiency: $\eta_{\text{alg}}^i \geq \beta(\kappa_{\mathcal{T}}, d, J) \|\nabla(u_J - u_J^i)\|, \quad 0 < \beta(\kappa_{\mathcal{T}}, d, J) < 1 \quad (\text{E})$

Theorem 2 (p -robust error contraction of the multilevel solver)

Let $u_J^i \in V_J^p$ be **arbitrary**, let u_J^{i+1} be constructed from u_J^i using one step of the multilevel solver. Then there holds

$$\|\nabla(u_J - u_J^{i+1})\| \leq \alpha(\kappa_{\mathcal{T}}, d, J) \|\nabla(u_J - u_J^i)\|, \quad 0 < \alpha(\kappa_{\mathcal{T}}, d, J) < 1 \quad (\text{C})$$

Corollary 1 (Equivalence of the two main results)

Under the assumptions of Theorems 1 and 2, (E) holds if and only if (C) holds.

SKETCH OF THE PROOF OF THEOREM 1 : $\eta_{\text{alg}}^i \geq \beta \|\nabla(u_J - u_J^i)\|$

► Due to the definition of η_{alg}^i , it is enough to show:

$$\text{if } \rho_{J,\text{alg}}^i \neq 0 : \quad \eta_{\text{alg}}^i = \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|} \geq \beta \|\nabla(u_J - u_J^i)\|,$$

$$\text{if } \rho_{J,\text{alg}}^i = 0 : \quad \eta_{\text{alg}}^i = 0 = \|\nabla(u_J - u_J^i)\|.$$

Our approach consists in giving a:

- ❶ lower bound on $(f, \rho_{J,\text{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\text{alg}}^i)$: *the damping proves to be crucial*
 - ❷ upper bound on $\|\nabla \rho_{J,\text{alg}}^i\|^2$: *rather straightforward*
 - ❸ upper bound on $\|\nabla(u_J - u_J^i)\|^2$: *more delicate*²
- by the **splitting** $\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2$.

Leading to:

$$\|\nabla(u_J - u_J^i)\|^2 \geq (\eta_{\text{alg}}^i)^2 \stackrel{\text{❶}}{\gtrsim} \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2 \stackrel{\text{❸}}{\gtrsim} \|\nabla(u_J - u_J^i)\|^2$$

²Schöberl et al. "Additive Schwarz preconditioning for p -version triangular and tetrahedral finite elements". 2008.

SKETCH OF THE PROOF OF THEOREM 1 : $\eta_{\text{alg}}^i \geq \beta \|\nabla(\mathbf{u}_J - \mathbf{u}_J^i)\|$

► Due to the definition of η_{alg}^i , it is enough to show:

$$\text{if } \rho_{J,\text{alg}}^i \neq 0 : \quad \eta_{\text{alg}}^i = \frac{(f, \rho_{J,\text{alg}}^i) - (\nabla \mathbf{u}_J^i, \nabla \rho_{J,\text{alg}}^i)}{\|\nabla \rho_{J,\text{alg}}^i\|} \geq \beta \|\nabla(\mathbf{u}_J - \mathbf{u}_J^i)\|,$$

$$\text{if } \rho_{J,\text{alg}}^i = 0 : \quad \eta_{\text{alg}}^i = 0 = \|\nabla(\mathbf{u}_J - \mathbf{u}_J^i)\|.$$

Our approach consists in giving a:

- ❶ lower bound on $(f, \rho_{J,\text{alg}}^i) - (\nabla \mathbf{u}_J^i, \nabla \rho_{J,\text{alg}}^i)$: *the damping proves to be crucial*
 - ❷ upper bound on $\|\nabla \rho_{J,\text{alg}}^i\|^2$: *rather straightforward*
 - ❸ upper bound on $\|\nabla(\mathbf{u}_J - \mathbf{u}_J^i)\|^2$: *more delicate*²
- by the **splitting** $\|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2$.

Corollary 2 (Equivalence error-splitting)

$$\|\nabla(\mathbf{u}_J - \mathbf{u}_J^i)\|^2 \approx \|\nabla \rho_0^i\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2$$

²Schöberl et al. “Additive Schwarz preconditioning for p -version triangular and tetrahedral finite elements”. 2008.

NUMERICAL RESULTS

Consider the following problem:

L-shape domain problem: $u(r, \theta) = r^{2/3} \sin(2\theta/3)$; $\Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$.

We focus on testing numerically the p -robust behavior of our solver, a common choice for the **stopping criterion** is

$$\frac{\|F_J - \mathbb{A}_J U_J^{i_{\text{stop}}}\|}{\|F_J\|} \leq 10^{-5} \frac{\|F_J - \mathbb{A}_J U_J^0\|}{\|F_J\|}.$$

We expect a p -robust solver

- ▶ to reach the above stopping criterion in a *similar number of iterations* i_{stop}
- ▶ to have *similar error contraction factors* $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^i)\|$ at all iterations for different polynomial degrees p , given a fixed J number of mesh levels.

NUMERICAL RESULTS: L-SHAPE PROBLEM

Comparing the number of iterations i_{stop} to reach the stopping criterion for **multigrid** with *Jacobi* and *Gauss-Seidel* smoothing.

J	p	DoF	"small" patches		"big" patches	MG(0,1)	
			dAS i_{stop}	wRAS i_{stop}	wRAS i_{stop}	Jacobi i_{stop}	GS i_{stop}
3	1	5057	76	17	8	44	9
	3	46 273	26	12	5	-	49
	6	185 857	23	10	5	-	228
	9	418 753	21	10	5	-	586
4	1	20 481	95	18	8	-	9
	3	185 857	29	12	5	-	42
	6	744 961	27	10	5	-	186
	9	1 677 313	25	9	5	-	454
5	1	82 433	112	17	8	-	8
	3	744 961	32	12	5	-	35
	6	2 982 913	31	9	5	-	147
	9	6 713 857	28	8	4	-	333

$$1 \leq j \leq J:$$

$$\rho_{J,\text{alg}}^j = \rho_0^j + \sum_{j=1}^J \frac{\sum_{\mathbf{a} \in \mathcal{V}_j} \rho_{j,\mathbf{a}}^j}{J(d+1)} \quad (\text{dAS})$$

$$\rho_{J,\text{alg}}^j = \rho_0^j + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_j} \mathcal{I}_j^p(\psi_j^{\mathbf{a}} \rho_{j,\mathbf{a}}^j), \quad (\text{wRAS})$$

► \mathcal{I}_j^p is the \mathbb{P}^p Lagrange interpolation operator on mesh level j

► For vertex $\mathbf{a} \in \mathcal{V}_j$, we denote the associated hat function by $\psi_j^{\mathbf{a}}$

COMPARISON WITH OTHER MULTILEVEL SOLVERS

We compare our methods with 4 well-established options (motivated by literature³⁴⁵) in terms of the number of iterations (and CPU times⁶).

J	p	DoF	wRAS		wRAS ₁		PCG(MG (3,3)-bJ)		MG(1,1)- PCG(iChol)		MG(0,1)- bGS		MG(3,3)- GS	
			i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time
3	1	5057	17	0.0 s	17	0.0 s	7	0.0 s	4	0.1 s	9	0.0 s	3	0.0 s
	3	46 273	12	0.2 s	18	0.2 s	3	0.2 s	14	0.5 s	8	1.0 s	4	0.1 s
	6	185 857	10	1.5 s	15	1.7 s	2	2.0 s	21	7.6 s	7	2.4 s	9	1.6 s
	9	418 753	10	7.2 s	14	7.7 s	2	10.5 s	63	1.2m	6	7.4 s	9	4.3 s
4	1	20 481	18	0.0 s	18	0.0 s	8	0.1 s	7	0.1 s	9	0.0 s	3	0.0 s
	3	185 857	12	1.0 s	18	1.0 s	3	0.8 s	29	4.1 s	8	4.3 s	4	0.3 s
	6	744 961	10	8.4 s	15	7.5 s	3	11.4 s	49	58.9 s	7	11.9 s	5	2.9 s
	9	1 677 313	9	29.7 s	13	36.1 s	2	30.3 s	167	12.5m	6	29.2 s	8	16.0 s
5	1	82 433	17	0.2 s	17	0.2 s	8	0.3 s	19	0.8 s	8	0.1 s	3	0.1 s
	3	744 961	12	3.4 s	17	3.6 s	3	3.6 s	77	57.7 s	8	16.1 s	4	1.5 s
	6	2 982 913	9	24.3 s	14	26.8 s	3	38.9 s	129	11.6m	7	44.5 s	4	10.0 s
	9	6 713 857	8	2.2m	12	2.2m	2	3.5m	+200	+1.0 h	6	2.1m	8	1.2m

³ Antonietti and Pennesi. "V-cycle multigrid algorithms for discontinuous Galerkin methods on non-nested polytopic meshes". 2019.

⁴ Botti et al. "h-multigrid agglomeration based solution strategies for discontinuous Galerkin discretizations of incompressible flow problems". 2017.

⁵ Schöberl. "C++11 Implementation of Finite Elements in NGSolve". 2014.

⁶ The experiments were run on one **Dell C6220** dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée "NEF" computation cluster, however, in a sequential Matlab script.

CONCLUSIONS: In this work, we presented

- ▶ a multilevel construction of the *algebraic residual lifting*
- ▶ an *a posteriori estimator* on the algebraic error and a *linear iterative solver*
- ▶ the proof of *p-robust efficiency* of the *a posteriori estimator* and *p-robust error contraction* of the solver
- ▶ numerical tests which agree with these theoretical findings

OUTLOOK: In future work, we aim to

- ▶ better understand the role of the mesh levels J .
- ▶ use *adaptivity* based on the derived splitting (equivalent to algebraic error estimator).
- ▶ apply our method to more involved problems.

THANK YOU FOR YOUR ATTENTION!