Contractive local adaptive smoothing based on Dörfler marking in a-posteriori-steered *p*-robust multigrid solvers*

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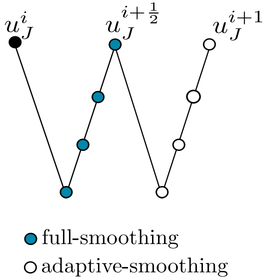
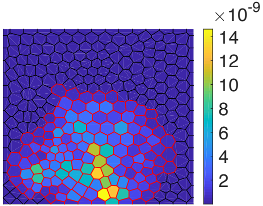
OVERVIEW

- Setting: finite element method of *order p* for a second-order elliptic diffusion problem.
- Adaptive approach in *a-posteriori-steered* multigrid solver based on
  1. a *full-smoothing* substep:
     - construction of a *localized a posteriori algebraic error estimator*
     - coinciding with a step of a *multigrid solver*
  2. a *local adaptive-smoothing* substep:
     - *Dörfler's marking* using the *localization* of our estimator
     - a step of multigrid by *smoothing locally* only marked regions
- Main results:
  1. The iterative solver **contracts the error *p*-robustly** on **each** of the substeps.
  2. The a posteriori estimator is a **two-sided *p*-robust bound** on the algebraic error associated to **each** of the substeps.
- Numerical results





FINITE ELEMENT DISCRETIZATION, ALGEBRAIC SYSTEM

Setting: $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, an open bounded polytope, $f \in L^2(\Omega)$ a source term, and $\mathbf{K} \in [L^\infty(\Omega)]^{d\times d}$ a symmetric and positive definite diffusion coefficient

Problem: find $u \in H_0^1(\Omega)$ such that $(\mathbf{K}\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega)$.

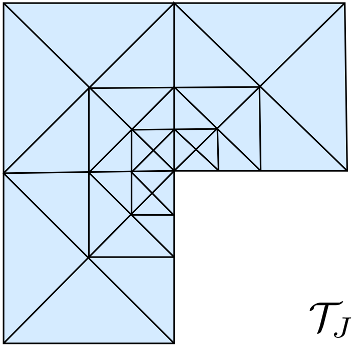
Fix $p \geq 1$ and define

$$V_J^p = \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega),$$

where $\mathbb{P}_p(\mathcal{T}_J) = \{v_J \in L^2(\Omega), v_J|_K \in \mathbb{P}_p(K)\ \forall K \in \mathcal{T}_J\}$.

Discrete problem: Find $u_J \in V_J^p$ such that

$$(\mathbf{K}\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \qquad \text{(FE)}$$

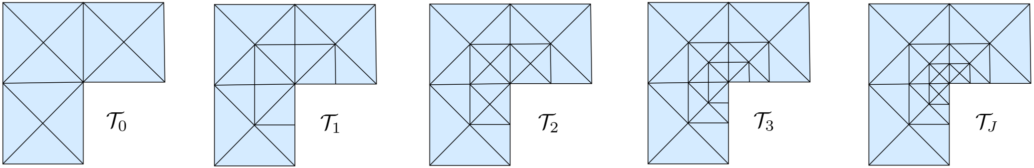


A HIERARCHY OF MESHES AND SPACES

Assumptions on $\{\mathcal{T}_j\}_{0 \leq j \leq J}$

- *Shape regularity*: The ratio element diameter over the diameter of the largest ball inscribed in the element is bounded for all elements by $\kappa_{\mathcal{T}} > 0$.
- *Refinement strength and mesh quasi-uniformity* **or** *local quasi-uniformity of bisection-generated meshes*.

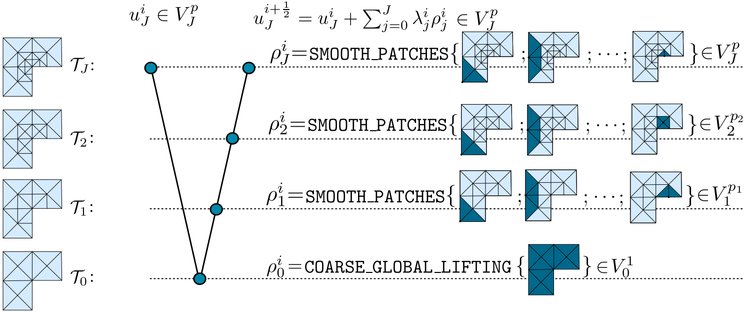
Example: A mesh hierarchy with $J = 4$ refinements and associated spaces of increasing polynomial degrees p_j, $j \in \{1, \dots, J\}$: $1 = p_0 \leq p_1 \leq \dots \leq p_J = p$.



$V_0^1 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$ $V_1^{p_1} = \mathbb{P}_{p_1}(\mathcal{T}_1) \cap H_0^1(\Omega)$ $V_2^{p_2} = \mathbb{P}_{p_2}(\mathcal{T}_2) \cap H_0^1(\Omega)$ $V_3^{p_3} = \mathbb{P}_{p_3}(\mathcal{T}_3) \cap H_0^1(\Omega)$ $V_J^{p} = \mathbb{P}_{p}(\mathcal{T}_J) \cap H_0^1(\Omega)$

A-posteriori-steering: we define an a posteriori error estimator and a solver update through

- the *same* V-cycle multigrid with zero pre- and a single post-smoothing step
- the smoother is *additive Schwarz* / *block Jacobi* associated to patches
- the optimal (*line search*) level-wise step-sizes are used in the multigrid error correction stage



This approach leads to the following connection between **solver** and **estimator**:

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+\frac{1}{2}})\right\|^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 - \left(\eta_{\mathrm{alg}}^i\right)^2,$$

with $\left(\eta_{\mathrm{alg}}^i\right)^2 = \sum_{j=0}^J \left(\lambda_j^i \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|\right)^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\right\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\right\|_{\omega_j^{\mathbf{a}}}^2.$

CAN WE PREDICT THE DISTRIBUTION OF THE ALGEBRAIC ERROR?

Dörfler's bulk-chasing criterion: $\theta^2 \left(\|\mathbf{K}^{\frac{1}{2}} \nabla \rho_0^i\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}} \right) \leq \sum_{j \in \mathcal{M}} \lambda_j^i \sum_{\mathbf{a} \in \mathcal{M}_j} \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}.$

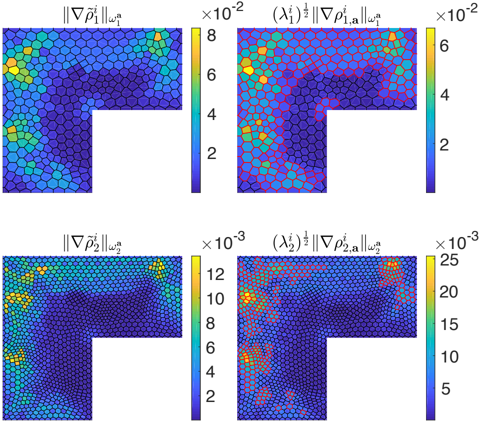
Hierarchy: uniform refinement, $J = 2$, $p_1 = p_2 = 3$. We compare:

- our local algebraic error indicators $\|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}$
- the local algebraic error distribution $\|\mathbf{K}^{\frac{1}{2}} \nabla \tilde{\rho}_j^i\|_{\omega_j^{\mathbf{a}}}$

where $\tilde{\rho}_j^i \in V_j^{p_j}$ is the level-wise orthogonal decomposition of the algebraic error with $\tilde{\rho}_0^i = \rho_0^i$ and, for $j \in \{1, \ldots, J\}$,

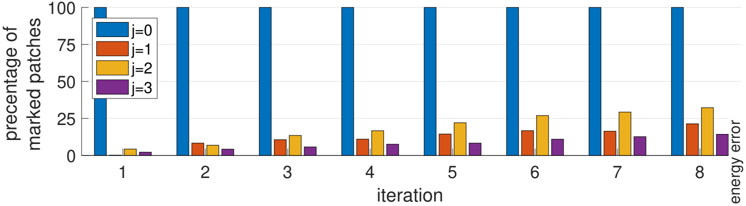
$$(\mathbf{K}\nabla\tilde{\rho}_j^i, \nabla v_j) = (f, v_j) - (\mathbf{K}\nabla u_J^i, \nabla v_j) - \sum_{k=0}^{j-1} (\mathbf{K}\nabla\tilde{\rho}_k^i, \nabla v_j) \quad \forall v_j \in V_j^{p_j},$$

so that $\sum_{j=0}^{J} \tilde{\rho}_j^i = u_J - u_J^i$.

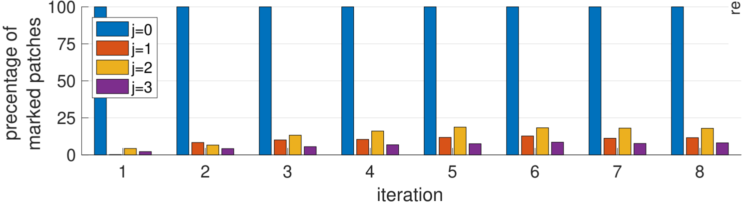


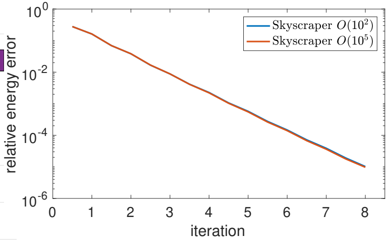
DOES THE ADAPTIVITY PAY OFF?

Skyscraper $O(10^2)$ test case (non-adaptive 15 iterations):



Skyscraper $O(10^5)$ test case (non-adaptive 15 iterations):





Hierarchy: $J = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$

CONCLUSIONS: In this work, we presented:

- A *p-robust contractive* multigrid solver steered by a *p-robustly efficient* a posteriori algebraic error estimator.
- Optimal level-wise *step-sizes* used in the multigrid error correction stage.
- Solver *adaptivity* using local smoothing based on the *localized* a posteriori error estimator and a bulk-chasing criterion.
- The adaptive-smoothing substep also *contracts* the algebraic error *p*-robustly.
- Numerical tests which confirm our theoretical results.

THANK YOU FOR YOUR ATTENTION!