Optimal step-sizes and adaptive number of smoothing steps in *p*-robust multigrid solvers*

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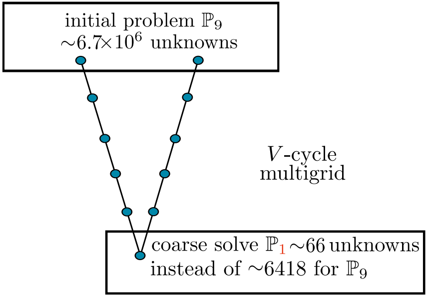
CONTEXT

- We address the issue of large linear systems of type $Ax = b$ arising from finite element method of *order p* discretizations.
- The approach is of *geometric multigrid-type* : V-cycle MG(ν_1, ν_2), where ν_1, ν_2, are the pre- and post-smoothing steps (ex. Jacobi, Gauss-Seidel, block Jacobi etc.).



References

- Hackbusch. “Multi-grid methods and applications”. 1985.
- Pavarino. “Additive Schwarz methods for the *p*-version finite element method”. 1994.
- Schöberl et al. “Additive Schwarz preconditioning for *p*-version triangular and tetrahedral finite elements”. 2008.
- Kanschat. “Robust smoothers for high-order discontinuous Galerkin discretizations of advection-diffusion problems”. 2008.
- Antonietti et al. “A uniform additive Schwarz preconditioner for high-order discontinuous Galerkin approximations of elliptic problems”. 2017.
- Antonietti and Pennesi. “V-cycle multigrid algorithms for discontinuous Galerkin methods on non-nested polytopic meshes”. 2019.
- Sundar, Stadler, and Biros. “Comparison of multigrid algorithms for high-order continuous finite element discretizations”. 2015.

MODEL PROBLEM, FINITE ELEMENT DISCRETIZATION, AND ALGEBRAIC SYSTEM

Setting: $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, an open bounded polytope, $f \in L^2(\Omega)$ a source term, and $\mathbf{K} \in [L^\infty(\Omega)]^{d\times d}$ a symmetric and positive definite diffusion coefficient

Problem: find $u \in H_0^1(\Omega)$ such that $(\mathbf{K}\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega)$.

Fix $p \geq 1$ and define

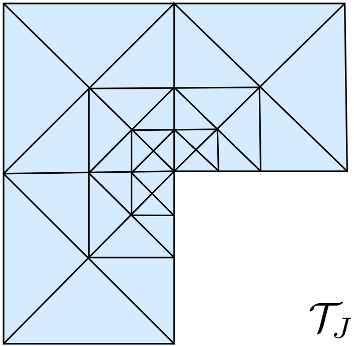
$$V_J^p = \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega),$$

where $\mathbb{P}_p(\mathcal{T}_J) = \{v_J \in L^2(\Omega), v_J|_K \in \mathbb{P}_p(K)\ \forall K \in \mathcal{T}_J\}$.

Discrete problem: Find $u_J \in V_J^p$ such that

$$(\mathbf{K}\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \qquad \text{(FE)}$$

Introducing a basis of V_J^p, then the problem can be rewritten as $\mathbb{A}_J \mathsf{U}_J = \mathsf{F}_J$.



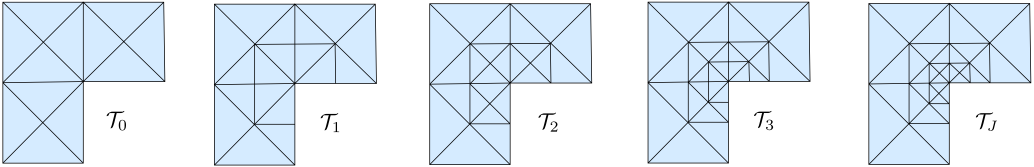
We work with the *basis-independent* functional formulation (FE).

A HIERARCHY OF MESHES AND SPACES

Assumptions: the mesh hierarchy $\{\mathcal{T}_j\}_{0\leq j\leq J}$ can be quasi-uniform or highly graded, satisfying:

- the initial coarsest mesh $\mathcal{T}_0$ is quasi-uniform,
- all the meshes of the hierarchy are shape-regular,
- the maximum strength of refinement from one mesh to the next is bounded.

Example: A mesh hierarchy with $J = 4$ refinements and associated spaces of increasing polynomial degrees p_j, $j \in \{1, \ldots, J\}$: $1 = p_0 \leq p_1 \leq \ldots \leq p_J = p$.



$V_0^1 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$ $V_1^{p_1} = \mathbb{P}_{p_1}(\mathcal{T}_1) \cap H_0^1(\Omega)$ $V_2^{p_2} = \mathbb{P}_{p_2}(\mathcal{T}_2) \cap H_0^1(\Omega)$ $V_3^{p_3} = \mathbb{P}_{p_3}(\mathcal{T}_3) \cap H_0^1(\Omega)$ $V_J^{p} = \mathbb{P}_{p}(\mathcal{T}_J) \cap H_0^1(\Omega)$

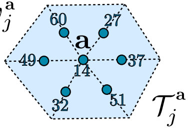
Patches

Let $\mathcal{V}_j$ be the set of vertices of the mesh $\mathcal{T}_j$, $j \in \{1, \ldots, J\}$. Given a vertex $\mathbf{a} \in \mathcal{V}_j$, we denote

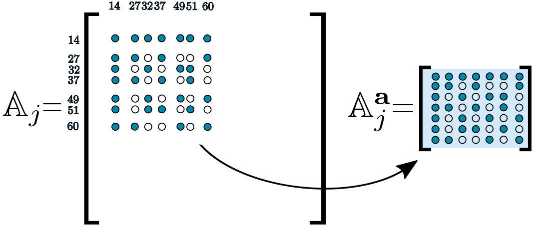
- $\mathcal{T}_j^{\mathbf{a}}$ the patch of elements sharing vertex **a**
- $\omega_j^{\mathbf{a}}$ the corresponding patch subdomain
- $V_j^{\mathbf{a}}$ the associated local space

Example: Geometric (left) and algebraic (right) representation of localizing the problem for $p_j = 2$, $j \in \{1, \ldots, J-1\}$ and a patch composed of 6 elements:

patch subdomain $\omega_j^{\mathbf{a}}$
for a vertex $\mathbf{a} \in \mathcal{V}_j$

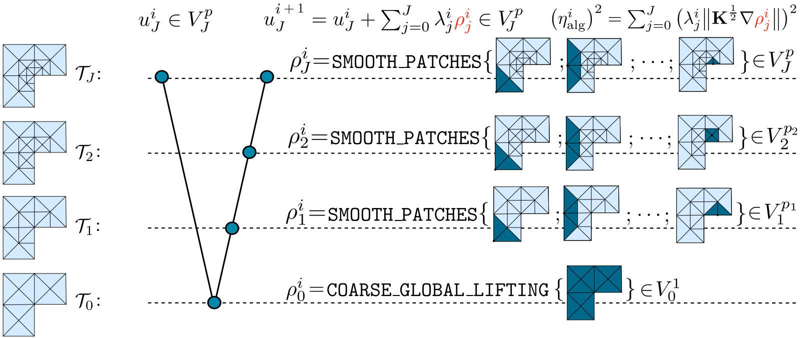


$$V_j^{\mathbf{a}} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$$



A-posteriori-steering approach: define an a posteriori algebraic error estimator and a solver update by

- the *same* V-cycle multigrid with zero pre- and a single post-smoothing step
- with *additive Schwarz* / *block Jacobi* smoothing associated to patches
- and optimal (*line search*) level-wise step-sizes used in the multigrid error correction stage



$u_J^i \in V_J^p$ $\quad$ $u_J^{i+1} = u_J^i + \sum_{j=0}^{J} \lambda_j^i \rho_j^i \in V_J^p$ $\quad$ $(\eta_{\mathrm{alg}}^i)^2 = \sum_{j=0}^{J} \left(\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|\right)^2$

$\mathcal{T}_J$: $\quad \rho_J^i = \texttt{SMOOTH\_PATCHES}\{\;;\;;\cdots;\;\} \in V_J^p$

$\mathcal{T}_2$: $\quad \rho_2^i = \texttt{SMOOTH\_PATCHES}\{\;;\;;\cdots;\;\} \in V_2^{p_2}$

$\mathcal{T}_1$: $\quad \rho_1^i = \texttt{SMOOTH\_PATCHES}\{\;;\;;\cdots;\;\} \in V_1^{p_1}$

$\mathcal{T}_0$: $\quad \rho_0^i = \texttt{COARSE\_GLOBAL\_LIFTING}\{\;\} \in V_0^1$

This approach leads to the following crucial connection between **estimator** and **solver**:

$$\|\mathbf{K}^{\frac{1}{2}} \nabla (u_J - u_J^{i+1})\|^2 = \|\mathbf{K}^{\frac{1}{2}} \nabla (u_J - u_J^{i})\|^2 - (\eta_{\mathrm{alg}}^i)^2.$$

LINK ESTIMATOR AND SOLVER: $\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2 - \left(\eta_{\mathrm{alg}}^i\right)^2$

Consequences:

- the a posteriori estimator η_{alg}^i is a *guaranteed lower bound* for the algebraic error

$$\left(\eta_{\mathrm{alg}}^i\right)^2 \leq \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2,$$

- proving the *efficiency* of the a posteriori estimator η_{alg}^i is equivalent to proving that the solver *contracts* the algebraic error:

$$\left(\eta_{\mathrm{alg}}^i\right)^2 \geq \beta^2\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2 \quad \text{(estimator efficiency)}$$

$$\Leftrightarrow \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2 - \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 \geq \beta^2\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2$$

$$\Leftrightarrow \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 \leq (1-\beta^2)\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2 \quad \text{(solver contraction).}$$

MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL

Let $u_J^i \in V_J^p$ be *arbitrary*. We construct its associated *level-wise algebraic residual liftings* $\{\rho_j^i\}_{j=0}^J$ and *level-wise step-sizes* $\{\lambda_j^i\}_{j=0}^J$ as follows:

Coarse solve: Define $\rho_0^i \in V_0$ by: $(\mathbf{K}\nabla\rho_0^i, \nabla v_0) = (f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0), \quad \forall v_0 \in V_0$ and set $\lambda_0^i := 1$.

Level-wise local solves: For $j = 1 : J$, for all $\mathbf{a} \in \mathcal{V}_j$, define $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$ by :

$$(\mathbf{K}\nabla\rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\mathbf{K}\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \sum_{k=0}^{j-1} \lambda_k^i (\mathbf{K}\nabla\rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}, \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

Level-wise algebraic residual liftings: Define $\rho_j^i \in V_j^{p_j}$ by: $\rho_j^i := \sum_{\mathbf{a}\in\mathcal{V}_j} \rho_{j,\mathbf{a}}^i$.

Level-wise step-sizes: If $\rho_j^i \neq 0$, set $\lambda_j^i := \dfrac{(f, \rho_j^i) - (\mathbf{K}\nabla(u_J^i + \sum_{k=0}^{j-1}\lambda_k^i\rho_k^i), \nabla\rho_j^i)}{\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|^2}$, otherwise set $\lambda_j^i := 1$.

A POSTERIORI ESTIMATOR AND SOLVER

Definition 1 (A posteriori estimator of the algebraic error)

Let $u_J^i \in V_J^p$ be *arbitrary*. Let $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ be constructed as above. Define the a posteriori estimator of the algebraic error associated to u_J^i as

$$\eta_{\mathrm{alg}}^i := \Big(\sum_{j=0}^{J} (\lambda_j^i \| \mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i \|)^2 \Big)^{\frac{1}{2}}.$$

Definition 2 (A posteriori-steered solver)

Initialize $u_J^0 = 0$ and let $i = 0$. Perform the following steps:

1. Construct $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ as detailed above.
2. Update the current approximation $u_J^{i+1} := u_J^i + \sum_{j=0}^J \lambda_j^i \rho_j^i$.
3. If $u_J^{i+1} = u_J^i$, then stop the solver; otherwise increase $i := i + 1$ and go to step 1.

Theorem 1 (Pythagorean error representation of one solver step)

For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be the next iterate constructed from u_J^i by our solver. Then

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 - \sum_{j=0}^{J}\left(\lambda_j^i \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|\right)^2.$$

Proof: Going from the finest level to the coarsest and by construction of the *optimal* step-sizes λ_j^i:

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla\left(\left(u_J - u_J^i - \sum_{j=0}^{J-1}\lambda_j^i\rho_j^i\right) - \lambda_J^i\rho_J^i\right)\right\|^2$$

$$= \left\|\mathbf{K}^{\frac{1}{2}}\nabla\left(u_J - u_J^i - \sum_{j=0}^{J-1}\lambda_j^i\rho_j^i\right)\right\|^2 - 2\lambda_J^i\left[\left(f, \rho_J^i\right) - \left(\mathbf{K}^{\frac{1}{2}}\nabla\left(u_J^i + \sum_{j=0}^{J-1}\lambda_j^i\rho_j^i\right), \nabla\rho_J^i\right)\right] + \left(\lambda_J^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_J^i\right\|\right)^2$$

$$= \left\|\mathbf{K}^{\frac{1}{2}}\nabla\left(u_J - u_J^i - \sum_{j=0}^{J-1}\lambda_j^i\rho_j^i\right)\right\|^2 - \left(\lambda_J^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_J^i\right\|\right)^2 = \ldots = \left\|\mathbf{K}^{\frac{1}{2}}\nabla\left(u_J - (u_J^i + \rho_0^i)\right)\right\|^2 - \sum_{j=1}^{J}\left(\lambda_j^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|\right)^2$$

$$= \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 - \sum_{j=0}^{J}\left(\lambda_j^i\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|\right)^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 - \left(\eta_{\mathrm{alg}}^i\right)^2.$$

This also proves the desired *link between the a posteriori estimator and solver.*

MAIN RESULTS[1,2,3]

Theorem 2 (p-robust reliable and efficient bound on the algebraic error)

Let $u_J^i \in V_J^p$ be *arbitrary*. Let η_{alg}^i be the associated a posteriori estimator on the algebraic error. Then, in addition to $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\| \geq \eta_{\mathrm{alg}}^i$, there holds:

$$\eta_{\mathrm{alg}}^i \geq \beta\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad 0 < \beta(\kappa_{\mathcal{T}}, J, d, \mathbf{K}) < 1.$$

Theorem 3 (*p*-robust error contraction of the multilevel solver)

For $u_J^i \in \mathbf{V}_J^p$, let $u_J^{i+1} \in \mathbf{V}_J^p$ be constructed from u_J^i using one step of the solver. There holds:

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\| \leq \alpha\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad \alpha = \sqrt{1-\beta^2}.$$

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[1] Miraçi, Papež, and Vohralík. “A multilevel algebraic error estimator and the corresponding iterative solver with *p*-robust behavior”. *SIAM J. Numer. Anal.* 2020.

[2] Schöberl et al. “Additive Schwarz preconditioning for p-version triangular and tetrahedral finite elements”. *IMA J. Numer. Anal.* 2008.

[3] Xu, Chen, and Nochetto. “Optimal multilevel methods for H(grad), H(curl), and H(div) systems on graded and unstructured grids”. *Springer.* 2009.

NUMERICAL RESULTS

- We focus on testing numerically the *p*-robust behavior of our solver
- **stopping criterion**

$$\frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{i_s}\|}{\|\mathsf{F}_J\|} \le 10^{-5} \frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{0}\|}{\|\mathsf{F}_J\|}.$$

We expect a *p*-robust solver to reach the above stopping criterion in a *similar number of iterations* i_s for different polynomial degrees p, given a fixed J number of mesh levels. Consider the test cases:

Sine: $u(x,y) = \sin(2\pi x)\sin(2\pi y), \ \Omega := (-1,1)^2,$

Peak: $u(x,y) = x(x-1)y(y-1)e^{-100((x-0.5)^2-(y-0.117)^2)}; \ \Omega := (0,1)^2,$

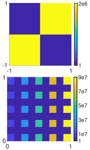
L-shape: $u(r,\theta) = r^{2/3}\sin(2\theta/3); \quad \Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0]),$

Checkerboard[4]**:** $u(r,\varphi) = r^{\gamma}\mu(\varphi); \quad \Omega := (-1,1)^2$

with jump in the diffusion coeffient $\mathcal{J}(\mathbf{K}) = O(10^6)$ or no jump,

Skyscraper: unknown analytic solution; $\quad \Omega := (0,1)^2$

with jump in the diffusion coeffient $\mathcal{J}(\mathbf{K}) = O(10^7)$ or $\mathcal{J}(\mathbf{K}) = O(1)$.



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[4] Kellogg. “On the Poisson equation with intersecting interfaces”. *Appl. Anal.* 1975.

NUMERICAL CONFIRMATION OF p-ROBUSTNESS

The mesh hierarchies here are obtained from J uniform refinements of an initial Delaunay mesh $\mathcal{T}_0$.

			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
p_j			1	p	1	p	1	p	1	p	1	p	1	p	1	p
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	2e^4	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	1e^5	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	6e^5	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	1e^6	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	6e^4	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	6e^5	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	2e^6	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	5e^6	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Numerical **K**- and J-robustness is observed even in low-regularity cases.

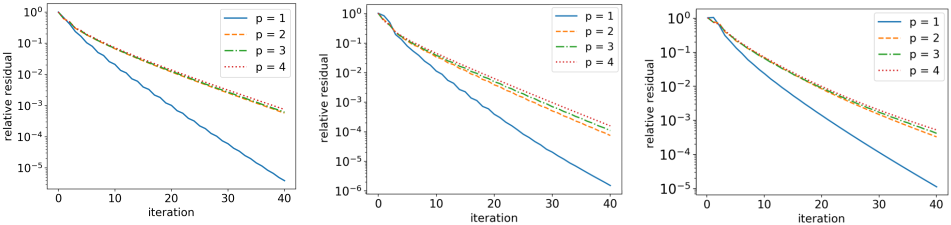
NUMERICAL TESTS IN THREE SPACE DIMENSIONS[5]

In the following tests we have $\mathbf{K} = I$ except in areas of the domain explicitly specified below, and, when available, exact solution u:

Cube: $u(x, y, z) = x(x-1)y(y-1)z(z-1),\ \Omega := (0,1)^3$,

Nested cubes: unknown analytic solution, $\Omega := (-1,1)^3$, $\mathbf{K} = 10^5 * I$ in $(-0.5, 0.5)^3$,

Checkers cubes: unknown analytic solution, $\Omega := (0,1)^3$, $\mathbf{K} = 10^6 * I$ in $(0, 0.5)^3 \cup (0.5, 1)^3$.



Left: cube. Center: nested cubes. Right: checkers cubes. DoFs: O(10^3) for $p = 1$, O(10^4) for $p = 2$, O(10^5) for $p = 3$, O(10^5) for $p = 4$. Mesh refinement is uniform, $p_j = 1$, $j \in \{1, \ldots, J\}$, and $J = 4$.

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[5] Schöberl. "C++11 Implementation of Finite Elements in NGsolve". *Tech. report.* 2014.

NUMERICAL ADVANTAGES OF OPTIMAL STEP-SIZES

Level-wise optimal step-sizes determined by line search[6]:

- *analytically*: **Pythagorean formula** for the algebraic error
- *numerically*: advantages of using even a single global[7] step-size on level J

		Sine		Peak		L-shape	
J	p	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J
3	1	21	-	19	68	17	44
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	10	-
4	1	23	-	20	-	18	-
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	9	-
5	1	22	-	20	-	17	-
	3	15	-	15	-	12	-
	6	13	-	14	-	9	-
	9	13	-	13	-	8	-

When $p = 1$, **wRAS** and **MG(0,1)-J** only differ by the use of the global optimal step-size.

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[6] Heinrichs. "Line relaxation for spectral multigrid methods". *J. Comput. Phys.* 1988.

[7] Miraçi, Papež, and Vohralík. "A multilevel algebraic error estimator and the corresponding iterative solver with p-robust behavior". *SIAM J. Numer. Anal.* 2020.

ADAPTIVE NUMBER OF POST-SMOOTHING STEPS

Starting point: $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\|^2 = \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\|^2 - \sum_{j=0}^{J}\left(\lambda_j^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|\right)^2.$

Algorithm: A-posteriori-steered multigrid with adaptive number of post-smoothing steps

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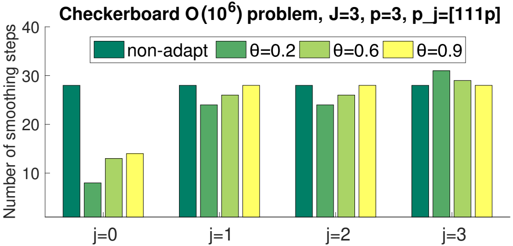
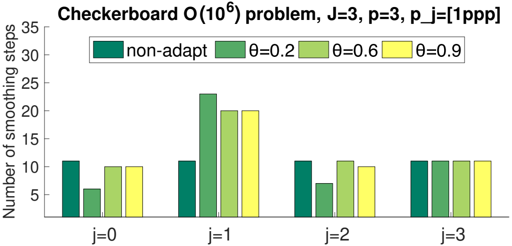
Input: [$p$, $J$, adaptivity parameter $\theta$, maximum smoothing parameter $\nu_{\max}$, tolerance tol]
$u_J^{-1} := 1$;  $i := 0$;  $u_J^i := 0$;
while $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J^i - u_J^{i-1})\| \geq tol$ do
    $i := i+1$; $u_J^i := u_J^{i-1}$;
    for $j = 0 : J$ do
        $\nu := 1$ ; $(\lambda_{j,\nu}^i, \rho_{j,\nu}^i)$ :=CONSTRUCT_LEVEL_LIFTING $(j, u_J^i)$;  $u_J^i = u_J^i + \lambda_{j,\nu}^i\rho_{j,\nu}^i$ ;
        while $\left[\nu < \nu_{\max} \,\&\, (\lambda_{j,\nu}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\nu}^i\|)^2 \geq \theta^2\left(\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu_k^i}(\lambda_{k,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{k,\ell}^i\|)^2 + \sum_{\ell=1}^{\nu-1}(\lambda_{j,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\ell}^i\|)^2\right)\right]$ do
            $\nu = \nu + 1$ ; $(\lambda_{j,\nu}^i, \rho_{j,\nu}^i)$ :=CONSTRUCT_LEVEL_LIFTING $(j, u_J^i)$;  $u_J^i = u_J^i + \lambda_{j,\nu}^i\rho_{j,\nu}^i$ ;
        end
        $\nu_j^i := \nu$ ;
    end
end
$i_{\mathrm{stop}} := i$; $u_J^{i_{\mathrm{stop}}} = u_J^i$;
Output: [ $u_J^{i_{\mathrm{stop}}}$ ]
```

Remark: if $\nu_{\max} = 1$, we obtain the non-adaptive a-posteriori-steered solver.

NUMERICAL TESTS

Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $\theta = 0.2$, $p = 3$, $J = 3$, and mesh hierarchies with $p_j = 1$ and $p_j = p$, $j \in \{1, \dots, J-1\}$.





	$p_j = p$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1

COMPARISON WITH OTHER MULTILEVEL SOLVERS

We compare our methods with [8,9,10] in terms of the number of iterations (and CPU times[11]).

		$\sim$MG(0,1)-bJ $1 \to 1, p$		$\sim$MG(0,1)-bJ $1, p \to p$		$\sim$MG(0, adapt)-bJ $1, p \to p$		$\sim$MG(0,adapt)-bJ (wRAS) $1 \nearrow p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		MG(0,1)-bGS $1 \to 1, p$		MG(3,3)-GS $1 \nearrow p$	
J	p	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time
3	1	18	0.05 s	18	0.07 s	8	0.04 s	8	0.04 s	10	0.07 s	6	0.39 s	10	0.04 s	4	0.02 s
	3	28	0.96 s	11	0.50 s	6	0.43 s	6	0.41 s	3	0.57 s	22	3.43 s	11	2.62 s	6	0.34 s
	6	25	9.88 s	10	5.43 s	6	5.24 s	5	2.90 s	2	5.24 s	44	51.38 s	9	7.35 s	11	5.91 s
	9	23	45.87 s	9	27.01 s	6	25.25 s	4	13.86 s	2	36.95 s	80	5.22m	8	32.53 s	11	19.72 s
4	1	19	0.12 s	19	0.12 s	9	0.11 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	27	3.85 s	11	2.07 s	6	1.89 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	24	41.79 s	9	20.19 s	6	20.69 s	4	12.54 s	3	38.40 s	80	6.87m	9	34.78 s	6	14.44 s
	9	23	3.63m	9	2.13m	6	2.09m	3	49.84 s	2	2.24m	80	23.08m	8	1.72m	9	1.21m

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[8] Antonietti and Pennesi. “V-cycle multigrid algorithms for discontinuous Galerkin methods on non-nested polytopic meshes”. *J. Sci. Comput.* 2019.

[9] Botti et al. “*h*-multigrid agglomeration based solution strategies for discontinuous Galerkin discretizations of incompressible flow problems”. *J. Comput. Phys.* 2017.

[10] Schöberl. “C++11 Implementation of Finite Elements in NGSolve”. *Tech. report.* 2014.

[11] The experiments were run on one Dell C6220 dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée “NEF” computation cluster, however, in a sequential Matlab script.

CONTRIBUTIONS

- A *p-robust contractive* multigrid solver steered by a *p-robustly efficient* a posteriori algebraic error estimator.
- Optimal level-wise *step-sizes* used in the multigrid error correction stage.
- A simple and efficient *adaptive* strategy for deciding the number of smoothing steps in multigrid solvers.
- Numerical tests which confirm our theoretical results and illustrate the advantages of our approaches.

PERSPECTIVES

- The theoretical study the robustness with respect to the jumps of the diffusion coefficient $\mathbf{K}$.
- The use of our a-posteriori-steered multigrid solver as an inexact solver in a setting where the hierarchy is obtained through *hp*-refinement.
- Applications to more involved problems.

THANK YOU FOR YOUR ATTENTION!