

A-posteriori-steered and adaptive p -robust multigrid solvers

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joint work with

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Introduction

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A-posteriori-steered multigrid

Main results

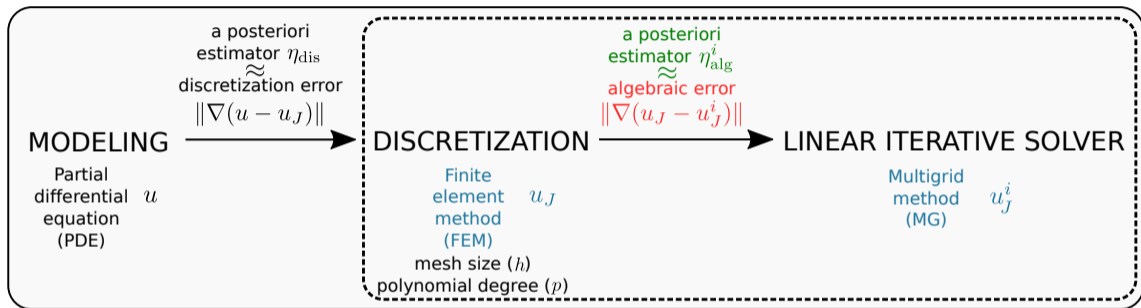
Numerical results

Adaptivity in a-posteriori-steered solvers

Conclusion

Introduction

Numerical simulations of physical phenomena typically follow the steps:



- Design a posteriori estimators of the algebraic error:
 - *guaranteed* lower bounds: $\eta_{\text{alg}}^i \leq \|\nabla(u_J - u_J^i)\|$,
 - *efficient*: $\eta_{\text{alg}}^i \geq \beta \|\nabla(u_J - u_J^i)\|$,
 - *p-robust*: $\beta \in (0, 1)$ is independent of p .
- Design *p-robust a-posteriori-steered* multigrid solvers: the estimator is used to define the solver.
- Develop adaptive solver strategies

References

- 1 Hackbusch. "Multi-grid methods and applications". *Springer*. 1985.
- 2 Oswald. "Multilevel finite element approximation". *Teubner*. 1994.
- 3 Gander. "Schwarz methods over the course of time". *Electron. Trans. Numer. Anal.* 2008.
- 4 Dolean, Jolivet, and Nataf. "An introduction to domain decomposition methods". *SIAM*. 2015.

p -robustness results

- 1 Quarteroni and Sacchi Landriani. "Domain decomposition preconditioners for the spectral collocation method". *J. Sci. Comput.* 1988.
- 2 Pavarino. "Additive Schwarz methods for the p -version finite element method". *Numer. Math.* 1994.
- 3 Schöberl, Melenk, Pechstein, and Zaglmayr. "Additive Schwarz preconditioning for p -version triangular and tetrahedral finite elements". *IMA J. Numer. Anal.* 2008.
- 4 Antonietti, Mascotto, and Verani. "A multigrid algorithm for the p -version of the virtual element method". *ESAIM Math. Model. Numer. Anal.* 2018.

Key technical tool: stable decomposition

Two key blocks in the theory of the thesis

- one-level p -robust stable decomposition
Schöberl, Melenk, Pechstein, and Zaglmayr. "Additive Schwarz preconditioning for p -version triangular and tetrahedral finite elements". *IMA J. Numer. Anal.* 2008.
- multilevel piecewise affine stable decomposition on graded meshes
Xu, Chen, and Nochetto. "Optimal multilevel methods for $H(\text{grad})$, $H(\text{curl})$, and $H(\text{div})$ systems on graded and unstructured grids". *Springer*. 2009.

give us a **multilevel p -robust stable decomposition**, *crucial* for our analysis.

Setting

Setting: domain $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, source term $f \in L^2(\Omega)$, s.p.d. diffusion coefficient $\mathbf{K} \in [L^\infty(\Omega)]^{d \times d}$.

Model problem: find $u \in H_0^1(\Omega)$ such that $(\mathbf{K}\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$.

Fix $p \geq 1$, let $\mathbb{P}_p(\mathcal{T}_J) := \{v_J \in L^2(\Omega), v_J|_K \in \mathbb{P}_p(K) \quad \forall K \in \mathcal{T}_J\}$,

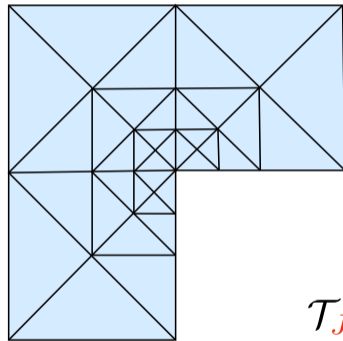
and define

$$V_J^p := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$$

Discrete problem: Find $u_J \in V_J^p$ such that

$$(\mathbf{K}\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \quad (\text{FE})$$

By introducing a basis of V_J^p : $\mathbb{A}_J \mathbf{U}_J = \mathbf{F}_J$. We work with the *basis-independent* functional formulation (FE).



\mathcal{T}_J

Algebraic residual functional: $v_J \mapsto (f, v_J) - (\mathbf{K}\nabla u_J^i, \nabla v_J) \in \mathbb{R}, \quad v_J \in V_J^p$.

Example: Two different hierarchies with $J = 3$ refinements.

Assumptions: The meshes $\{\mathcal{T}_j\}_{1 \leq j \leq J}$ can be generated through *uniform* or *adaptive* refinement, satisfying:

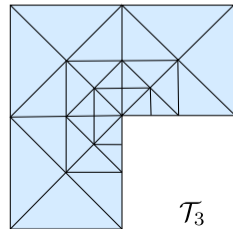
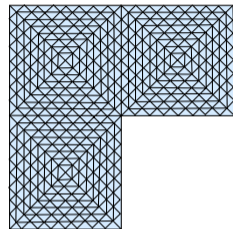
- (C_{qu}) -quasi-uniform \mathcal{T}_0 ,
- $(\kappa_{\mathcal{T}})$ -shape-regularity,
- (C_{ref}) -maximum strength of refinement.

For given p and J , choose *increasing* polynomial degrees p_j ,
 $j \in \{1, \dots, J\}$,

$$1 = p_0 \leq p_1 \leq p_2 \leq \dots \leq p_J = p,$$

and define the spaces

$$V_j^{p_j} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\Omega).$$



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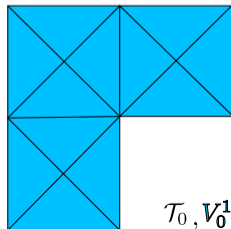
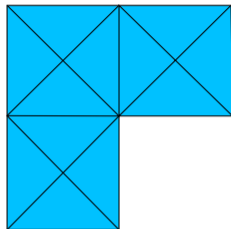
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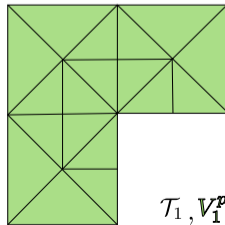
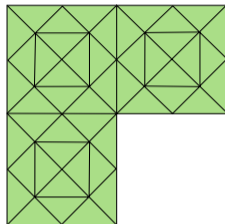
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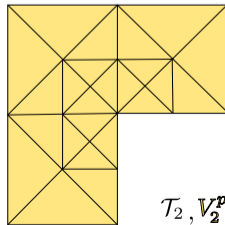
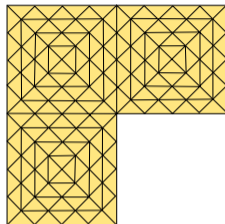
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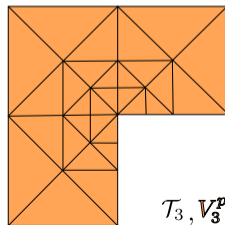
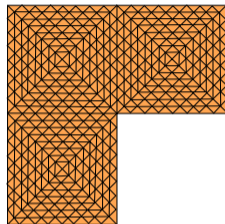
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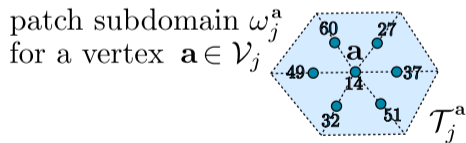
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 $\mathcal{T}_3, V_3^{p_3}$

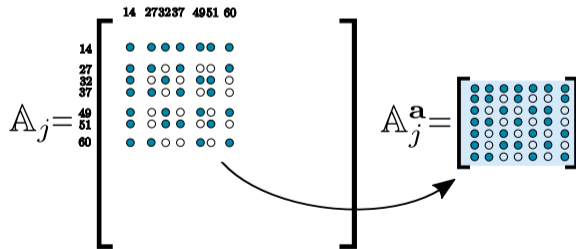
Let \mathcal{V}_j be the set of vertices of the mesh \mathcal{T}_j , $j \in \{1, \dots, J\}$. Given a vertex $\mathbf{a} \in \mathcal{V}_j$, we denote

- $\mathcal{T}_j^{\mathbf{a}}$ the patch of elements sharing vertex \mathbf{a}
- $\omega_j^{\mathbf{a}}$ the corresponding patch subdomain
- $V_j^{\mathbf{a}}$ the associated local space

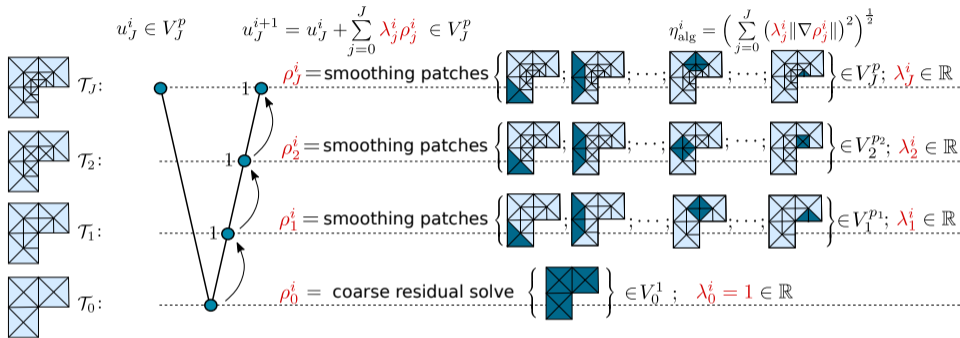
Example: Representation of localizing the problem for $p_j = 2$, $j \in \{1, \dots, J - 1\}$: geometric perspective and algebraic perspective



$$V_j^{\mathbf{a}} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$$



A-posteriori-steered multigrid



- V-cycle of geometric multigrid: coarse grid solve and level-wise smoothing
- **zero** pre- and a **single** post-smoothing step
- *additive Schwarz / block Jacobi* smoothing: fully *parallel* on each level
- level-wise step-sizes in multigrid error correction stage: optimally chosen by *line search*¹

¹Heinrichs. "Line relaxation for spectral multigrid methods". *J. Comput. Phys.* 1988.

Let $u_J^i \in V_J^p$ be arbitrary. We construct its associated *level-wise algebraic residual liftings* $\{\rho_j^i\}_{j=0}^J$ and *level-wise step-sizes* $\{\lambda_j^i\}_{j=0}^J$ as follows:

Coarse solve: Define $\rho_0^i \in V_0$ by: $(\mathbf{K}\nabla\rho_0^i, \nabla v_0) = (f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0)$, $\forall v_0 \in V_0$ and set $\lambda_0^i := 1$.

Level-wise local solves: For $j = 1 : J$, for all $\mathbf{a} \in \mathcal{V}_j$, define $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$ by :

$$(\mathbf{K}\nabla\rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\mathbf{K}\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \sum_{k=0}^{j-1} \lambda_k^i (\mathbf{K}\nabla\rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}, \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

Level-wise algebraic residual liftings: Define $\rho_j^i \in V_j^{p_j}$ by: $\rho_j^i := \sum_{\mathbf{a} \in \mathcal{V}_j} \rho_{j,\mathbf{a}}^i$.

Level-wise step-sizes: If $\rho_j^i \neq 0$, set $\lambda_j^i := \frac{(f, \rho_j^i) - (\mathbf{K}\nabla(u_J^i + \sum_{k=0}^{j-1} \lambda_k^i \rho_k^i), \nabla\rho_j^i)}{\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|^2}$, otherwise set $\lambda_j^i := 1$.

Definition (A posteriori estimator of the algebraic error)

Let $u_J^i \in V_J^p$ be *arbitrary*. Let $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ be constructed as above. Define the a posteriori estimator of the algebraic error associated to u_J^i as

$$\eta_{\text{alg}}^i := \left(\sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2 \right)^{\frac{1}{2}}.$$

Definition (A posteriori-steered solver)

Initialize $u_J^0 = 0$ and let $i = 0$. Perform the following steps:

- 1** Construct $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ as detailed above.
- 2** Update the current approximation $u_J^{i+1} := u_J^i + \sum_{j=0}^J \lambda_j^i \rho_j^i$.
- 3** If $u_J^{i+1} = u_J^i$, then stop the solver; otherwise increase $i := i + 1$ and go to step 1.

Proposition (Pythagorean error representation of one solver step)

For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be the next iterate constructed from u_J^i by our solver. Then

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\|^2 = \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - \underbrace{\sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2}_{= (\eta_{\text{alg}}^i)^2}.$$

Proof: Going from the finest level to the coarsest and by construction of the **optimal** step-sizes λ_j^i :

$$\begin{aligned} \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\|^2 &= \left\| \mathbf{K}^{\frac{1}{2}} \nabla \left(u_J - \left(u_J^i + \sum_{j=0}^J \lambda_j^i \rho_j^i \right) \right) \right\|^2 \\ &= \left\| \mathbf{K}^{\frac{1}{2}} \nabla \left(u_J - u_J^i - \sum_{j=0}^{J-1} \lambda_j^i \rho_j^i \right) \right\|^2 - 2\lambda_J^i \left[(f, \rho_J^i) - \left(\mathbf{K}^{\frac{1}{2}} \nabla \left(u_J^i + \sum_{j=0}^{J-1} \lambda_j^i \rho_j^i \right), \nabla \rho_J^i \right) \right] + \left(\lambda_J^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_J^i\| \right)^2 \\ &= \left\| \mathbf{K}^{\frac{1}{2}} \nabla \left(u_J - u_J^i - \sum_{j=0}^{J-1} \lambda_j^i \rho_j^i \right) \right\|^2 - (\lambda_J^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_J^i\|)^2 = \dots = \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - \sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2 \\ &= \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - (\eta_{\text{alg}}^i)^2. \end{aligned}$$

Corollary (Guaranteed lower bound on the algebraic error)

There holds:

$$\eta_{\text{alg}}^i \leq \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|.$$

Main results

Theorem (p -robust reliable and efficient bound on the algebraic error)

Let $u_J^i \in V_J^p$ be arbitrary. Let η_{alg}^i be the associated a posteriori estimator on the algebraic error. Then, in addition to $\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i$, there holds:

$$\eta_{\text{alg}}^i \geq \beta \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|, \quad 0 < \beta(\kappa_{\mathcal{T}}, J, d, \mathbf{K}) < 1.$$

Theorem (p -robust error contraction of the multilevel solver)

For $u_J^i \in \mathbf{V}_J^p$, let $u_J^{i+1} \in \mathbf{V}_J^p$ be constructed from u_J^i using one step of the solver. There holds:

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\| \leq \alpha \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|, \quad \alpha = \sqrt{1 - \beta^2}.$$

- Remark:**
- The dependence on J is at most *linear* under minimal H^1 -regularity.
 - Complete *independence* from J is obtained in H^2 -regularity setting.

Corollary (Equivalence of the two main results)

Proving the efficiency of the a posteriori estimator η_{alg}^i is equivalent to proving the solver contraction.

Proof: By using the *link between solver and estimator* given by the Pythagorean formula, there holds:

$$\begin{aligned} (\eta_{\text{alg}}^i)^2 &\geq \beta^2 \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \quad (\text{estimator efficiency}) \\ \Leftrightarrow \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\|^2 &\geq \beta^2 \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \\ \Leftrightarrow \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\|^2 &\leq (1 - \beta^2) \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \quad (\text{solver contraction}). \end{aligned}$$

Corollary (Equivalence of error–global estimator–local estimators)

Let the assumptions of Theorem 2 hold. Then

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2 = \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_0^i\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2.$$

Numerical results

Consider the test cases:

Sine: $u(x, y) = \sin(2\pi x) \sin(2\pi y), \quad \Omega := (-1, 1)^2,$

Peak: $u(x, y) = x(x-1)y(y-1)e^{-100((x-0.5)^2 - (y-0.117)^2)}; \quad \Omega := (0, 1)^2,$

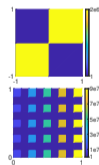
L-shape: $u(r, \theta) = r^{2/3} \sin(2\theta/3); \quad \Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]),$

Checkerboard²: $u(r, \varphi) = r^\gamma \mu(\varphi); \quad \Omega := (-1, 1)^2$

with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^6)$ or no jump,

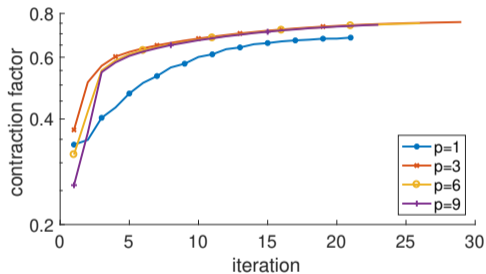
Skyscraper: unknown analytic solution; $\Omega := (0, 1)^2$

with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^7)$ or $\mathcal{J}(\mathbf{K}) = O(1)$.

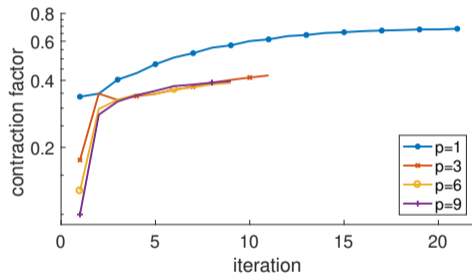


²Kellogg. "On the Poisson equation with intersecting interfaces". *Appl. Anal.* 1975.

L-shape problem, $J = 3$, and mesh hierarchy $p_j = 1$ (left) and $p_j = p$ (right), $j \in \{1, \dots, J - 1\}$



$1 \rightarrow 1, p$



$1, p \rightarrow p$

Stopping criterion:
$$\frac{\|F_J - \mathbb{A}_J U_J^{i_s}\|}{\|F_J\|} \leq 10^{-5} \frac{\|F_J - \mathbb{A}_J U_J^0\|}{\|F_J\|}.$$

The mesh hierarchies here are obtained from J uniform refinements of an initial Delaunay mesh \mathcal{T}_0 .

		H^2 -regular								H^1 -regular											
		Sine $\mathbf{K}=I$				Peak $\mathbf{K}=I$				L-shape $\mathbf{K}=I$				Checkerboard $\mathbf{K}=I$				Skyscraper $\mathcal{J}(\mathbf{K})=O(1)$			
		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$	
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19	19	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13	31	13	31	13	31
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11	28	11	28	11	28
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10	26	10	26	10	26
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11	30	11	30	11	30
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10	27	10	27	10	27
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9	25	9	25	9	25

Numerical \mathbf{K} - and J -robustness is observed even in low-regularity cases.

Peak, $1, p \rightarrow p$

J	p	i_s	J	p	i_s	J	p	i_s
4	1	14	8	1	16	16	1	16
	3	11		3	9		3	9
	6	9		6	8		6	8
	9	8		9	8		9	9

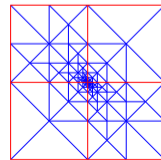
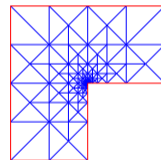
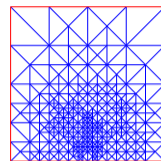
 L-shape, $\mathbf{K} = I$, $1, p \rightarrow p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4

 Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \rightarrow p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15

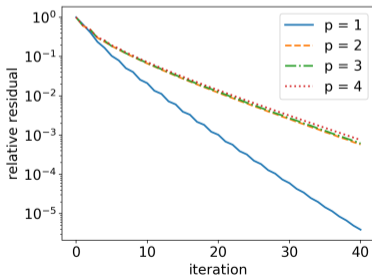
These low-regularity test cases indicate the possibility of J -dependence, in accordance with the theoretical results.



Test cases: exact solution u when available; $\mathbf{K} = I$ except where explicitly specified, uniform mesh refinement, $p_j = 1$, $j \in \{1, \dots, J\}$, and $J = 4$.

Cube: $\Omega := (0, 1)^3$,

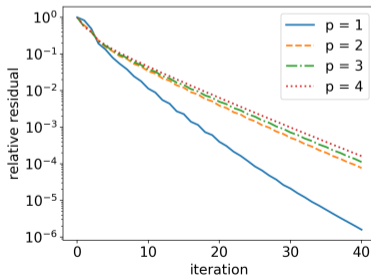
$$u(x, y, z) = x(x-1)y(y-1)z(z-1).$$



Nested cubes: $\Omega := (-1, 1)^3$,

unknown analytic solution,

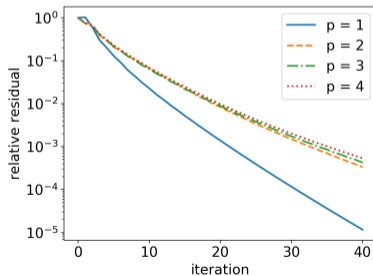
$\mathbf{K} = 10^5 * I$ in $(-0.5, 0.5)^3$.



Checkers cubes: $\Omega := (0, 1)^3$,

unknown analytic solution,

$\mathbf{K} = 10^6 * I$ in $(0, 0.5)^3 \cup (0.5, 1)^3$.



Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \approx (\eta_{\text{alg}}^i)^2 = \underbrace{\sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2}_{\textcircled{1} \text{ localization by levels}} = \underbrace{\|\mathbf{K}^{\frac{1}{2}} \nabla \rho_0^i\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2}_{\textcircled{2} \text{ localization by patches}}$$

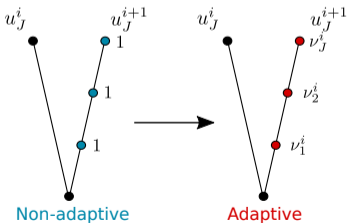
① Adaptive number of post-smoothing steps

② Adaptive local smoothing

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \approx (\eta_{\text{alg}}^i)^2 = \underbrace{\sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2}_{\text{① localization by levels}} = \underbrace{\|\mathbf{K}^{\frac{1}{2}} \nabla \rho_0^i\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2}_{\text{② localization by patches}}$$

① Adaptive number of post-smoothing steps

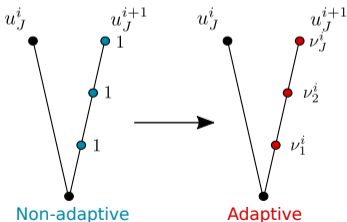


② Adaptive local smoothing

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

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① Adaptive number of post-smoothing steps



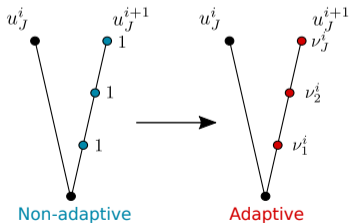
② Adaptive local smoothing



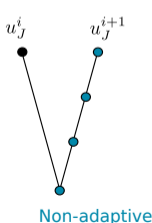
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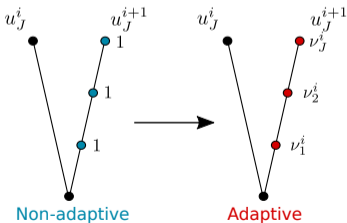
② Adaptive local smoothing



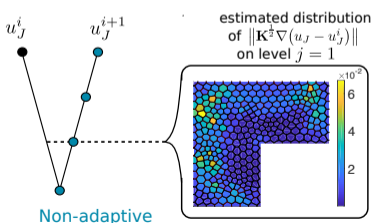
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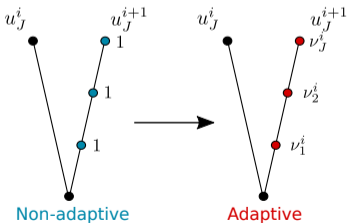
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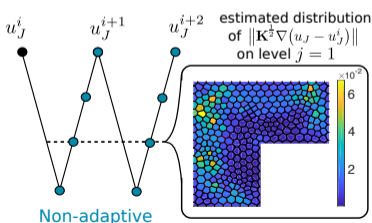
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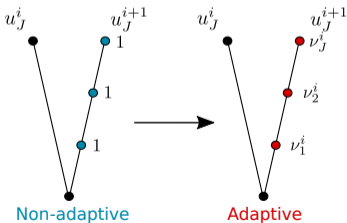
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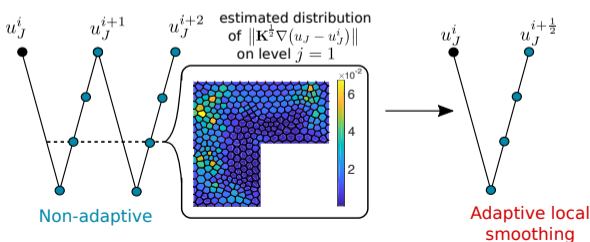
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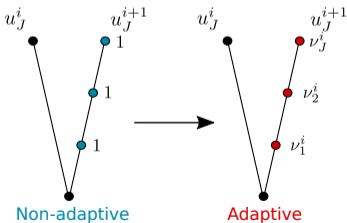
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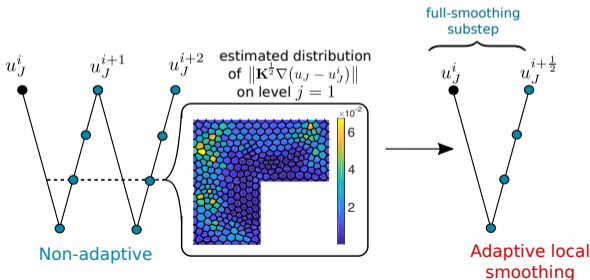
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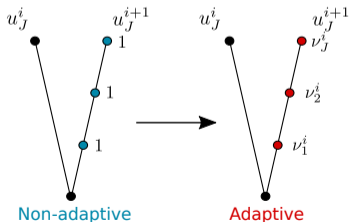
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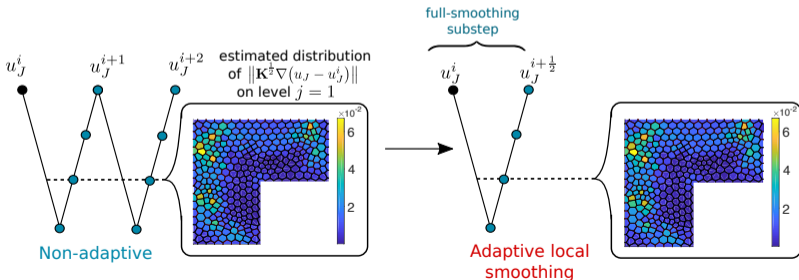
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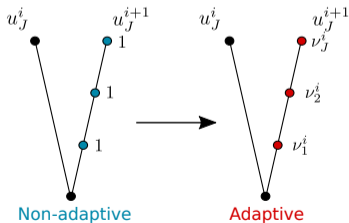
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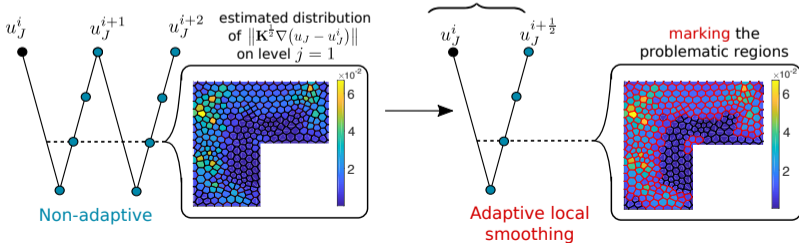
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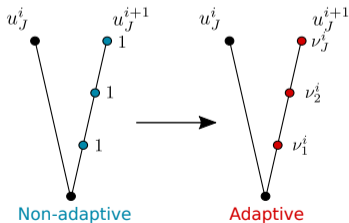
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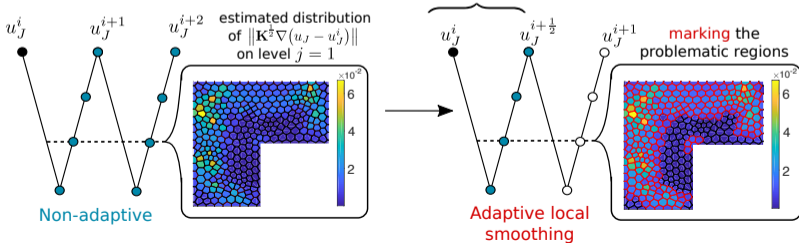
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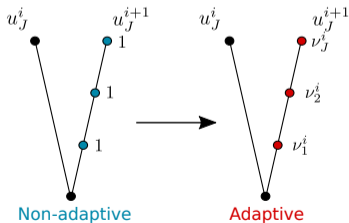
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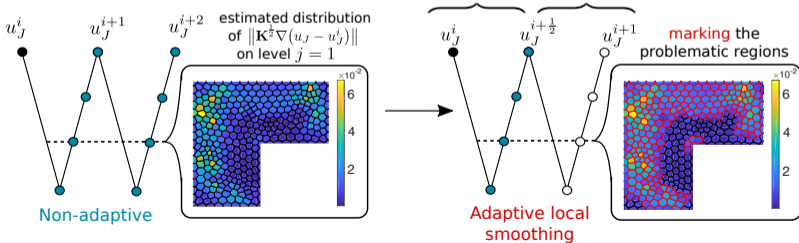
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① Adaptive number of post-smoothing steps



② Adaptive local smoothing



Conclusion

We presented:

- A **p -robustly efficient** a posteriori algebraic error estimator.
- A **p -robust contractive** multigrid solver which is steered by the a posteriori estimator.
- Optimal level-wise **step-sizes** used in the multigrid error correction stage.
- Two **adaptive** variants of a **p -robust contractive** a-posteriori-steered multigrid solver:
 - ▶ Approach 1: **adaptive number of smoothing steps** on each level.
 - ▶ Approach 2: applying **adaptive local smoothing** on all patches with increased algebraic error estimator.

Future work would explore:

- how to gain robustness with respect to the number of mesh levels J .
- how to use our approach to construct **hp -robust** algebraic solvers.

Thank you for your attention!

- 📄 Miraçi, Papež, and Vohralík. A multilevel algebraic error estimator and the corresponding iterative solver with p -robust behavior. *SIAM J. Numer. Anal.* (2020)
- 📄 Miraçi, Papež, and Vohralík. Contractive local adaptive smoothing based on Dörfler's marking in a-posteriori-steered p -robust multigrid solvers. *Comput. Methods Appl. Math.* (2021)
- 📄 Miraçi, Papež, and Vohralík. A-posteriori-steered p -robust multigrid with optimal step-sizes and adaptive number of smoothing steps. *SIAM J. Sci. Comput.* (2021). DOI 10.1137/20M1349503.

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Workgroup on Numerics of PDEs