

# A-posteriori-steered and adaptive $p$ -robust multigrid solvers

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joint work with

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A-posteriori-steered multigrid

Main results

Numerical results

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Conclusion

# Setting

**Setting:** domain  $\Omega \subset \mathbb{R}^d$ ,  $1 \leq d \leq 3$ , source term  $f \in L^2(\Omega)$ , s.p.d. diffusion coefficient  $\mathbf{K} \in [L^\infty(\Omega)]^{d \times d}$ .

**Model problem:** find  $u \in H_0^1(\Omega)$  such that  $(\mathbf{K}\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$ .

Fix  $p \geq 1$ , let  $\mathbb{P}_p(\mathcal{T}_J) := \{v_J \in L^2(\Omega), v_J|_K \in \mathbb{P}_p(K) \quad \forall K \in \mathcal{T}_J\}$ ,

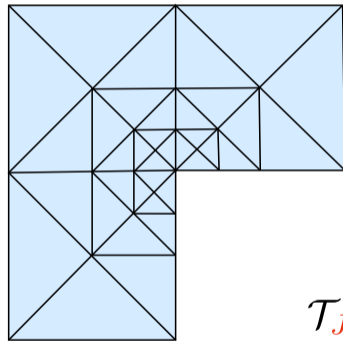
and define

$$V_J^p := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$$

**Discrete problem:** Find  $u_J \in V_J^p$  such that

$$(\mathbf{K}\nabla u_J, \nabla v_J) = (f, v_J) \quad \forall v_J \in V_J^p. \quad (\text{FE})$$

By introducing a basis of  $V_J^p$ :  $\mathbb{A}_J \mathbf{U}_J = \mathbf{F}_J$ . We work with the *basis-independent* functional formulation (FE).



$\mathcal{T}_J$

**Algebraic residual functional:**  $v_J \mapsto (f, v_J) - (\mathbf{K}\nabla u_J^i, \nabla v_J) \in \mathbb{R}, \quad v_J \in V_J^p$ .

**Example:** Two different hierarchies with  $J = 3$  refinements.

**Assumptions:** The meshes  $\{\mathcal{T}_j\}_{1 \leq j \leq J}$  can be generated through *uniform* or *adaptive* refinement, satisfying:

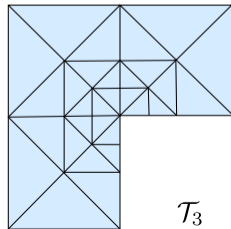
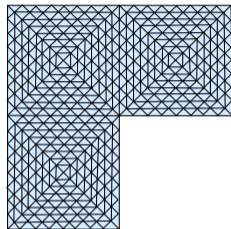
- $(C_{\text{qu}})$ -quasi-uniform  $\mathcal{T}_0$ ,
- $(\kappa_{\mathcal{T}})$ -shape-regularity,
- $(C_{\text{ref}})$ -maximum strength of refinement.

For given  $p$  and  $J$ , choose *increasing* polynomial degrees  $p_j$ ,  $j \in \{1, \dots, J\}$ ,

$$1 = p_0 \leq p_1 \leq p_2 \leq \dots \leq p_J = p,$$

and define the spaces

$$V_j^{p_j} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\Omega).$$



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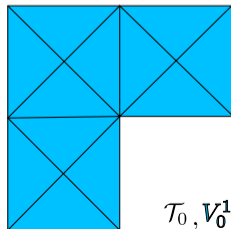
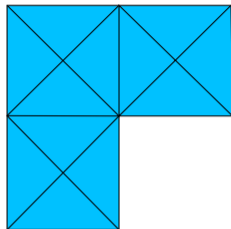
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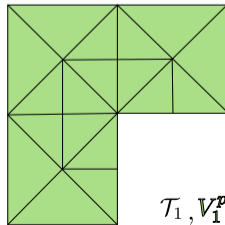
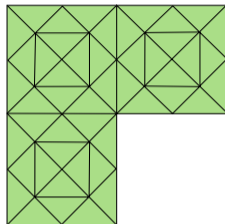
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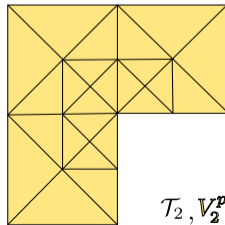
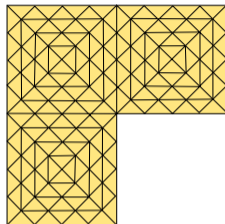
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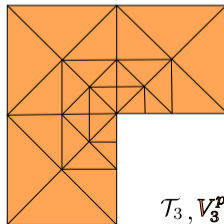
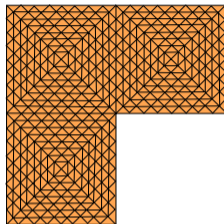
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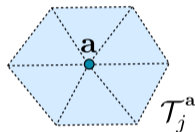
$$V_j^{p_j} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\Omega).$$



Let  $\mathcal{V}_j$  be the set of vertices of the mesh  $\mathcal{T}_j$ ,  $j \in \{1, \dots, J\}$ .

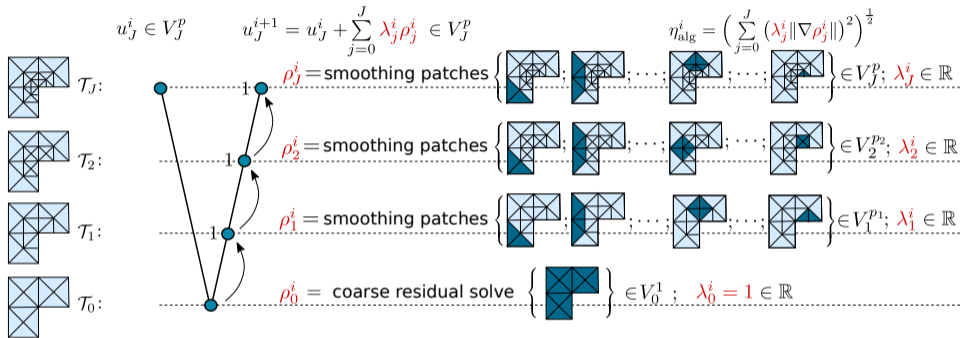
Given a vertex  $\mathbf{a} \in \mathcal{V}_j$ , we denote:

- $\mathcal{T}_j^{\mathbf{a}}$  the patch of elements sharing vertex  $\mathbf{a}$
- $\omega_j^{\mathbf{a}}$  the corresponding patch subdomain
- $V_j^{\mathbf{a}} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$  the associated local space



patch subdomain  $\omega_j^{\mathbf{a}}$   
for a vertex  $\mathbf{a} \in \mathcal{V}_j$

## A-posteriori-steered multigrid



- V-cycle of geometric multigrid: coarse grid solve and level-wise smoothing
- **zero** pre- and a **single** post-smoothing step
- *additive Schwarz / block Jacobi* smoothing: fully *parallel* on each level
- level-wise step-sizes in multigrid error correction stage: optimally chosen by *line search*<sup>1</sup>

<sup>1</sup>Heinrichs. "Line relaxation for spectral multigrid methods". *J. Comput. Phys.* 1988.

Let  $u_J^i \in V_J^p$  be arbitrary. We construct its associated *level-wise algebraic residual liftings*  $\{\rho_j^i\}_{j=0}^J$  and *level-wise step-sizes*  $\{\lambda_j^i\}_{j=0}^J$  as follows:

**Coarse solve:** Define  $\rho_0^i \in V_0$  by:  $(\mathbf{K}\nabla\rho_0^i, \nabla v_0) = (f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0)$ ,  $\forall v_0 \in V_0$  and set  $\lambda_0^i := 1$ .

**Level-wise local solves:** For  $j = 1 : J$ , for all  $\mathbf{a} \in \mathcal{V}_j$ , define  $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$  by :

$$(\mathbf{K}\nabla\rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\mathbf{K}\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \sum_{k=0}^{j-1} \lambda_k^i (\mathbf{K}\nabla\rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}, \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

**Level-wise algebraic residual liftings:** Define  $\rho_j^i \in V_j^{p_j}$  by:  $\rho_j^i := \sum_{\mathbf{a} \in \mathcal{V}_j} \rho_{j,\mathbf{a}}^i$ .

**Level-wise step-sizes:** If  $\rho_j^i \neq 0$ , set  $\lambda_j^i := \frac{(f, \rho_j^i) - (\mathbf{K}\nabla(u_J^i + \sum_{k=0}^{j-1} \lambda_k^i \rho_k^i), \nabla\rho_j^i)}{\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|^2}$ , otherwise set  $\lambda_j^i := 1$ .

## Proposition (Pythagorean error representation of one solver step)

For  $u_J^i \in V_J^p$ , let  $u_J^{i+1} \in V_J^p$  be the next iterate constructed from  $u_J^i$  by our solver. Then

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\|^2 = \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - \underbrace{\sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2}_{= (\eta_{\text{alg}}^i)^2}.$$

**Proof:** Going from the finest level to the coarsest and by construction of the **optimal** step-sizes  $\lambda_j^i$ :

$$\begin{aligned} \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\|^2 &= \left\| \mathbf{K}^{\frac{1}{2}} \nabla \left( u_J - \left( u_J^i + \sum_{j=0}^J \lambda_j^i \rho_j^i \right) \right) \right\|^2 \\ &= \left\| \mathbf{K}^{\frac{1}{2}} \nabla \left( u_J - u_J^i - \sum_{j=0}^{J-1} \lambda_j^i \rho_j^i \right) \right\|^2 - 2\lambda_J^i \left[ (f, \rho_J^i) - \left( \mathbf{K}^{\frac{1}{2}} \nabla \left( u_J^i + \sum_{j=0}^{J-1} \lambda_j^i \rho_j^i \right), \nabla \rho_J^i \right) \right] + \left( \lambda_J^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_J^i\| \right)^2 \\ &= \left\| \mathbf{K}^{\frac{1}{2}} \nabla \left( u_J - u_J^i - \sum_{j=0}^{J-1} \lambda_j^i \rho_j^i \right) \right\|^2 - (\lambda_J^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_J^i\|)^2 = \dots = \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - \sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2 \\ &= \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - (\eta_{\text{alg}}^i)^2. \end{aligned}$$

## Main results

### Theorem ( $p$ -robust reliable and efficient bound on the algebraic error)

Let  $u_J^i \in V_J^p$  be arbitrary. Let  $\eta_{\text{alg}}^i$  be the associated a posteriori estimator on the algebraic error. Then, there holds:

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i \quad \text{and} \quad \eta_{\text{alg}}^i \geq \beta \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|, \quad 0 < \beta(\kappa_{\mathcal{T}}, J, d, \mathbf{K}) < 1.$$

### Theorem ( $p$ -robust error contraction of the multilevel solver)

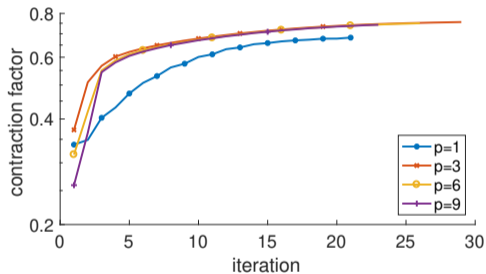
For  $u_J^i \in \mathbf{V}_J^p$ , let  $u_J^{i+1} \in \mathbf{V}_J^p$  be constructed from  $u_J^i$  using one step of the solver. There holds:

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\| \leq \alpha \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|, \quad \alpha = \sqrt{1 - \beta^2}.$$

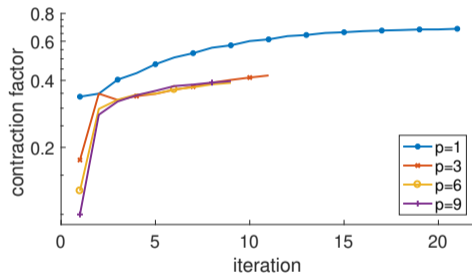
- Remark:**
- The dependence on  $J$  is at most *linear* under minimal  $H^1$ -regularity.
  - Complete *independence* from  $J$  is obtained in  $H^2$ -regularity setting.
  - $\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^{i+1})\|^2 = \|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 - (\eta_{\text{alg}}^i)^2$ .

## Numerical results

**L-shape problem**,  $J = 3$ , and mesh hierarchy  $p_j = 1$  (left) and  $p_j = p$  (right),  $j \in \{1, \dots, J - 1\}$



$1 \rightarrow 1, p$



$1, p \rightarrow p$

Stopping criterion: 
$$\frac{\|F_J - \mathbb{A}_J U_J^{i_s}\|}{\|F_J\|} \leq 10^{-5} \frac{\|F_J - \mathbb{A}_J U_J^0\|}{\|F_J\|}.$$

The mesh hierarchies here are obtained from  $J$  uniform refinements of an initial Delaunay mesh  $\mathcal{T}_0$ .

		$H^2$ -regular								$H^1$ -regular																			
		Sine $\mathbf{K}=I$				Peak $\mathbf{K}=I$				L-shape $\mathbf{K}=I$				Checkerboard $\mathbf{K}=I$				Skyscraper $\mathcal{J}(\mathbf{K})=O(10^6)$				Skyscraper $\mathcal{J}(\mathbf{K})=O(1)$				Skyscraper $\mathcal{J}(\mathbf{K})=O(10^7)$			
		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$		$1 \rightarrow 1, p$		$1, p \rightarrow p$					
$J$	$p$	DoF	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$	$i_s$				
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19	19	19	19	19	19	19	19	19	19				
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13	31	13	31	13	31	13	31	13					
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11	28	11	28	11	28	11	28	11					
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10	26	10	26	10	26	10	26	10					
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19				
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11	30	11	30	11	30	11	30	11					
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10	27	10	27	10	27	10	27	10					
	9	$5e^6$	<b>32</b>	14	31	15	23	<b>9</b>	22	9	23	9	25	9	25	9	25	9	25	9	25	9	25	9					

Numerical  $\mathbf{K}$ - and  $J$ -robustness is observed even in low-regularity cases.

We compare our methods with [2,3,4] in terms of the number of iterations (and CPU times<sup>5</sup>).

$J$	$p$	~MG(0,1) -bJ $1, p \rightarrow p$		~MG(0,adapt) -bJ (wRAS) $1 \nearrow p$		PCG(MG (3,3)-bJ) $p \rightarrow p$		MG(1,1)- PCG(iChol) $1 \nearrow p$		MG(0,1)- bGS $1 \rightarrow 1, p$		MG(3,3)- GS $1 \nearrow p$	
		$i_s$	time	$i_s$	time	$i_s$	time	$i_s$	time	$i_s$	time	$i_s$	time
4	1	19	0.12 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	11	2.07 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	9	20.19 s	4	12.54 s	3	38.40 s	>80	>6.87m	9	34.78 s	6	14.44 s
	9	9	2.13m	3	49.84 s	2	2.24m	>80	>23.08m	8	1.72m	9	1.21m

}
not  $p$ -robust
}
not  $p$ -robust

<sup>2</sup>Antonietti et al. *J. Sci. Comput.* 2017.

<sup>3</sup>Botti et al. *J. Comput. Phys.* 2017.

<sup>4</sup>Schöberl. "C++11 Implementation of Finite Elements in NGSolve". *Tech. report.* 2014.

<sup>5</sup>The experiments were run on one Dell C6220 dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée "NEF" computation cluster, however, in a sequential Matlab script.

Peak,  $1, p \rightarrow p$

$J$	$p$	$i_s$	$J$	$p$	$i_s$	$J$	$p$	$i_s$
4	1	14	8	1	16	16	1	16
	3	11		3	9		3	9
	6	9		6	8		6	8
	9	8		9	8		9	9

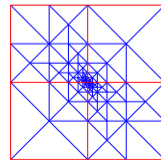
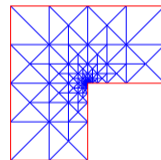
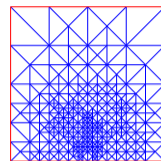
L-shape,  $\mathbf{K} = I$ ,  $1, p \rightarrow p$

$J$	$p$	$i_s$	$J$	$p$	$i_s$	$J$	$p$	$i_s$
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4

Checkerboard,  $\mathcal{J}(\mathbf{K}) = O(10^6)$ ,  $1, p \rightarrow p$

$J$	$p$	$i_s$	$J$	$p$	$i_s$	$J$	$p$	$i_s$
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15

These low-regularity test cases indicate the possibility of  $J$ -dependence, in accordance with the theoretical results.



## Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \approx (\eta_{\text{alg}}^i)^2 = \underbrace{\sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2}_{\textcircled{1} \text{ localization by levels}} = \underbrace{\|\mathbf{K}^{\frac{1}{2}} \nabla \rho_0^i\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2}_{\textcircled{2} \text{ localization by patches}}$$

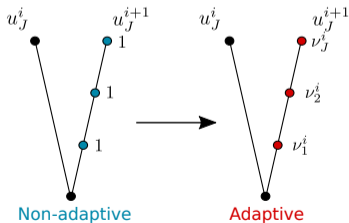
① Adaptive number of post-smoothing steps

② Adaptive local smoothing

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$$\|\mathbf{K}^{\frac{1}{2}} \nabla(u_J - u_J^i)\|^2 \approx (\eta_{\text{alg}}^i)^2 = \underbrace{\sum_{j=0}^J (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|)^2}_{\text{① localization by levels}} = \underbrace{\|\mathbf{K}^{\frac{1}{2}} \nabla \rho_0^i\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2}_{\text{② localization by patches}}$$

① Adaptive number of post-smoothing steps

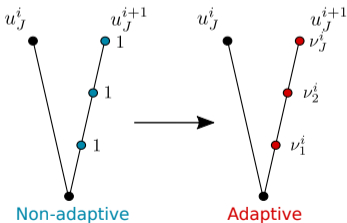


② Adaptive local smoothing

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

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① Adaptive number of post-smoothing steps

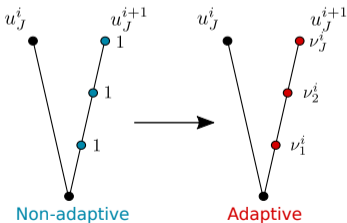


② Adaptive local smoothing

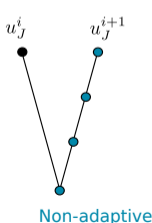
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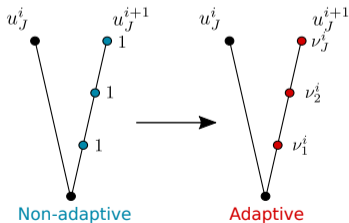
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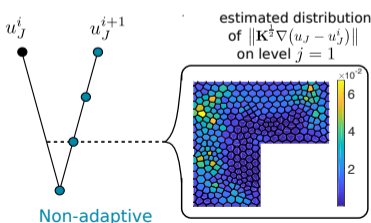
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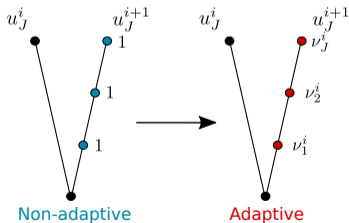
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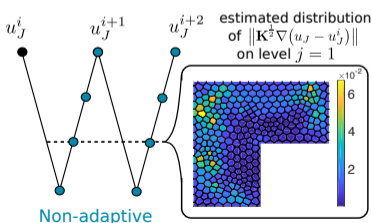
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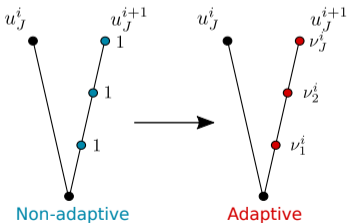
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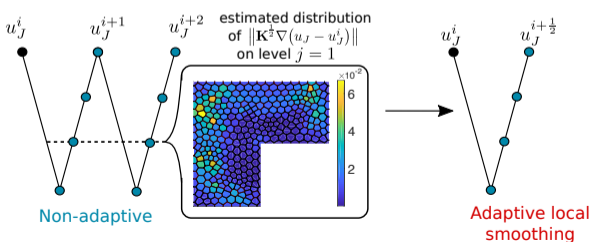
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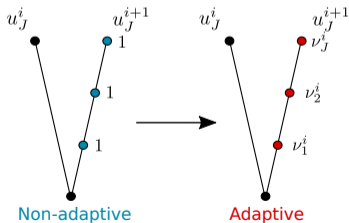
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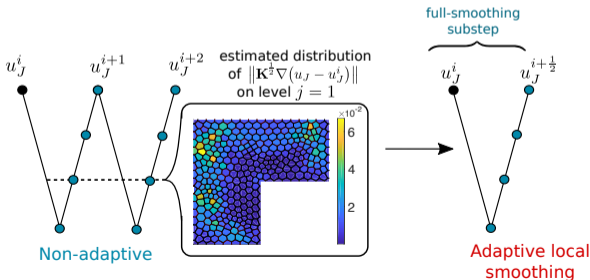
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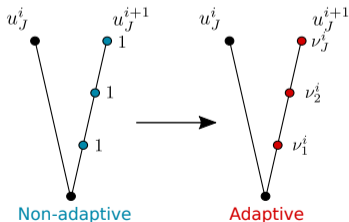
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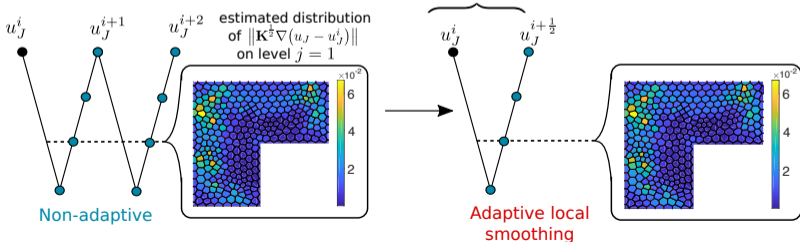
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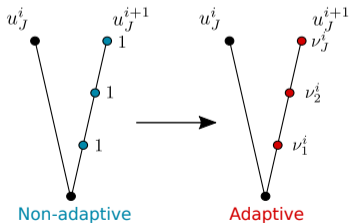
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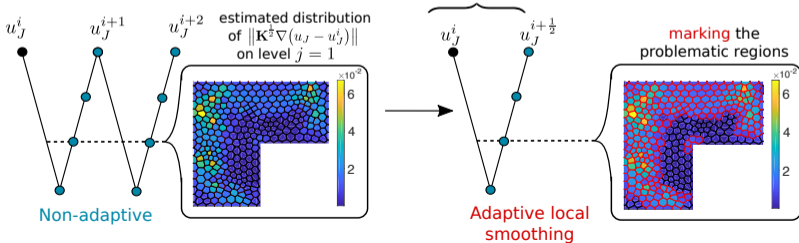
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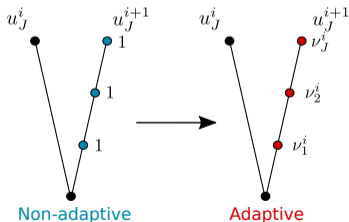
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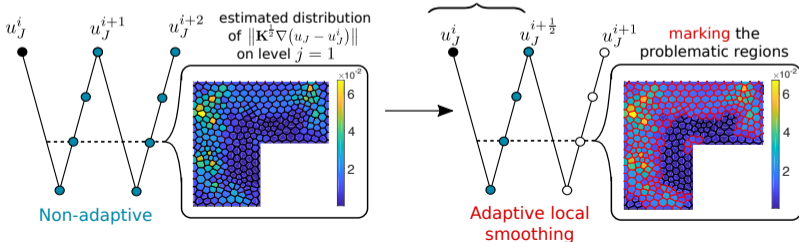
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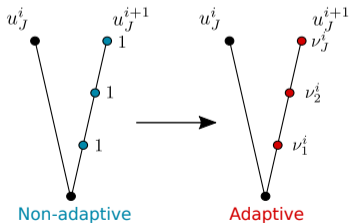
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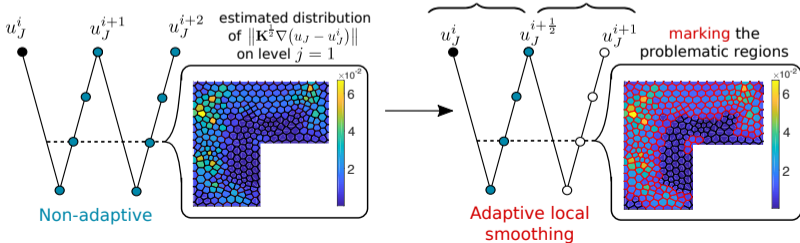
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② Adaptive local smoothing



## Conclusion

We presented:

- A  **$p$ -robustly efficient** a posteriori algebraic error estimator.
- A  **$p$ -robust contractive** multigrid solver which is steered by the a posteriori estimator.
- Optimal level-wise **step-sizes** used in the multigrid error correction stage.
- Two **adaptive** variants of a  **$p$ -robust contractive** a-posteriori-steered multigrid solver:
  - ▶ Approach 1: **adaptive number of smoothing steps** on each level.
  - ▶ Approach 2: applying **adaptive local smoothing** on all patches with increased algebraic error estimator.

Future work would explore:

- how to gain robustness with respect to the number of mesh levels  $J$ .
- how to use our approach to construct  **$hp$ -robust** algebraic solvers.

# Thank you for your attention!

- 📄 Miraçi, Papež, and Vohralík. A multilevel algebraic error estimator and the corresponding iterative solver with  $p$ -robust behavior. *SIAM J. Numer. Anal.* (2020)
- 📄 Miraçi, Papež, and Vohralík. A-posteriori-steered  $p$ -robust multigrid with optimal step-sizes and adaptive number of smoothing steps. *SIAM J. Sci. Comput.* (2021). DOI 10.1137/20M1349503.
- 📄 Miraçi, Papež, and Vohralík. Contractive local adaptive smoothing based on Dörfler's marking in a-posteriori-steered  $p$ -robust multigrid solvers. *Comput. Methods Appl. Math.* (2021)

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Institute of Analysis  
and Scientific Computing

NumPDEs group

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<https://www.asc.tuwien.ac.at/~amiraci>

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